A Brief Summary of Body of My Works

My main interests lie in Representation Theory of finite dimensional semisimple groups and their Kac-Moody analogs and the geometry and topology of their flag varieties. In addition, I have been interested in the moduli of semistable principal G-bundles over curves in its connection to Verlinde formula for the dimension of the space of conformal blocks and also the G-analog of the classical Hermitian eigenvalue problem, where G is any complex semisimple group.

The following is a brief description of some of my main results.

I extended the Laplacian calculation of Kostant (in the finite case) and Garland (in the affine case) to any symmetrizable Kac-Moody algebras and used this to develop a Hodge theory for their flag varieties (1984).

Motivated by Hodge theory, I jointly with Kostant introduced a ring (now commonly known as the *Kostant-Kumar nil-Hecke ring* or just the *nil-Hecke ring*) and used this to give a purely algebraic model for the cohomology of flag varieties associated to any semisimple group, more generally, any Kac-Moody group (1986). In particular, we gave an expression for the cup product of any two Schubert cohomology classes. This is the only known 'explicit' formula for the cup product which works in general. This work has been used in several important works including in uniformly determining the quantum cohomology of flag varieties. (Arabia extended this work to the equivariant cohomology in 1989.)

Jointly with Kostant I obtained similar results for the equivariant K-theory of any flag variety (1990). Again, this work has extensively been used in later developments in the subject. More recently, I came back to the study of equivariant K-theory of flag varieties and made a 'positivity' conjecture jointly with Graham (2008) for the dual Schubert basis. This conjecture has been established by Anderson-Griffeth-Miller (2011) in the finite case. In the general symmetrizable Kac-Moody case, the conjecture has been established by me (2017). I also proved an analogous positivity result for the Schubert basis jointly with Baldwin in the general symmetrizable Kac-Moody case (2017). In particular, this settles a conjecture due to Lam-Schilling-Shimozono (2010).

I proved the Demazure character formula for an arbitrary Kac-Moody group G by a new algebro-geometric method (1987). Recall that the Demazure character formula explicitly gives the character of the B-submodule generated by any extremal weight vector in an integrable highest weight G-module, where B is a Borel subgroup of G. This work has numerous applications, e.g., this is fundamental to the proof of Verlinde formula (see below). Moreover, I used this character formula to extend the celebrated Weyl-Kac character formula for symmeterizable Kac-Moody algebras to an arbitrary (not necessarily symmeterizable) Kac-Moody algebras.

In the nineteen sixties, Parthasarathy-Ranga Rao-Varadarajan gave an important conjecture on the decomposition of tensor product of two representations. Their conjecture asserted that for any two irreducible G-modules $V(\lambda), V(\mu)$ with highest weights λ, μ respectively, and any Weyl group element w, the tensor product $V(\lambda) \otimes V(\mu)$ has a component with extremal weight $\lambda + w\mu$, where G is a complex semisimple group. Subsequently, Kostant strengthened this conjecture by identifying the 'first ocurrence' of this piece in the tensor product. Now, I proved this conjecture by employing a mixture of algebro-geometric and representation theoretic techniques (1988). This result has found many applications by several mathematicians in different areas.

I (in collaboration with Ginzburg) determined the cohomology of quantized enveloping algebras at roots of unity (1993). This result has been used in several subsequent works including a fundamental work by Arkhipov-Bezrukavnikov-Ginzburg.

I (jointly with Vergne) introduced and studied the equivariant cohomology of manifolds with generalized coefficients (1993). We proved a version of Kunneth theorem and localization theorem for this new cohomology. This new cohomology has been used by others, especially in the study of the index of transversally elliptic operators.

A very important conjectural formula came out of Mathematical Physics, known as the Verlinde formula given by E. Verlinde (1988). The Verlinde formula attracted a lot of attention from mathematicians when it was heuristically realized that for the Wess-Zumino-Witten model associated to a simple algebraic group G over $\mathbb{C}$ (which is a particular Rational Conformal Field Theory), the space of conformal blocks admits an interpretation as the space of generalized theta functions, which is the space of holomorphic sections of the theta bundle on the moduli space $\mathcal{M}_G(\Sigma)$ of semistable principal G-bundles on a smooth projective curve Σ. This interpretation was rigorously established by Kumar-Narasimhan-Ramanathan (1994) (and also independently by Beauville-Laszlo and Faltings). Now, by a result of Tsuchiya-Ueno-Yamada, the dimension of the space of conformal blocks is given by the Verlinde formula. Thus, by the above result of Kumar-Narasimhan-Ramanathan (and others), Verlinde's conjectural formula for the dimension of the space of generalized theta functions gets established. A Séminaire Bourbaki talk by C. Sorger on 'La formule de Verlinde' during November, 1994 was devoted to these works. Building upon the above work, I (jointly with Narasimhan in 1997 and in another paper with Boysal in 2005) determined precisely the Picard group of the projective variety $\mathcal{M}_G(\Sigma)$.

I jointly with Hong have begun a systematic study to extend the theory of conformal blocks to a 'twisted' setting where the curve Σ is replaced by a finite Galois cover of Σ and the affine Kac-Moody group by the twisted affine Kac-Moody group (2019). Specifically, we study the spaces of twisted conformal blocks attached to a Γ-curve Σ with marked Γ-orbits and an action of Γ on a simple Lie algebra $\mathfrak{g}$, where Γ is a finite group. We prove that if Γ stabilizes a Borel subalgebra of $\mathfrak{g}$, then Propagation Theorem and Factorization Theorem hold. We endow a projectively flat connection on the sheaf of twisted conformal blocks attached to a smooth family of pointed Γ-curves; in particular, it is locally free. We also prove that the sheaf of twisted conformal blocks on the stable compactification of Hurwitz stack is locally free. Let $\mathscr{G}$ be the parahoric Bruhat-Tits group scheme on the quotient curve Σ/Γ obtained via the Γ-invariance of Weil restriction associated to Σ and the simply-connected simple algebraic group G with Lie algebra $\mathfrak{g}$. We prove that the space of twisted conformal blocks can be identified with the space of generalized theta functions on the moduli stack of quasi-parabolic $\mathscr{G}$-torsors on Σ/Γ when the level c is divisible by $|\Gamma|$ (establishing a conjecture due to Pappas-Rapoport). Further, we (jointly with J. Hong) proved that the space of twisted conformal blocks is isomorphic to the space associated to a quotient group of Γ acting on $\mathfrak{g}$ by diagram automorphisms and acting on a quotient of Σ (2022). Under some mild conditions on ramification types, we prove that calculating the dimension of twisted conformal blocks can be reduced to the situation when Γ acts on $\mathfrak{g}$ by diagram automorphisms and covers of $\mathbb{P}^1$ with 3 marked points. Assuming a twisted analogue of Teleman's vanishing theorem of Lie algebra homology, we derive an analogue of the Kac-Walton formula and the Verlinde formula for general Γ-curves (with mild restrictions on ramification types). In particular, if the Lie algebra $\mathfrak{g}$ is not of type D_4, there are no restrictions on ramification types.

I determined the precise singular locus of any Schubert variety in any flag variety in terms of the nil-Hecke ring (1996). Partial results in this direction were given earlier by several mathematicians. However, my criterion is a uniform criterion to determine such a locus and it has widely been used. It was extended by Juteau-Williamson in char. p.

An important breakthrough in understanding the geometry of Schubert varieties was the introduction by Mehta-Ramanathan of the notion of Frobenius split varieties and the result that the flag varieties G/P are Frobenius split. I (jointly with Lauritzen and Thomsen) have shown that the cotangent bundle of the flag varieties is Frobenius split for any good prime p (1999). This has provided several uniform and sharp results in the area.

In addition, I (in collaboration with Littelmann) have given a complete and self-contained representation theoretic approach to the Frobenius splitting method for G/P (2002). This new approach provides the Frobenius splitting very explicitly at the level of representations. The geometric Frobenius method (in char. $k = p > 0$) has been replaced by Lusztig's Frobenius maps for quantum groups at roots of unity (which exist not only for primes but any integer $\ell > 1$).

I (jointly with Thomsen) gave a conjectural generalization (2003) of the famous $n!$ theorem due to Haiman for any simple Lie algebra in terms of the geometry of principal nilpotent pairs.

The classical Hermitian eigenvalue problem asks the possible eigenvalues of the sum $A + B$ of two Hermitian matrices A and B under the constraint that the eigenvalues of A and B are fixed. The first nontrivial result towards this problem was obtained by H. Weyl (1912). This problem continued to attract the attention of several mathematicians during the last century and it was finally solved by combining the works of Horn (1962), Klyachko (1998), Knutson-Tao (1999) and Belkale (2001). This work was generalized for other complex reductive groups by Berenstein-Sjamaar (2000) and Kapovich-Leeb-Millson (2005). However, their work did not provide an optimal solution to the problem. Their system of inequalities had redundancies for any group G of type different from A_n. Now, I (jointly with Belkale) came up with a new product in the cohomology of flag varieties (now known as the *Belkale-Kumar product*) and used this to give a (in general much smaller) set of inequalities solving the eigenvalue problem (2006). It was shown by Ressayre that these smaller set of inequalities given by Belkale-Kumar provides an optimal solution of the eigenvalue problem for any reductive group G (2010). Thus, in some sense, my work with Belkale concluded this fundamental problem at a theoretical level, though an 'explicit' determination of the eigencone for general G is not yet fully achieved. Vergne, Berline and Walter have made some progress in this direction. Continuing our work, I (in collaboration with P. Belkale) determined the eigencone for the symplectic and odd orthogonal groups in terms of that of the ambient special linear group (2010). Moreover, I (jointly with Belkale and Ressayre) extended Fulton's conjecture to an arbitrary group (2011). There was a Séminaire Bourbaki talk in November, 2011 by M. Brion on 'Restriction de reprsentations et projection d'orbites co-adjointes' (d'après Belkale, Kumar, Ressayre) covering some of these works. I (jointly with Belkale) also solved the corresponding multiplicative eigenvalue problem for any compact group by giving an optimal set of inequalities to determine the corresponding polytope in terms of the quantum cohomology of flag varieties (and its deformed version introduced by us) (2016).

Cachazo-Douglas-Seiberg-Witten gave a conjecture on the structure of the conformal algebra (associated to any complex simple Lie algebra $\mathfrak{g}$) arising in Supersymmetric Gauge Theory (2002). They proved their conjecture for the special linear Lie algebra $\mathfrak{g} = sl_N$ and subsequently Witten and Etingof proved the conjecture for other classical Lie algebras. Now, I proved a substantial part of the conjecture for any simple $\mathfrak{g}$ (2008). I came up with a uniform proof for any simple Lie algebra $\mathfrak{g}$ using the geometry and topology of loop groups. I further generalized the result to the symmetric spaces (2010).

In an attempt to understand Valiant's conjecture and its strengthened version due to Mulmuley-Sohoni in Geometric Complexity Theory, I studied the geometry of the orbit closure of determinant and permanent and showed that they are *not* normal (2013). I further studied the representations supported by these orbit closures and connected its study to a famous 'Latin Square Conjecture' from 1992 due to Alon-Tarsi (2015).

I began a study to connect the cohomology of flag varieties G/P under cup product with the representation ring of L, where G is any semisimple group and P is any parabolic subgroup with L as its Levi component (2016). This provides a substantial generalization of the classical result connecting the cohomology algebra of the Grassmannians with the representation ring of general linear groups. I (in collaboration with Rogers) have determined this connection explicitly for the symplectic as well as odd orthogonal groups and their maximal parabolic subgroups (2020). Further, in collaboration with Jiale Xie, I have determined this connection explicitly for the even orthogonal groups as well as all the exceptional groups (except for E_8) and their maximal parabolic subgroups (2022).

In a joint work with Brown, I initiated a study of the saturated tensor cone for the integrable highest weight modules of symmetrizable Kac-Moody Lie algebras (2014). We solved the problem for affine SL(2) and gave a set E of inequalities for general symmetrizable Kac-Moody Lie algebras and conjectured that these provide an irredundent set of inequalities determining the saturated tensor cone. Our set E of inequalities has been shown to be sufficient by Ressayre (2017) for affine Kac-Moody Lie algebras. Further, in a joint work with Ressayre, I proved that the set E of inequalities provide a necessary system of inequalities for any symmetrizable Kac-Moody Lie algebras (2019). Thus, combining the above works, the problem of determining the saturated tensor cone for the affine Kac-Moody Lie algebras gets solved.

In a joint work with Rimanyi and Weber (2020), we introduced a new notion in elliptic Schubert calculus: the (twisted) Borisov-Libgober classes of Schubert varieties in general homogeneous spaces G/P. Our approach leads to a simple recursions for the elliptic classes of Schubert varieties. Comparing this recursion with R-matrix recursions of the so-called elliptic weight functions of Rimanyi-Tarasov-Varchenko, we prove that weight functions represent elliptic classes of Schubert varieties.

Let $\mathfrak{g}$ be an affine Kac-Moody Lie algebra and let λ, μ be two dominant integral weights for $\mathfrak{g}$. In a joint work with Jeralds (2021), we prove that under some mild restriction, for any positive root β, $V(\lambda) \otimes V(\mu)$ contains $V(\lambda + \mu - \beta)$ as a component, where $V(\lambda)$ denotes the integrable highest weight (irreducible) $\mathfrak{g}$-module with highest weight λ. This extends my corresponding result from the case of finite dimensional semisimple Lie algebras to the affine Kac-Moody Lie algebras. Then, we prove the corresponding geometric results including the higher cohomology vanishing on the $\mathcal{G}$-Schubert varieties in the product partial flag variety $\mathcal{G}/\mathcal{P} \times \mathcal{G}/\mathcal{P}$ with coefficients in certain sheaves coming from the ideal sheaves of $\mathcal{G}$-sub Schubert varieties, where $\mathcal{G}$ is the affine Kac-Moody group associated to the Lie algebra $\mathfrak{g}$ and $\mathcal{P}$ is a standard parabolic subgroup.

I determined the Lie subalgebra $\mathfrak{g}_{\rm nil}$ of any Borcherds symmetrizable generalized Kac-Moody Lie algebra $\mathfrak{g}$ generated by ad-locally nilpotent elements and proved that it is 'essentially' the same as the Levi subalgebra of $\mathfrak{g}$ with its simple roots precisely the real simple roots of $\mathfrak{g}$ (2021).

G. Lusztig asked certain questions (which he characterized as his 'expectations') relating the modular representation theory of a simple algebraic group over char. p with the representation theory of multi-loop algebra of $\mathfrak{g}$, where $\mathfrak{g}$ is the complex Lie algebra associated to the group G. In a recent work (2022) I prove that these questions have negative answers.

I have authored three books: *Kac-Moody groups, their flag varieties and representation theory* (2002); *Frobenius splitting methods in geometry and representation theory* (2004) (jointly with M. Brion); and *Conformal Blocks, Generalized Theta Functions and the Verlinde Formula* (published January, 2022 by the Cambridge University Press). All three of these are the very first books on the subject. The first two books have become standard references.