CHARACTER OF IRREDUCIBLE REPRESENTATIONS RESTRICTED TO FINITE ORDER ELEMENTS - AN ASYMPTOTIC FORMULA

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Abstract: Let G be a connected reductive group over the complex numbers and let $T \subset G$ be a maximal torus. For any $t \in T$ of finite order and any irreducible representation $V(\lambda)$ of G of highest weight λ, we determine the character $\mathrm{ch}(t, V(\lambda))$ by using the Lefschetz Trace Formula due to Atiyah-Singer and explicitly determining the connected components and their normal bundles of the fixed point subvariety $(G/P)^t \subset G/P$ (for any parabolic subgroup P). This together with Wirtinger's theorem gives an asymptotic formula for $\mathrm{ch}(t, V(n\lambda))$ when n goes to infinity.

1. Introduction

Let (G, B, T) be a triple consisting of a connected reductive algebraic group G over $\mathbb{C}$, a Borel subgroup B containing a maximal torus T. Their $\mathbb{C}$-points are denoted by the same symbols, i.e., $T = T(\mathbb{C})$ etc. Let $\lambda : T \to \mathbb{C}^\times$ be a dominant integral weight, and $(\pi_\lambda, V(\lambda))$ the associated highest weight representation of G with character $\Theta_\lambda : G \to \mathbb{C}$. The character $\Theta_\lambda : G \to \mathbb{C}$ is a function of λ as well as that of $g \in G$ which is determined by its restriction to T. In this paper, we study the asymptotic behaviour of $\Theta_\lambda(t)$ as a function of λ for a fixed $t \in T$ of finite order. For example, if $t = 1$, $\Theta_\lambda(1) = \dim \pi_\lambda$, which by the Weyl dimension formula is a polynomial function in λ of degree equal to $\dim(G/\mathrm{B})$. At the other extreme, if $t \in T$ is regular (and of finite order), then $\Theta_\lambda(t)$ is a piecewise constant function of λ, an assertion which the reader will immediately recognize as a consequence of the Weyl character formula. The aim of this paper is to prove, more generally, that if $t \in T$ is an element of finite order then $\Theta_\lambda(t)$ is a piecewise polynomial function in λ of degree which is bounded above by the dimension of the maximal unipotent subgroup of the centralizer $G^t = Z_G(t)$, a possibly disconnected reductive subgroup of G. Furthermore, in some situations, we prove that the bound is achieved (see Corollary 1.2).

The paper is inspired by some works of the second author with other collaborators, cf. [P], [NPP], as well as [AK] for classical groups, in which they calculate $\Theta_\lambda(t)$ for $t \in T$, a power of the Coxeter conjugacy class (which is the unique regular conjugacy class in the derived group of G of minimal order in the adjoint group, equal to the Coxeter number of the corresponding Weyl group), and find that $\Theta_\lambda(t)$ is either zero or is, up to a sign, the dimension of an irreducible representation of the identity component $(G^t)^o = Z_G(t)^o$ of the centralizer $G^t = Z_G(t)$. The present paper is less precise on concrete character values, but gives an asymptotic formula for $\Theta_\lambda(t)$ for all $t \in T$ of finite order (see Corollary 1.2). Precise calculation of the character values at elements of order 2 (for classical groups and G_2) is made in a recent Ph. D. thesis of Karmakar, see [Ka].

Let $\mathcal{L}(\lambda) = \mathcal{L}_P(\lambda) := G \times^P \mathbb{C}_{-\lambda} \to X_P$ be the homogeneous line bundle over $X_P = G/P$ associated to a character λ of P, where $P \supset B$ is a (standard) parabolic subgroup.

The present paper uses the Lefschetz Trace Formula due to Atiyah-Singer for the action of $t \in T$ on a homogeneous line bundle over G/B, and more generally on X_P to calculate the character $\Theta_\lambda(t)$. Specifically, our main theorem of this note is the following result (cf. Theorem 3.1).

Fix a dominant character λ of T and let $P = P_\lambda$ be the unique standard parabolic subgroup of G such that $\mathcal{L}(\lambda)$ is an ample line bundle over X_P. For any $t \in T$ of finite order, let S_t be the (finite) subgroup of T generated by t. In the following, W^P denotes the set of minimal coset representative in W/W_P, where W is the Weyl group of G and W_P is the Weyl group of P (which is by definition the Weyl group of its Levi component). Let $(G^t)^o$ denote the identity component of the centralizer G^t.

Theorem 1.1. *For any $t \in T$ of finite order and any integer $n \geqslant 0$,*

$$\Theta_{n\lambda}(t^{-1}) = \sum_{v \in Y_t} D_v^t(n, \lambda),$$

where

$$D_v^t(n,\lambda) = \sum_{k \geqslant 0} \int_{X_P^t(v)} \frac{c_1\left(\mathcal{L}(\lambda)_{|_{X_P^t(v)}}\right)^k \frac{n^k}{k!} e^{-nv\lambda}(t) \cdot \mathrm{td}(X_P^t(v))}{\prod_{\alpha \in (vR_P^+) \setminus R((G^t)^o)} [1 - e^{\alpha}(t) e^{-c_1(\overline{\mathcal{L}}(\alpha)_{|_{X_P^t(v)}})}]},$$

where (as in Proposition 2.1) $Y_t = Y_{S_t}$ denotes the set $\left\{v \in W^P : B^{S_t} \cdot vP/P = vP/P\right\}$, R_P^+ is the set of roots of the unipotent radical of P, td *is the Todd genus of the tangent bundle, the connected component $X_P^t(v) = X_P^{S_t}(v)$ of $X_P^{S_t}$ corresponds to the Weyl group element $v \in Y_{S_t}$ via Proposition 2.1, and $\overline{\mathcal{L}}(\alpha)_{|_{X_P^t(v)}}$ is the $(G^t)^o$-equivariant line bundle over $X_P^t(v)$ such that the fiber over vP/P has T-weight $-\alpha$.*

This theorem takes a simpler form when $P = B$ since in this case each $X_P^t(v)$ is isomorphic with the full flag variety of $Z_G(t)^o$ (cf. Corollary 3.6).

Let d_v be the complex dimension of $X_P^t(v)$ and let r be the order of $t \in T$. Consider the function

$$\chi_\lambda^t : \mathbb{Z}_{\geqslant 0} \to \mathbb{C}, \quad n \mapsto \Theta_{n\lambda}(t^{-1}).$$

As an immediate consequence of the above theorem and Wirtinger Theorem, we obtain the following corollary (cf. Corollaries 3.3 and 3.5).

Corollary 1.2. *For any fixed $0 \leq p < r$, $\chi^t_{\lambda_{|\{n \equiv p (mod\, r)\}}}$ is a polynomial function in n of degree $\leq$* $\max_{v \in Y_t} \{d_v\}$.

Moreover, the coefficient of n^{d_v} in $D_v^t(n,\lambda)$ restricted to $\{n \equiv p (mod\, r)\}$ is equal to

$$\frac{1}{d_v!} \int_{X_P^t(v)} c_1\left(\mathcal{L}(\lambda)_{|_{X_P^t(v)}}\right)^{d_v} \frac{e^{-pv\lambda}(t)}{\prod_{\alpha \in \left(vR_P^+\right) \setminus R\left((G^t)^o\right)} (1 - e^{\alpha}(t))}. \tag{1}$$

In particular, if there is a unique $v_o \in Y_t$ such that $d_{v_o} = \max_{v \in Y_t} \{d_v\}$. Then, for any fixed $0 \leq p < r$,

$$(\chi_\lambda^t)_{|_{n \equiv p (mod\ r)}}$$

is a polynomial function of degree exactly equal to d_{v_o}.

As a consequence of the above corollary, we obtain the following result (cf. Theorem 4.1).

Theorem 1.3. *Assume that $t \in T$ is of order 2. Then, the function $\left(\chi_\lambda^t\right)_{|\{n \in 2\mathbb{Z}_{\geq 0}\}}$ is a polynomial function of degree exactly equal to $d := max_{v \in Y_t} \{d_v\}$.*

To prove the above results, we prove several assertions of independent interest regarding the relationship of the fixed points of the action of $t \in T$ on G/P with the flag varieties of the identity component $Z_G(t)^o$ of $Z_G(t)$: each connected component of the fixed point set is a homogeneous space for $Z_G(t)^o$ (cf. Section 2, especially Proposition 2.1 and Corollary 2.2). In the case $P = B$, each connected component of the fixed point set X^t is isomorphic with the full flag variety of $Z_G(t)^o$ and there are exactly $\frac{\#W}{\#W(Z_G(t)^o)}$ many connected components (cf. Corollary 2.2 and Lemma 2.5).

After we completed this work, G. Lusztig pointed out his article 'Michael Atiyah and Representation Theory' in [H], where he mentioned an application of the Lefschetz trace formula due to Atiyah-Singer [AS] to get an explicit formula for the character of any irreducible representation V of G at an element t of finite order of G as a sum of contributions of the connected components of the fixed point set of t on the flag manifold of G. In this note we determine the connected components explicitly and thus work out such an explicit formula for the character $\mathrm{ch}(t, V)$.

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2. Fixed points of a flag variety under the action of a subgroup of torus

Let $T \subset B \subset P \subset G$ be a maximal torus, a Borel subgroup, a standard parabolic in a connected reductive group G over $\mathbb{C}$. Let $S \subset T$ be a subgroup (not necessarily connected) and let $X_P = G/P$ be the partial flag variety. Then, S acts on X_P via left multiplication. Let X_P^S be the fixed subvariety. Then, X_P^S is smooth. Let W^P be the set of minimal coset representative in W/W_P, where W is the Weyl group of G and W_P is the Weyl group of P (which is by definition the Weyl group of its Levi component). Let $(G^S)^o$ be the identity component of the fixed subgroup G^S under the conjugation action of S on G.

Proposition 2.1. *With the notation as above, the set of connected components of X_P^S is bijectively parameterized by*

$$Y_S := \left\{v \in W^P : \ B^S \cdot vP/P = vP/P\right\}.$$

For $v \in Y_S$, let $X_P^S(v)$ be the corresponding connected component of X_P^S. Then, the morphism

$$\hat{\phi}_v : (G^S)^o \longrightarrow X_P^S, \quad g \mapsto g \cdot vP/P$$

descends to a $(G^S)^o$-equivariant variety isomorphism

$$\phi_v : (G^S)^o / \left((G^S)^o \cap (vPv^{-1})\right) \xrightarrow{\sim} X_P^S(v).$$

Observe that the Borel subgroup $B^S \subset (G^S)^o \cap (vPv^{-1})$, since $v \in Y_S$. Moreover, since X_P^S is smooth, its connected components coincide with its irreducible components.

Proof. Take a connected component $\mathcal{Z} \subset X_P^S$. It is clearly B^S-stable and hence, by Borel Fixed Point Theorem, contains a B^S-fixed point vP/P (for some $v \in W^P$) since a B^S-fixed point, in particular, is a T-fixed point. We claim that the map

$$\hat{\phi}_v : \ (G^S)^o \to X_P^S, \quad g \mapsto g \cdot vP/P$$

induces an isomorphism

$$\phi_v : (G^S)^o / \left((G^S)^o \cap (vPv^{-1})\right) \xrightarrow{\sim} \mathcal{Z} :$$

By the Bruhat decomposition, we get

$$X_P = \sqcup_{w \in W^P} BwP/P,$$

and hence

$$X_P^S = \sqcup_{w \in W^P} B^S wP/P. \tag{2}$$

Since $vP/P \in \mathcal{Z}$, we get

$$(G^S)^o \cdot vP/P \subset \mathcal{Z}.$$

Moreover, the isotropy subgroup of v in $(G^S)^o$ is clearly $(G^S)^o \cap (vPv^{-1}) \supset B^S$, thus we get a $(G^S)^o$-equivariant embedding

$$\phi_v : (G^S)^o / \Big((G^S)^o \cap (vPv^{-1})\Big) \hookrightarrow \mathcal{Z}. \tag{3}$$

We now prove that any $(G^S)^o$-orbit in X_P^S is closed: By identity (2), any $(G^S)^o$-orbit A in X_P^S is a certain union $\sqcup_{w\in\theta(A)} B^S wP/P$, for some subset $\theta(A) \subset W^P$. Take $v_o \in \theta(A)$ of smallest length. Then, we prove that

$$B^S \cdot v_o P/P = v_o P/P. \tag{4}$$

For if (4) were false, take $\mathcal{U}_\alpha \subset B^S \subset (G^S)^o$ such that

$$\mathcal{U}_\alpha \cdot v_o P/P \supsetneq v_o P/P,$$

where $\mathcal{U}_\alpha$ is the one-parameter additive group corresponding to a positive root α. In particular

$$v_o^{-1}\mathcal{U}_\alpha v_o \not\subset P \text{ and hence } v_o^{-1}\mathcal{U}_{-\alpha} v_o \subset \mathcal{U}^P, \tag{5}$$

where $\mathcal{U}^P$ is the unipotent radical of P, which gives

$$\mathcal{U}_{-\alpha} v_o P/P = v_o P/P.$$

By (5), $v_o^{-1}\alpha$ is a negative root and hence

$$s_\alpha v_o < v_o \quad \text{by [K, Lemma 1.3.13]}, \tag{6}$$

where $s_\alpha \in W$ is the reflection through α. Since $\mathcal{U}_\alpha \subset B^S \subset (G^S)^o$, then so is $\mathcal{U}_{-\alpha} \subset (G^S)^o$ and hence the subgroup $\langle \mathcal{U}_\alpha, \mathcal{U}_{-\alpha}\rangle \subset G$ generated by $\mathcal{U}_\alpha$ and $\mathcal{U}_{-\alpha}$ is contained in $(G^S)^o$. Thus, $s_\alpha v_o P/P \in (G^S)^o \cdot v_o P/P$. This is a contradiction to the minimal choice of v_o in the $(G^S)^o$-orbit. Thus (4) is true, proving that any $(G^S)^o$-orbit in X_P^S is closed. Further, by (2), there are only finitely many $(G^S)^o$-orbits in X_P^S.

We return to the embedding ϕ_v as in (3). We now show that ϕ_v is surjective: Since $\mathcal{Z}$ is stable under the action of $(G^S)^o$, it is a (finite) union of $(G^S)^o$-orbits. But, each $(G^S)^o$-orbit is closed in X_P^S and hence $\mathcal{Z}$ is a disjoint union of finitely many closed $(G^S)^o$-orbits. But, $\mathcal{Z}$ being connected, it is a single $(G^S)^o$-orbit, proving that ϕ_v is a bijective isomorphism.

Consider the map

$$\psi : Y_S \to \Big\{\text{set of the connected components of } X_P^S\Big\},$$

which takes

$$v \in Y_S \mapsto (G^S) \cdot vP/P.$$

By the above proof $(G^S)^o \cdot vP/P$ is indeed a connected component of X_P^S and hence ψ is well defined. Moreover, ψ is injective since any compact homogeneous variety of a reductive connected algebraic group over $\mathbb{C}$ has a unique fixed point under any Borel subgroup, which follows from the Bruhat decomposition.

Further, as proved above, ψ is surjective. This proves the proposition. □

Corollary 2.2. *With the notation as in Proposition 2.1, assume that $P = B$. Then, each connected component of X_B^S is isomorphic, as a $(G^S)^o$-variety, with $(G^S)^o/B^S$. In particular, each connected component of X_B^S is of the same dimension.*

Proof. By the above proposition applied to the case of $P = B$, we get that any connected component $\mathcal{Z}$ of X_B^S is isomorphic with $(G^S)^o/\Big((G^S)^o \cap (vBv^{-1})\Big)$, for some $v \in Y_S \subset W$. Moreover, $B^S \subset (G^S)^o \cap (vBv^{-1})$. But, vBv^{-1} being a solvable group and B^S being a maximal solvable subgroup of $(G^S)^o$, we get that $B^S = (G^S)^o \cap (vBv^{-1})$. This proves the Corollary. □

Example 2.3. For a general parabolic P, X_P^S does not necessarily have components of the same dimension. For example, take $G = \mathrm{GL}_n(\mathbb{C})$ and P a maximal parabolic subgroup such that $X_P = \mathbb{P}^{n-1}$. Take $S \subset T$ be the subgroup of order 2 generated by the diagonal element $((+1)^m, (-1)^{n-m})$, which is $+1$ along the first m entries of the diagonal and -1 in the following $(n-m)$ entries. In this case,

$$X_P^S = \mathbb{P}^{m-1} \sqcup \mathbb{P}^{n-m-1}.$$

This provides a counter example if $n \neq 2m$.

Remark 2.4. (1) Following the notation and assumptions of Proposition 2.1,

$$Y_S = \left\{ v \in W^P : \mathcal{U}_v^S = e \right\},$$

where $\mathcal{U}_v \subset G$ is the unipotent group with Lie algebra

$$\bigoplus_{\alpha \in R^+ \cap v R_P^-} \mathfrak{g}_\alpha = \bigoplus_{\alpha \in R^+ \cap v R^-} \mathfrak{g}_\alpha,$$

where R^+ is the set of positive roots of G and R_P^- is the set of roots of the opposite unipotent radical of P. This follows since

$$B^S v\, P/P = \mathcal{U}_v^S \cdot vP/P.$$

(2) If S is generated by a finite order $t \in T$ which is regular, i.e., $e^\alpha(t) \neq 1$ for any root α of G, then $B^S = T$ (and hence $(G^S)^o = T$), $Y_S = W^P$ and the connected components of X_P^S are all points (use Proposition 2.1).

Lemma 2.5. *Following Proposition 2.1, we get the following:*

$$\sum_{v \in Y_S} \# \left(W((G^S)^o) / W\left((G^S)^o \cap (vPv^{-1}) \right) \right) = \# W^P, \tag{7}$$

where $W(H)$ is the Weyl group of H.

In particular,

$$\# Y_S \geq \frac{\# W^P}{\# W((G^S)^o)}. \tag{8}$$

For $P = B$, in fact we have the equality

$$\# Y_S = \frac{\# W}{\# W((G^S)^o)}. \tag{9}$$

Proof. The identity (7) follows from Proposition 2.1 since $wP/P \in X_P^S$ for any $w \in W^P$ and $\phi_v : (G^S)^o / \left((G^S)^o \cap vPv^{-1} \right) \xrightarrow{\sim} X_P^S(v)$ is a $(G^S)^o$-equivariant isomorphism for any $v \in Y_S$ and, moreover, $\sqcup_{v \in Y_S} X_P^S(v) = X_P^S$.

Inequality (8), of course, follows immediately from identity (7).

To prove identity (9), observe that for $P = B$, $(G^S)^o \cap vPv^{-1} = B^S$ for any $v \in Y_S$ (cf. proof of Corollary 2.2).

□

3. Character of any representation at finite order elements- An asymptotic formula via Lefschetz Theorem

We apply the Lefschetz Trace Formula due to Atiyah-Singer (cf. [AS, Theorem 4.6]) for the case of the complex manifold $X = X_P = G/P$, the automorphism of X_P is given by the left multiplication of a finite order element $t \in T$ and the vector bundle is a homogeneous line bundle $\mathcal{L}(\lambda) := G \times^P \mathbb{C}_{-\lambda} \to G/P$ associated to a character λ of P.

Fix a connected component $X^t_P(v) = X^S_P(v)$ (for $v \in Y_S$) as in Proposition 2.1, where $S \subset T$ is the finite subgroup generated by t. Then, the S-equivariant line bundle.

$$\mathcal{L}(\lambda)_{|_{X^t_P(v)}} \approx e^{-v\lambda}_{|_S} \otimes \hat{\mathcal{L}}(\lambda)_{|_{X^t_P(v)}}, \tag{10}$$

where $\hat{\mathcal{L}}(\lambda)_{|_{X^t_P(v)}}$ is the same line bundle as $\mathcal{L}(\lambda)_{|_{X^t_P(v)}}$ but with the trivial action of S.

We next determine the normal bundle N^t_v of $X^t_P(v)$ in X_P:

First of all the tangent space $T_{vP/P}(X_P)$ is given by the derivative of the curves (at $z = 0$) $\gamma_\alpha : \mathbb{C} \to X_P$, $z \mapsto v\mathrm{Exp}(zy_\alpha)P/P$, where α runs over the (negative) roots in R^-_P (see Remark 2.4(1)) and y_α is a fixed root vector of the root space $\mathfrak{g}_{-\alpha}$. For any $s \in T$, the action of T on the above curve is given by

$$\begin{aligned} sv\mathrm{Exp}(zy_\alpha)P/P &= v\left(v^{-1}sv\right)\mathrm{Exp}(zy_\alpha)(v^{-1}s^{-1}v)P/P \\ &= v\,\mathrm{Exp}(\mathrm{Ad}(v^{-1}sv)\cdot zy_\alpha)P/P. \end{aligned}$$

From this we see that the derivative $\dot{\gamma}_\alpha(0)$ is transformed by the T-action via $v \cdot y_\alpha$. Thus, the tangent space (as a T-module) is given by

$$T_v(X_P) = \bigoplus_{\alpha \in vR^-_P} \mathfrak{g}_\alpha. \tag{11}$$

Similarly, considering the curve $\mathbb{C} \ni z \mapsto \mathrm{Exp}(zy_\alpha)vP/P$ in $X^t_P(v)$, we get that

$$T_v\left(X^t_P(v)\right) = \bigoplus_{\alpha \in (vR^-_P) \cap R((G^t)^o)} \mathfrak{g}_\alpha, \tag{12}$$

where $R((G^t)^o) \subset R(G)$ denotes the set of all the roots of $(G^t)^o$ (here $G^t := G^S$). Thus, the normal bundle N^t_v over $X^t_P(v)$ is given by

$$N^t_v \approx \bigoplus_{\alpha \in (vR^+_P) \setminus R((G^t)^o)} \overline{\mathcal{L}}(\alpha)_{|_{X^t_P(v)}}, \tag{13}$$

where $\overline{\mathcal{L}}(\alpha)_{|_{X^t_P(v)}}$ is the $(G^t)^o$-equivariant line bundle over $X^t_P(v)$ such that the fiber over vP/P has T-weight $-\alpha$. Observe that

$$R^-((G^t)^o) \subset v \cdot R^-, \text{ thus } \quad R^+((G^t)^o) \subset v \cdot R^+. \tag{14}$$

If (14) were false, take $\alpha \in R^-((G^t)^o)$ such that $v^{-1} \cdot \alpha \in R^+$. This gives $\alpha \in (v \cdot R^+) \cap R^-$. But, since $\mathcal{U}^t_v = (e)$ (see Remark 2.4(1)), t acts nontrivially on $\mathfrak{g}_\alpha$ and hence $\alpha \notin R((G^t)^o)$. This contradicts the choice of α and hence (14) is proved.

Let $V(\lambda)$ be the irreducible representation of G with highest weight $\lambda \in \mathfrak{t}^*$ and let $t \in T$ be an element of finite order. Let $\mathrm{ch}(t, V(\lambda))$ be the trace of the action of t on $V(\lambda)$.

We now apply [AS, Theorem 4.6] to get the following theorem. Let $P = P_\lambda$ be the unique standard parabolic subgroup of G such that $\mathcal{L}(\lambda)$ is an ample line bundle over X_P.

Theorem 3.1. *With the notation as above, for any integer $n \geqslant 0$,*

$$\mathrm{ch}\left(t^{-1}, V(n\lambda)\right) = \sum_{v \in Y_S} D_v^t(n, \lambda),$$

where

$$D_v^t(n,\lambda) := \sum_{k \geqslant 0} \int_{X_P^t(v)} \frac{c_1\left(\mathcal{L}(\lambda)_{|_{X_P^t(v)}}\right)^k \frac{n^k}{k!} e^{-nv\lambda}(t) \cdot \mathrm{td}(X_P^t(v))}{\prod_{\alpha \in (vR_P^+) \setminus R((G^t)^o)} [1 - e^{\alpha}(t) e^{-c_1(\overline{\mathcal{L}}(\alpha)_{|_{X_P^t(v)}})}]},$$

as earlier R_P^+ is the set of roots of the unipotent radical of P and td *is the Todd genus of the tangent bundle,*

Proof. Use [AS, Theorem 4.6] together with the identities (7) and (13) for the line bundle $\mathcal{L}(n\lambda)$ over X_P. By the Borel-Weil-Bott Theorem,

$$\begin{aligned} & H^i\left(X_P, \mathcal{L}(n\lambda)\right) = 0, \quad \text{for all } i > 0 \\ \text{and} \quad & H^0\left(X_P, \mathcal{L}(n\lambda)\right) = V(n\lambda)^*. \end{aligned}$$

Thus,

$$\begin{aligned} \sum_{i \geq 0} (-1)^i \mathrm{Trace}\left(t,\ H^i\left(X_P, \mathcal{L}(n\lambda)\right)\right) &= \mathrm{ch}\left(t, V(n\lambda)^*\right) \\ &= \mathrm{ch}\left(t^{-1}, V(n\lambda)\right). \end{aligned}$$

□

Remark 3.2. The above theorem remains true (by the same proof) for any standard parabolic subgroup P replacing P_λ as long as λ extends to a character of P; in particular, for $P = B$. Applying the above theorem for $P = B$ and $n = 1$, we get that $\mathrm{ch}\left(t^{-1}, V(\lambda)\right)$ is a piecewise polynomial function in λ. In fact, it is a polynomial function restricted to the dominant elements of any coset $X(T)/d \cdot X(T)$ (d being the order of t), where $X(T)$ is the character group of T.

Let d_v be the complex dimension of $X_P^t(v)$ and let r be the order of $t \in T$. Consider the function

$$\chi_\lambda^t : \mathbb{Z}_{\geqslant 0} \to \mathbb{C}, \quad n \mapsto \mathrm{ch}\left(t^{-1}, V(n\lambda)\right).$$

As a corollary of Theorem (3.1), we get the following.

Corollary 3.3. *For any fixed $0 \leq p < r$, $\chi^t_{\lambda_{|\{n \equiv p (mod\, r)\}}}$ is a polynomial function in n of degree $\leq$* $\max_{v \in Y_S} \{d_v\}$.

Moreover, the coefficient of n^{d_v} in $D_v^t(n, \lambda)$ restricted to $\{n \equiv p(mod\, r)\}$ is equal to

$$\frac{1}{d_v!} \int_{X_P^t(v)} c_1\left(\mathcal{L}(\lambda)_{|_{X_P^t(v)}}\right)^{d_v} \frac{e^{-pv\lambda}(t)}{\prod_{\alpha \in \left(vR_P^+\right) \setminus R\left((G^t)^o\right)} (1 - e^{\alpha}(t))}. \tag{15}$$

Proof. The polynomial behavior of $\chi_{\lambda|\{n \equiv p (mod r)\}}$ follows immediately from Theorem 3.1. To prove (15), again use Theorem 3.1 together with the fact that the constant term of the Todd genus of any manifold is 1 (cf. [F, Example 3.2.4]). □

Remark 3.4. Since $\mathcal{L}(\lambda)$ is an ample line bundle over $X_P = G/P$, by Wirtinger theorem (cf. [GH, Chapter 0, §2]), for any $v \in Y_S$,

$$\int_{X_P^t(v)} c_1\left(\mathcal{L}(\lambda)|_{X_P^t(v)}\right)^{d_v} > 0.$$

As a corollary of Theorem 3.1, Corollary 3.3 and Remark 3.4, we immediately get the following result.

Corollary 3.5. *Following the notation and assumptions as in Theorem 3.1, assume further that there is a unique* $v_o \in Y_S$ *such that* $d_{v_o} = \max_{v \in Y_S}\{d_v\}$. *Then, for any fixed* $0 \leq p < r$,

$$\operatorname{ch}\left(t^{-1}, V(n\lambda)_{|_{n \equiv p (mod\ r)}}\right)$$

is a polynomial function of degree exactly equal to d_{v_o}.

When $P = B$, i.e., λ is a dominant regular highest weight, then Theorem 3.1 specializes to the following.

Corollary 3.6. *With the notation and assumptions as in Theorem 3.1 and with the additional assumption that* $P = B$,

$$\operatorname{ch}\left(t^{-1}, V(n\lambda)\right) = \sum_{v \in Y_S} \sum_{k \geqslant 0} \int_{\overset{o\,t}{X}} \frac{c_1\left(\mathcal{L}(v\lambda)_{|_{\overset{o\,t}{X}}}\right)^k \frac{n^k}{k!} e^{-nv\lambda}(t)\,\operatorname{td} \overset{o\,t}{X}}{\prod_{\alpha \in (vR^+)\setminus \overset{o\,t}{R}} \left[1 - e^{\alpha}(t) \cdot e^{-c_1\left(\mathcal{L}(\alpha)_{|_{\overset{o\,t}{X}}}\right)}\right]},$$

where $\overset{o\,t}{X} := (G^t)^o/B^t$ *and* $\overset{o\,t}{R}$ *denotes the set of roots of* $(G^t)^o$.

Proof. To deduce the corollary from Theorem 3.1, we need to observe that under the isomorphism of Proposition 2.1 (for any $v \in Y_S$):

$$\phi_v : \overset{o\,t}{X} \to X_B^t(v), \quad gB^t \mapsto gvB/B,$$

the line bundle $\mathcal{L}(\mu)_{|_{X_B^t(v)}}$ pulls back to

$$\phi_v^*\left(\mathcal{L}(\mu)_{|_{X_B^t(v)}}\right) \simeq \mathcal{L}(v\mu)_{|_{\overset{o\,t}{X}}}$$

under the isomorphism

$$\left[g, \mathbb{1}_{-v\mu}\right] \mapsto \left[gv, \mathbb{1}_{-\mu}\right], \quad \text{for } g \in (G^t)^o,$$

where $\mathbb{1}_{-v\mu}$ is a basis of $\mathbb{C}_{-v\mu}$. Moreover,

$$\phi_v^*\left(\overline{\mathcal{L}}(\mu)_{|_{X_B^t(v)}}\right) = \mathcal{L}(\mu)_{|_{\overset{o\,t}{X}}}.$$

□

Remark 3.7. The Todd genus of any flag variety G/P is determined by Brion [B, §3].

Example 3.8. Let the assumption be as in Theorem 3.1. Assume further that $t \in T$ is a regular element (i.e., $e^{\alpha}(t) \neq 1$ for any root α of G) of finite order r. Then, the function $n \mapsto \operatorname{ch}\left(t^{-1}, V(n\lambda)\right)$ is the constant function 1, when n is restricted to $r\mathbb{Z}_{\geq 0}$.

Even though it follows easily from the Weyl character formula, but we deduce it from Corollary 3.3:

Since t is regular, $Y_S = W^P$ and $B^t = T$, thus $X_P^t(v) = \{vP/P\}$ for any $v \in W^P$. Thus, each $d_v = 0$ and by Corollary 3.3, we get (for $n \in r\mathbb{Z}_{\geq 0}$)

$$\begin{aligned}
\mathrm{ch}\left(t^{-1}, V(n\lambda)\right) &= \sum_{v\in W^P} \frac{1}{\prod_{\alpha\in vR_P^+}(1-e^{\alpha}(t))} \\
&= \sum_{v\in W^P} \frac{1}{\prod_{\alpha\in\left(vR_P^+\right)\cap R^+}(1-e^{\alpha}(t))\cdot\prod_{\alpha\in(vR^+)\cap R^-}(1-e^{\alpha}(t))}, \\
&\quad \text{since } v\left(R^+\backslash R_P^+\right)\subset R^+ \\
&= \sum_{v\in W^P}(-1)^{\ell(v)} \frac{\prod_{\alpha\in vR^-\cap R^+} e^{\alpha}(t)}{\prod_{\alpha\in\left(vR_P^+\right)\cap R^+}(1-e^{\alpha}(t))\cdot\prod_{\alpha\in vR^-\cap R^+}(1-e^{\alpha}(t))} \\
&= \sum_{v\in W^P}(-1)^{\ell(v)} \frac{e^{\rho-v\rho}(t)}{\prod_{\alpha\in R^+\backslash v\left(R^+\backslash R_P^+\right)}(1-e^{\alpha}(t))} \\
&= 1, \text{ by the parabolic analogue of the Weyl denominator formula.}
\end{aligned}$$

4. Specialization of results for an involution t

In this section we consider elements $t \in T$ of order 2. As a consequence of Corollary 3.3 in the case of involution t, we get the following.

Theorem 4.1. *Follow the notation and assumptions as in Corollary 3.3 and assume further that $t \in T$ is of order 2. Then, the function $\chi^t_{\lambda|\{n\in 2\mathbb{Z}_{\geq 0}\}}$ is a polynomial function of degree exactly equal to $d := max_{v\in Y_S}\{d_v\}$.*

Proof. By Corollary 3.3, the coefficient of n^{d_v} in $D_v^t(n,\lambda)$ restricted to $2\mathbb{Z}_{\geq 0}$ is equal to

$$\frac{1}{d_v!}\int_{X_P^t(v)} c_1\left(\mathcal{L}(\lambda)_{|_{X_P^t(v)}}\right)^{d_v}\cdot\frac{1}{\prod_{\alpha\in\left(vR_P^+\right)\backslash R\left((G^t)^o\right)}(1-e^{\alpha}(t))}.$$

But since t is of order 2, $e^{\alpha}(t) = \pm 1$. Moreover, for $\alpha \notin R\left((G^t)^o\right)$, $e^{\alpha}(t) = -1$. Thus, the above sum reduces to

$$\frac{1}{d_v!}\int_{X_P^t(t)} c_1\left(\mathcal{L}(\lambda)_{|_{X_P^t(v)}}\right)^{d_v}\cdot 2^{-\#\left(vR_P^+\backslash R\left((G^t)^o\right)\right)}.$$

By Wirtingers Theorem (cf. Remark 2.3), the integral $\int_{X_P^t(v)} c_1\left(\mathcal{L}(\lambda)_{|_{X_P^t(v)}}\right)^{d_v} > 0$.

Since, by Theorem 3.1,

$$\chi_\lambda^t(n) = \sum_{v\in Y_S} D_v^t(n,\lambda),$$

the theorem follows. □

Example 4.2. *Consider $G = \mathrm{GL}_n(\mathbb{C})$ and $t = ((+1)^m, (-1)^{n-m})$ as in Example 2.3. In the case $n = 2m$ and m odd, using Theorem 3.1, it can be seen that for*

$$\lambda := (\lambda_1 = \lambda_2 = \cdots = \lambda_m \geq \lambda_{m+1} = \lambda_{m+2} = \cdots = \lambda_n)$$

with λ_1 and λ_{m+1} of opposite parity,

$$\mathrm{ch}(t, V(\lambda)) = 0.$$

To prove this observe that, by Example 2.3, $X_P^t = \mathbb{P}^{m-1}\sqcup\mathbb{P}^{m-1}$. Moreover, in this case, $Y_S = \{1, v_o\}$, where v_o is the cycle $(1, n, n-1, n-2, \ldots, 2)$. Further, $e^{-\lambda}(t) = -e^{-v_o\lambda}(t)$.

A similar result can be obtained for $\mathrm{Sp}(2n)$ *and* $\mathrm{SO}(n)$.

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