3

Errata for the book 'Conformal Blocks, Generalized Theta Functions and the Verlinde Formula' by Shrawan Kumar

From the beginning of Section 3.5 till the end of Definition 3.5.5 in the above book should be replaced by the following.

3.5 Local freeness of the sheaf of conformal blocks

As in earlier Sections, let $s \geq 1$ and let $c \geq 1$ be the level. Let $\vec{\lambda} = (\lambda_1, \ldots, \lambda_s) \in D_c^s$. For preliminaries on stacks, we refer to the Appendix C. In this Section, we consider families of s-pointed curves with formal parameters over $\mathbb{D}_\tau := \operatorname{Spec} \mathbb{C}[[\tau]]$ instead of over smooth irreducible varieties as in Section 3.3.

Let $(\Sigma_o, \vec{p}_o)$ be an s-pointed curve with a node at q (and possibly other nodes). Let $\tilde{\Sigma}_o$ be the normalization of Σ_o at the point q. The nodal point q splits into two smooth points q', q'' in $\tilde{\Sigma}_o$. Let z' (resp. z'') be a local parameter at q' (resp. q'') for the curve $\tilde{\Sigma}_o$. The following lemma shows that there exists a 'canonical' smoothing deformation of $(\Sigma_o, \vec{p}_o)$ over the formal disc $\mathbb{D}_\tau$. We denote by $\mathbb{D}_\tau^\times$ the associated punctured formal disc $\operatorname{Spec} \mathbb{C}((\tau))$.

A proof of the following lemma is sketched in [Looijenga-2013, §6]. For more details, we refer to [Damiolini-2020, §6.1].

Lemma 3.5.1 [1]

There exist a family of s-pointed curves with formal parameters $\mathcal{F}_\Sigma = (\Sigma, \vec{p}, \vec{t})$ *over* $\mathbb{D}_\tau$*, such that the following properties hold:*

(1) Over the closed point $o \in \mathbb{D}_\tau$, $\mathcal{F}_\Sigma|_{\tau=o} = (\Sigma_o, \vec{p}_o)$.

(2) $\mathcal{F}_\Sigma|_{\mathbb{D}_\tau^\times}$ *is a smooth family over the punctured formal disc* $\mathbb{D}_\tau^\times$ *if q is the unique node in* Σ_o.

[1] This is a modified version of Lemma 3.5.1 published in the book. The lemma stated there was erroneous (as pointed out by O. Gabber).

(3) The completed local ring $\hat{\mathscr{O}}_{\Sigma,q}$ *of* $\mathscr{O}_\Sigma$ *at q is isomorphic to*

$$\mathbb{C}[[z',z'',\tau]]/\langle \tau - z'z''\rangle \simeq \mathbb{C}[[z',z'']].$$

(4) There exists an isomorphism of $\mathbb{C}[[\tau]]$*-algebras:*

$$\kappa : \hat{\mathscr{O}}_{\Sigma\setminus\{q\},\Sigma_o\setminus\{q\}} \simeq \mathscr{O}_{\Sigma_o\setminus\{q\}}[[\tau]], \tag{1}$$

where the $\mathbb{C}[[\tau]]$*-algebra structure on* $\hat{\mathscr{O}}_{\Sigma\setminus\{q\},\Sigma_o\setminus\{q\}}$ *is obtained from the projection* $\Sigma\setminus\{q\}\to\mathbb{D}_\tau$.

(5) The scheme $(\Sigma_o\setminus\{q\})\times \operatorname{Spec}\mathbb{C}[[\tau]]$ *is an open subscheme of* Σ.

Proof We give only a very brief sketch. For further details, we refer to [Damiolini-2020, §6.1]. We first take $\mathbb{C}[\tau]_n := \frac{\mathbb{C}[\tau]}{\langle\tau^{n+1}\rangle}$ for any $n\geq 0$. Let $U\subset\Sigma_o$ be an open subset. If $q\notin U$, set $\mathscr{O}_{\Sigma_n}(U) := \frac{\mathscr{O}_{\Sigma_o}(U)[\tau]}{\langle\tau^{n+1}\rangle}$. If $q\in U$, set

$$\mathscr{O}_{\Sigma_n}(U) := \operatorname{Ker}\Big(\frac{\mathbb{C}[[z',z'']][\tau]}{\langle z'z''-\tau,\tau^{n+1}\rangle}\times\mathscr{O}_{\Sigma_n}(U\setminus\{q\}) \xrightarrow{\alpha_n-\beta_n} \frac{\mathbb{C}((z'))[\tau]}{\langle\tau^{n+1}\rangle}\times\frac{\mathbb{C}((z''))[\tau]}{\langle\tau^{n+1}\rangle}\Big),$$

where

$$\alpha_n : \frac{\mathbb{C}[[z',z'']][\tau]}{\langle z'z''-\tau,\tau^{n+1}\rangle}\to\frac{\mathbb{C}((z'))[\tau]}{\langle\tau^{n+1}\rangle}\times\frac{\mathbb{C}((z''))[\tau]}{\langle\tau^{n+1}\rangle}$$

is the $\mathbb{C}[\tau]_n$-linear ring homomorphism given by $z'\mapsto(z',\frac{\tau}{z''})$ and $z''\mapsto(\frac{\tau}{z'},z'')$ and

$$\beta_n : \mathscr{O}_{\Sigma_n}(U\setminus\{q\})\to\frac{\mathbb{C}((z'))[\tau]}{\langle\tau^{n+1}\rangle}\times\frac{\mathbb{C}((z''))[\tau]}{\langle\tau^{n+1}\rangle}$$

takes $f\in\mathscr{O}_{\Sigma_n}(U\setminus\{q\})\mapsto(f',f'')$, f' (resp. f'') being the expansion of f at the point q' (resp. q'') in the local parameter z' (resp. z'') using the identification

$$\mathscr{O}_{\Sigma_n}(U\setminus\{q\}) = \frac{\mathscr{O}_{\Sigma_o}(U\setminus\{q\})[\tau]}{\langle\tau^{n+1}\rangle} = \frac{\mathscr{O}_{\tilde{\Sigma}_o}(\tilde{U}\setminus\{q',q''\})[\tau]}{\langle\tau^{n+1}\rangle}.$$

Here $\tilde{U}$ is the inverse image of U in the normalization $\tilde{\Sigma}_o$ of Σ_o at q.

Observe that, for all $n>0$, there are natural morphisms $X_{n-1}\to X_n$ induced by the identity on topological spaces and by the projection $\mathbb{C}[\tau]_n\to\mathbb{C}[\tau]_{n-1}$ on the structure sheaves. Let X_∞ be the direct limit of the family $(X_n)_{n\geq 0}$. Then, as shown in [Damiolini-2020, §6.1], X_∞ is algebraizable by using Grothendieck's Existence Theorem. Let X denote the 'algebraization' of X_∞. Then, X satisfies all the properties of the lemma. □

Let $\hat{\mathfrak{g}}'$ (resp. $\hat{\mathfrak{g}}''$)) be the affine Kac-Moody Lie algebra attached to the point q' (resp. q'') in $\tilde{\Sigma}_o$ with respect to the parameters z' and z'' respectively. For any $\mu\in D_c$, let $\mathscr{H}'(\mu^*)$ (resp. $\mathscr{H}''(\mu)$) be the highest-weight integrable representation of $\hat{\mathfrak{g}}'$ (resp. $\hat{\mathfrak{g}}''$) with highest weights μ^* (resp. μ). We will abbreviate

$\mathscr{H}'(\mu^*)$ and $\mathscr{H}''(\mu)$, respectively by $\mathscr{H}(\mu^*)$ and $\mathscr{H}(\mu)$ and understand that $\mathscr{H}(\mu^*)$ (resp. $\mathscr{H}(\mu)$) is a representation of $\hat{\mathfrak{g}}'$ (resp. $\hat{\mathfrak{g}}''$).

Lemma 3.5.2 *There exists a nondegenerate pairing* $\langle\,,\,\rangle : \mathscr{H}(\mu^*)\times\mathscr{H}(\mu) \to \mathbb{C}$ *such that for any* $h_1 \in \mathscr{H}(\mu^*)$, $h_2 \in \mathscr{H}(\mu)$, *and* $x[z'^n] \in \hat{\mathfrak{g}}'$,

$$\langle x[z'^n]\cdot h_1, h_2\rangle + \langle h_1, x[z''^{-n}]\cdot h_2\rangle = 0. \tag{1}$$

Proof This follows immediately from the explicit construction of $\mathscr{H}(\mu^*)$ as in Definition 3.1.1 by taking the standard pairing $\mathscr{H}(\mu)^\vee \times \mathscr{H}(\mu) \to \mathbb{C}$. □

Definition 3.5.3 There exist direct sum decompositions by the negative of the d-degree (cf. identity (1) of the proof of Theorem 1.2.10) (putting the d-degree of the highest-weight vectors at 0):

$$\mathscr{H}(\mu^*) = \bigoplus_{k=0}^{\infty} \mathscr{H}(\mu^*)_k,\ \mathscr{H}(\mu) = \bigoplus_{k=0}^{\infty} \mathscr{H}(\mu)_k.$$

Choose any basis $\{f_j^k\}_{j\in S_\mu^k}$ of $\mathscr{H}(\mu^*)_k$ and let $\{v_j^k\}_{j\in S_\mu^k}$ be the dual basis of $\mathscr{H}(\mu)_k$ under the above pairing $\langle\,,\,\rangle$. (Observe that $\langle\mathscr{H}(\mu^*)_k, \mathscr{H}(\mu)_{k'}\rangle = 0$ unless $k = k'$.) Set, for any $k \geq 0$,

$$\Delta_{\mu,k} := \sum_{j\in S_\mu^k} f_j^k \otimes v_j^k \in \mathscr{H}(\mu^*)_k \otimes \mathscr{H}(\mu)_k.$$

For $k < 0$, we set $\Delta_{\mu,k} = 0$.

Then, $\Delta_{\mu,0} = \mathrm{I}_\mu$, where I_μ is the element introduced in Definition 3.1.1. In view of Lemma 3.5.2, $\Delta_{\mu,k}$ satisfies the following property (for any $k, n \in \mathbb{Z}$):

$$(x[z'^n]\otimes 1)\cdot\Delta_{\mu,k+n} + (1\otimes x[z''^{-n}])\cdot\Delta_{\mu,k} = 0, \text{ for any } x[z'^n] \in \hat{\mathfrak{g}}'. \tag{1}$$

To prove (1), take any $f^{k+n} \in \mathscr{H}(\mu^*)_{k+n}$ and $v^k \in \mathscr{H}(\mu)_k$. Then, under the standard tensor product bilinear form,

$$\begin{aligned}
&\langle (x[z'^n]\otimes 1)\cdot\Delta_{\mu,k+n} + (1\otimes x[z''^{-n}])\cdot\Delta_{\mu,k}, v^k\otimes f^{k+n}\rangle \\
&\quad= \sum_{j\in S_\mu^{k+n}} \langle x[z'^n]\cdot f_j^{k+n}, v^k\rangle\langle f^{k+n}, v_j^{k+n}\rangle + \sum_{j_1\in S_\mu^k} \langle f_{j_1}^k, v^k\rangle\langle f^{k+n}, x[z''^{-n}]\cdot v_{j_1}^k\rangle \\
&\quad= -\sum_{j\in S_\mu^{k+n}} \langle f_j^{k+n}, x[z''^{-n}]\cdot v^k\rangle\langle f^{k+n}, v_j^{k+n}\rangle - \sum_{j_1\in S_\mu^k} \langle f_{j_1}^k, v^k\rangle\langle x[z'^n]\cdot f^{k+n}, v_{j_1}^k\rangle, \\
&\qquad\qquad \text{by Lemma 3.5.2} \\
&\quad= -\langle f^{k+n}, x[z''^{-n}]\cdot v^k\rangle - \langle x[z'^n]\cdot f^{k+n}, v^k\rangle \\
&\quad= 0, \text{ by Lemma 3.5.2.}
\end{aligned}$$

This proves (1).

We now construct the following 'gluing' tensor element:

$$\Delta_\mu := \sum_{k\geq 0} \Delta_{\mu,k}\tau^k \in (\mathscr{H}(\mu^*)\otimes\mathscr{H}(\mu))\,[[\tau]].$$

Let $\theta = (\theta', \theta'')$ be the map as defined in the Proof of Lemma 3.5.1:

$$\theta' : \hat{\mathscr{O}}_{\Sigma,q} \to \mathbb{C}((z'))[[\tau]], \text{ and } \theta'' : \hat{\mathscr{O}}_{\Sigma,q} \to \mathbb{C}((z''))[[\tau]],$$

where, as earlier, $\hat{\mathscr{O}}_{\Sigma,q}$ is the completion of $\mathscr{O}_\Sigma$ along q. Then, for any $f(z', z'') = \sum_{i\geq 0, j\geq 0} a_{i,j} z'^i z''^j \in \hat{\mathscr{O}}_{\Sigma,q}$ (cf. Lemma 3.5.1(3)), we have

$$\theta'(f) = f(z', \tau/z') = \sum_{j\geq 0}\Big(\sum_{i\geq 0} a_{i,j} z'^{i-j}\Big)\tau^j$$

and

$$\theta''(f) = f(\tau/z'', z'') = \sum_{i\geq 0}\Big(\sum_{j\geq 0} a_{i,j} z''^{j-i}\Big)\tau^i.$$

The morphisms θ', θ'' induce a $\mathbb{C}[[\tau]]$-linear Lie algebra morphism

$$\theta : \mathfrak{g}\otimes\hat{\mathscr{O}}_{\Sigma,q} \to \mathfrak{g}((z'))[[\tau]] \oplus \mathfrak{g}((z''))[[\tau]], \quad x\otimes f \mapsto x\otimes\theta'(f) + x\otimes\theta''(f),$$

where τ acts on $\hat{\mathscr{O}}_{\Sigma,q}$ via

$$\tau\cdot f(z', z'') = z'z'' f(z', z'').$$

Thus, we get an injective map of $\mathfrak{g}\otimes\hat{\mathscr{O}}_{\Sigma,q}$ into $\hat{\mathfrak{g}}'[[\tau]]\oplus\hat{\mathfrak{g}}''[[\tau]]$ (but *not* a Lie algebra homomorphism). The latter acts canonically on $(\mathscr{H}(\mu^*)\otimes\mathscr{H}(\mu))\,[[\tau]]$. Thus, we get a $\mathbb{C}[[\tau]]$-linear projective representation of $\mathfrak{g}\otimes\hat{\mathscr{O}}_{\Sigma,q}$ in $(\mathscr{H}(\mu^*)\otimes\mathscr{H}(\mu))\,[[\tau]]$.

Lemma 3.5.4 *The element $\Delta_\mu \in (\mathscr{H}(\mu^*)\otimes\mathscr{H}(\mu))\,[[\tau]]$ defined above is annihilated by $\mathfrak{g}\otimes\hat{\mathscr{O}}_{\Sigma,q}$ via the morphism θ defined as above.*

Proof For any $x[z'^i z''^j] \in \mathfrak{g}\otimes\hat{\mathscr{O}}_{\Sigma,q}$,

$$\begin{aligned} x[z'^i z''^j]\cdot\Delta_\mu &= \sum_{k\in\mathbb{Z}} (x[z'^{i-j}]\otimes 1)\cdot\Delta_{\mu,k}\tau^{k+j} + \sum_{k\in\mathbb{Z}} (1\otimes x[z''^{j-i}])\cdot\Delta_{\mu,k}\tau^{k+i} \\ &= -\sum_{k\in\mathbb{Z}} (1\otimes x[z''^{j-i}])\cdot\Delta_{\mu,k+j-i}\tau^{k+j} \\ &\quad + \sum_{k\in\mathbb{Z}} (1\otimes x[z''^{j-i}])\cdot\Delta_{\mu,k}\tau^{k+i}, \quad \text{by (1) of Definition 3.5.3} \\ &= 0. \end{aligned}$$

From this it is easy to see that $x[f]\cdot\Delta_\mu = 0$ for any $x[f] \in \mathfrak{g}\otimes\hat{\mathscr{O}}_{\Sigma,q}$. This proves the lemma. □

Definition 3.5.5 We follow the notation from Lemma 3.5.1. Let $\mathscr{H}(\lambda_i)$ denote the integrable $\hat{\mathfrak{g}}$-module with highest weight λ_i and let $\mathscr{H}(\vec{\lambda})$ denote their tensor product. Define

$$\mathscr{H}(\vec{\lambda})_{\mathbb{D}_\tau} := \mathscr{H}(\vec{\lambda})[[\tau]].$$

Then, as in Definition 3.3.2, we get an action of $\mathfrak{g}[\Sigma \backslash \vec{p}]$ on $\mathscr{H}(\vec{\lambda})_{\mathbb{D}_\tau}$ (induced from the embedding φ as in (6) of Definition 3.3.2).

We now construct a morphism of $\mathbb{C}[[\tau]]$-modules:

$$F_{\vec{\lambda}} : \mathscr{H}(\vec{\lambda})_{\mathbb{D}_\tau} \longrightarrow \bigoplus_{\mu \in D_c} \Big(\mathscr{H}(\vec{\lambda}) \otimes \mathscr{H}(\mu^*) \otimes \mathscr{H}(\mu) \Big) [[\tau]]$$

given by

$$\sum_{j=0}^{\infty} h_j \tau^j \mapsto \sum_{j,k=0}^{\infty} (h_j \otimes \Delta_{\mu,k}) \tau^{j+k}, \quad \text{for } h_j \in \mathscr{H}(\vec{\lambda}).$$

Since $\vec{p}_o$ is disjoint from q and Σ is the product in the complement of q (cf. Lemma 3.5.1(5)), $\vec{p}_o$ gives rise to the canonical sections $\vec{p}$ over $\mathbb{D}_\tau$. Consider the following canonical homomorphisms (obtained by restrictions):

$$\mathscr{O}_{\Sigma \backslash \vec{p}} \to \mathscr{O}_{\Sigma \backslash \{\vec{p}, q\}} \to \hat{\mathscr{O}}_{\Sigma \backslash \{\vec{p}, q\}, \Sigma_o \backslash \{\vec{p}_o, q\}} \overset{\kappa'}{\simeq} \mathscr{O}_{\Sigma_o \backslash \{\vec{p}_o, q\}}[[\tau]] \simeq \mathscr{O}_{\tilde{\Sigma}_o \backslash \{\vec{p}_o, q', q''\}}[[\tau]],$$

where the isomorphism κ' in the above is induced from the isomorphism κ of Lemma 3.5.1 (see the isomorphism (1) there). This gives rise to a Lie algebra homomorphism (depending upon the isomorphism κ'):

$$\kappa_{\vec{p}} : \mathfrak{g}[\Sigma \backslash \vec{p}] \to \Big(\mathfrak{g} \otimes \mathscr{O}_{\tilde{\Sigma}_o \backslash \{\vec{p}_o, q', q''\}} \Big) [[\tau]].$$

Hence, the Lie algebra $\mathfrak{g}[\Sigma \backslash \vec{p}]$ acts on $\Big(\mathscr{H}(\vec{\lambda}) \otimes \mathscr{H}(\mu^*) \otimes \mathscr{H}(\mu) \Big) [[\tau]]$ via the action of $\mathfrak{g} \otimes \mathscr{O}_{\tilde{\Sigma}_o \backslash \{\vec{p}_o, q', q''\}}$ on $\mathscr{H}(\vec{\lambda}) \otimes \mathscr{H}(\mu^*) \otimes \mathscr{H}(\mu)$ at the points $\{\vec{p}_o, q', q''\}$ (cf. Definition 3.1.1) and extending it $\mathbb{C}[[\tau]]$-linearly.

There is *no change* in the remaining part of this Section 3.5 starting with Theorem 3.5.6.