NONEXISTENCE OF REGULAR MAPS BETWEEN HOMOGENEOUS PROJECTIVE VARIETIES

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1. Introduction

The base field is the field of complex numbers $\mathbb{C}$. We study non-existence of non-constant regular maps from a partial flag variety $X = G/P$ to another partial flag variety $X' = G'/P'$, where G (resp. G') is a connected simple algebraic group and $P \subset G$ (resp. $P' \subset G'$) is a parabolic subgroup. Observe that there is no uniqueness of the presentation $X \cong G/P$, e.g., the projective space $\mathbb{P}^{2n-1}$ is the homogeneous space of $\mathrm{SL}(2n)$ as well as $\mathrm{Sp}(2n)$ (see Remark 4 for an exhaustive list).

The main result of this note is the following theorem (cf. Theorem 1):

Theorem: *Let G, G' be as above and P (resp. B') be a parabolic subgroup of G different from its Borel subgroup (resp. a Borel subgroup of G'). Then, there does not exist any non-constant regular map from G/P to G'/B'.*

In Section 3, we recall some results on the non-existence of non-constant regular maps from a partial flag variety X to another partial flag variety X' (but, by no means, an exhaustive list). Also, we make the following conjecture (cf. Conjecture 5):

Conjecture: *Let X, X' be two homogeneous projective varieties as above. Then,*

(a) Assume that X is different from $\mathbb{P}^{2n}$ (for $n \geq 1$) and

$$\textit{minss rank}\ X > \textit{maxss rank}\ X',$$

where minss rank and maxss rank are defined in Definition 3. Then, there does not exist any non-constant regular map from X to X'.

(b) If $X = \mathbb{P}^{2n}$ (for $n \geq 1$) and there exists a non-constant regular map from $X \to X'$, then

$$\textit{minss rank}\ \mathbb{P}^{2n-1} = n - 1 \leq \textit{maxss rank}\ X'.$$

Acknowledgements: The above theorem was proved (partially supported by then NSF grant number DMS-0070679) and communicated to J. Landsberg in a letter dated September 17, 2002. We decided to publish this note now due to its renewed interest. We thank J. Landsberg and A. Parameswaran for some helpful conversations.

THERE IS NO CONFLICT OF INTEREST. MOREOVER, ALL THE DATA IS INCLUDED IN THE MANUSCRIPT.

2. The main theorem and its proof

We prove the following theorem, which is the main result of this note.

Theorem 1. *Let G, G' be simple and connected algebraic groups and let P (resp. B') be a parabolic subgroup of G different from its Borel subgroup (resp. a Borel subgroup of G'). Then, there does not exist any non-constant regular map from G/P to G'/B'.*

Proof. Let $H^*(G'/B')$ be the singular cohomology of the full flag variety G'/B' with real coefficients. Then, by the Bruhat decomposition of G'/B', $H^*(G'/B')$ has a Schubert basis $\{\epsilon^w\}_{w\in W'}$, where W' is the Weyl group of G' and $\epsilon^w \in H^{2\ell(w)}(G'/B')$ ($\ell(w)$ being the length of w). We declare a cohomology class ϵ *non-negative* if $\epsilon = \sum_w a_w \epsilon^w$ has all its coefficients a_w non-negative and write it as $\epsilon \geq 0$. Observe that by a classical positivity theorem (cf., e.g., [Ku1, Corollary 11.4.12]), for any $v, w \in W'$, the cup product of two non-negative classes is non-negative.

Let $f : G/P \to G'/B'$ be a regular map. We first prove the theorem in the case P is a maximal parabolic subgroup of G but not a Borel subgroup (in particular, G is not isogeneous to SL(2); thus G/P is not isomorphic with $\mathbb{P}^1$). Consider the induced map $f^* : H^2(G'/B') \to H^2(G/P)$. For any $x \in H^2(G'/B')$, since $H^2(G/P)$ is one dimensional (spanned by a Schubert class denoted ϵ_o), $f^*(x)^2$ is a non-negative class. Choose a basis $\{x_i\}$ of $H^2(G'/B') \simeq \mathfrak{h}'^*_{\mathbb{R}}$ (under the Borel isomorphism [Ku1, Definition 11.3.5]) such that $\sum_i x_i^2$ is W'-invariant (use the W'-invariant positive definite form on $\mathfrak{h}^*_{\mathbb{R}}$). Here $\mathfrak{h}'^*_{\mathbb{R}}$ is the real span of the weight lattice of G' (with respect to the fixed choice of a maximal torus in G'). Thus,

$$\sum_i x_i^2 = 0 \in H^4(G/B),$$

since $H^*(G'/B')$ does not have any W'-invariant except in $H^0(G'/B')$. Write $f^*(x_i) = d_i \epsilon_o$, for $d_i \in \mathbb{R}$. Hence,

$$0 = f^*(\sum_i x_i^2) = \sum_i (f^*(x_i))^2 = \sum_i d_i^2 \epsilon_o^2 \geq 0. \tag{1}$$

Now, $\epsilon_o^2 \in H^4(G/P)$ is non-zero by Wirtinger Theorem (cf. [GH, Page 31]). (This is here we have used the assumption that G/P is not isomorphic with $\mathbb{P}^1$.) Hence, by the above equation, each $d_i = 0$. Thus, $f^*(H^2(G'/B')) = 0$. This forces f to be a constant since

$$H^2(G/B) \simeq \mathrm{Pic}(G/B) \otimes_{\mathbb{Z}} \mathbb{R}$$

(and similarly for G'/B') and a very ample line bundle on G'/B' pulls-back via f to a non-trivial line bundle if f is non-constant. This completes the proof of the theorem in the case P is a maximal parabolic subgroup of G.

We come now to the case when P is not a maximal parabolic of G (and not a Borel subgroup). Take any parabolic subgroup $Q \supset P$ of G such that P is a maximal parabolic subgroup of Q and Q/P is not isomorphic with $\mathbb{P}^1$ (i.e., the unique extra simple root contained in the Levi subgroup of Q is connected to one of the simple roots for the Levi subgroup of P in the Dynkin diagram of G). Then, the variety Q/P is isomorphic with a variety of the form H/L, where H is a simple and connected algebraic group and L is a maximal parabolic subgroup of H different from its Borel subgroup. Thus, from the maximal parabolic case proved above, we get that f restricted to Q/P is constant and so is f restricted to gQ/P for any $g \in G$. Hence, f factors through G/Q as a regular map f_Q, i.e., we have the following commutative triangle:

$$\begin{array}{ccc} G/P & \xrightarrow{\ \pi\ } & G/Q \\ & {\scriptstyle f}\searrow \quad \swarrow{\scriptstyle f_Q} & \\ & G'/B', & \end{array}$$

where π is the canonical projection. Continuing this way, we get that f descends to a regular map $\bar{f} : G/\bar{P} \to G'/B'$, where $\bar{P}$ is a maximal parabolic subgroup of G. Thus, the theorem follows from the case of maximal parabolic subgroups of G proved above.

□

3. Review of some related results and a conjecture

We recall the following results on the existence (or non-existence) of regular maps between projective homogeneous varieties:

Theorem 2. *Let $G(r, n)$ denote the Grassmannian of r-dimensional subspaces of $\mathbb{C}^n$.*

(a) (due to Tango [T]) There does not exist any non-constant regular map from $\mathbb{P}^m \to G(r, n)$ for $m \geq n$.

(b) (due to Paranjape-Srinivas [PS]) There exists a finite surjective regular map $f : G(r, n) \to G(s, m)$ if and only if $r = s$ and $n = m$. Moreover, in this case, f is a biregular isomorphism.

(c) (due to Hwang-Mok [HM]) Let G be a simple and connected algebraic group and let P be a maximal parabolic subgroup of G and let $f : G/P \to Y$ be a surjective regular map to a smooth projective variety Y of positive dimension. Then, either G/P is biregular isomorphic to the projective space $\mathbb{P}^n$, $n = \dim G/P$, or f is a biregular isomorphism.

(d) (due to J. Landsberg- unpublished) There does not exist any non-constant regular map from $\mathbb{P}^5 \to G(3, 6)$. Also, there is no non-constant regular map from $\mathbb{P}^6 \to \mathrm{SO}(10)/P(5)$, where $P(5)$ is the maximal parabolic subgroup of $\mathrm{SO}(10)$ obtained from deleting the 5-th simple root (following the convention in [Bo, Planche IV]).

(e) (due to Naldi-Occhetta [NO]) Any regular map from $f : G(r, n) \to G(s, m)$ for $n > m$ is constant.

(f) (due to Muñoz-Occhetta-Solà Conde [MOS]) Let G be a simple and connected algebraic group of classical type and P a parabolic subgroup different from its Borel subgroup. Let M be a smooth complex projective variety such that e.d.$(M) >$ e.d.(G/P), where e.d. (effective good divisibility) is defined in loc cit., §1. Then, there are no nonconstant regular maps from $M \to G/P$.

(g) (due to Bakshi-Parameswaran [BP]) Let P_i be the minimal parabolic subgroup of $\mathrm{SL}(n)$ such that its Levi subgroup has (only) one simple root α_i (following the convention in [Bo, Planche I]. Then, any regular map $f : \mathbb{P}^3 \to \mathrm{SL}(n)/P_i$ is constant for $i \in \{1, n-1\}$. Moreover, any regular map $f : \mathbb{P}^4 \to \mathrm{SL}(n)/P_i$ is constant for any i. Further, there exist non-constant regular maps $f : \mathbb{P}^3 \to \mathrm{SL}(n)/P_i$ for $i \in \{2, \dots, n-2\}$.

Definition 3. Let $X = G/P, X' = G'/P'$ be as in the beginning of Introduction. Define the *minimum* (resp. *maximum*) *semisimple stabilizer rank* of X as the minimum (resp. maximum) of the ranks of the semisimple part of the Levi component of P (for all possible realizations of X as G/P, with G a simple and connected algebraic group and P a parabolic subgroup). Denote these ranks by *minss rank* and *maxss rank* respectively.

Observe that the stabilizer rank of X is equal to the rank of G - rank of the Picard group of X.

Remark 4. The list of non-isogeneous simple and connected G, G' such that $X = G/P \simeq G'/P'$ (for some parabolic subgroups $P \subset G$ and $P' \subset G'$) is as follows (cf. [O] or [D, §2][1]):

1. $G = \mathrm{SL}(2n), G' = \mathrm{Sp}(2n), X = \mathbb{P}^{2n-1}$, (for $n \geq 2$)
2. $G = \mathrm{SO}(7), G' = G_2, X$ is the quadric of dimension 5
3. $G = \mathrm{SO}(2n+2), G' = \mathrm{SO}(2n+1)$ (for $n \geq 2$), X is the variety of isotropic subspaces of dimension n in $\mathbb{C}^{2n+1}$, where $\mathbb{C}^{2n+1}$ is equipped with a non-degenerate quadratic form.

We make the following conjecture:

Conjecture 5. *Let X, X' be two homogeneous projective varieties as in the above Definition. Then,*

(a) Assume that X is different from $\mathbb{P}^{2n}$ (for $n \geq 1$) and

$$\textit{minss rank } X > \textit{maxss rank } X'.$$

Then, there does not exist any non-constant regular map from X to X′.

(b) If $X = \mathbb{P}^{2n}$ (for $n \geq 1$) and there exists a non-constant regular map from $X \to X'$, then

$$\textit{minss rank } \mathbb{P}^{2n-1} = n - 1 \leq \textit{maxss rank } X'.$$

[1]We thank V. Popov for the reference [O] and M. Brion for the reference [D]

Example 6. *(due to J. Landsberg)* Consider the map

$$\mathrm{SL}(4p)/P_1 = \mathbb{P}^{4p-1} \to \mathrm{SO}(6p)/P_1 = Q^{6p-2},\ \ [x_0^1, x_1^1, \ldots, x_0^{2p}, x_1^{2p}] \mapsto$$
$$[(x_0^1)2, (x_1^1)^2, x_0^1 x_1^1, \ldots, (x_0^{2p})^2, x_1^{2p})^2, x_0^{2p} x_1^{2p}],$$

where Q^{6p-2} is the smooth quadric of dimension $6p-2$. Now, restrict this map to a general hyperplane $\mathbb{P}^{4p-2}$ to get a non-constant regular map from $\mathbb{P}^{4p-2} \to Q^{6p-2}$. Observe that minss rank $\mathbb{P}^{4p-2} = 4p-3$ and maxss rank $Q^{6p-2} = 3p-1$.

Remark 7. Let $\{V_t\}$ be a family of rank two vector bundles on $\mathbb{P}^3$ parametrized by the formal disc of one dimension. Assume that the general member of the family is a trivial vector bundle. Then, is the special member V_0 also a trivial vector bundle? This question is a slightly weaker version of a question by Kollár and Peskine on complete intersections of a family of smooth curves in $\mathbb{P}^3$. An affirmative answer of the above question on vector bundles is equivalent to the non-existence of non-constant regular maps from $\mathbb{P}^3 \to \mathcal{X}$, where $\mathcal{X}$ is the infinite Grassmannian associated to affine SL(2) (cf. [Ku2]).

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