

COMPONENTS OF $V(\rho) \otimes V(\rho)$ AND DOMINANT WEIGHT POLYHEDRA FOR AFFINE KAC–MOODY LIE ALGEBRAS

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ABSTRACT. Kostant asked the following question: Let \mathfrak{g} be a simple Lie algebra over the complex numbers. Let λ be a dominant integral weight. Then, $V(\lambda)$ is a component of $V(\rho) \otimes V(\rho)$ if and only if $\lambda \leq 2\rho$ under the usual Bruhat–Chevalley order on the set of weights. In an earlier work with R. Chirivì and A. Maffei the second author gave an affirmative answer to this question up to a saturation factor. The aim of the current work is to extend this result to untwisted affine Kac–Moody Lie algebra \mathfrak{g} associated to any simple Lie algebra \mathfrak{g} (up to a saturation factor). In fact, we prove the result for affine sl_n without any saturation factor. Our proof requires some additional techniques including the Goddard–Kent–Olive construction and study of the characteristic cone of non-compact polyhedra.

1. INTRODUCTION

Let \mathfrak{g} be a symmetrizable Kac–Moody Lie algebra over \mathbb{C} . In particular, we could let \mathfrak{g} be a (finite-dimensional) semisimple Lie algebra, or an untwisted affine Kac–Moody Lie algebra. Given two dominant integral weights λ and μ of \mathfrak{g} , the tensor decomposition problem seeks to understand the irreducible components of $V(\lambda) \otimes V(\mu)$, where $V(\lambda)$ is the irreducible representation of \mathfrak{g} with highest weight λ , and similarly for μ . This is a classic problem, with approaches and applications in representation theory, geometry, conformal field theory, and algebraic combinatorics, just to name a few.

In his study of the exterior algebra $\bigwedge \mathfrak{g}$ of semisimple Lie algebras, Kostant [Kos97] found that as \mathfrak{g} -representations, $\bigwedge \mathfrak{g}$ decomposes as

$$\bigwedge \mathfrak{g} \cong 2^l (V(\rho) \otimes V(\rho)),$$

where l is the rank of \mathfrak{g} and ρ is half the sum of positive roots. Because of the role $\bigwedge \mathfrak{g}$ plays in the computation of Lie algebra homology, and also in the structure of the Clifford algebra $\mathcal{C}(\mathfrak{g})$ as was Kostant’s motivation, one would like to understand the decomposition of the tensor product $V(\rho) \otimes V(\rho)$ into irreducible components. In this direction, Kostant made the following conjecture, first recorded in the work of Berenstein–Zelevinsky, where they also gave a proof of this conjecture when $\mathfrak{g} = sl_{n+1}$ [BZ92].

Conjecture 1.1. *Let \mathfrak{g} be a semisimple Lie algebra over \mathbb{C} . Let $\lambda \in \mathcal{P}^+$ be a dominant integral weight such that $\lambda \leq 2\rho$ in dominance order. Then $V(\lambda) \subset V(\rho) \otimes V(\rho)$.*

Of course, the opposite implication clearly holds. Later, Chirivì–Kumar–Maffei [CKM17] proved a weaker version of Kostant’s conjecture, which recovers Berenstein–Zelevinsky’s result for sl_{n+1} , by showing that the expected components appear in the tensor product decomposition “up to saturation,” in the language of the saturated tensor cone; see Section 4 for the precise definitions.

Theorem 1.2 ([CKM17, Theorem 3]). *Let \mathfrak{g} be a (finite-dimensional) semisimple Lie algebra over \mathbb{C} . Let $\lambda \in \mathcal{P}^+$ be a dominant integral weight such that $\lambda \leq 2\rho$ in dominance order. Then for any saturation factor $d \geq 1$ of \mathfrak{g} , $V(d\lambda) \subset V(d\rho) \otimes V(d\rho)$. In particular, for $\mathfrak{g} = sl_n$, $V(\lambda) \subset V(\rho) \otimes V(\rho)$.*

The proof of Theorem 1.2 in loc. cit. makes use of a set of inequalities due to Berenstein–Sjamaar [BS00] which controls the possible irreducible components of $V(\lambda) \otimes V(\mu)$ up to saturation. The tensor decomposition problem is a particular example of a larger class of branching problems, which

consider the restriction of representations of \mathfrak{g} to an embedded subalgebra $\mathfrak{g}_1 \hookrightarrow \mathfrak{g}$. In the tensor decomposition case, this corresponds to the diagonal embedding $\mathfrak{g} \hookrightarrow \mathfrak{g} \oplus \mathfrak{g}$. A possible representation-theoretic generalization of Conjecture 1.1 is to consider the branching of the representation $V(\rho)$ of \mathfrak{g} to a subalgebra \mathfrak{g}_1 . This was studied by Nadimpalli–Pattanayak [NP22], guided by the approach in [CKM17], generalizing Theorem 1.2 via the corresponding inequalities of the related “branching cone” (see [Kum14] for a survey on the cones and inequalities for the tensor product and more general branching setting).

We now propose a different representation-theoretic generalization; in particular, we consider the tensor decomposition problem for any untwisted affine Kac–Moody Lie algebra \mathfrak{g} . In this setting, one can again try to determine the irreducible components of the tensor product $V(\rho) \otimes V(\rho)$, where ρ now is the sum of fundamental weights (see Section 2 for conventions). However, as the highest weight irreducible representations in this case are infinite-dimensional, and the tensor product $V(\rho) \otimes V(\rho)$ has infinitely-many components, the problem is more subtle. Nevertheless, we prove the following theorem, which is a verbatim generalization of Theorem 1.2 (cf. Theorems 5.6 and 6.2 in the text).

Theorem 1.3. *Let \mathfrak{g} be an untwisted affine Kac–Moody Lie algebra, $\lambda \in \mathcal{P}^+$ a dominant integral weight such that $\lambda \leq 2\rho$ in dominance order. Then,*

- (1) *for any saturation factor $d \geq 1$ of \mathfrak{g} , we have $V(d\lambda) \subset V(d\rho) \otimes V(d\rho)$.*
- (2) *if $\mathfrak{g} = A_n^{(1)} = \widehat{\mathfrak{sl}}_{n+1}$, then $V(\lambda) \subset V(\rho) \otimes V(\rho)$.*

We note that, unlike in the semisimple case, Theorem 1.3(2) does not follow from 1.3(1), as the smallest saturation factor d for $\widehat{\mathfrak{sl}}_{n+1}$ is $d = 2$. While a similar approach via inequalities as in [CKM17] and [NP22] in the affine Kac–Moody setting would be effective to prove Theorem 1.3(1) (see Remark 5.3), such an approach would not address 1.3(2).

Instead, because of its utility in the $\mathfrak{g} = \widehat{\mathfrak{sl}}_{n+1}$ case, our primary tool in proving Theorem 1.3 is the action of the Virasoro algebra on $V(\rho) \otimes V(\rho)$ via the Goddard–Kent–Olive (or GKO) construction, which we recall in Section 4.2. The applicability of the GKO construction to the tensor decomposition problem allows us to restrict attention to certain “maximal” components $V(\nu)$ of $V(\rho) \otimes V(\rho)$, whose existence demonstrates the appearance of the components $V(\nu - k\delta)$ for all $k \geq 0$. This technique originates with Kac–Wakimoto [KW88] and has seen applications in the study of the affine tensor semigroup (see Section 4.1 for references). More recently, the GKO action was crucially used in a similar fashion in our previous work on root components for affine Kac–Moody Lie algebras [JK22].

To apply the GKO construction effectively, we must first understand the dominant weights $\lambda \leq 2\rho$. To do this, we introduce the notion of the *dominant weight polyhedron* D_μ for a highest weight integrable (irreducible) representation $V(\mu)$ of an affine Kac–Moody Lie algebra. In the finite case, these are compact polytopes and are completely describable by their vertices. This description was used in [CKM17, Proposition 9] to express those $\lambda \leq 2\rho$ in a particularly useful form. However, in the affine case, these polyhedra are no longer compact, so cannot be completely determined by their vertices. Taking this into account, we give in Proposition 3.9 an a priori larger set of points that contains the vertices of D_μ for regular dominant μ ; in fact, in Lemma 3.10 we show that these points are precisely the vertices when μ is regular dominant. By a general result for the structure of polyhedra, we then give an explicit decomposition of D_μ for μ regular dominant. With this result, we recover a decomposition result for $\lambda \leq 2\rho$ in Proposition 5.1 analogous to the one in the semisimple case as in [CKM17].

Finally, we apply the GKO construction to the specific problem of determining the components of $V(\rho) \otimes V(\rho)$ up to a saturation factor d as in Theorem 1.3. We further propose an exact analogue of Kostant’s conjecture (without any saturation factor) for affine Kac–Moody Lie algebras in Conjecture 6.3.

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2. NOTATION AND CONVENTIONS FOR AFFINE KAC–MOODY LIE ALGEBRAS

We fix here briefly the key notation and conventions used throughout the paper. For a full treatment, we refer to [Kac90, Chapters 6, 7], whose conventions we adopt.

By \mathbb{Z}_+ (resp. \mathbb{Q}_+) we mean the nonnegative integers (resp., rational numbers).

Throughout, we denote by \mathfrak{g} the untwisted affine Kac–Moody Lie algebra, associated to a finite-dimensional simple Lie algebra \mathfrak{g} . We fix a choice of Cartan subalgebra and Borel subalgebra $\mathfrak{h} \subset \mathfrak{b} \subset \mathfrak{g}$ based on a choice of Cartan subalgebra \mathfrak{h} and a Borel subalgebra $\mathfrak{b} \supset \mathfrak{h}$ of \mathfrak{g} . Relative to this choice, we choose a set of simple roots $\Delta = \{\alpha_0, \alpha_1, \dots, \alpha_l\} = \{\alpha_i\}_{i \in I}$, where $l = \text{rank } \mathfrak{g}$ and $I := \{0, 1, \dots, l\}$. We set Φ the set of roots of \mathfrak{g} and denote by Φ^+ (resp., Φ^-) the set of positive roots (resp., negative roots) for this choice. The set of roots Φ can further be partitioned into the set of real roots Φ_{Re} and imaginary roots Φ_{Im} , with basic imaginary root $\delta := \alpha_0 + \theta$, where θ is the highest root of \mathfrak{g} . We denote by $d, K \in \mathfrak{h}$ the derivation and central elements of \mathfrak{g} , respectively.

The Weyl group of \mathfrak{g} is denoted by W , and has simple reflections $\{s_0, s_1, \dots, s_l\}$ associated to the simple roots.

We let

$$\mathcal{P}^+ := \{\lambda \in \mathfrak{h}^* : \langle \lambda, \alpha_i^\vee \rangle \in \mathbb{Z}_+ \forall i \in I\}$$

be the set of dominant integral weights of \mathfrak{g} . The fundamental weights $\Lambda_i \in \mathcal{P}^+$ are defined uniquely by $\langle \Lambda_i, \alpha_j^\vee \rangle = \delta_{ij}$, the Kronecker delta, for all simple coroots α_j^\vee and $\langle \Lambda_i, d \rangle = 0$. Then the corresponding Weyl vector $\rho \in \mathcal{P}^+$ is given by $\rho = \sum_{i=0}^l \Lambda_i$ (so that $\langle \rho, d \rangle = 0$). For any $\lambda \in \mathcal{P}^+$, the value of $\langle \lambda, K \rangle$ is called the level of λ ; throughout, we will restrict to those $\lambda \in \mathcal{P}^+$ with **positive** level.

The weight lattice \mathcal{P} and root lattice \mathcal{Q} are given by

$$\mathcal{P} := \bigoplus_{i=0}^l \mathbb{Z}\Lambda_i \oplus \mathbb{C}\delta, \quad \mathcal{Q} := \bigoplus_{i=0}^l \mathbb{Z}\alpha_i.$$

Finally, the similarly defined objects for \mathfrak{g} will be denoted by a circle; i.e., the root lattice $\mathring{\mathcal{Q}}$ of \mathfrak{g} , etc. We fix also a normalized invariant form $(\cdot|\cdot)$ on \mathfrak{g} such that $(\theta|\theta) = 2$, where again $\theta \in \mathring{\Phi}^+$ is the highest root of the underlying semisimple Lie algebra \mathfrak{g} ; we will always use this choice of normalization.

3. IRREDUCIBLE HIGHEST WEIGHT REPRESENTATIONS AND DOMINANT WEIGHT POLYHEDRA

In this section, we recall some basic results on the structure of irreducible highest weight representations of \mathfrak{g} . While many of the results mentioned hold more generally at the level of symmetrizable Kac–Moody algebras, we focus only on statements for the affine Lie algebra setting. We use [Kac90, Chapters 11, 12] as our primary sources.

3.1. Weights of irreducible highest weight modules. Let $\lambda \in \mathcal{P}^+$ be a dominant integral weight of \mathfrak{g} . Then we denote by $V(\lambda)$ the corresponding irreducible, highest weight representation of \mathfrak{g} with highest weight λ . As an \mathfrak{h} -module via restriction, $V(\lambda)$ has a decomposition into weight spaces

$$V(\lambda) = \bigoplus_{\mu \in \mathcal{P}} V(\lambda)_\mu,$$

where $V(\lambda)_\mu := \{v \in V(\lambda) : h.v = \langle \mu, h \rangle v \forall h \in \mathfrak{h}\}$. We set $\mathcal{P}(\lambda)$ to be the set of weights of the representation $V(\lambda)$; that is, $\mathcal{P}(\lambda) = \{\mu \in \mathcal{P} : V(\lambda)_\mu \neq 0\}$. This is a (typically infinite) W -invariant set of weights in \mathcal{P} .

For any two weights $\lambda, \mu \in \mathcal{P}$, we say that $\lambda \geq \mu$ if $\lambda - \mu \in \mathcal{Q}^+ := \bigoplus_{i=0}^l \mathbb{Z}_+ \alpha_i$. This defines a partial order on the weight lattice \mathcal{P} , the *dominance order*. As $V(\lambda)$ is a highest weight representation, for

any $\mu \in \mathcal{P}(\lambda)$, necessarily $\mu \leq \lambda$. While it is not immediately clear from an algebraic perspective for which $\mu \leq \lambda$ we have $\mu \in \mathcal{P}(\lambda)$, this question has a nice combinatorial answer. We record now the following standard proposition, taken from [Kac90, Proposition 12.5(a)(b)], which gives a concise description of $\mathcal{P}(\lambda)$.

Proposition 3.1. *For $\lambda \in \mathcal{P}^+$, let $V(\lambda)$ be the irreducible highest weight representation with highest weight λ , and set of weights $\mathcal{P}(\lambda)$. Then*

$$\mathcal{P}(\lambda) = W \cdot \{\mu \in \mathcal{P}^+ : \mu \leq \lambda\} = (\lambda + \mathcal{Q}) \cap \text{conv}_{\mathbb{Q}}\{w\lambda : w \in W\},$$

where $\text{conv}_{\mathbb{Q}}$ denotes the rational convex hull.

3.2. δ -maximal weights. Let $\lambda \in \mathcal{P}^+$ be a dominant integral weight, and consider the associated highest weight irreducible representation $V(\lambda)$ with set of weights $\mathcal{P}(\lambda)$. We recall the following definition.

Definition 3.2. *A weight $\mu \in \mathcal{P}(\lambda)$ is called δ -maximal if $\mu + k\delta \notin \mathcal{P}(\lambda)$ for any $k > 0$. We denote by $\mathcal{P}_{max}(\lambda)$ the set of all δ -maximal weights.*

Since δ is fixed by all Weyl group elements $w \in W$, the set of δ -maximal weights in $\mathcal{P}(\lambda)$ is W -invariant. But as any weight in $\mathcal{P}(\lambda)$ can be W -translated to a dominant weight (cf. [Kum02, Proposition 1.4.2(c)] together with Proposition 3.1), it suffices to understand the set of δ -maximal dominant weights; we denote this set by $\mathcal{P}_{max}^+(\lambda)$. The study of δ -maximal dominant weights plays a crucial role in applications of representation theory to modular forms and conformal field theory, via understanding (up to renormalization) the “string functions”

$$t \mapsto \dim(V(\lambda)_{\mu-t\delta})$$

for $\mu \in \mathcal{P}_{max}^+(\lambda)$ (see [Kac90, Chapter 12]). As these will play an important role in what is to come, we record the following lemma on the “ δ -strings” $\{\mu - k\delta\}_{k \geq 0} \subset \mathcal{P}(\lambda)$, which is a reformulation of [Kac90, Prop. 12.5(e)].

Lemma 3.3. *Let $\lambda \in \mathcal{P}^+$ and $\mu \in \mathcal{P}(\lambda)$. Then $\mu - k\delta \in \mathcal{P}(\lambda)$ for all $k \geq 0$.*

Combining Lemma 3.3 and Proposition 3.1, we can get the following description of δ -maximal dominant weights, taken from [BK14, Proposition 4.4].

Proposition 3.4. *Let $\lambda \in \mathcal{P}^+$. Then $\mathcal{P}_{max}^+(\lambda) = \{\mu \in \mathcal{P}^+ : \mu \leq \lambda \text{ \& } \lambda - \mu = \sum_{i \in I} c_i \alpha_i, \exists i \in I : c_i < a_i\}$, where $\{a_i\}$ are defined by $\delta = \sum_{i \in I} a_i \alpha_i$.*

Proof. Suppose for contradiction that $\mu \in \mathcal{P}_{max}^+(\lambda)$ and $\lambda - \mu = \sum_{i \in I} c_i \alpha_i$ with $c_i \geq a_i$ for all i . Set $\mu' := \mu + \delta \in \mathcal{P}^+$. Then we have

$$\lambda - \mu' = \lambda - \mu - \delta = \sum_{i \in I} (c_i - a_i) \alpha_i.$$

Note that since $c_i - a_i \geq 0$ for all i by assumption, we get that $\lambda - \mu' \in \mathcal{Q}^+$, or that $\mu' \leq \lambda$ in dominance order. But then by Proposition 3.1, we get that $\mu' \in \mathcal{P}(\lambda)$, which contradicts μ being δ -maximal. The converse is clear. \square

3.3. Dominant weight polyhedra. We next want to better understand the dominant weights $\mu \in \mathcal{P}^+$ such that $\mu \in \mathcal{P}(\lambda)$. By Lemma 3.3, it would suffice to understand those weights in $\mathcal{P}_{max}^+(\lambda)$. However, this is in general a subtle issue, as it is not straightforward to describe uniformly the set $\mathcal{P}_{max}^+(\lambda)$ in terms of λ . We instead consider a larger set, which we refer to as the *dominant weight polyhedron* associated to λ , and denote it by D_λ . This will be a rational, (generically) unbounded polyhedron in the sense of [Sch99], which we take as our standard reference.

We restrict for our purposes to the case when $\lambda \in \mathcal{P}^+$ is regular dominant, so that $\langle \lambda, \alpha_i^\vee \rangle > 0$ for all $i \in I := \{0, 1, \dots, l\}$. We will also assume that $\langle \lambda, d \rangle = 0$, up to replacing λ with $\lambda - \langle \lambda, d \rangle \delta$. Let $\mathcal{C}_{\mathbb{Q}}$ denote the rational dominant chamber, defined by

$$\mathcal{C}_{\mathbb{Q}} := \{\mu \in \mathfrak{h}_{\mathbb{Q}}^* : \langle \mu, \alpha_i^\vee \rangle \geq 0 \forall i \in I\},$$

where $\mathfrak{h}_{\mathbb{Z}}^* := (\oplus_{i=0}^l \mathbb{Z}\Lambda_i) \oplus \mathbb{Z}\delta$ and $\mathfrak{h}_{\mathbb{Q}}^* := \mathbb{Q} \otimes_{\mathbb{Z}} \mathfrak{h}_{\mathbb{Z}}^*$. We now fix the following definition.

Definition 3.5. *For a dominant integral weight $\lambda \in \mathcal{P}^+$, the dominant weight polyhedron D_λ is defined by*

$$D_\lambda := \mathcal{C}_{\mathbb{Q}} \cap \text{conv}_{\mathbb{Q}}\{w\lambda : w \in W\}.$$

Thus, by Proposition 3.1, the points $\mu \in D_\lambda$ correspond to rational dominant weights such that $N\mu \in \mathcal{P}(N\lambda) \cap \mathcal{P}^+$ for some scaling factor $N \in \mathbb{Z}_{>0}$. Again by Proposition 3.1, the points $\mu \in D_\lambda$ are determined by the set of inequalities for $\mu \in \mathfrak{h}_{\mathbb{Q}}^*$:

$$\begin{cases} \langle \mu, \alpha_i^\vee \rangle \geq 0 \quad \forall i \in I, \\ \langle \lambda - \mu, x_i \rangle \geq 0 \quad \forall i \in I, \\ \langle \lambda - \mu, K \rangle = 0, \end{cases}$$

where x_i is the fundamental coweight defined by $\langle \alpha_j, x_i \rangle = \delta_{i,j}$. The faces of D_λ are given by taking a subset of the above inequalities with equality. To this end, let $J \subset I$ and consider the sets

$$A_J := \sum_{k \in J} \mathbb{Q}_+ \Lambda_k,$$

$$\bar{A}_J := A_J + \mathbb{Q}_+(-\delta),$$

$$B_J := \sum_{k \in J} \mathbb{Q}_+ \alpha_k,$$

$$C_J = C_J(\lambda) := \lambda - B_J.$$

We isolate $\bar{A} = \bar{A}_I = \sum_{k \geq 0} \mathbb{Q}_+ \Lambda_k + \mathbb{Q}_+(-\delta)$, $B = B_I = \sum_{k \geq 0} \mathbb{Q}_+ \alpha_k$, and $C = C_I = \lambda - B$. Then, $\bar{A}_H \cap C_K$ are faces of D_λ for $H, K \subset I$. By construction, we have that

$$(1) \quad D_\lambda = \bar{A} \cap C;$$

note that as any $\mu \in D_\lambda$ satisfies $\langle \mu, d \rangle \leq \langle \lambda, d \rangle$ since $N\mu \leq N\lambda$ (for some positive integer N) in dominance order, the restriction to $\mathbb{Q}_+(-\delta)$ in the definition of \bar{A} suffices.

Unlike when considering a finite-dimensional semisimple Lie algebra \mathfrak{g} , the polyhedra D_λ are typically unbounded. First, we want to determine the ‘‘infinite directions’’ of the polyhedron D_λ . We introduce the following definition from [Sch99, Section 8.2].

Definition 3.6. *For a polyhedron P , the characteristic cone $\text{char. cone}(P)$ is given by*

$$\text{char. cone}(P) := \{y : x + y \in P \quad \forall x \in P\}.$$

Lemma 3.7. *For $\lambda \in \mathcal{P}^+$, we have $\text{char. cone}(D_\lambda) = \mathbb{Q}_+(-\delta)$.*

Proof. As in (1), $D_\lambda = \bar{A} \cap C$. Clearly $\mathbb{Q}_+(-\delta) \subseteq \text{char. cone}(\bar{A} \cap C)$. Let $y \in \mathfrak{h}_{\mathbb{Q}}^*$ such that $y \in \text{char. cone}(\bar{A} \cap C)$. Since $\lambda \in \bar{A} \cap C$, and by the definition of C , we immediately conclude $y \in -\sum_{k \geq 0} \mathbb{Q}_+ \alpha_k$.

Assume that $\langle y, \alpha_j^\vee \rangle < 0$ for some j . Then for any $x \in \bar{A} \cap C$, we have $x + Ny \in \bar{A} \cap C$ for any $N \in \mathbb{Z}_+$. Taking N sufficiently large, we get that

$$\langle x + Ny, \alpha_j^\vee \rangle = \langle x, \alpha_j^\vee \rangle + N\langle y, \alpha_j^\vee \rangle < 0.$$

Contradiction, since $x + Ny \in \bar{A}$, and any $z \in \bar{A}$ satisfies $\langle z, \alpha_j^\vee \rangle \geq 0$ for all j . Thus, $\langle y, \alpha_j^\vee \rangle \geq 0$ for all j .

Next, suppose that $\langle y, \alpha_j^\vee \rangle > 0$ for some j . Then, for the canonical central element K of \mathfrak{g} , since we can write $K = \sum_{k \geq 0} a_k^\vee \alpha_k^\vee$ for the appropriate positive labels a_k^\vee , and as y pairs nonnegatively with all α_k^\vee , we get

$$\langle y, K \rangle > 0;$$

contradiction, since $y \in -\sum_{k \geq 0} \mathbb{Q}_+ \alpha_k$, and α_k pairs to zero with K for all $k \geq 0$.

Thus, $\langle y, \alpha_k^\vee \rangle = 0$ for all $k \geq 0$. Now, using that $\mathfrak{h}_\mathbb{Q}^* = \mathfrak{h}_\mathbb{Q}^* \oplus (\mathbb{Q}(\delta) \oplus \mathbb{Q}(\Lambda_0))$, we get that

$$y \in \mathfrak{h}_\mathbb{Q}^* \oplus \mathbb{Q}(\delta)$$

and is orthogonal to all of $\mathfrak{h}_\mathbb{Q}^*$; thus necessarily $y \in \mathbb{Q}(\delta)$. In particular, this forces $y \in \mathbb{Q}_+(-\delta)$, as desired. \square

We are next interested in the minimal faces of D_λ . By the general theory of polyhedra [Sch99, Section 8.5(22)], the minimal faces of a polyhedron P are translates of the ‘‘lineality space’’

$$\text{lin. space}(P) := \text{char. cone}(P) \cap -\text{char. cone}(P).$$

Then by Lemma 3.7, in the case of D_λ , we get that $\text{lin. space}(D_\lambda) = \{0\}$. Thus the minimal faces of D_λ are zero-dimensional, hence vertices. To understand these vertices, we will examine the various face intersections $A_H \cap C_K$ and $\overline{A}_H \cap C_K$, for subsets $H, K \subseteq I$, and analyze those which intersect in single points. First, in the next lemma we introduce the possible candidates for the vertices of D_λ .

Lemma 3.8. *Let $\lambda \in \mathcal{P}^+$ be a regular dominant integral weight. For any subset $J \subsetneq I$, let $W_J = \langle s_j : j \in J \rangle \subset W$ be the corresponding (finite) parabolic subgroup corresponding to the simple roots $\{\alpha_j\}_{j \in J}$. Define*

$$\hat{b}_J(\lambda) := \frac{1}{|W_J|} \sum_{w \in W_J} w(\lambda).$$

Then $\hat{b}_J(\lambda) \in D_\lambda$. In fact, for each $J \subsetneq I$ we have $\hat{b}_J(\lambda) \in \overline{A}_{I \setminus J}$, and $\hat{b}_J(\lambda) \neq \hat{b}_{J'}(\lambda)$ for $J \neq J'$.

Proof. First, we show that $\hat{b}_J(\lambda) \in D_\lambda$. To this end, by its definition, $\hat{b}_J(\lambda) \in \text{conv}_\mathbb{Q}\{w\lambda : w \in W\}$. It remains to show that $\hat{b}_J(\lambda) \in \mathcal{C}_\mathbb{Q}$. Let α_i^\vee be a simple coroot. We consider two cases: first, suppose that $i \in J$. Then we have that

$$(2) \quad |W_J| \langle \hat{b}_J(\lambda), \alpha_i^\vee \rangle = \left\langle \sum_{w \in W_J} w(\lambda), \alpha_i^\vee \right\rangle = \left\langle \lambda, \sum_{w \in W_J} w^{-1}(\alpha_i^\vee) \right\rangle = 0,$$

since $i \in J$ so that $\sum_{w \in W_J} w^{-1}(\alpha_i^\vee) = 0$. Else if $i \notin J$, then for all $w \in W_J$, we have $w^{-1}(\alpha_i^\vee)$ is a positive coroot, so that

$$(3) \quad |W_J| \langle \hat{b}_J(\lambda), \alpha_i^\vee \rangle = \left\langle \sum_{w \in W_J} w(\lambda), \alpha_i^\vee \right\rangle = \left\langle \lambda, \sum_{w \in W_J} w^{-1}(\alpha_i^\vee) \right\rangle > 0,$$

since λ is regular dominant. Thus, $\hat{b}_J(\lambda) \in \mathcal{C}_\mathbb{Q}$, as desired. In total, this says $\hat{b}_J(\lambda) \in D_\lambda$.

Further, since by (2) and (3), $\hat{b}_J(\lambda)$ is a rational dominant weight that vanishes precisely on $\{\alpha_j^\vee\}_{j \in J}$, we get that $\hat{b}_J(\lambda) \in \overline{A}_{I \setminus J}$ by definition, since $\langle w\lambda, d \rangle \leq \langle \lambda, d \rangle = 0$.

Finally, if $j \in I \setminus J'$ we get that $\langle \hat{b}_J(\lambda), \alpha_j^\vee \rangle = 0$ while $\langle \hat{b}_{J'}(\lambda), \alpha_j^\vee \rangle > 0$, by (2) and (3), so that $\hat{b}_J(\lambda) \neq \hat{b}_{J'}(\lambda)$. \square

Proposition 3.9. *Let $\lambda \in \mathcal{P}^+$ be a regular dominant integral weight, and assume that $\langle \lambda, d \rangle = 0$. Then the vertices of D_λ are contained in the set $\{\hat{b}_J(\lambda)\}_{J \subsetneq I}$.*

Proof. As shown above, the minimal faces of D_λ are points, hence vertices. We show that they are contained in the set $\{\hat{b}_J(\lambda)\}_{J \subsetneq I}$. This is done by considering the following ten claims.

Claim 1. *If $H \not\subseteq K$, then $\overline{A_{I \setminus H}} \cap C_K = \emptyset$.*

Take $y \in \overline{A_{I \setminus H}} \cap C_K$ and write $y = \lambda - x$ for some $x \in B_K$. Write

$$(4) \quad x = \lambda - y = \lambda - \sum_{h \notin H} a_h \Lambda_h + b\delta$$

where $x \in \sum_{k \in K} \mathbb{Q}_+ \alpha_k$, $a_h, b \in \mathbb{Q}_+$. Take $j \in H \setminus K$. Evaluating (4) at α_j^\vee , we get a contradiction.

Claim 2. *If $H \subsetneq K$ and $0 \notin K$, then $A_{I \setminus H} \cap C_K$ contains $\hat{b}_H(\lambda)$ and $\hat{b}_K(\lambda)$.*

Clearly $\hat{b}_H(\lambda), \hat{b}_K(\lambda) \in C_K$. By Lemma 3.8, we know that $\hat{b}_H(\lambda), \hat{b}_K(\lambda) \in \overline{A_{I \setminus H}}$. Now, since $0 \notin K$ (and thus also not in H), we get that $\langle \hat{b}_K(\lambda), d \rangle = \langle \hat{b}_H(\lambda), d \rangle = 0$, so that $\hat{b}_K(\lambda), \hat{b}_H(\lambda) \in A_{I \setminus H}$.

Claim 3. *For any subsets $H, K \subset I$, $A_{I \setminus H} \cap C_K = A_{I \setminus H} \cap C_{K \setminus \{0\}}$.*

This follows easily since for any $x \in A_{I \setminus H}$, $\langle x, d \rangle = 0$, whereas $\langle \alpha_0, d \rangle = 1$.

Claim 4. *For $H \subsetneq K \subsetneq I$ and $0 \in H$, $\hat{b}_H(\lambda), \hat{b}_K(\lambda) \in \overline{A_{I \setminus H}} \cap C_K$ and $A_{I \setminus H} \cap C_K = \emptyset$.*

Of course, $\hat{b}_H(\lambda), \hat{b}_K(\lambda) \in C_K$, and by Lemma 3.8, $\hat{b}_H(\lambda), \hat{b}_K(\lambda) \in \overline{A_{I \setminus H}}$. For the second part, by Claim 3, $A_{I \setminus H} \cap C_K = A_{I \setminus H} \cap C_{K \setminus \{0\}} = \emptyset$, by Claim 1 since $0 \in H$.

Claim 5. *For $H \subsetneq K \subsetneq I$ and $0 \in K \setminus H$, $\hat{b}_H(\lambda), \hat{b}_{K \setminus \{0\}}(\lambda) \in A_{I \setminus H} \cap C_K$, even if $H = K \setminus \{0\}$.*

By Claim 3, $A_{I \setminus H} \cap C_K = A_{I \setminus H} \cap C_{K \setminus \{0\}}$. Further $H \subset K \setminus \{0\}$ since $0 \notin H$. Thus, the claim follows from Claim 2. (Since $0 \notin H$, it is easy to see that $\hat{b}_H(\lambda) \in A_{I \setminus H}$.)

Claim 6. *For $H \subsetneq K = I$ and $H \subset I \setminus \{i_1, i_2\}$, none of $A_{I \setminus H} \cap C_I$ or $\overline{A_{I \setminus H}} \cap C_I$ contain exactly one point.*

If $i_1 = 0$, $A_{I \setminus H} \cap C_I = A_{I \setminus H} \cap C_{I \setminus \{0\}}$ contains at least two points by Claim 2 and hence so does $\overline{A_{I \setminus H}} \cap C_I$. If neither of i_1, i_2 is 0 (i.e., we can assume that $0 \in H$), then

$$A_{I \setminus H} \cap C_I = A_{I \setminus H} \cap C_{I \setminus \{0\}} = \emptyset,$$

where the equalities follow from Claims 3 and 1 respectively. Further,

$$\overline{A_{I \setminus H}} \cap C_I \supset \overline{A_{I \setminus H}} \cap C_{I \setminus \{i_1\}} \supset \{\hat{b}_H(\lambda), \hat{b}_{I \setminus \{i_1\}}(\lambda)\}$$

by Claim 4.

Claim 7. *For $H = I \setminus \{i_0\}$, $K = I$: if $i_0 = 0$, then $A_{I \setminus H} \cap C_I$ contains the point $\hat{b}_{I \setminus \{0\}}(\lambda)$. If $i_0 \neq 0$, then $A_{I \setminus H} \cap C_I = \emptyset$ and $\overline{A_{I \setminus H}} \cap C_I$ contains the point $\hat{b}_{I \setminus \{i_0\}}(\lambda)$.*

Let $i_0 = 0$. By Lemma 3.8, we have $\hat{b}_{I \setminus \{0\}}(\lambda) \in \overline{A_{I \setminus \{0\}}}$. But since $H = I \setminus \{0\}$, necessarily $\langle \hat{b}_{I \setminus \{0\}}(\lambda), d \rangle = \langle \lambda, d \rangle = 0$, so in fact $\hat{b}_{I \setminus \{0\}}(\lambda) \in A_{I \setminus \{0\}}$. Of course, $\hat{b}_{I \setminus \{0\}}(\lambda) \in C_I$.

For $i_0 \neq 0$, by Claim 3, $A_{I \setminus H} \cap C_I = A_{I \setminus H} \cap C_{I \setminus \{0\}}$, and since $0 \in H$, by Claim 1 this is empty. Now, again by Lemma 3.8, $\hat{b}_{I \setminus \{i_0\}}(\lambda) \in \overline{A_{I \setminus H}} \cap C_I$ for $i_0 \neq 0$.

Claim 8. *For any H such that $0 \notin H$, $A_{I \setminus H} \cap C_H = \overline{A_{I \setminus H}} \cap C_H$ contains the point $\hat{b}_H(\lambda)$.*

The identity $A_{I \setminus H} \cap C_H = \overline{A_{I \setminus H}} \cap C_H$ follows since, for any $y \in C_H$, $\langle y, d \rangle = 0$. Further, $\hat{b}_H(\lambda) \in C_H$ by construction and $\hat{b}_H(\lambda) \in \overline{A_{I \setminus H}}$ by Lemma 3.8.

Claim 9. *For any H such that $0 \in H \subsetneq I$, $A_{I \setminus H} \cap C_H = \emptyset$ and $\overline{A_{I \setminus H}} \cap C_H$ contains the point $\hat{b}_H(\lambda)$.*

$A_{I \setminus H} \cap C_H = A_{I \setminus H} \cap C_{H \setminus \{0\}}$, by Claim 3. But by Claim 1, this is empty. Finally, $\hat{b}_H(\lambda) \in C_H$ by construction, and by Lemma 3.8, $\hat{b}_H(\lambda) \in \overline{A_{I \setminus H}}$.

Claim 10. $\overline{A}_{\Gamma \setminus I} \cap C_I = \emptyset$.

We need to find $\{a_i\}_{i \in I}$, $a_i \geq 0$, such that

$$\lambda - \sum_{i \in I} a_i \alpha_i \in \mathbb{Q}_+(-\delta).$$

Now, the above implies $\langle \lambda, \alpha_j^\vee \rangle = \sum_{i \in I} a_i \langle \alpha_i, \alpha_j^\vee \rangle$ for all α_j^\vee . But this can be rewritten as

$$(5) \quad \Lambda = A \cdot C,$$

where $\Lambda = (\lambda_0, \lambda_1, \lambda_2, \dots, \lambda_l)$, for $\lambda_j := \langle \lambda, \alpha_j^\vee \rangle > 0$, $A = (a_0, a_1, \dots, a_l)$, and $C = (\alpha_i(\alpha_j^\vee))_{i,j \geq 0}$ is the affine Cartan matrix. But (5) is contradicted by [Kac90, Theorem 4.3].

Combining all the cases covered by Claims 1–10, we get that if any of $A_{\Gamma \setminus H} \cap C_K$ or $\overline{A}_{\Gamma \setminus H} \cap C_K$ for any $H, K \subseteq I$ is a single point, then that point is one of the points from the set $\{\hat{b}_H(\lambda)\}_{H \subsetneq I}$. \square

Lemma 3.10. *Following the notation and assumptions as in Proposition 3.9, the set of vertices of D_λ is equal to (not just contained in) $\{\hat{b}_H(\lambda)\}_{H \subsetneq I}$.*

Proof. Take any $H \subsetneq I$ and let $\mu \in \overline{A}_{\Gamma \setminus H} \cap C_H$. Then, we can write

$$\mu = \sum_{i \notin H} m_i \Lambda_i - m\delta = \lambda - \sum_{k \in H} n_k \alpha_k, \text{ for some } m_i, m, n_k \in \mathbb{Q}_+.$$

Evaluating the above at α_i^\vee , for $i \in H$, we get:

$$(6) \quad \langle \lambda, \alpha_i^\vee \rangle - \sum_{k \in H} n_k \langle \alpha_k, \alpha_i^\vee \rangle = 0, \text{ for any } i \in H.$$

Consider the row matrices $\vec{\lambda}_H := (\langle \lambda, \alpha_i^\vee \rangle)_{i \in H}$; $\vec{n} := (n_k)_{k \in H}$ and the $|H| \times |H|$ -matrix $C_H := (\langle \alpha_k, \alpha_i^\vee \rangle)_{k,i \in H}$. Since H is a proper subset of I , C_H is the Cartan matrix of a semisimple Lie algebra. In particular, it is invertible. Hence the above equation (6) gives

$$\vec{\lambda}_H = \vec{n} \cdot C_H, \text{ i.e., } \vec{\lambda}_H \cdot C_H^{-1} = \vec{n}.$$

Thus, \vec{n} is unique if it exists satisfying $n_k \in \mathbb{Q}_+$ for all $k \in H$. This shows that $\overline{A}_{\Gamma \setminus H} \cap C_H$ has at most one point. However, as shown in the Claims 8 and 9 of the proof of Proposition 3.9, it has at least one point $\hat{b}_H(\lambda)$. This proves the lemma. \square

Specializing to $\lambda = 2\rho$, we can simplify the above to get the following corollary.

Corollary 3.11. *The vertices of $D_{2\rho}$ are given by the set $\{\hat{b}_J(2\rho)\}_{J \subsetneq I}$, where*

$$\hat{b}_J(2\rho) = \rho + w_0^J(\rho),$$

with $w_0^J \in W_J$ the longest element.

Proof. By definition, we have

$$\hat{b}_J(2\rho) = \frac{1}{|W_J|} \sum_{w \in W_J} w(2\rho).$$

This we rewrite as

$$\hat{b}_J(2\rho) = 2\rho + \frac{1}{|W_J|} \sum_{w \in W_J} (w(2\rho) - 2\rho) = 2\rho + \frac{2}{|W_J|} \sum_{w \in W_J} w(\rho) - \rho.$$

Now, we can write

$$w(\rho) - \rho = - \sum_{\beta \in \Phi_J(w)} \beta,$$

where $\Phi_J(w) := \{\beta \in \Phi_J^+ : w^{-1}(\beta) \in -\Phi_J^+\}$ and Φ_J^+ are the positive roots in the sub-root system corresponding to J . Every root $\beta \in \Phi_J^+$ appears in precisely half of the sets $\Phi_J(w)$ as w varies across W_J (since for any $\beta \in \Phi_J^+$ and $w \in W_J$, β belongs to exactly one of $\Phi_J(w)$ or $\Phi_J(w w_0^J)$); thus we get

$$\hat{b}_J(2\rho) = 2\rho - \frac{2}{|W_J|} \sum_{\beta \in \Phi_J^+} \frac{|W_J|}{2} \beta = 2\rho - \sum_{\beta \in \Phi_J^+} \beta = 2\rho + (w_0^J(\rho) - \rho) = \rho + w_0^J(\rho),$$

as desired. \square

Finally, by the decomposition theorem for polyhedra in terms of minimal faces and the characteristic cone (cf. [Sch99, Section 8.9]), we get the following crucial proposition.

Proposition 3.12. *For any $\lambda \in \mathcal{P}^+$ regular dominant, the dominant weight polyhedron D_λ decomposes as*

$$D_\lambda = \text{conv}_{\mathbb{Q}}\{\hat{b}_J(\lambda)\}_{J \subseteq I} + \mathbb{Q}_+(-\delta).$$

Remark 3.13. *In the case of a finite-dimensional semisimple Lie algebra, the vertices of the corresponding dominant weight polytope for 2ρ were worked out by Kostant, as reported in [BZ92]. For any dominant weight, formulas for the vertices resembling those in Proposition 3.9 were given in [BJK21]. In this case, the vertices completely determine the polytope, as it is compact. While we will not need it here, we conjecture the following stronger results.*

Conjecture 3.14. *Let $\lambda \in \mathcal{P}^+$ be an arbitrary (not necessarily regular) dominant integral weight with positive level. Then the vertices of D_λ are given precisely by the set $\{\hat{b}_J(\lambda)\}_{J \subseteq I}$ (which may have repeated elements), and D_λ decomposes as*

$$D_\lambda = \text{conv}_{\mathbb{Q}}\{\hat{b}_J(\lambda)\} + \mathbb{Q}_+(-\delta).$$

4. TENSOR PRODUCT OF IRREDUCIBLE REPRESENTATIONS AND THE GODDARD–KENT–OLIVE CONSTRUCTION

In this section, we will review some of the key features of the tensor decomposition problem, which seeks to understand the irreducible components of $V(\lambda) \otimes V(\mu)$ for $\lambda, \mu \in \mathcal{P}^+$. We will also briefly introduce an action of the Virasoro algebra on $V(\lambda) \otimes V(\mu)$, given by the *Goddard–Kent–Olive* (or GKO) construction, and its application to the tensor decomposition problem.

Throughout, we will take $\lambda, \mu \in \mathcal{P}^+$ with $\lambda(d) = \mu(d) = 0$, without loss of generality, as we could replace λ (and similarly μ) up to twisting by $k\delta$ via $V(\lambda) \otimes V(k\delta) = V(\lambda + k\delta)$.

4.1. The tensor semigroup. We return now to the tensor decomposition problem, as discussed in the introduction. To this end, consider $\lambda, \mu \in \mathcal{P}^+$ and the tensor product decomposition

$$V(\lambda) \otimes V(\mu) \cong \bigoplus_{\nu \in \mathcal{P}^+} V(\nu)^{m_{\lambda, \mu}^{\nu}},$$

where $m_{\lambda, \mu}^{\nu} \in \mathbb{Z}_+$ is the multiplicity of the subrepresentation $V(\nu)$ in $V(\lambda) \otimes V(\mu)$. While the problem of determining for which ν we have $m_{\lambda, \mu}^{\nu} \neq 0$ (or in fact determining $m_{\lambda, \mu}^{\nu}$ precisely) has a long history which has developed exact methods to give exact solutions (crystals, the Littelmann path model, among others), predicting components which appear in $V(\lambda) \otimes V(\mu)$ based only off of the pair (λ, μ) is challenging. One such example of components, known as the Parthasarathy–Ranga Rao–Varadarajan or PRV components, is given in the following theorem originally due to Kumar [Kum88] for semisimple Lie algebras. Its extension to any symmetrizable Kac–Moody Lie algebra was proved in [Kum89] and Mathieu [Mat89].

Theorem 4.1. *Let $\lambda, \mu \in \mathcal{P}^+$ be two dominant integral weights and $w, v \in W$ two Weyl group elements such that $\nu := w\lambda + v\mu \in \mathcal{P}^+$. Then $V(\nu) \subset V(\lambda) \otimes V(\mu)$.*

As above, we are interested in the set of triples $\{(\lambda, \mu; \nu) \in (\mathcal{P}^+)^3 : m_{\lambda, \mu}^\nu \neq 0\}$. This set in fact forms a semigroup, which we refer to as the tensor semigroup for \mathfrak{g} . This follows, for example, from the analogue of the Borel–Weil theorem for symmetrizable Kac–Moody Lie algebras ([Kum02, Corollary 8.3.12]). The tensor semigroup and its generators are difficult to understand, even in the finite-dimensional semisimple case. We consider a simpler object, known as the saturated tensor semigroup or saturated tensor cone, in the following definition.

Definition 4.2. *The saturated tensor cone associated to \mathfrak{g} , denoted $\Gamma(\mathfrak{g})$, is given by*

$$\Gamma(\mathfrak{g}) := \{(\lambda, \mu; \nu) \in (\mathcal{P}_{\mathbb{Q}}^+)^3 : \exists N \geq 1 \text{ with } m_{N\lambda, N\mu}^{N\nu} \neq 0\};$$

that is, $V(N\nu) \subset V(N\lambda) \otimes V(N\mu)$ for some $N \geq 1$, where

$$\mathcal{P}_{\mathbb{Q}}^+ := \{\lambda \in \mathfrak{h}^* : \langle \lambda, \alpha_i^\vee \rangle \in \mathbb{Q}_+ \forall i \in I\}.$$

For a finite-dimensional semisimple Lie algebra \mathfrak{g} , the saturated tensor cone $\Gamma(\mathfrak{g})$ has been studied extensively using geometric methods; we refer to the survey [Kum14]. In this case, $\Gamma(\mathfrak{g})$ is a closed polyhedral convex cone with inequalities determined by data coming from the cohomology of the associated flag varieties. An important notion in this topic is that of a saturation factor for $\Gamma(\mathfrak{g})$, which we define below.

Definition 4.3. *We say that a positive integer $d \geq 1$ is a saturation factor of \mathfrak{g} (or equivalently of $\Gamma(\mathfrak{g})$) if, for any $(\lambda, \mu; \nu) \in \Gamma(\mathfrak{g}) \cap (\mathcal{P}^+)^3$, we have $V(d\nu) \subset V(d\lambda) \otimes V(d\mu)$.*

For example, the saturation theorem of Knutson and Tao [KT99] says that $d = 1$ is a saturation factor for $\Gamma(\mathfrak{sl}_{n+1})$. The saturation conjecture of Kapovich and Millson [KM06] posits that $d = 1$ should also be a saturation factor for any simply-laced simple Lie algebra; it is known that the minimal saturation factor for the non-simply laced types satisfy $d > 1$.

When \mathfrak{g} is an affine Kac–Moody Lie algebra, $\Gamma(\mathfrak{g})$ is no longer closed nor polyhedral (the semigroup is infinitely generated); however, a similar set of inequalities governing the points of $\Gamma(\mathfrak{g})$ were conjectured by Brown and Kumar [BK14] and later proven by Ressayre [Res22] in this setting. Further, it is not a priori clear that a saturation factor d (analogously defined as in Definition 4.3) should exist for $\Gamma(\mathfrak{g})$. Nevertheless, in loc. cit. Ressayre provides saturation factors in this setting, again building on the work of Brown and Kumar, who computed the saturation factors for $\mathfrak{g} = A_1^{(1)}$. In the following sections, we will rely only on the existence of a saturation factor for $\Gamma(\mathfrak{g})$, and not make explicit use of their specific values apart from the case of $d = 1$ for $\Gamma(\mathfrak{sl}_{n+1})$.

Finally, we recall the definition of a δ -maximal component $V(\nu) \subset V(\lambda) \otimes V(\mu)$, taken from [BK14, Definition 2.2] or [JK22, Definition 2.2].

Definition 4.4. *Let $V(\nu) \subset V(\lambda) \otimes V(\mu)$ be an irreducible component of the tensor product. We say that $V(\nu)$ is δ -maximal if $V(\nu + k\delta) \not\subset V(\lambda) \otimes V(\mu)$ for any $k > 0$.*

The δ -maximal components are natural tensor product analogues of the δ -maximal dominant weights $\mathcal{P}_{max}^+(\lambda)$ of an irreducible representation $V(\lambda)$. Likewise, we can often recover arbitrary components in $V(\lambda) \otimes V(\mu)$ by focusing on just the δ -maximal components. Motivated by the similarities, we would like to understand the associated “ δ -strings” in $\Gamma(\mathfrak{g})$, which should correspond to subrepresentations $V(\nu - k\delta) \subset V(\lambda) \otimes V(\mu)$. Unfortunately, the exact analogue of Lemma 3.3 does not hold in general for the appearance of components $V(\nu - k\delta)$. However, it is not too far from being correct; the goal of the next section is to give the precise analogue of Lemma 3.3.

4.2. The Goddard–Kent–Olive construction. We next introduce the key technical tool which will allow us to study the components of $V(\rho) \otimes V(\rho)$, which is the Goddard–Kent–Olive (or GKO, for short) construction of the Virasoro algebra. The GKO construction, which is particularly applicable in the study of branching problems and the tensor decomposition problem, is a “relative” version of

the earlier Sugawara coset construction. For further details, we refer to [KRR13, Lecture 10]. First, we recall the definition of the Virasoro algebra in the following definition.

Definition 4.5. *The Virasoro algebra Vir is a Lie algebra over \mathbb{C} with basis $\{c, L_k : k \in \mathbb{Z}\}$ with commutator relations*

$$[L_k, L_j] = (k - j)L_{k+j} + \frac{1}{12}(k^3 - k)\delta_{k,-j}c, \quad [Vir, c] = 0.$$

Given a finite-dimensional semisimple Lie algebra \mathfrak{g} , the Sugawara construction realizes Vir as a subalgebra of $\widehat{U}(\mathfrak{g})$, a particular completion of the universal enveloping algebra of the associated affine Kac–Moody Lie algebra. The utility of the construction is to examine highest weight irreducible representations $V(\lambda)$ of \mathfrak{g} as representations of the Virasoro algebra. When we consider this realization of the Virasoro algebra, we will denote the basis by $\{L_k^{\mathfrak{g}}, c^{\mathfrak{g}}\}$.

The GKO construction considers an embedding $\mathfrak{g}_1 \hookrightarrow \mathfrak{g}_2$ of finite-dimensional semisimple (or more generally, reductive) Lie algebras, and remembers both the actions of their affinizations \mathfrak{g}_1 and \mathfrak{g}_2 on \mathfrak{g}_2 -modules. In particular, we can take the diagonal embedding $\mathfrak{g} \hookrightarrow \mathfrak{g} \oplus \mathfrak{g}$. Then the Virasoro algebra realization on the $\mathfrak{g} \oplus \mathfrak{g}$ -module $V(\lambda) \otimes V(\mu)$, for $\lambda, \mu \in \mathcal{P}^+$ given by the GKO construction is

$$\begin{aligned} L_k^{GKO} &= L_k^{\mathfrak{g} \oplus \mathfrak{g}} - L_k^{\mathfrak{g}}, \\ c^{GKO} &= c^{\mathfrak{g} \oplus \mathfrak{g}} - c^{\mathfrak{g}}. \end{aligned}$$

The following proposition, taken from [KRR13], encapsulates the key properties of this Virasoro action on a tensor product $V(\lambda) \otimes V(\mu)$.

Proposition 4.6 ([KRR13], Proposition 10.3). *Let \mathfrak{g} be the affine Kac–Moody Lie algebra associated to a simple Lie algebra \mathfrak{g} and $\lambda, \mu \in \mathcal{P}^+$ with levels $\langle \lambda, K \rangle = l$, $\langle \mu, K \rangle = m$. Then*

(1) $V(\lambda) \otimes V(\mu)$ is a unitarizable Virasoro representation with nonnegative central charge

$$(\dim \mathfrak{g}) \left(\frac{l}{l + h^\vee} + \frac{m}{m + h^\vee} - \frac{l + m}{l + m + h^\vee} \right),$$

where h^\vee is the dual Coxeter number of \mathfrak{g} ([Kac90, Section 6.1]).

(2) L_0^{GKO} acts on $V(\lambda) \otimes V(\mu)$ by

$$\frac{1}{2} \left(\frac{(\lambda|\lambda + 2\rho)}{l + h^\vee} + \frac{(\mu|\mu + 2\rho)}{m + h^\vee} - \frac{\Omega}{l + m + h^\vee} \right),$$

where Ω is the Casimir operator of \mathfrak{g} ([Kum02, Section 1.5]) and $(\cdot|\cdot)$ is the normalized form on \mathfrak{h}^* as in [Kum02, Lemma 13.1.8]. Recall (*loc. cit.*) that Ω acts on a component $V(\nu) \subset V(\lambda) \otimes V(\mu)$ via the scalar $(\nu|\nu + 2\rho)$.

(3) For all k , $[L_k^{GKO}, \mathfrak{g}'] = 0$; *i.e.*, the L_k^{GKO} are intertwining operators for the representation of \mathfrak{g}' on $V(\lambda) \otimes V(\mu)$.

In particular, given a δ -maximal component $V(\nu) \subset V(\lambda) \otimes V(\mu)$, Proposition 4.6(3) says that L_k^{GKO} stabilizes the \mathfrak{g}' isotypical component

$$\sum_{n \geq 0} V(\nu - n\delta)^{m_{\lambda, \mu}^{\nu - n\delta}} \subset V(\lambda) \otimes V(\mu),$$

with $L_k^{GKO} \cdot v_\nu$ either zero or a highest weight vector for the \mathfrak{g} -action, where v_ν is the highest weight vector of $V(\nu)$. We refer to Kac–Wakimoto [KW88] who considered the connection between the action of the GKO construction on these components and the nonvanishing of the multiplicities $m_{\lambda, \mu}^{\nu - k\delta}$. This was further used in the contexts of the affine tensor semigroup and affine tensor cone by Brown–Kumar [BK14] and Ressarye [Res22]. These methods were adapted and crucially used in our previous work [JK22] on the existence of root components in the tensor decomposition problem. The following

proposition encapsulates the technical application of this approach; we refer to [JK22, Section 3] for a more in-depth treatment and proofs.

Proposition 4.7. *Let $\lambda, \mu \in \mathcal{P}^+$ with positive levels l, m , respectively. Let $V(\nu) \subset V(\lambda) \otimes V(\mu)$ be a δ -maximal component. Then*

- (1) *If $\frac{(\lambda|\lambda+2\rho)}{l+h^\vee} + \frac{(\mu|\mu+2\rho)}{m+h^\vee} - \frac{(\nu|\nu+2\rho)}{l+m+h^\vee} \neq 0$, then $V(\nu - k\delta) \subset V(\lambda) \otimes V(\mu)$ for all $k \geq 0$.*
- (2) *Else, $V(\nu - k\delta) \subset V(\lambda) \otimes V(\mu)$ for $k = 0, k \geq 2$, and $V(\nu - \delta) \not\subset V(\lambda) \otimes V(\mu)$.*

Thus, the key takeaway is that by computing the action of L_0^{GKO} on a potential δ -maximal component $V(\nu) \subset V(\lambda) \otimes V(\mu)$, we can show the existence of all submodules of the form $V(\nu - k\delta)$ by demonstrating the appearance of $V(\nu)$ in the tensor product. We note, however, that this does *not* give a way to construct the δ -maximal components themselves; this typically must be done independently of the Virasoro action considerations. These constructions, as well as verifying the positivity of the L_0^{GKO} action, will form the basis of our approach when restricting to $\lambda = \mu = \rho$ in the following sections.

5. COMPONENTS OF $V(d\rho) \otimes V(d\rho)$

In this section \mathfrak{g} is the affine Kac-Moody Lie algebra associated to a simple Lie algebra $\hat{\mathfrak{g}}$.

With the previous details at our disposal, we can now consider the components of $V(d\rho) \otimes V(d\rho)$ for a saturation factor $d \geq 1$ of \mathfrak{g} . In particular, recall that by definition (as in Definition 4.3) if $\lambda \leq 2\rho$ is a dominant integral weight such that $V(N\lambda) \subset V(N\rho) \otimes V(N\rho)$ for some $N \geq 1$, then for any saturation factor $d \geq 1$ we have $V(d\lambda) \subset V(d\rho) \otimes V(d\rho)$. Our approach is to first show, using the convexity of the saturated tensor semigroup, that such an N exists for all dominant weights $\lambda \leq 2\rho$. First, we have the following proposition which is an adaptation of [CKM17, Proposition 9] to the affine setting.

Proposition 5.1. *Let $\lambda \in \mathcal{P}^+$ with $\lambda \leq 2\rho$. Then we can write $\lambda = \rho + \beta$ for some weight $\beta \in \mathcal{P}(\rho)$.*

Proof. Since $\lambda \leq 2\rho$, we have that $\lambda - \rho \in \rho + \mathcal{Q}$. So, by Proposition 3.1, it suffices to show that $\lambda - \rho \in \text{conv}_{\mathbb{Q}}\{w\rho : w \in W\}$. Now, by Proposition 3.12 and Corollary 3.11, we can write

$$\lambda = \sum_{J \subsetneq I} c_J \hat{b}_J(2\rho) + c_\delta(-\delta),$$

where as in Corollary 3.11, $\hat{b}_J(2\rho) = \rho + w_0^J(\rho)$, and $c_J, c_\delta \in \mathbb{Q}_+$ with $\sum_J c_J = 1$. Thus we have

$$\lambda - \rho = \sum_{J \subsetneq I} c_J w_0^J(\rho) + c_\delta(-\delta).$$

Clearing denominators, we can write

$$N(\lambda - \rho) = \sum_{J \subsetneq I} c'_J w_0^J(\rho) + c'_\delta(-\delta),$$

where $N \in \mathbb{Z}_{\geq 1}$, and $c'_J, c'_\delta \in \mathbb{Z}_+$ with $\sum_J c'_J = N$. Now,

$$\sum_{J \subsetneq I} c'_J w_0^J(\rho) = \sum_{J \subsetneq I} \frac{c'_J}{N} w_0^J(N\rho) \in \mathcal{P}(N\rho),$$

by Proposition 3.1 since

$$N\rho - \sum_{J \subsetneq I} c'_J w_0^J(\rho) = - \sum_{J \subsetneq I} c'_J (w_0^J(\rho) - \rho) \in \mathcal{Q}.$$

By Lemma 3.3, we have that subtracting $c'_\delta(\delta)$ will remain a weight of $\mathcal{P}(N\rho)$ so that in total we get $N(\lambda - \rho) \in \mathcal{P}(N\rho)$. Since $N(\lambda - \rho)$ is an integral weight in $\mathcal{P}(N\rho)$, by Proposition 3.1 we can write

$$N(\lambda - \rho) = \sum_{w \in W} a_w w(N\rho) = N \sum_{w \in W} a_w w(\rho)$$

for some $a_w \in \mathbb{Q}_+$ with $\sum_{w \in W} a_w = 1$. Dividing by N and using Proposition 3.1 again gives the result since

$$\left(\sum_{w \in W} a_w w(\rho) \right) - \rho = \lambda - 2\rho \in Q \text{ by assumption.}$$

□

Remark 5.2. *By the same proof, since the candidates for vertices of the dominant weight polyhedron depend linearly on the highest weight, we have that for any integer $M \geq 1$, if $\lambda \in \mathcal{P}^+$ such that $\lambda \leq 2M\rho$, then we can write $\lambda = M\rho + \beta$ for some weight $\beta \in \mathcal{P}(M\rho)$.*

Remark 5.3. *Using Proposition 5.1, one could alternatively adapt the proof of Theorem 1.2 as in [CKM17] in a straightforward way to prove our Theorem 1.3(1) using the results of Ressayre [Res22] on the inequalities for the affine tensor semigroup. We do not take that approach, however, and instead highlight the use of the GKO construction, which will be key to the case $\mathfrak{g} = A_n^{(1)}$.*

Now, we can use Proposition 5.1 and Remark 5.2 for $\lambda \leq 2M\rho$ to simplify the proof of the following computational proposition. This, in turn, will allow us to apply the GKO construction to reduce the proof of Theorem 1.3 to the δ -maximal components by using Proposition 4.7.

Proposition 5.4. *Let $M \geq 1$ be an integer, and let $\lambda \in \mathcal{P}^+$ with $\lambda \leq 2M\rho$. Then*

$$\frac{2(M\rho|(M+2)\rho)}{(M+1)h^\vee} - \frac{(\lambda|\lambda+2\rho)}{(2M+1)h^\vee} > 0.$$

Proof. Write $\lambda = M\rho + \beta$ for some $\beta \in \mathcal{P}(M\rho)$. Substituting this for λ and by careful simplification, we get

$$\begin{aligned} \frac{2(M\rho|(M+2)\rho)}{(M+1)h^\vee} - \frac{(\lambda|\lambda+2\rho)}{(2M+1)h^\vee} &= \frac{1}{(M+1)(2M+1)h^\vee} \left((2M+4)((M\rho|M\rho) - (M\rho|\beta)) \right. \\ &\quad \left. + (M+1)((M\rho|M\rho) - (\beta|\beta)) + 2(M\rho|M\rho) + (2\rho|M\rho - \beta) \right). \end{aligned}$$

But by [Kac90, Proposition 11.4], we have that $(M\rho|M\rho) - (M\rho|\beta) \geq 0$ and $(M\rho|M\rho) - (\beta|\beta) \geq 0$, since $\beta \in \mathcal{P}(M\rho)$. Finally, one can check that $2(M\rho|M\rho) = 2M^2(\rho|\rho) > 0$, and $(2\rho|M\rho - \beta) \geq 0$, since $M\rho - \beta \in \mathcal{Q}^+$. Thus the expression of the proposition is strictly positive. □

With this in hand, we can apply Proposition 4.7 to conclude the following. Observe that, by definition, $h^\vee := \langle \rho, K \rangle$.

Corollary 5.5. *Let M be a positive integer. Take $\lambda \in \mathcal{P}^+$ such that $V(\lambda) \subset V(M\rho) \otimes V(M\rho)$ is a δ -maximal component. Then $V(\lambda - k\delta) \subset V(M\rho) \otimes V(M\rho)$ for any $k \geq 0$.*

We can now prove the following theorem, which is the first major result of this paper.

Theorem 5.6. *Take $\lambda \in \mathcal{P}^+$ such that $\lambda \leq 2\rho$. Then $V(d\lambda) \subset V(d\rho) \otimes V(d\rho)$ for any saturation factor d of \mathfrak{g} .*

Proof. As in the proof of Proposition 5.1, write

$$\lambda = \sum_{J \in I} c_J \hat{b}_J(2\rho) + c_\delta(-\delta), \text{ for } c_J, c_\delta \in \mathbb{Q}_+.$$

Again clearing denominators, we write (for some integer $N \geq 1$)

$$N\lambda = \sum_{J \in I} c'_J \hat{b}_J(2\rho) + c'_\delta(-\delta)$$

with each $c'_J, c'_\delta \in \mathbb{Z}_+$ and $\sum_J c'_J = N$. But since $c'_J \hat{b}_J(2\rho) = c'_J \rho + w_0^J(c'_J \rho)$ (cf. Corollary 3.11), via the PRV components of Theorem 4.1 we get that $V(c'_J \hat{b}_J(2\rho)) \subset V(c'_J \rho) \otimes V(c'_J \rho)$ for all J . (Observe that $\rho + w_0^J(\rho)$ is a dominant weight.) Thus by additivity in the tensor semigroup, we get

$$V(\Lambda) \subset V(N\rho) \otimes V(N\rho),$$

where $\Lambda := \sum_{J \in I} c'_J \hat{b}_J(2\rho)$. Finally, by Corollary 5.5, we get (by replacing Λ with the corresponding δ -maximal component) that

$$V(N\lambda) = V(\Lambda - c'_\delta(\delta)) \subset V(N\rho) \otimes V(N\rho).$$

Thus by the existence of a saturation factor for the affine tensor cone, we have that $V(d\lambda) \subset V(d\rho) \otimes V(d\rho)$ for any saturation factor d , as desired. \square

Remark 5.7. Assume that $\lambda \in \mathcal{P}^+$, $\lambda \leq 2\rho$ is such that

$$V(d\lambda) \subset V(d\rho) \otimes V(d\rho), \quad \text{for some } d \in \mathbb{Z}_{>0}.$$

Then, for any $k \in \mathbb{Z}$ and $\lambda' \in \mathcal{P}^+$ such that $\lambda' \leq 2(\rho + k\delta)$,

$$V(d\lambda') \subset V(d(\rho + k\delta)) \otimes V(d(\rho + k\delta)).$$

To prove this, observe that for any $\mu \in \mathcal{P}^+$ and $k \in \mathbb{Z}$,

$$V(\mu + k\delta) \simeq V(\mu) \otimes V(k\delta).$$

6. THE CASE $\mathfrak{g} = A_n^{(1)}$ AND CONJECTURE FOR OTHER TYPES

In this section, we restrict our attention to the case when $\mathfrak{g} = A_n^{(1)} = \widehat{sl}_{n+1}$. By Theorem 5.6, we know that for any saturation factor d for \mathfrak{g} we have that $V(d\lambda) \subset V(d\rho) \otimes V(d\rho)$ for all dominant $\lambda \leq 2\rho$. However, unlike in the corresponding finite-dimensional simple Lie algebra $\mathfrak{g} = sl_{n+1}$, the saturation factor d cannot be taken to be $d = 1$. It is known ([Res22, Theorem 3]) that in type $A_n^{(1)}$, any $d \geq 2$ is a saturation factor.

While the method of proof of Theorem 5.6, applied to a finite-dimensional semisimple Lie algebra, give a new proof of Theorem 1.2, we will assume the validity of Theorem 1.2 and use it to prove the corresponding result for $A_n^{(1)}$. By Corollary 5.5, it suffices to consider (to prove the following Theorem 6.2) the case when $\lambda \in \mathcal{P}_{max}^+(2\rho)$. Since for $A_n^{(1)}$ we have that $\delta = \alpha_0 + \alpha_1 + \cdots + \alpha_n$, by Proposition 3.4, we know that any δ -maximal dominant weight $\lambda \leq 2\rho$ satisfies

$$\text{supp}(2\rho - \lambda) \not\subseteq I = \{0, 1, \dots, n\},$$

where we say for any $\gamma \in \mathcal{Q}^+$ the support $\text{supp}(\gamma) = \{i \in I : c_i \neq 0 \text{ where } \gamma = \sum_{i \in I} c_i \alpha_i\}$. We first give the following lemma, for certain such λ , following a similar approach as in [JK22, Proof of Proposition 6.4].

Lemma 6.1. Let $\lambda \in \mathcal{P}_{max}^+(2\rho)$ such that $2\rho - \lambda \in \mathring{\mathcal{Q}}^+$. Then $V(\lambda) \subset V(\rho) \otimes V(\rho)$.

Proof. Let $\mathring{V}(\rho)$ be the \mathfrak{g} -submodule of $V(\rho)$ generated by the highest weight vector $v_\rho \in V(\rho)$, and similarly for $\mathring{V}(\lambda)$. Then by Theorem 1.2, we know that

$$\mathring{V}(\lambda) \subset \mathring{V}(\rho) \otimes \mathring{V}(\rho).$$

Let $v_\lambda \in V(\rho) \otimes V(\rho)$ be a corresponding highest weight vector for $\mathring{\mathfrak{g}}$ that generates $\mathring{V}(\lambda)$. We claim that this is in fact a highest weight vector for \mathfrak{g} . All that remains to check is that $e_0.v_\lambda = 0$, where e_0 is a root vector corresponding to the positive root α_0 . But by convention, we have that $\rho(d) = 0$, so that no vector in $V(\rho) \otimes V(\rho)$ can have weight ν with $\nu(d) \geq 1$. And, since $2\rho - \lambda \in \mathring{\mathcal{Q}}^+$, we know that $\lambda(d) = 0$. But then if $e_0.v_\lambda \neq 0$, we have that $e_0.v_\lambda$ has weight satisfying $(\lambda + \alpha_0)(d) = \alpha_0(d) = 1$, a contradiction. Thus $e_0.v_\lambda = 0$, so that $V(\lambda) \subset V(\rho) \otimes V(\rho)$. \square

By the symmetry of the Dynkin diagram of type $A_n^{(1)}$, Lemma 6.1 actually suffices to handle the case of all δ -maximal weights $\lambda \in \mathcal{P}_{\max}^+(2\rho)$. Together with Corollary 5.5 for $M = 1$, we arrive at the following theorem.

Theorem 6.2. *Let $\mathfrak{g} = A_n^{(1)}$, and let $\lambda \in \mathcal{P}^+$ with $\lambda \leq 2\rho$. Then $V(\lambda) \subset V(\rho) \otimes V(\rho)$.*

Proof. First, let $\lambda \in \mathcal{P}_{\max}^+(2\rho)$ be a δ -maximal weight. We know that $\text{supp}(2\rho - \lambda) \subsetneq I$. Consider any diagram automorphism σ of the Dynkin diagram of $A_n^{(1)}$ such that $\text{supp}(\sigma(2\rho - \lambda)) \subset I \setminus \{0\}$. Then σ induces an automorphism of \mathfrak{g} and also of $V(\rho)$ and $V(\rho) \otimes V(\rho)$ (since ρ is stable under the diagram automorphism). If $\lambda' := \sigma(\lambda) \in \mathcal{P}^+$ is the corresponding dominant integral weight, we thus have $\lambda' \leq 2\rho$ and $\text{supp}(2\rho - \lambda') \subset I \setminus \{0\}$, so by Lemma 6.1 we get that $V(\lambda') \subset V(\rho) \otimes V(\rho)$. Applying σ^{-1} , we get that $V(\lambda) \subset V(\rho) \otimes V(\rho)$. Now, the theorem follows by applying Corollary 5.5. \square

For \mathfrak{g} of other types, we do not have the utility of diagram automorphisms that allow us the reduction to those λ with $2\rho - \lambda \in \mathring{Q}^+$. Even if this were the case, as Conjecture 1.1 is only known to hold in types A_n and the exceptional types, the proof method used for Theorem 6.2 to leverage the result for $\mathring{\mathfrak{g}}$ is not easily adaptable.

However, direct computer-verified computations via the software Sage [TSD17] show that Theorem 6.2 extends for low rank examples like $\mathfrak{g} = B_2^{(1)}, G_2^{(1)}, B_3^{(1)}$, and $D_4^{(1)}$. Thus, we conclude by proposing the following analogue of Kostant's conjecture for affine Kac–Moody Lie algebras.

Conjecture 6.3. *Let \mathfrak{g} be any untwisted affine Kac–Moody Lie algebra, and $\lambda \in \mathcal{P}^+$ such that $\lambda \leq 2\rho$. Then $V(\lambda) \subset V(\rho) \otimes V(\rho)$.*

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