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Conjectural Positivity of Chern–Schwartz–MacPherson Classes for Richardson Cells

Shrawan Kumar*

Department of Mathematics, University of North Carolina at Chapel Hill, Chapel Hill, NC 27599-3250, USA

**Correspondence to be sent to: e-mail: shrawan@email.unc.edu*

Following some work of Aluffi–Mihalcea–Schürmann–Su for the CSM classes of Schubert cells and some elaborate computer calculations by Rimanyi and Mihalcea, I conjecture that the CSM classes of the Richardson cells expressed in the Schubert basis have nonnegative coefficients. This conjecture was principally motivated by a new product $\square$ coming from the Segre-SM classes in the cohomology of flag varieties (such that the associated Gr of this product is the standard cup product) and the conjecture that the structure constants of this new product $\square$ in the standard Schubert basis have alternating sign behavior. I prove that this conjecture on the sign of the structure constants of $\square$ would follow from my above positivity conjecture about the CSM classes of Richardson cells.

1 Introduction

Let G be a connected simple algebraic group over $\mathbb{C}$, and let B be a Borel subgroup and $T \subset B$ a maximal torus. Let W be the associated Weyl group with its length function ℓ. For any $u \in W$, let $\mathring{X}_u = BuB/B$ be the Schubert cell, and let $\mathring{X}^u := B^- uB/B$ be the opposite Schubert cell, where B^- is the opposite Borel subgroup of G. For $u, v \in W$, let $\mathring{X}_u^v := \mathring{X}_u \cap \mathring{X}^v$ be the Richardson cell. Let $\left\{[X_u] \in H^{2(\dim(G/B)-\ell(u))}(G/B, \mathbb{Z})\right\}_{u\in W}$ be the Schubert cohomology basis of $H^*(G/B, \mathbb{Z})$, where $[X_u]$ is the Poincaré dual of the fundamental homology class μ_{X_u} of $X_u := \overline{\mathring{X}_u}$ in $H_{2\ell(u)}(G/B, \mathbb{Z})$.

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The aim of this paper is to study the Chern–Schwartz–MacPherson (CSM) class $c_{\rm SM}(\mathring{X}_u^v; G/B)$ of Richardson cell $\mathring{X}_u^v \subset G/B$ and its connection to some conjecture I learned from A. Knutson on a new product $\square$ in the cohomology $H^*(G/B,\mathbb{Z})$ of G/B.

We first prove the following theorem (cf. Theorem 4).

Theorem A. For any $u, v \in W$, write

$$c_{\rm SM}(\mathring{X}_u^v; G/B) = \sum_{w\in W} c_w^{u,v}[X_w], \text{ for } c_w^{u,v} \in \mathbb{Z}.$$

Then, $c_w^{u,v} = 0$ unless $\ell(w) + \ell(u) + \ell(v)$ is even.

We further conjecture the following, which is the main conjecture of this note (cf. Conjecture 5). A similar positivity conjecture has also been formulated by Rui Xiong (cf. Remark 6(b)).

Conjecture B. *With the notation as above in Theorem A,* $c_w^{u,v} \geq 0, \quad$ *for all* u, v, w.

We make the following conjecture (cf. Conjecture 7) that is a weaker form of the above conjecture (as proved in Theorem 18).

Conjecture C. Write, for any $u, v \in W$, $c_{\rm SM}(\mathring{X}_u^v; G/B) = \sum_w d_w^{u,v} c_{\rm SM}(\mathring{X}_w; G/B)$. Then, $(-1)^{\ell(w)-\ell(u)-\ell(v)} \cdot d_w^{u,v} \geq 0$.

Consider the following new product $\square$ in $H^*(G/B,\mathbb{Z})$ coming from the Segre-SM classes, taking the gr of which we recover the standard cup product.

$$\epsilon^u \square \epsilon^v = \sum_{\ell(w)\geq \ell(u)+\ell(v)} \chi\left(\mathring{X}(u,v,w;g)\right) \epsilon^w,$$

where $\{\epsilon^w\}_{w\in W}$ is the basis of $H^*(G/B,\mathbb{Z})$ Kronecker dual to the homology basis $\{\mu_{X_v}\}_{v\in W}$, that is, $\epsilon^w(\mu_{X_v}) = \delta_{v,w}$, $\mathring{X}(u,v,w;g) := \mathring{X}_{w_o u} \cap \mathring{X}^v \cap g\mathring{X}_w$ and $g \in G$ is a general element such that the intersection $X(u,v,w;g) := X_{w_o u} \cap X^v \cap gX_w$ satisfies the stratified transverse intersection property (X^v being $\overline{\mathring{X}^v}$). We recall the following conjecture (cf. Conjecture 12) made independently by Knutson and Zinn-Zustin, and Mihalcea (cf. the remark just above Conjecture 12).

Conjecture D. *For any* $u, v, w \in W$,

$$(-1)^{\dim \mathring{X}(u,v,w;g)} \chi\left(\mathring{X}(u,v,w;g)\right) \geq 0,$$

for a general element $g \in G$ such that $X(u, v, w; g)$ satisfies the stratified transverse intersection property.

We prove the following result showing that Conjecture D is equivalent to the Conjecture C (cf. Proposition 17). In fact, we have the following sharper result.

Proposition E. For any fixed $u, v \in W$,

$$(-1)^{\dim \mathring{X}(u,v,w;g)} \chi(\mathring{X}(u, v, w; g)) \geq 0 \,\forall w \in W \iff \text{ Conjecture C is true for } \mathring{X}^{v}_{w_o u}.$$

Here (as in Conjecture D) $g \in G$ is a general element such that $X(u, v, w; g)$ satisfies the stratified transverse intersection property.

Thus, the validity of Conjecture B implies Conjecture C (cf. Theorem 18).

2 Main Conjecture on CSM Classes and Some Results

We take varieties to be defined over $\mathbb{C}$ and reduced and irreducible.

Let G be a connected simple algebraic group over $\mathbb{C}$, and let B be a Borel subgroup and $T \subset B$ a maximal torus. Let W be the associated Weyl group with its length function ℓ. For any $u \in W$, define

$$\mathring{X}_u = BuB/B, \quad \mathring{X}^u := B^- uB/B, \quad X_u = \overline{BuB/B}, \quad X^u := \overline{B^- uB/B},$$

where B^- is the opposite Borel subgroup of G (i.e., $B^- \cap B = T$). Then, $\mathring{X}_u$ (resp. X_u) is called the *Schubert cell* (resp. *Schubert variety*) and $\mathring{X}^u$ (resp. X^u) is called the *opposite Schubert cell* (resp. *opposite Schubert variety*).

Let $\{\epsilon^u\}_{u \in W}$ be the Schubert cohomology basis of $H^*(G/B, \mathbb{Z})$, that is, $\epsilon^u\left(\mu_{X_v}\right) = \delta_{u,v}$, where μ_{X_v} denotes the fundamental homology class of X_v in $H_{2\ell(v)}(G/B, \mathbb{Z})$. Thus, $\epsilon^u \in H^{2\ell(u)}(G/B, \mathbb{Z})$.

For any $u, v \in W$, define the *Richardson cell* $\mathring{X}^v_u := \mathring{X}_u \cap \mathring{X}^v$, and let $X^v_u := X_u \cap X^v$ be the Richardson variety. Then, it is non-empty if and only if $v \leq u$.

Definition 1. (CSM class) ([6], [3, Example 19.1.7]) Let X be an algebraic variety over the complex numbers $\mathbb{C}$, and let $\mathscr{C}(X)$ be the group of constructible functions on X. Thus, the elements of $\mathscr{C}(X)$ are finite sums of the form $\sum_i c_i \mathbb{I}_{X_i}$, where $c_i \in \mathbb{Z}$, $X_i \subset X$ are locally closed (irreducible) subvarieties and $\mathbb{I}_{X_i}$ is the characteristic function of the set X_i. For a proper morphism of varieties $X \to Y$, one defines a $\mathbb{Z}$-linear push-forward

$$f_* : \mathscr{C}(X) \to \mathscr{C}(Y), \text{ where } f_*(\mathbb{I}_W)(y) := \chi(f^{-1}(y) \cap W),$$

for a subvariety $W \subset X$ and $y \in Y$, where χ denotes the topological Euler–Poincaré characteristic. Then, MacPherson proved a conjecture of Deligne and Grothendieck stating that there exists a unique natural transformation $c_* : \mathscr{C} \to \bar{H}_*$ such that if X is smooth, then

$$c_*(\mathbb{I}_X) = c(T_X) \cap \mu_X,$$

where T_X is the tangent bundle of X, $c(T_X)$ is the total Chern class of T_X, μ_X is the fundamental class of X and $\bar{H}_*$ denotes the Borel–Moore homology with integral coefficients. Here, the naturality of c_* means that for any proper morphism $f : X \to Y$ between varieties,

$$f_*(c_*(\phi)) = c_*(f_*(\phi)) \in \bar{H}_*(Y), \text{ for any } \phi \in \mathscr{C}(X).$$

The *CSM class* of any subvariety W of a variety X is defined by

$$c_{\mathrm{SM}}(W; X) := c_*(\mathbb{I}_W) \in \bar{H}_*(X).$$

By the additivity of χ and using [3, Definition 1.4 and Example 19.1.7], for a complete variety X and any locally closed smooth subvariety W, we get

$$\int c_{\mathrm{SM}}(W; X) = \chi(W). \tag{2.1}$$

If X is a smooth variety of dimension d, then we can identify via the Poincaré duality

$$\bar{H}_i(X) \simeq H^{2d-i}(X, \mathbb{Z}), \text{ (cf. [4, Lemma 2 in §B.2]).} \tag{2.2}$$

Definition 2. (Segre-SM class) Define the *Segre-SM class* s_{SM} of any constructible set $Y \subset G/B$ by

$$s_{\mathrm{SM}}(Y; G/B) := c_{\mathrm{SM}}(Y; G/B) \cdot c(T_{G/B})^{-1} \in H^*(G/B, \mathbb{Z}),$$

where we view $c_{\mathrm{SM}}(Y; G/B)$ as a class in $H^*(G/B, \mathbb{Z})$ under the identification (2.2).

The following result is due to [2, Corollary 1.4 and Theorem 7.5].

Theorem 3. For $X = G/B$ and $u \in W$, write

$$c_{\mathrm{SM}}(\mathring{X}_u; G/B) = \sum_{w \in W} a_{u,w}[X_w].$$

Then, $a_{u,w} \geq 0, a_{u,u} = 1$, and $a_{u,w} = 0$ unless $w \leq u$, where w_o is the longest element of the Weyl group W and $[X_w] = \epsilon^{w_o w} \in H^{2(\ell(w_o)-\ell(w))}(G/B, \mathbb{Z})$ is the Poincaré dual of the fundamental class μ_{X_w}.

Moreover,

$$s_{\mathrm{SM}}(\mathring{X}_u; G/B) = \sum_{w \in W} (-1)^{\ell(u)-\ell(w)} \quad a_{u,w}[X_w].$$

Theorem 4. For any $u, v \in W$, write

$$c_{\mathrm{SM}}(\mathring{X}_u^v; G/B) = \sum_{w \in W} c_w^{u,v}[X_w], \text{ for } c_w^{u,v} \in \mathbb{Z}.$$

Then, $c_w^{u,v} = 0$ unless $\ell(w) + \ell(u) + \ell(v)$ is even.

Proof. By [8, Theorem 1.2] and Definition 2,

$$\begin{aligned} c_{\mathrm{SM}}(\mathring{X}_u^v; G/B) &= s_{\mathrm{SM}}(\mathring{X}_u; G/B) \cdot c_{\mathrm{SM}}(\mathring{X}^v; G/B) \\ &= \left(\sum (-1)^{\ell(u)-\ell(\theta_1)} a_{u,\theta_1}[X_{\theta_1}]\right) \cdot \left(\sum a_{w_o v,\theta_2}[X_{\theta_2}]\right), \text{ by Theorem 3} \\ &= (A_+ - A_-) \cdot (B_+ + B_-), \end{aligned}$$

$$\begin{aligned} \text{where} \quad A_+ &:= \sum_{\ell(u)-\ell(\theta_1) \in 2\mathbb{Z}} a_{u,\theta_1}[X_{\theta_1}], \quad A_- := \sum_{\ell(u)-\ell(\theta_1)-1 \in 2\mathbb{Z}} a_{u,\theta_1}[X_{\theta_1}] \\ B_+ &:= \sum_{\ell(w_o v)-\ell(\theta_2) \in 2\mathbb{Z}} a_{w_o v,\theta_2}[X_{\theta_2}], \quad B_- := \sum_{\ell(w_o v)-\ell(\theta_2)-1 \in 2\mathbb{Z}} a_{w_o v,\theta_2}[X_{\theta_2}]. \end{aligned}$$

Thus,

$$c_{\mathrm{SM}}\left(\mathring{X}_u^v; G/B\right) = (A_+B_+ - A_-B_-) + (A_+B_- - A_-B_+). \tag{2.3}$$

Similarly, $c_{\mathrm{SM}}(\mathring{X}_u^v; G/B) = c_{\mathrm{SM}}(\mathring{X}_u; G/B) \cdot s_{\mathrm{SM}}(\mathring{X}^v; G/B)$. Hence,

$$\begin{aligned} c_{\mathrm{SM}}(\mathring{X}_u^v; G/B) &= (A_+ + A_-) \cdot (B_+ - B_-) \\ &= (A_+B_+ - A_-B_-) + (A_-B_+ - A_+B_-). \end{aligned} \tag{2.4}$$

Comparing the equations (2.3) and (2.4), we get $A_+B_- - A_-B_+ = 0$, and hence

$$\begin{aligned} c_{\mathrm{SM}}(\mathring{X}_u^v; G/B) &= A_+B_+ - A_-B_- \\ &= \sum_{\substack{(-1)^{\ell(\theta_1)}=(-1)^{\ell(u)} \\ (-1)^{\ell(\theta_2)}=(-1)^{\ell(w_o v)}}} a_{u,\theta_1} \cdot a_{w_o v,\theta_2} [X_{\theta_1}][X_{\theta_2}] \\ &\quad - \sum_{\substack{(-1)^{\ell(\theta_1)}=(-1)^{\ell(u)+1} \\ (-1)^{\ell(\theta_2)}=(-1)^{\ell(w_o v)+1}}} a_{u,\theta_1} \cdot a_{w_o v,\theta_2} [X_{\theta_1}][X_{\theta_2}] \\ &= \sum_{\ell(w)=\ell(u)+\ell(v) (mod\ 2)} c_w^{u,v}[X_w]. \end{aligned}$$

This proves the theorem. ■

We make the following *main conjecture* of the paper.

Conjecture 5. *With the notation as above in Theorem* (4), $c_w^{u,v} \geq 0, \quad$ for all u, v, w.

Remark 6. (a) The above conjecture is valid for SL_5 (and hence for $\mathrm{SL}_4, \mathrm{SL}_3, \mathrm{SL}_2$) as the elaborate computer calculation done by Rimanyi shows. Also, the above conjecture is valid for all rank−2 groups as shown by Mihalcea via computer calculation.

(b) Mihalcea informed me on August 4, 2022 that Rui Xiong recently wrote to him mentioning a similar positivity conjecture.

As before in Theorem 3, write

$$c_{\mathrm{SM}}(\mathring{X}_u; G/B) = \sum_{w \in W} a_{u,w}[X_w].$$

Then, by Theorem 3, $a_{u,w} = 0$ unless $w \leq u$ and $a_{u,u} = 1$. Thus, $\{c_{\mathrm{SM}}(\mathring{X}_u; G/B)\}_{u \in W}$ provides another basis of $\bar{H}_*(G/B, \mathbb{Z}) \simeq H^{2\dim(G/B)-*}(G/B, \mathbb{Z})$.

We make the following conjecture that is a weaker form of Conjecture 5 (as proved in Theorem 18).

Conjecture 7. *Write, for any* $u, v \in W$, $c_{SM}(\mathring{X}_u^v; G/B) = \sum_w d_w^{u,v} c_{SM}(\mathring{X}_w; G/B)$. *Then,* $(-1)^{\ell(w)-\ell(u)-\ell(v)} \cdot d_w^{u,v} \geq 0$.

Let $\Phi : H^*(G/B) \to H^*(G/B)$ be the ring automorphism such that $\Phi_{|H^{2i}(G/B)} = (-1)^i \mathrm{Id}_{H^{2i}(G/B)}$. The following lemma was observed by Mihalcea.

Lemma 8. Write

$$\Phi\left(s_{\mathrm{SM}}(\mathring{X}_u^v; G/B)\right) = \sum_{w\in W} f_w^{u,v}[X_w].$$

Then,

$$(-1)^{\ell(w_o u)+\ell(v)} f_w^{u,v} \geq 0, \text{ for all } w \in W.$$

Proof. By [8, Theorem 1.2],

$$\begin{aligned} s_{\mathrm{SM}}(\mathring{X}_u^v; G/B) &= s_{\mathrm{SM}}(\mathring{X}_u; G/B) \cdot s_{\mathrm{SM}}(\mathring{X}_{w_o v}; G/B) \\ &= (-1)^{\ell(w_o u)}\,\Phi\left(c_{\mathrm{SM}}(\mathring{X}_u; G/B)\right)\cdot(-1)^{\ell(v)}\Phi\left(c_{\mathrm{SM}}(\mathring{X}_{w_o v}; G/B)\right), \text{ by Theorem 3} \\ &= (-1)^{\ell(w_o u)+\ell(v)}\,\Phi\left(c_{\mathrm{SM}}(\mathring{X}_u; G/B)\cdot c_{\mathrm{SM}}(\mathring{X}_{w_o v}; G/B)\right), \text{ since } \Phi \text{ is a ring homomorphism.} \end{aligned}$$

This proves the lemma by using the 1st part of Theorem 3. ■

3 A New Product in $H^*(G/B)$

I learned of the deformed product $\square$ in $H^*(G/B, \mathbb{Z})$ defined below (motivated from the Segre-SM classes) from Knutson (cf. [1, 5]).

By Theorem 3, $\{s_{\mathrm{SM}}(\mathring{X}^u; G/B)\}_{u\in W}$ is a basis of $H^*(G/B, \mathbb{Z})$. Write, for $u, v \in W$,

$$s_{\mathrm{SM}}(\mathring{X}^u; G/B) \cdot s_{\mathrm{SM}}(\mathring{X}^v; G/B) = \sum_{w\in W} e_w^{u,v} s_{\mathrm{SM}}(\mathring{X}^w; G/B), \text{ for some unique } e_w^{u,v} \in \mathbb{Z}.$$

Thus, for any basis $\eta = \{\eta^u\}_{u\in W}$ of $H^*(G/B, \mathbb{Z})$, we can define a new *commutative and associative product* $\square_\eta$ in $H^*(G/B, \mathbb{Z})$ by

$$\eta^u \square_\eta \eta^v = \sum_{w\in W} e_w^{u,v}\, \eta^w. \tag{3.1}$$

(We thank one of the referees for this observation.)

Proposition 9. With the above notation, for any $u, v, w \in W$,

$$e_w^{u,v} = d_{w_o w}^{w_o u, v} = \chi\left(\mathring{X}(u, v, w; g)\right),$$

where $d_w^{u,v}$ is as in Conjecture 7, we abbreviate $\mathring{X}(u, v, w; g) := \mathring{X}_{w_o u} \cap \mathring{X}^v \cap g\mathring{X}_w$ and $g \in G$ is a general element of G such that the intersection $X(u, v, w; g) := X_{w_o u} \cap X^v \cap gX_w$ is transverse as complex Whitney stratified subsets, that is, the transversality holds at each strata of the triple intersection. We call such an intersection *stratified transverse intersection*.

In particular, $\chi\left(\mathring{X}(u,v,w;g)\right)$ does not depend upon the choice of a general element $g \in G$ such that $X(u,v,w;g)$ satisfies the stratified transverse intersection property. See also [2, Section 8.2] and [1, Corollary 10.3 and Remark 10.4]. (We thank L. Mihalcea for the two references.)

Proof. The identity $s_{\mathrm{SM}}(\mathring{X}^u; G/B) \cdot s_{\mathrm{SM}}(\mathring{X}^v; G/B) = \sum_{w\in W} e_w^{u,v} s_{\mathrm{SM}}(\mathring{X}^w; G/B)$ gives

$$s_{\mathrm{SM}}(\mathring{X}^u; G/B) \cdot c_{\mathrm{SM}}(\mathring{X}^v; G/B) = \sum_{w\in W} e_w^{u,v} c_{\mathrm{SM}}(\mathring{X}^w; G/B),$$

which gives

$$s_{\mathrm{SM}}(\mathring{X}_{w_o u}; G/B) \cdot c_{\mathrm{SM}}(\mathring{X}^v; G/B) = \sum_{w\in W} e_w^{u,v} c_{\mathrm{SM}}(\mathring{X}_{w_o w}; G/B). \tag{3.2}$$

Hence, by the 1st equality in the proof of Theorem 4, we get from the above equation,

$$c_{\mathrm{SM}}(\mathring{X}^v_{w_o u}; G/B) = \sum_{w\in W} e_w^{u,v} c_{\mathrm{SM}}(\mathring{X}_{w_o w}; G/B).$$

Thus, from the definition of $d_w^{u,v}$, we get

$$\begin{aligned} e_w^{u,v} &= d_{w_o w}^{w_o u, v} \\ &= \chi(\mathring{X}(u,v,w;g)), \text{ by subsequent equation (4.4).} \end{aligned}$$

■

Definition 10. Taking the basis $\{\eta^u = \epsilon^u\}_{u\in W}$ of $H^*(G/B,\mathbb{Z})$ in equation (3.1) and using Proposition 9, we get a new commutative and associative product $\square$ in $H^*(G/B,\mathbb{Z})$ defined by

$$\epsilon^u \square \epsilon^v = \sum_{\ell(w)\geq \ell(u)+\ell(v)} \chi\left(\mathring{X}(u,v,w;g)\right)\epsilon^w,$$

for a general element $g \in G$ such that the intersection $X(u,v,w;g)$ is stratified transverse.

Taking the gr of $\square$, we clearly recover the standard cup product.

Example 11. For $G = \mathrm{SL}_2$,

$$\epsilon^e \square \epsilon^e = \epsilon^e - \epsilon^s, \text{ where } s \text{ is the (only) simple reflection.}$$

By an easy explicit calculation in SL_2, we see that, for a general $g \in \mathrm{SL}_2$, the variety $\mathring{X}_s \cap \mathring{X}^e \cap g\mathring{X}_s$ is isomorphic with $\mathbb{C} \setminus \{\text{two points}\}$. Hence, $\chi\left(\mathring{X}(e,e,s;g)\right) = -1$.

I learned of the following nonnegativity conjecture from Knutson (cf. [5, Conjecture after Corollary 1 on Page 43], though they specifically conjectured it only for 4-step flag manifolds and proved it for $\leq$ 3-step flag manifolds; also 2022 ICM talk by Knutson). Further, Mihalcea informed me that he also has made this conjecture for general G/B and spoke about it in some of the conferences/colloquia.

Conjecture 12. *For any $u, v, w \in W$,*

$$(-1)^{\dim \mathring{X}(u,v,w;g)} \chi\left(\mathring{X}(u, v, w; g)\right) \geq 0,$$

for a general element $g \in G$ such that $X(u, v, w; g)$ satisfies the stratified transverse intersection property.

As a special case of Identity 2.1, we isolate the following.

Lemma 13.

$$\int_{G/B} c_{\mathrm{SM}}\left(\mathring{X}(u, v, w; g); G/B\right) = \chi\left(\mathring{X}(u, v, w; g)\right).$$

The following result is a special case of [8, Theorem 1.2]. (See an analogous result [7, Proposition 3.8].)

Theorem 14. Assuming the intersection $X(u, v, w; g)$ is stratified transverse, we have

$$\begin{aligned} s_{\mathrm{SM}}\left(\mathring{X}(u, v, w; g); G/B\right) &= s_{\mathrm{SM}}\left(\mathring{X}_{w_o u}; G/B\right) \cdot s_{\mathrm{SM}}\left(\mathring{X}^{v}; G/B\right) \cdot s_{\mathrm{SM}}\left(g\mathring{X}_{w}; G/B\right) \\ &= s_{\mathrm{SM}}\left(\mathring{X}_{w_o u}; G/B\right) \cdot s_{\mathrm{SM}}\left(\mathring{X}^{v}; G/B\right) \cdot s_{\mathrm{SM}}\left(\mathring{X}_{w}; G/B\right). \end{aligned}$$

Theorem 15. For any $u, v, w \in W$ and general $g \in G$, we have

$$\chi\left(\mathring{X}(u, v, w; g)\right) = (-1)^{\dim \mathring{X}(u,v,w;g)} \sum_{\ell(u_1)+\ell(v_1)+\ell(w_1)=2\ell(w_o)} (-1)^{\ell(u)-\ell(w_o u_1)} a_{w_o u, u_1} a_{w_o v, v_1} a_{w, w_1} \cdot \int_{G/B} [X_{u_1}] \cdot [X_{v_1}] \cdot [X_{w_1}],$$

where the notation $a_{u,w}$ is as in Theorem 3.

Proof. Let $\left[c_{\mathrm{SM}}(Y; G/B)\right]_{\mathrm{top}}$ denote the top degree component of $c_{\mathrm{SM}}(Y; G/B)$ for any locally closed subvariety $Y \subset G/B$, that is, writing

$$c_{\mathrm{SM}}(Y; G/B) = \sum_{w} a_w(Y)[X_w], \qquad \left[c_{\mathrm{SM}}(Y; G/B)\right]_{\mathrm{top}} = a_e(Y).$$

There is a similar meaning for $\left[s_{\mathrm{SM}}(Y; G/B)\right]_{\mathrm{top}}$.

Then, by Theorem 14 and Definition 2,

$$c_{\mathrm{SM}}\left(\mathring{X}(u,v,w;g);G/B\right) = c_{\mathrm{SM}}\left(\mathring{X}_{w_o u};G/B\right)\cdot s_{\mathrm{SM}}\left(\mathring{X}_{w_o v};G/B\right)\cdot s_{\mathrm{SM}}\left(\mathring{X}_{w};G/B\right)$$

$$= \left(\sum_{u_1\leq w_o u} a_{w_o u,u_1}[X_{u_1}]\right)\cdot\left(\sum_{v_1\leq w_o v}(-1)^{\ell(w_o v)-\ell(v_1)}a_{w_o v,v_1}[X_{v_1}]\right)$$

$$\cdot\left(\sum_{w_1\leq w}(-1)^{\ell(w)-\ell(w_1)}a_{w,w_1}[X_{w_1}]\right), \text{ by Theorem 3}$$

$$= \sum_{u_1,v_1,w_1}(-1)^{\ell(w_o v)-\ell(v_1)+\ell(w)-\ell(w_1)}\left(a_{w_o u,u_1}\, a_{w_o v,v_1}\, a_{w,w_1}[X_{u_1}]\cdot[X_{v_1}]\cdot[X_{w_1}]\right).$$

Thus, by Lemma 13,

$$\chi\left(\mathring{X}(u,v,w;g)\right) = \left[c_{\mathrm{SM}}\left(\mathring{X}(u,v,w;g);G/B\right)\right]_{\mathrm{top}}$$

$$= \sum_{\substack{u_1,v_1,w_1:\\ \ell(u_1)+\ell(v_1)+\ell(w_1)=2\ell(w_o)}}(-1)^{\ell(w)-\ell(u_1)+\ell(w_o v)}\, a_{w_o u,u_1}a_{w_o v,v_1}a_{w,w_1}[[X_{u_1}]\cdot[X_{v_1}]\cdot[X_{w_1}]]_{\mathrm{top}}$$

$$= (-1)^{\dim\mathring{X}(u,v,w;g)}\sum_{\substack{u_1,v_1,w_1:\\ \ell(u_1)+\ell(v_1)+\ell(w_1)=2\ell(w_o)}}(-1)^{\ell(w_o u)-\ell(u_1)}a_{w_o u,u_1}a_{w_o v,v_1}a_{w,w_1}[[X_{u_1}]\cdot[X_{v_1}]\cdot[X_{w_1}]]_{\mathrm{top}},$$

since

$$\dim\mathring{X}(u,v,w;g) = \ell(w)-\ell(u)-\ell(v). \tag{3.3}$$

To prove the above identity, observe that, being transverse intersection,

$$\operatorname{codim}\mathring{X}(u,v,w;g) = \operatorname{codim}\mathring{X}_{w_o u} + \operatorname{codim}\mathring{X}^{v} + \operatorname{codim}\mathring{X}_{w} = \ell(u)+\ell(v)+\ell(w_o)-\ell(w),$$

where the codim denotes the codimension in G/B. ■

4 Conjecture 5 implies Conjecture 7

Theorem 16. Assuming the validity of Conjecture 5, we get for any $u,v,w\in W$,

$$(-1)^{\dim\mathring{X}(u,v,w;g)}\chi(\mathring{X}(u,v,w;g))\geq 0,$$

for any g as in Conjecture 12. Thus, Conjecture 12 is valid assuming Conjecture 5.

Proof. By [8, Theorem 1.2],

$$c_{\mathrm{SM}}(\mathring{X}(u,v,w;g);G/B) \quad = \quad c_{\mathrm{SM}}(\mathring{X}^{v}_{w_o u};G/B)\cdot s_{\mathrm{SM}}(\mathring{X}_{w};G/B) \tag{4.1}$$

$$= \left(\sum_{\substack{\theta_1: \\ \ell(\theta_1)+\ell(w_o u)+\ell(v) \text{ is even}}} c_{\theta_1}^{w_o u, v} [X_{\theta_1}] \right) \cdot \sum_{\theta_2} (-1)^{\ell(w)-\ell(\theta_2)} a_{w,\theta_2} [X_{\theta_2}],$$

by Theorems 3 and 4

$$= \sum_{\substack{\theta_1,\theta_2: \\ \ell(\theta_1)+\ell(w_o u)+\ell(v) \text{ is even}}} (-1)^{\ell(w)-\ell(\theta_2)} c_{\theta_1}^{w_o u, v} \cdot a_{w,\theta_2} [X_{\theta_1}][X_{\theta_2}]. \tag{4.2}$$

Thus,

$$\begin{aligned} \chi(\mathring{X}(u,v,w;g)) &= \int_{G/B} c_{\text{SM}}(\mathring{X}(u,v,w;g); G/B), \quad \text{by Lemma 13} \\ &= \text{coefficient of } [X_e] \text{ in } c_{\text{SM}}(\mathring{X}(u,v,w;g); G/B) \\ &= \sum_{\substack{\theta_2 = w_o\theta_1 \\ \ell(\theta_1)+\ell(w_o u)+\ell(v) \text{ is even}}} (-1)^{\ell(w)-\ell(\theta_2)} c_{\theta_1}^{w_o u, v} a_{w,\theta_2} \\ &= (-1)^{\ell(u)+\ell(v)+\ell(w)} \left(\sum_{\substack{\theta_2 = w_o\theta_1 \\ \ell(\theta_1)+\ell(w_o u)+\ell(v) \text{ is even}}} c_{\theta_1}^{w_o u, v} a_{w,\theta_2} \right). \end{aligned} \tag{4.3}$$

This proves the theorem by the 1st part of Theorem 3 and the identity (3.3) assuming the validity of Conjecture 5. ■

Proposition 17. For any fixed $u, v \in W$,

$$(-1)^{\dim \mathring{X}(u,v,w;g)} \chi(\mathring{X}(u,v,w;g)) \geq 0 \, \forall w \in W \iff \text{Conjecture 7 is true for } \mathring{X}^v_{w_o u}.$$

Here (as in Conjecture 12), $g \in G$ is a general element such that $X(u,v,w;g)$ satisfies the stratified transverse intersection property.

Proof. By Lemma 13,

$$\begin{aligned} \chi(\mathring{X}(u,v,w;g)) &= \int_{G/B} c_{\text{SM}}(\mathring{X}(u,v,w;g); G/B) \\ &= \left\langle c_{\text{SM}}(\mathring{X}^v_{w_o u}; G/B), s_{\text{SM}}(\mathring{X}_w; G/B) \right\rangle, \text{ by (4.1),} \\ &\qquad \text{where } \langle \alpha, \beta \rangle \text{ is the coefficient of } [X_e] \text{ in } \alpha.\beta \\ &= \left\langle \sum_\theta d_\theta^{w_o u, v} c_{\text{SM}}(\mathring{X}_\theta; G/B), s_{\text{SM}}(\mathring{X}_w; G/B) \right\rangle, \text{ where } d_\theta^{w_o u, v} \text{ is as in Conjecture 7} \\ &= \sum_\theta d_\theta^{w_o u, v} \left\langle c_{\text{SM}}(\mathring{X}_\theta; G/B), s_{\text{SM}}(\mathring{X}_w; G/B) \right\rangle \\ &= d_{w_o w}^{w_o u, v}, \text{ by [1, Corollary 7.4] and Theorem 3.} \end{aligned} \tag{4.4}$$

Recall that, conjecturally, $\chi(\mathring{X}(u,v,w;g))$ has sign

$$(-1)^{\dim \mathring{X}(u,v,w;g)} = (-1)^{\ell(w)-\ell(u)-\ell(v)}, \text{ by equation (3.3).}$$

Also, by Conjecture 7, $d_{w_o w}^{w_o u, v}$ has sign $(-1)^{\ell(w_o w)-\ell(w_o u)-\ell(v)} = (-1)^{\ell(w)-\ell(u)-\ell(v)}$. ■

Combining Theorem 16 and Proposition 17, we get the following.

Theorem 18. Conjecture 7 is implied by the validity of Conjecture 5.

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References

[1] Aluffi, P., L. Mihalcea, J. Schürmann, and C. Su. "Shadows of characteristic cycles, Verma modules, and positivity of Chern–Schwartz–MacPherson classes of Schubert cells." (2022): preprint arXiv:1709.08697, v3.

[2] Aluffi, P., L. Mihalcea, J. Schürmann, and C. Su. "From motivic Chern classes of Schubert cells to their Hirzebruch and CSM classes." preprint arXiv:2212.12509, v1.

[3] Fulton, W. *Intersection Theory* 2nd ed. Springer, 1998.

[4] Fulton, W. *Young Tableaux*, vol. 35. London Mathematical Society Student Texts. Cambridge University Press, 1997.

[5] Knutson, A. and P. Zinn-Justin. "Schubert puzzles and integrability II: multiplying motivic Segre classes." (2021): preprint arXiv:2102.00563, v3.

[6] MacPherson, R. "Chern classes for singular algebraic varieties." *Ann. of Math. (2)* 100 (1974): 423–32. https://doi.org/10.2307/1971080.

[7] Ohmoto, T. "Singularities of maps and characteristic classes." *Adv. Stud. Pure Math.* 68 (2016): 191–265.

[8] Schürmann, J. "Chern Classes and Transversality for Singular Spaces." In *Singularities in Geometry, Topology, Foliations and Dynamics*, edited by J. Cisneros-Molina et al. Trends in Mathematics. 207–31. Birkhäuser, 2017.