REPRESENTATION RING OF LEVI SUBGROUPS VERSUS COHOMOLOGY RING OF FLAG VARIETIES III

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Abstract. For any reductive group G and a parabolic subgroup P with its Levi subgroup L, the first author [Ku2] introduced a ring homomorphism $\xi^P_\lambda : \mathrm{Rep}^{\mathbb{C}}_{\lambda-\mathrm{poly}}(L) \to H^*(G/P, \mathbb{C})$, where $\mathrm{Rep}^{\mathbb{C}}_{\lambda-\mathrm{poly}}(L)$ is a certain subring of the complexified representation ring of L (depending upon the choice of an irreducible representation $V(\lambda)$ of G with highest weight λ). In this paper we study this homomorphism for $G = \mathrm{SO}(2n)$ and its maximal parabolic subgroups P_{n-k} for any $2 \le k \le n-1$ (with the choice of $V(\lambda)$ to be the defining representation $V(\omega_1)$ in $\mathbb{C}^{2n}$). Thus, we obtain a $\mathbb{C}$-algebra homomorphism $\xi^D_{n,k} : \mathrm{Rep}^{\mathbb{C}}_{\omega_1-\mathrm{poly}}(\mathrm{SO}(2k)) \to H^*(\mathrm{OG}(n-k, 2n), \mathbb{C})$. We determine this homomorphism explicitly in the paper. We further analyze the behavior of $\xi^D_{n,k}$ when n tends to ∞ keeping k fixed and show that $\xi_{n,k}$ becomes injective in the limit. We also determine explicitly (via some computer calculation) the homomorphism ξ^P_λ for all the exceptional groups G (with a specific 'minimal' choice of λ) and all their maximal parabolic subgroups except E_8.

1. Introduction

This is a follow-up of the papers [Ku2] and [KR]. It may be recalled that the paper [Ku2] arose from some questions of M. Vergne.

Let G be a connected reductive algebraic group over $\mathbb{C}$ with Borel subgroup B and maximal torus $T \subset B$. Let P be a standard parabolic subgroup with Levi subgroup L containing T. Let W (resp. W_L) be the Weyl group of G (resp. L). Let $V(\lambda)$ be an irreducible almost faithful representation of G with highest weight λ, i.e., λ is a dominant integral weight and the corresponding map $\rho_\lambda : G \to \mathrm{GL}(V(\lambda))$ has finite kernel. Then, Springer defined an adjoint-equivariant regular map with Zariski dense image from the group to its Lie algebra, $\theta_\lambda : G \to \mathfrak{g}$, which depends on λ (cf. [BR]).

Using the Springer morphism θ_λ, Kumar [Ku2] defined the λ-polynomial representation ring $\mathrm{Rep}^{\mathbb{C}}_{\lambda-\mathrm{poly}}(L)$, which is a subring of the complexified representation ring $\mathrm{Rep}^{\mathbb{C}}(L)$. For $G = \mathrm{GL}(n)$ and $V(\lambda)$ the defining representation $V(\omega_1) = \mathbb{C}^n$, the ring $\mathrm{Rep}_{\omega_1-\mathrm{poly}}(G) := \mathrm{Rep}^{\mathbb{C}}_{\omega_1-\mathrm{poly}}(G) \cap \mathrm{Rep}(G)$ coincides with the standard polynomial representation ring $\mathrm{Rep}_{\mathrm{poly}}(G)$ of $\mathrm{GL}(n)$.

Kumar [Ku2] also defined a surjective $\mathbb{C}$-algebra homomorphism

$$\xi^P_\lambda : \mathrm{Rep}^{\mathbb{C}}_{\lambda-\mathrm{poly}}(L) \to H^*(G/P, \mathbb{C}). \tag{1}$$

This map generalizes (cf. [Ku2, Theorem 8]) the classical ring homomorphism

$$\phi_{n,r} : \mathrm{Rep}_{\mathrm{poly}}(\mathrm{GL}(r)) \to H^*(\mathrm{Gr}(r, n), \mathbb{Z}),$$

where $\mathrm{Gr}(r, n)$ is the Grassmannian of r-planes in $\mathbb{C}^n$, which is also the quotient $\mathrm{GL}(n)/P_r$ for the maximal parabolic subgroup P_r of $\mathrm{GL}(n)$ obtained from deleting the r-th node in the Dynkin diagram of $\mathrm{SL}(n)$.

In a subsequent work, Kumar–Rogers [KR] explicitly determined the homormorphism $\xi^P_{\omega_1}$ for the classical groups G of types B and C and the maximal parabolic subgroups, where ω_1 is the first fundamental weight. Moreover, in these cases, they further analyzed the map $\xi^P_{\omega_1}$ in the limit described below.

For type C_n, i.e., $G = \mathrm{Sp}(2n)$ (the symplectic group) and any positive integer $0 < k < n$, consider the standard maximal parabolic subgroup P^C_{n-k} corresponding to the $(n-k)$-th node of the Dynkin diagram of $\mathrm{Sp}(2n)$, and let L^C_{n-k} be its Levi subgroup. Then, the quotient $\mathrm{Sp}(2n)/P^C_{n-k}$ is the isotropic Grassmannian $\mathrm{IG}(n-k, 2n)$, and we have

$$L^C_{n-k} \simeq \mathrm{GL}(n-k) \times \mathrm{Sp}(2k).$$

Thus, following (1), we get a ring homomorphism

$$\xi^{P^C_{n-k}}_{\omega_1} : \mathrm{Rep}^{\mathbb{C}}_{\omega_1-\mathrm{poly}}(L^C_{n-k}) \to H^*(\mathrm{IG}(n-k, 2n), \mathbb{C}).$$

Restricting $\xi^{P^C_{n-k}}_{\omega_1}$ to the component $\mathrm{Sp}(2k)$, we get a ring homomorphism

$$\xi^C_{n,k} : \mathrm{Rep}^{\mathbb{C}}_{\omega_1-\mathrm{poly}}(\mathrm{Sp}(2k)) \to H^*(\mathrm{IG}(n-k, 2n), \mathbb{C}).$$

Define the stable cohomology ring

$$\mathbb{H}^*(\mathrm{IG}_k, \mathbb{C}) := \varprojlim_n H^*(\mathrm{IG}(n-k, 2n), \mathbb{C})$$

as the inverse limit under a certain embedding of $\mathrm{IG}(n-k, 2n) \hookrightarrow \mathrm{IG}(n+1-k, 2(n+1))$ (cf. [KR, Definition 15]). Then, the homomorphisms $(\xi^C_{n,k})_{n>k}$ combine to give a ring homomorphism

$$\xi^C_k : \mathrm{Rep}^{\mathbb{C}}_{\omega_1-\mathrm{poly}}(\mathrm{Sp}(2k)) \to \mathbb{H}^*(\mathrm{IG}_k, \mathbb{C}).$$

Kumar–Rogers proved that the homomorphism ξ^C_k is injective [KR, Theorem 16] (however it is not surjective). They also obtained parallel results for the classical groups of type B.

The aim of this paper is to complete the above results for the even orthogonal groups $SO(2n)$*, and also the exceptional groups of type* G_2, F_4, E_6 *and* E_7*. In principle, we can also handle* E_8*, but the results in this case are too long and complicated to reproduce here.*

For $n \geq 4$ and $2 \leq k \leq n-1$, let $\mathrm{OG}(n-k, 2n)$ be the set of $(n-k)$-dimensional isotropic subspaces in $V = \mathbb{C}^{2n}$. Then, $\mathrm{OG}(n-k, 2n)$ is the quotient $\mathrm{SO}(2n)/P^D_{n-k}$ by the standrad maximal parabolic subgroup P^D_{n-k} corresponding to the $(n-k)$-th node of the Dynkin diagram of type D_n. Let L^D_{n-k} be its Levi subgroup. Then,

$$L^D_{n-k} \simeq \mathrm{GL}(n-k) \times \mathrm{SO}(2k).$$

Thus, following (1), we get a ring homomorphism

$$\xi^{P^D_{n-k}}_{\omega_1} : \mathrm{Rep}^{\mathbb{C}}_{\omega_1-\mathrm{poly}}(L^D_{n-k}) \to H^*(\mathrm{OG}(n-k, 2n), \mathbb{C}).$$

Restricting $\xi^{P^D_{n-k}}_{\omega_1}$ to the component $\mathrm{SO}(2k)$, we get a ring homomorphism

$$\xi^D_{n,k} : \mathrm{Rep}^{\mathbb{C}}_{\omega_1-\mathrm{poly}}(\mathrm{SO}(2k)) \to H^*(\mathrm{OG}(n-k, 2n), \mathbb{C}).$$

We determine this ring homomorphism explicitly in Theorem 3.6.

Define the stable cohomology ring

$$\mathbb{H}^*(\mathrm{OG}_k, \mathbb{C}) := \varprojlim_n H^*(\mathrm{OG}(n-k, 2n), \mathbb{C})$$

as the inverse limit under the embedding $\pi_n : \mathrm{OG}(n-k, 2n) \hookrightarrow \mathrm{OG}(n+1-k, 2(n+1))$ (cf. the discussion after (23)). Then, the homomorphisms $(\xi^D_{n,k})_{n\geq k+2}$ combine to give a ring homomorphism

$$\xi^D_k : \mathrm{Rep}^{\mathbb{C}}_{\omega_1-\mathrm{poly}}(\mathrm{SO}(2k)) \to \mathbb{H}^*(\mathrm{OG}_k, \mathbb{C}).$$

Following is one of our main results of the paper (cf. Theorem 3.8 and Remark 3.9).

Theorem 1.1. *The above ring homomorphism* $\xi_k^D : Rep^{\mathbb{C}}_{\omega_1-poly}(SO(2k)) \to \mathbb{H}^*(OG_k, \mathbb{C})$ *is injective. However, it is not surjective.*

The proof of the above theorem has common features with the proof of [KR, Theorem 29] for $\mathrm{SO}(2k+1)$. However, there is one additional subtlety in the present proof (for $\mathrm{SO}(2k)$) due to the existence of an additional generator $\bar{h}_{1,k}$ in $\mathrm{Rep}^{\mathbb{Z}}_{\mathrm{poly}}(\mathrm{SO}(2k))$ (cf. equation (29)).

For the exceptional groups of type G_2, F_4, E_6 and E_7, we calculate the Springer morphism θ_λ restricted to T by using [R, Theorems 1, 2], where we take λ such that $V(\lambda)$ has the minimum Dynkin index. Further, for these exceptional groups and any maximal parabolic subgroup P we determine *explicitly* the generators of $\mathrm{Rep}^{\mathbb{C}}_{\lambda-\mathrm{poly}}(L)$ and their images in $H^*(G/P, \mathbb{C})$ under ξ^P_λ in the Schubert basis (cf. Sections 5-8), using a computer program [X], which is developed based on the algorithms in [Lee] and [D]. (Observe that there is no injectivity question for the exceptional groups, as the morphism ξ^P_λ maps to a finite dimensional vector space in the absence of a suitable "$n \to \infty$" limit.)

The proofs in the case of $\mathrm{SO}(2n)$ rely on some results of Buch-Kresch-Tamvakis [BKT1], work of Kumar [Ku2] and work of Kumar-Rogers [KR].

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2. Prelimanaries and Notation

We recall some notation and results from [Ku2].

Let G be a connected reductive group over $\mathbb{C}$ with a Borel subgroup B and maximal torus $T \subset B$. Let P be a standard parabolic subgroup with the Levi subgroup L containing T. We denote their Lie algebras by the corresponding Gothic characters: $\mathfrak{g}, \mathfrak{b}, \mathfrak{t}, \mathfrak{p}, \mathfrak{l}$ respectively. We denote by $\Delta = \{\alpha_1, \dots, \alpha_\ell\} \subset \mathfrak{t}^*$ the set of simple roots. The fundamental weights of $\mathfrak{g}$ are denoted by $\{\omega_1, \dots, \omega_\ell\} \subset \mathfrak{t}^*$. Let W (resp. W_L) be the Weyl group of G (resp. L). Then, W is generated by the simple reflections $\{s_i\}_{1\le i\le \ell}$. Let W^P denote the set of smallest coset representatives in the cosets in W/W_L. *Throughout the paper we follow the indexing convention as in [Bo, Planche I - IX].*

Let $X(T)$ be the group of characters of T and let $D \subset X(T)$ be the set of dominant characters (with respect to the given choice of B and hence positive roots, which are the roots of $\mathfrak{b}$). Then, the isomorphism classes of finite dimensional irreducible representations of G are bijectively parameterized by D under the correspondence $\lambda \in D \rightsquigarrow V(\lambda)$, where $V(\lambda)$ is the irreducible representation of G with highest weight λ. We call $V(\lambda)$ *almost faithful* if the corresponding map $\rho_\lambda : G \to \mathrm{Aut}(V(\lambda))$ has finite kernel.

Recall the Bruhat decomposition for the flag variety:

$$G/P = \sqcup_{w\in W^P} \Lambda^P_w, \text{ where } \Lambda^P_w := BwP/P.$$

Let $\bar{\Lambda}^P_w$ denote the closure of Λ^P_w in G/P. We denote by $[\bar{\Lambda}^P_w] \in H_{2\ell(w)}(G/P, \mathbb{Z})$ its fundamental class. Let $\{\epsilon^P_w\}_{w\in W^P}$ denote the Kronecker dual basis of the cohomology, i.e.,

$$\epsilon^P_w([\bar{\Lambda}^P_v]) = \delta_{w,v}, \text{ for any } v, w \in W^P.$$

Thus, ϵ^P_w belongs to the singular cohomology:

$$\epsilon^P_w \in H^{2\ell(w)}(G/P, \mathbb{Z}).$$

We abbreviate ϵ_w^B by ϵ_w. Then, for any $w \in W^P$, $\epsilon_w^P = \pi^*(\epsilon_w)$, where $\pi : G/B \to G/P$ is the standard projection.

Definition 2.1. Let $V(\lambda)$ be any almost faithful irreducible representation of G. Following Springer (cf. [BR, §9]), define the map

$$\theta_\lambda : G \to \mathfrak{g} \quad \text{(depending upon } \lambda)$$

as follows:

$$\begin{array}{ccc} G & \xrightarrow{\rho_\lambda} & \mathrm{Aut}(V(\lambda)) \subset \mathrm{End}(V(\lambda)) = \mathfrak{g} \oplus \mathfrak{g}^\perp \\ & \searrow^{\theta_\lambda} & \downarrow \pi \\ & & \mathfrak{g} \end{array}$$

where $\mathfrak{g}$ sits canonically inside $\mathrm{End}(V(\lambda))$ via the derivative $d\rho_\lambda$, the orthogonal complement $\mathfrak{g}^\perp$ is taken with respect to the standard conjugate $\mathrm{Aut}(V(\lambda))$-invariant form on $\mathrm{End}(V(\lambda))$: $\langle A, B\rangle := \mathrm{tr}(AB)$, and π is the projection to the $\mathfrak{g}$-factor. (By considering a compact form K of G, it is easy to see that $\mathfrak{g} \cap \mathfrak{g}^\perp = \{0\}$.)

Since $\pi \circ d\rho_\lambda$ is the identity map, θ_λ is a local diffeomorphism at 1 (and hence with Zariski dense image). Of course, by construction, θ_λ is an algebraic morphism. Moreover, since the decomposition $\mathrm{End}(V(\lambda)) = \mathfrak{g} \oplus \mathfrak{g}^\perp$ is G-stable, it is easy to see that θ_λ is G-equivariant under conjugation.

We recall the following lemma from [Ku2, Lemma 2].

Lemma 2.2. *The above morphism restricts to* $\theta_{\lambda|T} : T \to \mathfrak{t}$.

For any $\mu \in X(T)$, we have a G-equivariant line bundle $\mathcal{L}(\mu)$ on G/B associated to the principal B-bundle $G \to G/B$ via the one dimensional B-module μ^{-1}. (Any $\mu \in X(T)$ extends uniquely to a character of B.) The one dimensional B-module μ is also denoted by $\mathbb{C}_\mu$. Recall the surjective Borel homomorphism

$$\beta : S(\mathfrak{t}^*) \to H^*(G/B, \mathbb{C}),$$

which takes a character $\mu \in X(T)$ to the first Chern class of the line bundle $\mathcal{L}(\mu)$. (We realize $X(T)$ as a lattice in $\mathfrak{t}^*$ via taking derivative.) We then extend this map linearly over $\mathbb{C}$ to $\mathfrak{t}^*$ and extend further as a graded algebra homomorphism from $S(\mathfrak{t}^*)$ (doubling the degree). Under the Borel homomorphism,

$$\beta(\omega_i) = \epsilon_{s_i}, \quad \text{for any fundamental weight } \omega_i. \tag{2}$$

Fix a compact form K of G such that $T_o := K \cap T$ is a (compact) maximal torus of K. Then, $W \simeq N(T_o)/T_o$, where $N(T_o)$ is the normalizer of T_o in K. Recall that β is W-equivariant under the standard action of W on $S(\mathfrak{t}^*)$ and the W-action on $H^*(G/B, \mathbb{C})$ induced from the W-action on $G/B \simeq K/T_o$ via

$$(nT_o) \cdot (kT_o) := kn^{-1}T_o, \quad \text{for } n \in N(T_o) \text{ and } k \in K.$$

Thus, for any standard parabolic subgroup P with the Levi subgroup L containing T, restricting β, we get a surjective graded algebra homorphism:

$$\beta^P : S(\mathfrak{t}^*)^{W_L} \to H^*(G/B, \mathbb{C})^{W_L} \simeq H^*(G/P, \mathbb{C}),$$

where the last isomorphism, which is induced from the projection $G/B \to G/P$, can be found, e.g., in [Ku1, Corollary 11.3.14].

Now, the Springer morphism $\theta_{\lambda|T} : T \to \mathfrak{t}$ (restricted to T) gives rise to the corresponding W-equivariant injective algebra homomorphism on the affine coordinate rings:

$$(\theta_{\lambda|T})^* : \mathbb{C}[\mathfrak{t}] = S(\mathfrak{t}^*) \to \mathbb{C}[T].$$

Thus, on restriction to W_L-invariants, we get an injective algebra homomorphism

$$\theta_\lambda(P)^* : \mathbb{C}[\mathfrak{t}]^{W_L} = S(\mathfrak{t}^*)^{W_L} \to \mathbb{C}[T]^{W_L}.$$

(Since W_L-invariants depend upon the choice of the parabolic subgroup P, we have included P in the notation of $\theta_\lambda(P)^*$.) Now, let $\mathrm{Rep}(L)$ be the representation ring of L and let $\mathrm{Rep}^{\mathbb{C}}(L) := \mathrm{Rep}(L) \otimes_{\mathbb{Z}} \mathbb{C}$ be its complexification. Then, as it is well known,

$$\mathrm{Rep}^{\mathbb{C}}(L) \simeq \mathbb{C}[T]^{W_L} \tag{3}$$

obtained from taking the character of an L-module restricted to T.

Definition 2.3. We call a virtual character $\chi \in \mathrm{Rep}^{\mathbb{C}}(L)$ of L a *λ-polynomial character* if the corresponding function in $\mathbb{C}[T]^{W_L}$ is in the image of $\theta_\lambda(P)^*$. The set of all λ-polynomial characters of L, which is, by definition, a subalgebra of $\mathrm{Rep}^{\mathbb{C}}(L)$ isomorphic to the algebra $S(\mathfrak{t}^*)^{W_L}$, is denoted by $\mathrm{Rep}^{\mathbb{C}}_{\lambda-\mathrm{poly}}(L)$. Of course, the map $\theta_\lambda(P)^*$ induces an algebra isomorphism (still denoted by)

$$\theta_\lambda(P)^* : S(\mathfrak{t}^*)^{W_L} \simeq \mathrm{Rep}^{\mathbb{C}}_{\lambda-\mathrm{poly}}(L),$$

under the identification (3).

It is easy to see that

$$\mathrm{Rep}_{\omega_1-\mathrm{poly}}(\mathrm{GL}(n)) = \mathrm{Rep}_{\mathrm{poly}}(\mathrm{GL}(n)), \tag{4}$$

where $\mathrm{Rep}_{\mathrm{poly}}(\mathrm{GL}(n))$ denotes the subring of the representation ring $\mathrm{Rep}(\mathrm{GL}(n))$ spanned by the irreducible polynomial representations of $\mathrm{GL}(n)$.

We recall the following result from [Ku2, Theorem 5].

Theorem 2.4. *Let $V(\lambda)$ be an almost faithful irreducible G-module and let P be any standard parabolic subgroup. Then, the above maps (specifically $\beta^P \circ (\theta_\lambda(P)^*)^{-1}$) give rise to a surjective $\mathbb{C}$-algebra homomorphism*

$$\xi^P_\lambda : \mathrm{Rep}^{\mathbb{C}}_{\lambda-\mathrm{poly}}(L) \to H^*(G/P, \mathbb{C}).$$

Moreover, let Q be another standard parabolic subgroup with Levi subgroup R containing T such that $P \subset Q$ (and hence $L \subset R$). Then, we have the following commutative diagram:

$$\begin{array}{ccc}
\mathrm{Rep}^{\mathbb{C}}_{\lambda-\mathrm{poly}}(R) & \xrightarrow{\xi^Q_\lambda} & H^*(G/Q, \mathbb{C}) \\
\downarrow{\scriptstyle \gamma} & & \downarrow{\scriptstyle \pi^*} \\
\mathrm{Rep}^{\mathbb{C}}_{\lambda-\mathrm{poly}}(L) & \xrightarrow{\xi^P_\lambda} & H^*(G/P, \mathbb{C}),
\end{array}$$

where π^* is induced from the standard projection $\pi : G/P \to G/Q$ and γ is induced from the restriction of representations.

3. Injectivity Result for the even orthogonal Groups

In this section, we take $n \geq 4$ to avoid some trivial cases.

Let $V = \mathbb{C}^{2n}$ be equipped with the non-degenerate quadratic form $\langle\ ,\ \rangle$ so that its matrix $(\langle e_i, e_j\rangle)_{1\leq i,j\leq 2n}$ in the standard basis $\{e_1, \dots, e_{2n}\}$ is given by the antidiagonal matrix E_D with all its antidiagonal entries equal to 1. Then,

$$\mathrm{SO}(2n) = \{g \in \mathrm{SL}(2n) : (g^t)^{-1} = E_D g E_D^{-1}\}$$

is the special orthogonal group. Now, as in [Ku2, Lemma 10], the Springer morphism $\theta = \theta_{\omega_1}$ (with respect to the first fundamental weight ω_1) for $G = \mathrm{SO}(2n)$ is given by the Cayley Transform:

$$g \to \frac{g - E_D^{-1} g^t E_D}{2}, \quad \text{for } g \in G.$$

Take the maximal torus in $\mathrm{SO}(2n)$ defined by

$$T_D = \{\mathbf{t} = \mathrm{diag}(t_1, \dots, t_n, t_n^{-1}, \dots, t_1^{-1}) : t_i \in \mathbb{C}^*\}. \tag{5}$$

Its Lie algebra is given by

$$\mathfrak{t}_D = \{\bar{\mathbf{t}} = \mathrm{diag}(x_1, \dots, x_n, -x_n, \dots, -x_1) : x_i \in \mathbb{C}\}. \tag{6}$$

Restrict the Springer morphism θ to the torus to get

$$\theta(\mathbf{t}) = \mathrm{diag}(\bar{t}_1, \dots, \bar{t}_n, -\bar{t}_n, \dots, -\bar{t}_1), \quad \text{where } \bar{t}_i := \frac{t_i - t_i^{-1}}{2}. \tag{7}$$

The Weyl group W_G of G is a subgroup of S_{2n} consisting of permutations $(a_1 \cdots a_{2n})$ satisfying $a_i + a_{2n+1-i} = 2n+1$ for $1 \leq i \leq 2n$, and the cardinality of the set $\{i \mid 1 \leq i \leq n,\ a_i > n\}$ is even (cf. [BL, §3.5]). Recall the following result from [Ku2, Proposition 12].

Proposition 3.1. *Let $f : T \to \mathbb{C}$ be a regular map. Then, $f \in \mathrm{Rep}^{\mathbb{C}}_{\mathrm{poly}}(G) = \mathrm{Rep}^{\mathbb{C}}_{\omega_1-\mathrm{poly}}(G)$ if and only if there exist symmetric polynomials $P_f(x_1, \dots, x_n)$ and $Q_f(x_1, \dots, x_n)$ satisfying*

$$f(\mathbf{t}) = P_f(\bar{t}_1^2, \dots, \bar{t}_n^2) + \left(\bar{t}_1 \bar{t}_2 \cdots \bar{t}_n\right) Q_f(\bar{t}_1^2, \dots, \bar{t}_n^2) \tag{8}$$

where $\mathbf{t}$ is from (5) and $\bar{t}_i$ from (7).

Definition 3.2. For $1 \leq r \leq n-2$, we let $\mathrm{OG} = \mathrm{OG}(r, 2n)$ to be the set of r-dimensional isotropic subspaces in $V = \mathbb{C}^{2n}$. We take $B_D := B \cap \mathrm{SO}(2n)$, where B is the Borel subgroup of $\mathrm{SL}(2n)$ consisting of upper triangular matrices (of determinant 1). Then, $\mathrm{OG}(r, 2n)$ is the quotient $\mathrm{SO}(2n)/P_r^D$ by the standard maximal parabolic subgroup P_r^D with $\Delta \setminus \{\alpha_r\}$ as the set of simple roots of its Levi component L_r^D. Here L_r^D is the unique Levi subgroup of P_r^D containing T_D. We have

$$L_r^D \simeq \mathrm{GL}(r) \times \mathrm{SO}(2(n-r)). \tag{9}$$

By [Ku2, §9],

$$\mathrm{Rep}^{\mathbb{C}}_{\mathrm{poly}}(L_r^D) \simeq \mathbb{C}_{\mathrm{sym}}[\bar{t}_1, \dots, \bar{t}_r] \otimes_{\mathbb{C}} \left(R \oplus \bar{t}_{r+1} \cdots \bar{t}_n R\right), \quad \text{where } R = \mathbb{C}_{\mathrm{sym}}[\bar{t}_{r+1}^2, \dots, \bar{t}_n^2]. \tag{10}$$

Since the case of $r = n-1$ is parallel to $r = n$ due to the diagram automorphism of D_n taking the n-th node to the $(n-1)$-th node, we only consider $r = n$ here. We have $\mathrm{SO}(2n)/P_n^D$ identified with a connected component $\mathrm{OG}(n, 2n)_+$ of the set of n-dimensional isotropic subspaces of V. Moreover, its Levi subgroup is given by

$$L_n^D \simeq \mathrm{GL}(n). \tag{11}$$

In this case, by [Ku2, §9],

(12) $$\mathrm{Rep}^{\mathbb{C}}_{\mathrm{poly}}(L^D_n) \simeq \mathbb{C}_{\mathrm{sym}}[\bar{t}_1, \dots, \bar{t}_n].$$

Now, we describe the geometry of $\mathrm{OG}(r, 2n)$ following [BKT1]. Two subspaces E and F of V are said to be in the *same family* if $\dim(E \cap F) \equiv n (\mathrm{mod}\, 2)$. Fix once and for all an isotropic subspace L_n of V of dimension n. An isotropic flag is a complete flag $F_\bullet$ consisting of subspaces F_i of V, with $0 = F_0 \subsetneq F_1 \subsetneq F_2 \subsetneq \cdots \subsetneq F_{2n} = V$, such that $F_{n+i} = F^{\perp}_{n-i}$ for all $0 \leq i \leq n-1$, and with F_n and L_n in the same family. As the orthogonal space $F^{\perp}_{n-1}/F_{n-1}$ contains only two isotropic lines, to each such flag $F_\bullet$ there corresponds an alternate isotropic flag $\tilde{F}_\bullet$ such that $\tilde{F}_i = F_i$ for all $i < n$ but with $\tilde{F}_n$ in the opposite family from F_n.

Let $\mathrm{Fl} = \mathrm{Fl}_n = \mathrm{SO}(2n)/B_D$ be the full flag variety consisting of the isotropic flags. Then, the flags $F_\bullet$ give rise to a sequence of tautological vector bundles over Fl:

$$\mathcal{F}_1 \subset \mathcal{F}_2 \subset \cdots \subset \mathcal{F}_n \subset \mathcal{E}, \text{ with } rank(\mathcal{F}_i) = i,$$

where $\mathcal{E} : \mathrm{Fl} \times \mathbb{C}^{2n} \to \mathrm{Fl}$ is the trivial rank-$2n$ vector bundle.

Set $k = n - r \geq 2$, the Schubert varieties in $\mathrm{OG}(r, 2n)$ are indexed by a set $\tilde{\mathcal{P}}(k, n)$ defined as follows. First for any non-negative integer k we say a partition λ is *k-strict* if no part greater than k is repeated, i.e., $\lambda_j > k$ implies $\lambda_{j+1} < \lambda_j$. To any k-strict partition λ, there is associated a number in $\{0, 1, 2\}$ called the *type* of λ, denoted $\mathrm{type}(\lambda)$ (cf. [BKT1, §3]). For example, if λ has no part equal to k, then $\mathrm{type}(\lambda) = 0$. The elements of $\tilde{\mathcal{P}}(k, n)$ are the k-strict partitions contained in an $(n - k) \times (n + k - 1)$ rectangle. For any $\lambda \in \tilde{\mathcal{P}}(k, n)$, we define index set $P' = P'(\lambda) = \{p'_1(\lambda) < \cdots < p'_r(\lambda)\} \subset [1, 2n]$ by

$$\begin{aligned} p'_j(\lambda) =& n + k - 1 - \lambda_j + \#\{i < j \mid \lambda_i + \lambda_j \leq 2k - 1 + j - i\} \\ &+ \begin{cases} 1, & \text{if } \lambda_j > k, \text{ or } \lambda_j = k < \lambda_{j-1} \text{ and } n - 1 + j + \mathrm{type}(\lambda) \text{ is even}, \\ 2, & \text{otherwise.} \end{cases} \end{aligned}$$

Given any fixed isotropic flag $F_\bullet$, each $\lambda \in \tilde{\mathcal{P}}(k, n)$ indexes a codimension $|\lambda|$ Schubert variety $X_\lambda(F_\bullet)$ in $\mathrm{OG}(r, 2n)$, defined as the locus of $\Sigma \in \mathrm{OG}(r, 2n)$ such that

$$\dim(\Sigma \cap F_{p'_j(\lambda)}) \geq j, \text{ if } p'_j(\lambda) \neq n + 1,$$
$$\dim(\Sigma \cap \tilde{F}_n) \geq j, \text{ if } p'_j(\lambda) = n + 1$$

for all j with $1 \leq j \leq \ell(\lambda)$, where $\ell(\lambda)$ is the number of non-zero parts of λ. For each $\lambda \in \tilde{\mathcal{P}}(k, n)$, we let τ_λ denote the cohomology class in $H^{2|\lambda|}(\mathrm{OG}(r, 2n), \mathbb{Z})$ dual to the cycle determined by the Schubert variety indexed by λ.

The special Schubert varieties for $\mathrm{OG}(r, 2n)$ are $X_1, \dots, X_k, X'_k, X_{k+1}, \dots, X_{n+k-1}$. They are defined by a single Schubert condition as follows. For $p \neq k$, we have

$$X_p(F_\bullet) = \{\Sigma \in \mathrm{OG}(r, 2n) \mid \Sigma \cap F_{\epsilon(p)} \neq 0\}$$

where

(13) $$\epsilon(p) = n + k - p + \begin{cases} 2, & \text{if } p \leq k \\ 1, & \text{if } p > k. \end{cases}$$

For any n, define

$$X_k(F_\bullet) = \{\Sigma \in \mathrm{OG}(r, 2n) \mid \Sigma \cap F_n \neq 0\}$$

and

$$X'_k(F_\bullet) = \{\Sigma \in \mathrm{OG}(r, 2n) \mid \Sigma \cap \tilde{F}_n \neq 0\}.$$

We let τ_p denote the cohomology class of $X_p(F_\bullet)$ for $1 \le p \le n+k-1$ and τ'_k denote the cohomology class of $X'_k(F_\bullet)$. Our definition of τ_k, τ'_k is slightly different from that of [BKT1, §3.2] to make them compatible under taking limits as $n \to \infty$ (cf. the proof of Proposition 3.7). Their τ_k, τ'_k coincides with our τ_k, τ'_k respectively for odd n. However, for even n, their τ'_k (resp. τ_k) is our τ_k (resp. τ'_k).

We have a short exact sequence of vector bundles on $\mathrm{OG}(r = n-k, 2n)$:

$$0 \to \mathcal{S} \to V_D \to \mathcal{Q} \to 0. \tag{14}$$

Here V_D is the trivial bundle of rank $2n$, $\mathcal{S}$ is the taotological subbundle of rank $n-k$, and $\mathcal{Q}$ is the quotient bundle of rank $n+k$. Let $c_i = c_i(\mathcal{Q})$ for $1 \le i \le n+k$ be the i-th Chern class of the quotient bundle $\mathcal{Q}$. (Observe that $c_0 = 1$ and $c_i = 0$ for $i < 0$.) Then, we have the following presentation of $H^*(\mathrm{OG}(r, 2n), \mathbb{Z})$ from [BKT1, Theorem 3.2].

First, for each $s > 0$, let Δ_s denote the $s \times s$ Schur determinant:

$$\Delta_s = \det(c_{1+j-i})_{1\le i,j\le s}.$$

By [BKT1, Identity 40], we have

$$c_p(\mathcal{Q}) = \begin{cases} \tau_p, & \text{if } p < k,\\ \tau_k + \tau'_k, & \text{if } p = k,\\ 2\tau_p, & \text{if } p > k. \end{cases} \tag{15}$$

Then, we have:

Theorem 3.3. *The cohomology ring $H^*(\mathrm{OG}(n-k, 2n), \mathbb{Z})$ is presented as a quotient of the polynomial ring*

$$\mathbb{Z}[\tau_1, \dots, \tau_k, \tau'_k, \tau_{k+1}, \dots, \tau_{n+k-1}]$$

modulo the relations

$$\Delta_s = 0,\ n-k < s < n, \tag{16}$$

$$\tau_k \Delta_{n-k} = \tau'_k \Delta_{n-k} = \sum_{p=k+1}^{n} (-1)^{p+k+1} \tau_p \Delta_{n-p}, \tag{17}$$

$$\sum_{p=k+1}^{s} (-1)^p \tau_p \Delta_{s-p} = 0,\ n < s < n+k, \tag{18}$$

$$\tau_s^2 + \sum_{p=1}^{s} (-1)^p \tau_{s+p} c_{s-p} = 0,\ k+1 \le s < n, \tag{19}$$

$$\tau_k \tau'_k + \sum_{p=1}^{k} (-1)^p \tau_{k+p} \tau_{k-p} = 0, \tag{20}$$

where we set $\tau_0 = 1, \tau_p = 0$ for $p \ge n+k$.

Now, we can get a similar result as [KR, Proposition 11] for the image of ξ^P. First, we define

$$x_j = -c_1(\mathcal{F}_j/\mathcal{F}_{j-1}), \quad \text{for } 1 \le j \le n.$$

Lemma 3.4. *For $1 \leq j \leq n$, the Schubert divisor $\epsilon_{s_j} \in H^2(\mathrm{Fl}, \mathbb{Z})$ corresponding to the simple reflection s_j is given by*

$$\begin{aligned}\epsilon_{s_j} &= x_1 + \cdots + x_j, \ \textit{for } 1 \leq j \leq n-2\\ &= \frac{1}{2}(x_1 + \cdots + x_{n-1} - x_n), \ \textit{for } j = n-1\\ &= \frac{1}{2}(x_1 + \cdots + x_n), \ \textit{for } j = n.\end{aligned}$$

Moreover, we have $\xi^B(\bar{t}_j) = x_j$ for all $1 \leq j \leq n$.

Proof. By the identity (2), for any $1 \leq j \leq n$,

$$\beta(\omega_j) := c_1(\mathcal{L}(\omega_j)) = \epsilon_{s_j}, \tag{21}$$

where $\beta : S(\mathfrak{t}_D^*) \to H^*(\mathrm{Fl}, \mathbb{C})$ is the Borel homomorphism.

For $1 \leq i \leq n$, since $\mathcal{F}_i/\mathcal{F}_{i-1} \simeq \mathcal{L}(-\delta_i)$, where δ_i is the character of T_D taking $\mathrm{diag}(t_1, \ldots, t_n, t_n^{-1}, \ldots, t_1^{-1}) \mapsto t_i$,

$$x_i = -c_1(\mathcal{F}_i/\mathcal{F}_{i-1}) = \beta(\delta_i). \tag{22}$$

Now, by [Bo, Planche IV], combining the equations (21) and (22), we get

$$\begin{aligned}\epsilon_{s_j} &= \beta(\delta_1 + \cdots + \delta_j) = x_1 + \cdots + x_j, \ \text{ for } 1 \leq j \leq n-2\\ &= \beta\left(\frac{1}{2}(\delta_1 + \cdots + \delta_{n-1} - \delta_n)\right) = \frac{1}{2}(x_1 + \cdots + x_{n-1} - x_n), \ \text{ for } j = n-1\\ &= \beta\left(\frac{1}{2}(\delta_1 + \cdots + \delta_n)\right) = \frac{1}{2}(x_1 + \cdots + x_n), \ \text{ for } j = n.\end{aligned}$$

This proves the first part of the lemma.

Further, by [Ku2, Proposition 24(c)] (under the convention that $\epsilon_{s_0} = 0$),

$$\begin{aligned}&\xi^B(\bar{t}_j)\\ &= \epsilon_{s_j} - \epsilon_{s_{j-1}} = x_j, \ \text{ for } 1 \leq j \leq n-2\\ &= \epsilon_{s_{n-1}} + \epsilon_{s_n} - \epsilon_{s_{n-2}} =\\ &\frac{1}{2}(x_1 + \cdots + x_{n-1} - x_n) + \frac{1}{2}(x_1 + \cdots + x_{n-1} + x_n) - (x_1 + \cdots + x_{n-2}) = x_{n-1}, \text{ for } j = n-1\\ &= \epsilon_{s_n} - \epsilon_{s_{n-1}} = \frac{1}{2}(x_1 + \cdots + x_n) - \frac{1}{2}(x_1 + \cdots + x_{n-1} - x_n) = x_n, \ \text{ for } j = n.\end{aligned}$$

This proves the second part of the lemma. □

By Lemma 3.4, we get:

Lemma 3.5. *For $0 \leq j \leq n$,*

$$c(Q_j)c(Q_j^*) = \prod_{p=j+1}^{n} (1 - x_p)(1 + x_p) \in H^*(\mathrm{Fl}, \mathbb{Z}),$$

where $Q_j := \mathcal{E}/\mathcal{F}_j$ and $c(Q_j)$ is the total Chern class of Q_j.

Proof. By the definition, we have $1 - x_p = c(\mathcal{F}_p/\mathcal{F}_{p-1})$. Then,

$$\prod_{p=j+1}^{n} (1 - x_p)(1 + x_p) = \prod_{p=j+1}^{n} c(\mathcal{F}_p/\mathcal{F}_{p-1})c((\mathcal{F}_p/\mathcal{F}_{p-1})^*) = \frac{c(\mathcal{F}_n)}{c(\mathcal{F}_j)}\frac{c(\mathcal{F}_n^*)}{c(\mathcal{F}_j^*)}.$$

From the exact sequence $0 \to \mathcal{F}_j \to \mathcal{E} \to Q_j \to 0$, we get

$$c(Q_j)c(\mathcal{F}_j) = 1 \text{ and } c(Q_j^*)c(\mathcal{F}_j^*) = 1.$$

In particular, we have $c(Q_n)c(\mathcal{F}_n) = 1$. From the bilinear form we have $Q_n = (\mathcal{F}_n^{\perp})^*$. Further, from the definition, we have $\mathcal{F}_n^{\perp} = \mathcal{F}_n$. Thus, we have

$$c(\mathcal{F}_n)c(\mathcal{F}_n^*) = 1.$$

This proves the lemma. □

The above lemma allows us to prove the following crucial result of the paper in the case of $\mathrm{SO}(2n)$.

Theorem 3.6. *For $2 \le k \le n-1$, the map $\xi^{P_{n-k}} : Rep^{\mathbb{C}}_{\mathrm{poly}}(L_{n-k}) \to H^*(\mathrm{OG}(n-k, 2n), \mathbb{C})$ takes*

$$e_i((\bar{t}_{n-k+1})^2, \dots, (\bar{t}_n)^2) \mapsto c_i^2 + 2\sum_{j=1}^{i}(-1)^j c_{i+j}c_{i-j}, \text{ for any } 1 \le i \le k,$$

where $c_p = c_p(Q)$ is the p-th Chern class of the quotient bundle Q as in (14) *and e_i is the i-th elementary symmetric polynomial.*

Proof. This follows from Lemmas 3.4 and 3.5 by taking the degree $2i$ component in Lemma 3.5 for $j = n-k$, and using the projection $\mathrm{Fl} \to \mathrm{OG}(n-k, 2n)$. □

For any $k \ge 2$, consider the inverse system:

$$\cdots \leftarrow H^*(\mathrm{OG}(n-k, 2n), \mathbb{Z}) \xleftarrow{\pi_n^*} H^*(\mathrm{OG}(n+1-k, 2(n+1)), \mathbb{Z}) \leftarrow \cdots, \tag{23}$$

where $\pi_n : \mathrm{OG}(n-k, 2n) \hookrightarrow \mathrm{OG}(n+1-k, 2(n+1))$ is the embedding $S \to T_n(S) \oplus \mathbb{C}e_n$. Here, $T_n : \mathbb{C}^{2n} \to \mathbb{C}^{2n+2}$ is the linear embedding taking $e_i \mapsto e_i$ for $1 \le i \le n-1$, taking $e_i \mapsto e_{i+1}$ for $n \le i \le n+1$, and $e_i \mapsto e_{i+2}$ for $n+2 \le i \le 2n$. Notice that in the definition of the isotropic flag $F_\bullet$ we fix an isotropic subspace $L_n \subset \mathbb{C}^{2n}$ for each $n \ge 4$. To define a compatible isotropic subspace $L_{n+1} \subset \mathbb{C}^{2n+2}$ (under the embedding T_n), we set $L_{n+1} = T_n(L_n) \oplus \mathbb{C}e_n$.

Define the stable cohomology ring as

$$\mathbb{H}^*(\mathrm{OG}_k, \mathbb{Z}) = \varprojlim_n H^*(\mathrm{OG}(n-k, 2n), \mathbb{Z}).$$

Then, we have the following proposition (cf. [BKT2, §4.2]).

Proposition 3.7. *For any $k \ge 2$, the stable cohomology ring $\mathbb{H}^*(\mathrm{OG}_k, \mathbb{Z})$ admits a presentation as polynomial ring $\mathbb{Z}[\tau_1, \dots, \tau_{k-1}, \tau_k, \tau'_k, \tau_{k+1}, \dots]$ modulo the relations:*

$$\tau_s^2 + \sum_{p=1}^{s}(-1)^p \tau_{s+p}c_{s-p} = 0, \ s \ge k+1,$$

$$\tau_k\tau'_k + \sum_{p=1}^{k}(-1)^p \tau_{k+p}\tau_{k-p} = 0,$$

where we interpret $\tau_0 = 1$ and

$$c_p = \begin{cases} \tau_p, & \text{if } p < k, \\ \tau_k + \tau'_k, & \text{if } p = k, \\ 2\tau_p, & \text{if } p > k. \end{cases}$$

Further, the natural restriction homomorphism $\varphi_{k,n} : \mathbb{H}^*(\mathrm{OG}_k, \mathbb{Z}) \to H^*(\mathrm{OG}(n-k, 2n), \mathbb{Z})$ takes c_p to $c_p(Q)$. Moreover, it is surjective.

Proof. Recall from (15) that for $c_p(n) = c_p(Q)$, we have

$$c_p(n) = \begin{cases} \tau_p, & \text{if } p < k, \\ \tau_k + \tau'_k, & \text{if } p = k, \\ 2\tau_p, & \text{if } p > k, \end{cases}$$

where Q is the tautological quotient bundle over $\mathrm{OG}(n-k, 2n)$. (Here we use the notation $c_p(n)$ instead of c_p since we will need to vary n.)

From the functoriality of Chern classes, π_n^* takes $c_p(n+1) \to c_p(n)$ for $1 \le p \le n+k$, and $c_{n+k+1}(n+1) \mapsto 0$. Also, π_n^* takes τ_k to τ_k, and τ'_k to τ'_k:

To prove this, represent $\tau_k = \tau_k^n \in H^{2k}(\mathrm{OG}(n-k, 2n), \mathbb{Z})$ as the Schubert class $\epsilon_{w_n}^{P_{n-k}}$, where $w_n := s_{n-1} \dots s_{n-k}$ (cf. [BKT2, Proof of Proposition 6.3]) . Then, $w_n \cdot (\mathbb{C}e_1 \oplus \cdots \oplus \mathbb{C}e_{n-k}) = \mathbb{C}e_1 \oplus \cdots \oplus \mathbb{C}e_{n-k-1} \oplus \mathbb{C}e_n$, under the standard basis $\{e_i\}_{1 \le i \le 2n}$ of $\mathbb{C}^{2n}$. Moreover,

$$\pi_n\left((\mathbb{C}e_1 \oplus \cdots \oplus \mathbb{C}e_{n-k-1} \oplus \mathbb{C}e_n)\right) = \mathbb{C}e_1 \oplus \cdots \oplus \mathbb{C}e_{n-k-1} \oplus \mathbb{C}e_{n+1} \oplus \mathbb{C}e_n.$$

Consider the (ordered) basis $\tilde{\mathbf{e}} := \{e_1, \dots, e_{n-k-1}, e_n, e_{n-k}, \dots, e_{n-1}, \hat{e}_n, e_{n+1}, \dots, e_{2n+2}\}$ of $\mathbb{C}^{2n+2}$. Then,

$$w_{n+1} \cdot (\mathbb{C}e_1 \oplus \cdots \oplus \mathbb{C}e_{n-k-1} \oplus \mathbb{C}e_n \oplus \mathbb{C}e_{n-k}) = \mathbb{C}e_1 \oplus \cdots \oplus \mathbb{C}e_{n-k-1} \oplus \mathbb{C}e_n \oplus \mathbb{C}e_{n+1}.$$

Thus,

$$\pi_n\left(w_n \cdot (\mathbb{C}e_1 \oplus \cdots \oplus \mathbb{C}e_{n-k})\right) = w_{n+1} \cdot (\mathbb{C}e_1 \oplus \cdots \oplus \mathbb{C}e_{n-k-1} \oplus \mathbb{C}e_{n-k} \oplus \mathbb{C}e_n).$$

Further, it is easy to see that

$$\pi_n\left(B_n^- \cdot w_n \cdot (\mathbb{C}e_1 \oplus \cdots \oplus \mathbb{C}e_{n-k})\right) \subset \tilde{B}_{n+1}^- \cdot \pi_n\left(w_n \cdot (\mathbb{C}e_1 \oplus \cdots \oplus \mathbb{C}e_{n-k})\right), \tag{24}$$

where B_n^- is the Borel subgroup opposite to the standard Borel subgroup B_n of $\mathrm{SO}(2n)$ in the $\{e_i\}_{1 \le i \le 2n}$ basis of $\mathbb{C}^{2n}$ and $\tilde{B}_{n+1}^-$ is the Borel subgroup opposite to the standard Borel subgroup of $\mathrm{SO}(2n+2)$, but under the $\tilde{\mathbf{e}}$ basis of $\mathbb{C}^{2n+2}$. Now, since π_n is an embedding and the two sides of the equation (24) have the same dimension ($= k$), the inclusion in the equation (24) is an equality. This proves that τ_k^{n+1} restricts to τ_k^n under π_n^*. Since $\tau_k + \tau'_k = c_k$ and c_k restricts to c_k, we get that τ'_k also restricts to τ'_k.

Thus, from the presentation of the ring $H^*(\mathrm{OG}(n-k, 2n), \mathbb{Z})$ (cf. Theorem 3.3), the first three relations (16)–(18) disappear in the inverse limit. This proves the presentation of $\mathbb{H}^*(\mathrm{OG}_k, \mathbb{Z})$ as in the proposition.

For each $\lambda \in \tilde{\mathcal{P}}(k, n)$ we define a monomial τ^λ in terms of the special Schubert classes as follows. If $\mathrm{type}(\lambda) \neq 2$, then set $\tau^\lambda = \prod_i \tau_{\lambda_i}$. If $\mathrm{type}(\lambda) = 2$ then τ^λ is defined by the same product formula, but replacing each occurrence of τ_k with τ'_k. Then, $\{\tau^\lambda\}_{\lambda \in \tilde{\mathcal{P}}(k,n)}$ form a $\mathbb{Z}$-basis of $H^*(\mathrm{OG}(r, 2n), \mathbb{Z})$ (cf. [BKT1, Theorem 3.2(b)]).

From the description of π_n^*, the natural ring homomorphism $\varphi_{k,n} : \mathbb{H}^*(\mathrm{OG}_k, \mathbb{Z}) \to H^*(\mathrm{OG}(n-k, 2n), \mathbb{Z})$ takes τ^λ to τ^λ whenever $\lambda \in \tilde{\mathcal{P}}(k, n)$ and zero otherwise. In particular, $\varphi_{k,n}$ is surjective since $\{\tau^\lambda\}_{\lambda \in \tilde{\mathcal{P}}(k,n)}$ span $H^*(\mathrm{OG}(r, 2n), \mathbb{Z})$. □

Having described $\mathbb{H}^*(\mathrm{OG}_k, \mathbb{Z})$, we now construct a morphism $\mathrm{Rep}^{\mathbb{C}}_{\mathrm{poly}}(\mathrm{SO}(2k)) \to \mathbb{H}^*(\mathrm{OG}_k, \mathbb{C})$.

By [Ku2, Proposition 12],

$$\mathrm{Rep}^{\mathbb{C}}_{\mathrm{poly}}(\mathrm{SO}(2k)) \simeq \mathbb{C}_{\mathrm{sym}}[\bar{h}_1^2, \dots, \bar{h}_k^2] \oplus \left(\bar{h}_{1,k} \mathbb{C}_{\mathrm{sym}}[\bar{h}_1^2, \dots, \bar{h}_k^2]\right), \tag{25}$$

where $\bar{h}_{1,k} := \bar{h}_1 \bar{h}_2 \cdots \bar{h}_k$. Define (using the decomposition (9))

$$\iota_k^n : \mathrm{Rep}^{\mathbb{C}}_{\mathrm{poly}}(\mathrm{SO}(2k)) \to \mathrm{Rep}^{\mathbb{C}}_{\mathrm{poly}}(L^D_{n-k}) \tag{26}$$

by taking $f(\bar{\mathbf{h}}) \to 1 \otimes f(\bar{\mathbf{t}})$, where $\bar{\mathbf{h}} := (\bar{h}_1, \ldots, \bar{h}_k)$, $\bar{\mathbf{t}} := (\bar{t}_{n-k+1}, \ldots, \bar{t}_n)$ and $f(\bar{\mathbf{t}})$ is the same polynomial written in the $\bar{\mathbf{t}}$ variables under the transformation $\bar{t}_{n-k+i} = \bar{h}_i$. This gives rise to a $\mathbb{C}$-algebra homomorphism

$$\xi_{n,k} = \xi^{P_{n-k}} \circ \iota_k^n : \mathrm{Rep}^{\mathbb{C}}_{\mathrm{poly}}(\mathrm{SO}(2k)) \to H^*(\mathrm{OG}(n-k, 2n), \mathbb{C}).$$

Moreover, by virtue of Theorem 3.6 and the isomorphism (25), $(\xi_{n,k})_{|\mathbb{C}_{\mathrm{sym}}[\bar{h}_1^2, \ldots, \bar{h}_k^2]}$ commutes with the following inverse system:

$$\begin{array}{ccc}
 & & \uparrow \pi_{n-1}^* \\
\mathrm{Rep}^{\mathbb{C}}_{\mathrm{poly}}(\mathrm{SO}(2k)) & \xrightarrow{\xi_{n,k}} & H^*(\mathrm{OG}(n-k, 2n), \mathbb{C}) \\
 & \searrow^{\xi_{n+1,k}} & \uparrow \pi_n^* \\
 & & H^*(\mathrm{OG}(n-k+1, 2n+2), \mathbb{C}) \\
 & & \uparrow \pi_{n+1}^*
\end{array}$$

We next show that

$$\pi_n^* \circ \xi_{n+1,k}(\bar{h}_{1,k}) = \xi_{n,k}(\bar{h}_{1,k}). \tag{27}$$

To prove this, define a morphism $\hat{\pi}_n : \mathrm{Fl}_n \to \mathrm{Fl}_{n+1}$ by

$$\begin{aligned}
&\hat{\pi}_n\Big((0 = F_0 \subset F_1 \subset \cdots \subset F_{2n} = \mathbb{C}^{2n})\Big) \\
&= (0 \subset T_n(F_1) \subset \cdots \subset T_n(F_{n-k-1}) \subset \widehat{T_n(F_{n-k})} \subset T_n(F_{n-k}) \oplus \mathbb{C}e_n \\
&\subset \cdots \subset T_n(F_n) \oplus \mathbb{C}e_n \subset (T_n(F_{n-1}) \oplus \mathbb{C}e_n)^\perp \subset \cdots \subset (T_n(F_{n-k}) \oplus \mathbb{C}e_n)^\perp \subset \\
&(\widehat{T_n(F_{n-k})})^\perp \subset T_n(F_{n-k-1})^\perp \subset \cdots \subset T_n(F_1)^\perp \subset \mathbb{C}^{2n+2}),
\end{aligned}$$

where we map F_i successively to the right side omitting the two terms with $\widehat{\ }$ over them. Clearly, $T_n(F_n) \oplus \mathbb{C}e_n$ and L_{n+1} are in the same family. We have the following commutative diagram:

$$\begin{array}{ccc}
\mathrm{Fl}_n & \xrightarrow{\hat{\pi}_n} & \mathrm{Fl}_{n+1} \\
\downarrow & & \downarrow \\
OG(n-k, 2n) & \xrightarrow{\pi_n} & OG(n+1-k, 2(n+1)),
\end{array}$$

where the vertical maps are the canonical projections. From the above definition of $\hat{\pi}_n$, it is clear that the line bundle $\mathcal{F}_{j+1}/\mathcal{F}_j$ over Fl_{n+1} pulls back under $\hat{\pi}_n$ to the line bundle $\mathcal{F}_j/\mathcal{F}_{j-1}$ over Fl_n for any $n-k+1 \le j \le n$. Thus, by Lemma 3.4 and the injectivity of $H^*(OG(n-k, 2n), \mathbb{Z}) \to H^*(\mathrm{Fl}_n, \mathbb{Z})$, we get the validity of identity (27).

Thus, we get a $\mathbb{C}$-algebra homomorphism

$$\xi_k : \mathrm{Rep}^{\mathbb{C}}_{\mathrm{poly}}(\mathrm{SO}(2k)) \to \mathbb{H}^*(\mathrm{OG}_k, \mathbb{C}).$$

The following theorem is one of our main results for D_n-type groups.

Theorem 3.8. *Let $k \ge 2$. Then, the above $\mathbb{C}$-algebra homomorphism $\xi_k : Rep^{\mathbb{C}}_{\mathrm{poly}}(\mathrm{SO}(2k)) \to \mathbb{H}^*(\mathrm{OG}_k, \mathbb{C})$ takes the generator*

$$e_i((\bar{h}_1)^2, \ldots, (\bar{h}_k)^2) \mapsto c_i^2 + 2\sum_{j=1}^{i} (-1)^j c_{i+j} c_{i-j}, \ \textit{for any } 1 \le i \le k, \tag{28}$$

where c_i are as in Proposition 3.7.

Moreover, ξ_k is injective.

Proof. We will simply abbreviate $e_i((\bar{h}_1)^2, \dots, (\bar{h}_k)^2)$ by e_i. The first part of the theorem (i.e., (28)) follows directly from Theorem 3.6, Proposition 3.7 and the definition of i_k^n (as in equation (26)).

For the injectivity of ξ_k, since $\mathbb{C}$ is a torsionfree $\mathbb{Z}$-module, we only need to prove that

$$\xi_k^{\mathbb{Z}} : \mathrm{Rep}^{\mathbb{Z}}_{\mathrm{poly}}(\mathrm{SO}(2k)) \to \mathbb{H}^*(\mathrm{OG}_k, \mathbb{Z})$$

is injective (cf. [Sp, Chap. 5, Sec. 2, Lemma 5]), where

$$\mathrm{Rep}^{\mathbb{Z}}_{\mathrm{poly}}(\mathrm{SO}(2k)) := \mathbb{Z}_{\mathrm{sym}}[\bar{h}_1^2, \dots, \bar{h}_k^2] \oplus \left(\bar{h}_{1,k}\mathbb{Z}_{\mathrm{sym}}[\bar{h}_1^2, \dots, \bar{h}_k^2]\right). \tag{29}$$

First, we take the subring $\bar{\mathbb{H}}^*(\mathrm{OG}_k, \mathbb{Z})$ of $\mathbb{H}^*(\mathrm{OG}_k, \mathbb{Z})$ generated by $\{\tau_k - \tau'_k, c_1, c_2, \dots, c_k, \dots\}$. Consider the restricted homomorphism:

$$\eta_k^{\mathbb{Z}} : \mathbb{Z}_{\mathrm{sym}}[\bar{h}_1^2, \dots, \bar{h}_k^2] \to \bar{\mathbb{H}}^*(\mathrm{OG}_k, \mathbb{Z}),$$

which is well defined by (28). Observe that $\eta_k^{\mathbb{Z}}$ is a homomorphism of graded rings if we assign degree $4i$ to each e_i and the standard cohomological degree to $\bar{\mathbb{H}}^*(\mathrm{OG}_k, \mathbb{Z})$.

We now prove that $\eta_k^{\mathbb{Z}}$ is injective:

From Proposition 3.7, $\bar{\mathbb{H}}^*(\mathrm{OG}_k, \mathbb{Z})$ has the presentation as polynomial ring $\mathbb{Z}[\tau_k - \tau'_k, c_1, c_2, \dots, c_k, \dots]$ modulo the relations:

$$c_s^2 + 2\sum_{p=1}^{s} (-1)^p c_{s+p} c_{s-p} = 0, \ \ s \geq k+1,$$

$$c_k^2 - (\tau_k - \tau'_k)^2 + 2\sum_{p=1}^{k} (-1)^p c_{k+p} c_{k-p} = 0.$$

Since $\bar{\mathbb{H}}^*(\mathrm{OG}_k, \mathbb{Z})$ is free $\mathbb{Z}$-module of finite rank in each degree (since so is $\mathbb{H}^*(\mathrm{OG}_k, \mathbb{Z})$), we have an exact sequence (cf. [Sp, Chap. 5, Sec. 2, Lemma 5])

$$0 \to \mathbb{Z}_2 \otimes_{\mathbb{Z}} \bar{K} \to \mathbb{Z}_2 \otimes_{\mathbb{Z}} \mathbb{Z}_{\mathrm{sym}}[\bar{h}_1^2, \dots, \bar{h}_k^2],$$

where $\mathbb{Z}_2 := \mathbb{Z}/(2)$ and $\bar{K}$ is the kernel of $\eta_k^{\mathbb{Z}}$. We next prove that

$$\eta_k^{\mathbb{Z}_2} : \mathbb{Z}_2 \otimes_{\mathbb{Z}} \mathbb{Z}_{\mathrm{sym}}[\bar{h}_1^2, \dots, \bar{h}_k^2] \to \mathbb{Z}_2 \otimes_{\mathbb{Z}} \bar{\mathbb{H}}^*(\mathrm{OG}_k, \mathbb{Z})$$

is injective:

To prove this, observe that by the above presentation of $\bar{\mathbb{H}}^*(\mathrm{OG}_k, \mathbb{Z})$, We have

$$\mathbb{Z}_2 \otimes_{\mathbb{Z}} \bar{\mathbb{H}}^*(\mathrm{OG}_k, \mathbb{Z}) \simeq \mathbb{Z}_2[c_1, \dots, c_{k-1}] \otimes \frac{\mathbb{Z}_2[\tau_k - \tau'_k, c_k]}{\langle (\tau_k - \tau'_k)^2 - c_k^2 \rangle} \otimes \frac{\mathbb{Z}_2[c_{k+1}, c_{k+2}, \dots]}{\langle c_{k+1}^2, c_{k+2}^2, \dots \rangle}.$$

Moreover, by the equation (28), the ring homomorphism $\eta_k^{\mathbb{Z}_2}$ takes:

$$e_i \mapsto c_i^2, \ \text{for any } 1 \leq i \leq k-1, \tag{30}$$

$$e_k \mapsto (\tau_k - \tau'_k)^2 = c_k^2. \tag{31}$$

Thus, we conclude that $\eta_k^{\mathbb{Z}_2}$ is injective, so $\mathbb{Z}_2 \otimes_{\mathbb{Z}} \bar{K} = 0$. Since $\bar{K}$ is free $\mathbb{Z}$-module of finite rank in each degree, we have $\bar{K} = 0$, so $\eta_k^{\mathbb{Z}}$ is injective.

We are now ready to prove the injectivity of $\xi_k^{\mathbb{Z}}$ (and hence that of ξ_k):

Any element in $\mathrm{Rep}^{\mathbb{Z}}_{\mathrm{poly}}(\mathrm{SO}(2k))$ can be written as $f + \bar{h}_{1,k}g$, where $f, g \in \mathbb{Z}[e_1, \ldots, e_k]$. Now, assume that

$$\xi^{\mathbb{Z}}_k(f + \bar{h}_{1,k}g) = 0. \tag{32}$$

In particular,

$$\eta^{\mathbb{Z}}_k(f^2) = \xi^{\mathbb{Z}}_k(f^2) = \xi^{\mathbb{Z}}_k(\bar{h}^2_{1,k}g^2) = \eta^{\mathbb{Z}}_k(\bar{h}^2_{1,k}g^2). \tag{33}$$

Notice that we have $\bar{h}^2_{1,k} = e_k((\bar{h}_1)^2, \ldots, (\bar{h}_k)^2)$. Thus, we have

$$\eta^{\mathbb{Z}}_k(f^2(e_1, \ldots, e_k)) = \eta^{\mathbb{Z}}_k(e_k \cdot g^2(e_1, \ldots, e_k)).$$

If $g \neq 0$, let the highest degree of e_k in f, g be p, q respectively. Then, the highest degree of e_k in $f^2(e_1, \ldots, e_k)$ and $e_k g^2(e_1, \ldots, e_k)$ are $2p$ and $2q + 1$ respectively. In particular, $f^2 \neq e_k g^2$ and hence $\eta^{\mathbb{Z}}_k(f^2) \neq \eta^{\mathbb{Z}}_k(\bar{h}^2_{1,k}g^2)$ (due to the injectivity of $\eta^{\mathbb{Z}}_k$). This contradicts (33) and hence $g = 0$ (and hence so is $f = 0$) by (32). This proves the injectivity of $\xi^{\mathbb{Z}}_k$ completing the proof of the theorem. □

Remark 3.9. The homomorphism $\xi_k : \mathrm{Rep}^{\mathbb{C}}_{\mathrm{poly}}(\mathrm{SO}(2k)) \to \mathbb{H}^*(\mathrm{OG}_k, \mathbb{C})$ as in Theorem 3.8 is *not* surjective since the domain is finitely generated whereas the range is not.

4. Exceptional Groups

In the discussion of exceptional groups in the following Sections 4-8, we identify the maximal torus T of connected and simply-connected G by

$$T = \mathrm{Hom}_{\mathbb{Z}}(\mathfrak{t}^*_{\mathbb{Z}}, \mathbb{C}^*), \tag{34}$$

where $\mathfrak{t}_{\mathbb{Z}} := \oplus_{i=1}^{\ell} \mathbb{Z}\alpha_i^\vee$ ($\alpha_i^\vee$ being the simple coroots) and $\mathfrak{t}^*_{\mathbb{Z}} := \mathrm{Hom}_{\mathbb{Z}}(\mathfrak{t}_{\mathbb{Z}}, \mathbb{Z})$ is the weight lattice.

In the following sections, we use the coordinates $(\bar{t}_1, \ldots, \bar{t}_\ell)$ of any element $\bar{t} \in \mathfrak{t}$ in the basis $\{\alpha_1^\vee, \ldots, \alpha_\ell^\vee\}$.

We use the following in the calculations in the following sections.

• The Springer morphism θ_λ restricted to T is determined by using [R, Theorems 1, 2]. The actual calculation can be found in the link [X].

• Let L_r be the Levi subgroup (containing the maximal torus) of a maximal parabolic subgroup P_r of G and let W_r be its Weyl group. We determine the invariant ring $S(\mathfrak{t}^*)^{W_r}$ by first considering the action of the generators of W_r on $\mathfrak{t}^*$ in the basis consisting of the simple roots making use of [Ku2, Proposition 12] when L_r is of classical type. When L_r is of exceptional type, we make use of [L, §3]. Then, we change the simple root basis to the basis consisting of the fundamental weights by using the Cartan matrix.

• We explicitly determine the cup product in $H^*(G/B, \mathbb{Z})$ by using the following theorem of Duan. The actual calculation can be found in the link [X].

Write, for any $u, v \in W$,

$$\epsilon^B_u \cdot \epsilon^B_v = \sum_{w \in W} c^w_{u,v} \epsilon^B_w.$$

For any strictly upper triangular square matrix A of size $k \times k$, Duan *recursively* defines an operator (cf. [D, Definition 2])

$$T_A : \mathbb{Z}[x_1, \ldots, x_k]^{(k)} \to \mathbb{Z},$$

where the superscript (k) denotes the subspace of the homogeneous polynomials of degree k (assigning degree 1 to each x_i). Now, fix a reduced decomposition $s_{i_1} \ldots s_{i_k}$ of w in terms of simple reflections (of length k) and define the $k \times k$-matrix $A_w = (a_{p,q})$ by $a_{p,q} := 0$ if $p \geq q$ and $a_{p,q} := -\alpha_{i_p}(\alpha_{i_q}^\vee)$ for $p < q$.

Then, by [D, Theorem 2.3],

Theorem 4.1. *Take $u, v, w \in W$ with $k = \ell(w) = \ell(u)+\ell(v)$ and fix a reduced decomposition $s_{i_1} \dots s_{i_k}$ of w. Then,*

$$c_{u,v}^w = T_{A_w}\left(\left(\sum_{|L|=\ell(u), s_L=u} x_L\right)\left(\sum_{|K|=\ell(v), s_K=v} x_K\right)\right),$$

where $L, K \subset \{1, \dots, k\}$, $s_L := \prod_{j\in L} s_{i_j}$ (product in the ascending order of j), and $x_L := \prod_{j\in L} x_j$.

The result does not depend on the choice of the reduced decomposition of w.

In the following sections, we only consider the standard maximal parabolic subgroups P_r ($1 \leq r \leq \ell$), where its Levi subgroup L_r containing T has for its simple roots all the simple roots except α_r.

For our Springer morphism, we take the weight λ with the minimum Dynkin index. Then, $\lambda = \omega_1$ (for G_2); $\lambda = \omega_4$ (for F_4); $\lambda = \omega_1$ (for E_6); $\lambda = \omega_7$ (for E_7); $\lambda = \omega_8$ (for E_8) (cf. [Ku3, Corollary A.9]). With this choice, we abbreviate $\mathrm{Rep}^{\mathbb{C}}_{\lambda-\mathrm{poly}}(L_r)$ by $\mathrm{Rep}^{\mathbb{C}}_{\mathrm{poly}}(L_r)$ and $\xi^{P_r}_\lambda$ by ξ^{P_r}.

We follow the indexing convention as in [Bo, Planche V-IX].

5. G_2

We take the basis $x_1 = \omega_1, x_2 = \omega_2 - \omega_1$ of $\mathfrak{t}^*_{\mathbb{Z}}$ and use the coordinates (t_1, t_2) of $t \in T$ defined by (under the identification (34)) $t_i = t(x_i)$. Then, the Springer morphism θ_{ω_1} is given as follows:

Lemma 5.1.

(35)

$$\theta_{\omega_1}(t_1, t_2) = \left(\Theta_1(t) = \frac{1}{6}(2t_1 - 2t_1^{-1} + t_2 - t_2^{-1} + t_1t_2^{-1} - t_1^{-1}t_2), \Theta_2(t) = \frac{1}{2}(t_1 - t_1^{-1} - t_2^{-1} + t_2)\right).$$

It is easy to see that

$$s_1 \cdot x_1 = x_2,\ s_1 \cdot x_2 = x_1,\ s_2 \cdot x_1 = x_1,\ s_2 \cdot x_2 = x_1 - x_2.$$

Thus, s_1 switch x_1 and x_2, and s_2 fixes x_1 and $(x_2 - \frac{1}{2}x_1)^2$ and hence

$$S(\mathfrak{t}^*)^{W_1} = \mathbb{C}[x_1, (x_2 - \frac{1}{2}x_1)^2] \text{ and } S(\mathfrak{t}^*)^{W_2} = \mathbb{C}_{\mathrm{sym}}[x_1, x_2],$$

where W_i is the Weyl group of L_i.

Combining the equation (2) with Lemma 5.1 and using Theorem 4.1, we get the following:

Theorem 5.2. *The generators of $\mathrm{Rep}^{\mathbb{C}}_{\mathrm{poly}}(L_1)$ under ξ^{P_1} are mapped as follows:*

$$\Theta_1(t) \mapsto \epsilon^{P_1}_{s_1},\ (\Theta_2(t) - \frac{1}{2}\Theta_1(t))^2 \mapsto \frac{3}{4}\epsilon^{P_1}_{s_2s_1}.$$

Similarly, under ξ^{P_2} we get

$$\Theta_2(t) \mapsto \epsilon^{P_2}_{s_2},\ \Theta_1(t)\Theta_2(t) - \Theta_1^2(t) \mapsto \epsilon^{P_2}_{s_1s_2}.$$

6. F_4

We use $\{\omega_1, \omega_2, \omega_3, \omega_4\}$ as a basis for $\mathfrak{t}^*$. This gives rise to a coordinate system on T as earlier using the identity (34).

Lemma 6.1.

$$\theta_{\omega_4}(t_1, t_2, t_3, t_4) = (\Theta_1, \Theta_2, \Theta_3, \Theta_4)\,, \tag{36}$$

where

$$\begin{aligned}\Theta_1(t) =&\frac{1}{12}(2t_4 + 2t_3t_4^{-1} + 2t_2t_3^{-1} + 2t_1t_2^{-1}t_3 + 2t_1t_3^{-1}t_4 + 2t_1t_4^{-1}\\ &- 2t_1^{-1}t_4 - 2t_1^{-1}t_3t_4^{-1} - 2t_1^{-1}t_2t_3^{-1} - 2t_2^{-1}t_3 - 2t_3^{-1}t_4 - 2t_4^{-1}).\end{aligned}$$

$$\begin{aligned}\Theta_2(t) =&\frac{1}{12}(4t_4 + 4t_3t_4^{-1} + 4t_2t_3^{-1} + 2t_1t_2^{-1}t_3 + 2t_1^{-1}t_3 + 2t_1t_3^{-1}t_4\\ &+ 2t_1^{-1}t_2t_3^{-1}t_4 + 2t_1t_4^{-1} + 2t_1^{-1}t_2t_4^{-1} - 2t_1t_2^{-1}t_4 - 2t_1t_2^{-1}t_3t_4^{-1} - 2t_1^{-1}t_4\\ &- 2t_1^{-1}t_3t_4^{-1} - 2t_1t_3^{-1} - 2t_1^{-1}t_2t_3^{-1} - 4t_2^{-1}t_3 - 4t_3^{-1}t_4 - 4t_4^{-1}).\end{aligned}$$

$$\begin{aligned}\Theta_3(t) =&\frac{1}{12}(3t_4 + 3t_3t_4^{-1} + 2t_2t_3^{-1} + 2t_1t_2^{-1}t_3 + 2t_1^{-1}t_3 + t_1t_3^{-1}t_4\\ &+ t_1^{-1}t_2t_3^{-1}t_4 + t_1t_4^{-1} + t_2^{-1}t_3t_4 + t_1^{-1}t_2t_4^{-1} + t_2^{-1}t_3^2t_4^{-1} - t_2t_3^{-2}t_4\\ &- t_2t_3^{-1}t_4^{-1} - t_1t_2^{-1}t_4 - t_1t_2^{-1}t_3t_4^{-1} - t_1^{-1}t_4 - t_1^{-1}t_3t_4^{-1} - 2t_1t_3^{-1}\\ &- 2t_1^{-1}t_2t_3^{-1} - 2t_2^{-1}t_3 - 3t_3^{-1}t_4 - 3t_4^{-1}).\end{aligned}$$

$$\begin{aligned}\Theta_4(t) =&\frac{1}{12}(2t_4 + t_3t_4^{-1} + t_2t_3^{-1} + t_1t_2^{-1}t_3 + t_1^{-1}t_3 + t_1t_3^{-1}t_4\\ &+ t_1^{-1}t_2t_3^{-1}t_4 + t_2^{-1}t_3t_4 + t_3^{-1}t_4^2 - t_3t_4^{-2} - t_2t_3^{-1}t_4^{-1} - t_1t_2^{-1}t_3t_4^{-1}\\ &- t_1^{-1}t_3t_4^{-1} - t_1t_3^{-1} - t_1^{-1}t_2t_3^{-1} - t_2^{-1}t_3 - t_3^{-1}t_4 - 2t_4^{-1}).\end{aligned}$$

Combining the equation (2) with Lemma 6.1 and using Theorem 4.1, we get the following:

Theorem 6.2. *The generators of* $\mathrm{Rep}^{\mathbb{C}}_{\mathrm{poly}}(L_r)$ *under* ξ^{P_r} *are mapped as follows:*

(a) (L_1)**:** *Let*

$$y_1^{L_1} = -\Theta_1 + 2\Theta_4,\ y_2^{L_1} = -\Theta_1 + 2\Theta_3 - 2\Theta_4,\ y_3^{L_1} = -\Theta_1 + 2\Theta_2 - 2\Theta_3,\ y_4^{L_1} = \Theta_1.$$

Then,

$$\mathrm{Rep}^{\mathbb{C}}_{\mathrm{poly}}(L_1) = \mathbb{C}_{\mathrm{sym}}\left[(y_1^{L_1})^2, (y_2^{L_1})^2, (y_3^{L_1})^2\right] \otimes_{\mathbb{C}} \mathbb{C}[y_4^{L_1}].$$

Further, the generators of $\mathrm{Rep}^{\mathbb{C}}_{\mathrm{poly}}(L_1)$ *under* ξ^{P_1} *go to:*

$$\begin{aligned}e_1\left((y_1^{L_1})^2, (y_2^{L_1})^2, (y_3^{L_1})^2\right) &\mapsto -\epsilon^{P_1}_{s_2s_1},\\ e_2\left((y_1^{L_1})^2, (y_2^{L_1})^2, (y_3^{L_1})^2\right) &\mapsto -10\epsilon^{P_1}_{s_2s_3s_2s_1} + 28\epsilon^{P_1}_{s_4s_3s_2s_1},\\ e_3\left((y_1^{L_1})^2, (y_2^{L_1})^2, (y_3^{L_1})^2\right) &\mapsto 12\epsilon^{P_1}_{s_4s_1s_2s_3s_2s_1} - 16\epsilon^{P_1}_{s_3s_4s_2s_3s_2s_1},\\ y_4^{L_1} &\mapsto \epsilon^{P_1}_{s_1}.\end{aligned}$$

(b) (L_2)**:** *Let*

$$y_1^{L_2} = -3\Theta_1+2\Theta_2,\ y_2^{L_2} = 3\Theta_1-\Theta_2,\ y_3^{L_2} = 3\Theta_2-4\Theta_3,\ y_4^{L_2} = -\Theta_2+4\Theta_3-4\Theta_4,\ y_5^{L_2} = -\Theta_2+4\Theta_4.$$

Then,

$$\mathrm{Rep}^{\mathbb{C}}_{\mathrm{poly}}(L_2) = \mathbb{C}_{\mathrm{sym}}[y_1^{L_2}, y_2^{L_2}] \otimes_{\mathbb{C}} \mathbb{C}_{\mathrm{sym}}\left[y_3^{L_2}, y_4^{L_2}, y_5^{L_2}\right] \Big/ \left(y_1^{L_2} + y_2^{L_2} - y_3^{L_2} - y_4^{L_2} - y_5^{L_2}\right).$$

Further, the generators of $\mathrm{Rep}^{\mathbb{C}}_{\mathrm{poly}}(L_2)$ *under* ξ^{P_2} *go to:*

$$e_1(y_1^{L_2}, y_2^{L_2}) \mapsto \epsilon^{P_2}_{s_2},$$

$$e_2(y_1^{L_2}, y_2^{L_2}) \mapsto 7\epsilon^{P_2}_{s_1s_2} - 4\epsilon^{P_2}_{s_3s_2},$$

$$e_1(y_3^{L_2}, y_4^{L_2}, y_5^{L_2}) \mapsto \epsilon^{P_2}_{s_2},$$

$$e_2(y_3^{L_2}, y_4^{L_2}, y_5^{L_2}) \mapsto -5\epsilon^{P_2}_{s_1s_2} + 6\epsilon^{P_2}_{s_3s_2},$$

$$e_3(y_3^{L_2}, y_4^{L_2}, y_5^{L_2}) \mapsto -4\epsilon^{P_2}_{s_3s_1s_2} - 10\epsilon^{P_2}_{s_2s_3s_2} - 36\epsilon^{P_2}_{s_4s_3s_2}.$$

(c) (L_3): *Let*

$$y_1^{L_3} = 2\Theta_1 - \Theta_3,\ y_2^{L_3} = -2\Theta_1 + 2\Theta_2 - \Theta_3,\ y_3^{L_3} = -2\Theta_2 + 3\Theta_3,\ y_4^{L_3} = 2\Theta_3 - 3\Theta_4,\ y_5^{L_3} = -\Theta_3 + 3\Theta_4.$$

Then,

$$\mathrm{Rep}^{\mathbb{C}}_{\mathrm{poly}}(L_3) = \mathbb{C}_{\mathrm{sym}}[y_1^{L_3}, y_2^{L_3}, y_3^{L_3}] \otimes_{\mathbb{C}} \mathbb{C}_{\mathrm{sym}}\left[y_4^{L_3}, y_5^{L_3}\right] \Big/ \left(y_1^{L_3} + y_2^{L_3} + y_3^{L_3} - y_4^{L_3} - y_5^{L_3}\right).$$

Further, the generators of $\mathrm{Rep}^{\mathbb{C}}_{\mathrm{poly}}(L_3)$ *under* ξ^{P_3} *go to:*

$$e_1(y_1^{L_3}, y_2^{L_3}, y_3^{L_3}) \mapsto \epsilon^{P_3}_{s_3},$$

$$e_2(y_1^{L_3}, y_2^{L_3}, y_3^{L_3}) \mapsto 3\epsilon^{P_3}_{s_2s_3} - 5\epsilon^{P_3}_{s_4s_3},$$

$$e_3(y_1^{L_3}, y_2^{L_3}, y_3^{L_3}) \mapsto 11\epsilon^{P_3}_{s_1s_2s_3} - 5\epsilon^{P_3}_{s_3s_2s_3} - 2\epsilon^{P_3}_{s_4s_2s_3},$$

$$e_1(y_4^{L_3}, y_5^{L_3}) \mapsto \epsilon^{P_3}_{s_3},$$

$$e_2(y_4^{L_3}, y_5^{L_3}) \mapsto -2\epsilon^{P_3}_{s_2s_3} + 7\epsilon^{P_3}_{s_4s_3}.$$

(d) (L_4): *Let*

$$y_1^{L_4} = \Theta_1 - \Theta_4,\ y_2^{L_4} = -\Theta_1 + \Theta_2 - \Theta_4,\ y_3^{L_4} = -\Theta_2 + 2\Theta_3 - \Theta_4,\ y_4^{L_4} = \Theta_4.$$

Then,

$$\mathrm{Rep}^{\mathbb{C}}_{\mathrm{poly}}(L_4) = \mathbb{C}_{\mathrm{sym}}\left[(y_1^{L_4})^2, (y_2^{L_4})^2, (y_3^{L_4})^2\right] \otimes_{\mathbb{C}} \mathbb{C}[y_4^{L_4}].$$

Further, the generators of $\mathrm{Rep}^{\mathbb{C}}_{\mathrm{poly}}(L_4)$ *under* ξ^{P_4} *go to:*

$$e_1\left((y_1^{L_4})^2, (y_2^{L_4})^2, (y_3^{L_4})^2\right) \mapsto -\epsilon^{P_4}_{s_3s_4},$$

$$e_2\left((y_1^{L_4})^2, (y_2^{L_4})^2, (y_3^{L_4})^2\right) \mapsto 7\epsilon^{P_4}_{s_1s_2s_3s_4} - 5\epsilon^{P_4}_{s_3s_2s_3s_4},$$

$$e_3\left((y_1^{L_4})^2, (y_2^{L_4})^2, (y_3^{L_4})^2\right) \mapsto -2\epsilon^{P_4}_{s_2s_3s_1s_2s_3s_4} + 3\epsilon^{P_4}_{s_4s_3s_1s_2s_3s_4},$$

$$y_4^{L_4} \mapsto \epsilon^{P_4}_{s_4}.$$

7. E_6

We use $\{\omega_1, \ldots, \omega_6\}$ as a basis for $\mathfrak{t}^*$. This gives rise to a coordinate system on T as earlier using the identity (34).

Lemma 7.1.

$$\theta_{\omega_1}(t_1, \ldots, t_6) = (\Theta_1, \ldots, \Theta_6)\,, \tag{37}$$

where

$$\begin{aligned}\Theta_1(t) =& \frac{1}{6}(4t_1 + t_1^{-1}t_3 + t_3^{-1}t_4 + t_2t_4^{-1}t_5 + t_2^{-1}t_5 + t_2t_5^{-1}t_6 + t_2^{-1}t_4t_5^{-1}t_6 \\ &+ t_2t_6^{-1} + t_3t_4^{-1}t_6 + t_2^{-1}t_4t_6^{-1} + t_1t_3^{-1}t_6 + t_3t_4^{-1}t_5t_6^{-1} - 2t_1^{-1}t_6 + t_1t_3^{-1}t_5t_6^{-1} \\ &+ t_3t_5^{-1} - 2t_1^{-1}t_5t_6^{-1} + t_1t_3^{-1}t_4t_5^{-1} - 2t_1^{-1}t_4t_5^{-1} + t_1t_2t_4^{-1} - 2t_1^{-1}t_2t_3t_4^{-1} \\ &+ t_1t_2^{-1} - 2t_1^{-1}t_2^{-1}t_3 - 2t_2t_3^{-1} - 2t_2^{-1}t_3^{-1}t_4 - 2t_4^{-1}t_5 - 2t_5^{-1}t_6 - 2t_6^{-1}).\end{aligned}$$

$$\begin{aligned}\Theta_2(t) =& \frac{1}{6}(3t_1 + 3t_1^{-1}t_3 + 3t_3^{-1}t_4 + 3t_2t_4^{-1}t_5 + 3t_2t_5^{-1}t_6 + 3t_2t_6^{-1} - 3t_1t_2^{-1} - 3t_1^{-1}t_2^{-1}t_3 \\ &- 3t_2^{-1}t_3^{-1}t_4 - 3t_4^{-1}t_5 - 3t_5^{-1}t_6 - 3t_6^{-1}).\end{aligned}$$

$$\begin{aligned}\Theta_3(t) =& \frac{1}{6}(5t_1 + 5t_1^{-1}t_3 + 2t_3^{-1}t_4 + 2t_2t_4^{-1}t_5 + 2t_2^{-1}t_5 + 2t_2t_5^{-1}t_6 \\ &+ 2t_2^{-1}t_4t_5^{-1}t_6 + 2t_2t_6^{-1} + 2t_3t_4^{-1}t_6 + 2t_2^{-1}t_4t_6^{-1} - t_1t_3^{-1}t_6 + 2t_3t_4^{-1}t_5t_6^{-1} \\ &- t_1^{-1}t_6 - t_1t_3^{-1}t_5t_6^{-1} + 2t_3t_5^{-1} - t_1^{-1}t_5t_6^{-1} - t_1t_3^{-1}t_4t_5^{-1} - t_1^{-1}t_4t_5^{-1} \\ &- t_1t_2t_4^{-1} - t_1^{-1}t_2t_3t_4^{-1} - t_1t_2^{-1} - t_1^{-1}t_2^{-1}t_3 - 4t_2t_3^{-1} - 4t_2^{-1}t_3^{-1}t_4 \\ &- 4t_4^{-1}t_5 - 4t_5^{-1}t_6 - 4t_6^{-1}).\end{aligned}$$

$$\begin{aligned}\Theta_4(t) =& \frac{1}{6}(6t_1 + 6t_1^{-1}t_3 + 6t_3^{-1}t_4 + 3t_2t_4^{-1}t_5 + 3t_2^{-1}t_5 + 3t_2t_5^{-1}t_6 \\ &+ 3t_2^{-1}t_4t_5^{-1}t_6 + 3t_2t_6^{-1} + 3t_2^{-1}t_4t_6^{-1} - 3t_1t_2t_4^{-1} - 3t_1^{-1}t_2t_3t_4^{-1} - 3t_1t_2^{-1} \\ &- 3t_1^{-1}t_2^{-1}t_3 - 3t_2t_3^{-1} - 3t_2^{-1}t_3^{-1}t_4 - 6t_4^{-1}t_5 - 6t_5^{-1}t_6 - 6t_6^{-1}).\end{aligned}$$

$$\begin{aligned}\Theta_5(t) =& \frac{1}{6}(4t_1 + 4t_1^{-1}t_3 + 4t_3^{-1}t_4 + 4t_2t_4^{-1}t_5 + 4t_2^{-1}t_5 + t_2t_5^{-1}t_6 \\ &+ t_2^{-1}t_4t_5^{-1}t_6 + t_2t_6^{-1} + t_3t_4^{-1}t_6 + t_2^{-1}t_4t_6^{-1} + t_1t_3^{-1}t_6 + t_3t_4^{-1}t_5t_6^{-1} \\ &+ t_1^{-1}t_6 + t_1t_3^{-1}t_5t_6^{-1} - 2t_3t_5^{-1} + t_1^{-1}t_5t_6^{-1} - 2t_1t_3^{-1}t_4t_5^{-1} - 2t_1^{-1}t_4t_5^{-1} \\ &- 2t_1t_2t_4^{-1} - 2t_1^{-1}t_2t_3t_4^{-1} - 2t_1t_2^{-1} - 2t_1^{-1}t_2^{-1}t_3 - 2t_2t_3^{-1} - 2t_2^{-1}t_3^{-1}t_4 \\ &- 2t_4^{-1}t_5 - 5t_5^{-1}t_6 - 5t_6^{-1}).\end{aligned}$$

$$\begin{aligned}\Theta_6(t) =& \frac{1}{6}(2t_1 + 2t_1^{-1}t_3 + 2t_3^{-1}t_4 + 2t_2t_4^{-1}t_5 + 2t_2^{-1}t_5 + 2t_2t_5^{-1}t_6 \\ &+ 2t_2^{-1}t_4t_5^{-1}t_6 - t_2t_6^{-1} + 2t_3t_4^{-1}t_6 - t_2^{-1}t_4t_6^{-1} + 2t_1t_3^{-1}t_6 - t_3t_4^{-1}t_5t_6^{-1} \\ &+ 2t_1^{-1}t_6 - t_1t_3^{-1}t_5t_6^{-1} - t_3t_5^{-1} - t_1^{-1}t_5t_6^{-1} - t_1t_3^{-1}t_4t_5^{-1} - t_1^{-1}t_4t_5^{-1} \\ &- t_1t_2t_4^{-1} - t_1^{-1}t_2t_3t_4^{-1} - t_1t_2^{-1} - t_1^{-1}t_2^{-1}t_3 - t_2t_3^{-1} - t_2^{-1}t_3^{-1}t_4 \\ &- t_4^{-1}t_5 - t_5^{-1}t_6 - 4t_6^{-1}).\end{aligned}$$

Combining the equation (2) with Lemma 7.1 and using Theorem 4.1, we get the following:

Theorem 7.2. *The generators of* $\mathrm{Rep}^{\mathbb{C}}_{\mathrm{poly}}(L_r)$ *under* ξ^{P_r} *are mapped as follows:*

(a) (L_1)**:** *For* L_1*, we let*

$$y^{L_1}_1 = 3\Theta_1,\ y^{L_1}_2 = -\Theta_1 - 2\Theta_2 + 2\Theta_3,\ y^{L_1}_3 = -\Theta_1 + 2\Theta_2 + 2\Theta_3 - 2\Theta_4,$$
$$y^{L_1}_4 = -\Theta_1 + 2\Theta_4 - 2\Theta_5,\ y^{L_1}_5 = -\Theta_1 + 2\Theta_5 - 2\Theta_6,\ y^{L_1}_6 = -\Theta_1 + 2\Theta_6.$$

Then,

$$\mathrm{Rep}^{\mathbb{C}}_{\mathrm{poly}}(L_1) = \mathbb{C}[y^{L_1}_1] \otimes_{\mathbb{C}} \left[R_{6,1} \oplus (y^{L_1}_2 y^{L_1}_3 y^{L_1}_4 y^{L_1}_5 y^{L_1}_6)R_{6,1}\right],$$

where $R_{6,1} = \mathbb{C}_{\mathrm{sym}}\left[(y^{L_1}_2)^2, (y^{L_1}_3)^2, (y^{L_1}_4)^2, (y^{L_1}_5)^2, (y^{L_1}_6)^2\right]$.
Further, the generators of $\mathrm{Rep}^{\mathbb{C}}_{\mathrm{poly}}(L_1)$ *under* ξ^{P_1} *go to*

$$y^{L_1}_1 \to 3\epsilon^{P_1}_{s_1},$$
$$e_1\left((y^{L_1}_2)^2, (y^{L_1}_3)^2, (y^{L_1}_4)^2, (y^{L_1}_5)^2, (y^{L_1}_6)^2\right) \mapsto -3\epsilon^{P_1}_{s_3s_1},$$
$$e_2\left((y^{L_1}_2)^2, (y^{L_1}_3)^2, (y^{L_1}_4)^2, (y^{L_1}_5)^2, (y^{L_1}_6)^2\right) \mapsto -54\epsilon^{P_1}_{s_2s_4s_3s_1} + 42\epsilon^{P_1}_{s_5s_4s_3s_1},$$
$$e_3\left((y^{L_1}_2)^2, (y^{L_1}_3)^2, (y^{L_1}_4)^2, (y^{L_1}_5)^2, (y^{L_1}_6)^2\right) \mapsto -108\epsilon^{P_1}_{s_4s_5s_2s_4s_3s_1} + 174\epsilon^{P_1}_{s_6s_5s_2s_4s_3s_1},$$
$$e_4\left((y^{L_1}_2)^2, (y^{L_1}_3)^2, (y^{L_1}_4)^2, (y^{L_1}_5)^2, (y^{L_1}_6)^2\right) \mapsto 42\epsilon^{P_1}_{s_1s_3s_4s_5s_2s_4s_3s_1} + 291\epsilon^{P_1}_{s_6s_3s_4s_5s_2s_4s_3s_1} - 519\epsilon^{P_1}_{s_5s_6s_4s_5s_2s_4s_3s_1},$$
$$e_5\left((y^{L_1}_2)^2, (y^{L_1}_3)^2, (y^{L_1}_4)^2, (y^{L_1}_5)^2, (y^{L_1}_6)^2\right) \mapsto 333\epsilon^{P_1}_{s_5s_6s_1s_3s_4s_5s_2s_4s_3s_1} - 180\epsilon^{P_1}_{s_4s_5s_6s_3s_4s_5s_2s_4s_3s_1}.$$
$$e_5\left(y^{L_1}_2, y^{L_1}_3, y^{L_1}_4, y^{L_1}_5, y^{L_1}_6\right) \mapsto 6\epsilon^{P_1}_{s_5s_2s_4s_3s_1} - 21\epsilon^{P_1}_{s_6s_5s_4s_3s_1}$$

(b) (L_2)**:** *For* L_2*, we let*

$$y^{L_2}_1 = 3\Theta_1 - \Theta_2,\ y^{L_2}_2 = -3\Theta_1 - \Theta_2 + 3\Theta_3,\ y^{L_2}_3 = -\Theta_2 - 3\Theta_3 + 3\Theta_4,$$
$$y^{L_2}_4 = 2\Theta_2 - 3\Theta_4 + 3\Theta_5,\ y^{L_2}_5 = 2\Theta_2 - 3\Theta_5 + 3\Theta_6,\ y^{L_2}_6 = 2\Theta_2 - 3\Theta_6.$$

Then,

$$\mathrm{Rep}^{\mathbb{C}}_{\mathrm{poly}}(L_2) = \mathbb{C}_{\mathrm{sym}}[y^{L_2}_1, y^{L_2}_2, y^{L_2}_3, y^{L_2}_4, y^{L_2}_5, y^{L_2}_6].$$

Further, the generators of $\mathrm{Rep}^{\mathbb{C}}_{\mathrm{poly}}(L_2)$ *under* ξ^{P_2} *go to*

$$e_1\left(y^{L_2}_1, y^{L_2}_2, y^{L_2}_3, y^{L_2}_4, y^{L_2}_5, y^{L_2}_6\right) \mapsto 3\epsilon^{P_2}_{s_2},$$
$$e_2\left(y^{L_2}_1, y^{L_2}_2, y^{L_2}_3, y^{L_2}_4, y^{L_2}_5, y^{L_2}_6\right) \mapsto 6\epsilon^{P_2}_{s_4s_2},$$
$$e_3\left(y^{L_2}_1, y^{L_2}_2, y^{L_2}_3, y^{L_2}_4, y^{L_2}_5, y^{L_2}_6\right) \mapsto 34\epsilon^{P_2}_{s_3s_4s_2} - 20\epsilon^{P_2}_{s_5s_4s_2},$$
$$e_4\left(y^{L_2}_1, y^{L_2}_2, y^{L_2}_3, y^{L_2}_4, y^{L_2}_5, y^{L_2}_6\right) \mapsto 141\epsilon^{P_2}_{s_1s_3s_4s_2} - 42\epsilon^{P_2}_{s_5s_3s_4s_2} + 60\epsilon^{P_2}_{s_6s_5s_4s_2},$$
$$e_5\left(y^{L_2}_1, y^{L_2}_2, y^{L_2}_3, y^{L_2}_4, y^{L_2}_5, y^{L_2}_6\right) \mapsto 9\epsilon^{P_2}_{s_5s_1s_3s_4s_2} - 48\epsilon^{P_2}_{s_4s_5s_3s_4s_2} + 90\epsilon^{P_2}_{s_6s_5s_3s_4s_2},$$
$$e_6\left(y^{L_2}_1, y^{L_2}_2, y^{L_2}_3, y^{L_2}_4, y^{L_2}_5, y^{L_2}_6\right) \mapsto$$
$$-130\epsilon^{P_2}_{s_4s_5s_1s_3s_4s_2} + 411\epsilon^{P_2}_{s_6s_5s_1s_3s_4s_2} + 56\epsilon^{P_2}_{s_2s_4s_5s_3s_4s_2} - 76\epsilon^{P_2}_{s_6s_4s_5s_3s_4s_2}.$$

(c) (L_3)**:** *For* L_3*, we let*

$$y^{L_3}_1 = 5\Theta_1 - \Theta_3,\ y^{L_3}_2 = -5\Theta_1 + 4\Theta_3,\ y^{L_3}_3 = -4\Theta_2 + 3\Theta_3,\ y^{L_3}_4 = 4\Theta_2 + 3\Theta_3 - 4\Theta_4,$$
$$y^{L_3}_5 = -\Theta_3 + 4\Theta_4 - 4\Theta_5,\ y^{L_3}_6 = -\Theta_3 + 4\Theta_5 - 4\Theta_6,\ y^{L_3}_7 = -\Theta_3 + 4\Theta_6.$$

Then,

$$\mathrm{Rep}^{\mathbb{C}}_{\mathrm{poly}}(L_3) = \mathbb{C}_{\mathrm{sym}}[y^{L_3}_1, y^{L_3}_2] \otimes_{\mathbb{C}} \mathbb{C}_{\mathrm{sym}}[y^{L_3}_3, y^{L_3}_4, y^{L_3}_5, y^{L_3}_6, y^{L_3}_7]/R_{6,3},$$

where $R_{6,3} = y_1^{L_3} + y_2^{L_3} - y_3^{L_3} - y_4^{L_3} - y_5^{L_3} - y_6^{L_3} - y_7^{L_3}$.
Further, the generators of $\mathrm{Rep}^{\mathbb{C}}_{\mathrm{poly}}(L_3)$ *under* ξ^{P_3} *go to*

$$e_1\left(y_1^{L_3}, y_2^{L_3}\right) \mapsto 3\epsilon^{P_3}_{s_3},$$
$$e_2\left(y_1^{L_3}, y_2^{L_3}\right) \mapsto 21\epsilon^{P_3}_{s_1s_3} - 4\epsilon^{P_3}_{s_4s_3},$$
$$e_1\left(y_3^{L_3}, y_4^{L_3}, y_5^{L_3}, y_6^{L_3}, y_7^{L_3}\right) \mapsto 4\epsilon^{P_3}_{s_3},$$
$$e_2\left(y_3^{L_3}, y_4^{L_3}, y_5^{L_3}, y_6^{L_3}, y_7^{L_3}\right) \mapsto -6\epsilon^{P_3}_{s_1s_3} + 10\epsilon^{P_3}_{s_4s_3},$$
$$e_3\left(y_3^{L_3}, y_4^{L_3}, y_5^{L_3}, y_6^{L_3}, y_7^{L_3}\right) \mapsto -4\epsilon^{P_3}_{s_4s_1s_3} - 58\epsilon^{P_3}_{s_2s_4s_3} + 70\epsilon^{P_3}_{s_5s_4s_3},$$
$$e_4\left(y_3^{L_3}, y_4^{L_3}, y_5^{L_3}, y_6^{L_3}, y_7^{L_3}\right) \mapsto$$
$$-31\epsilon^{P_3}_{s_2s_4s_1s_3} - 38\epsilon^{P_3}_{s_3s_4s_1s_3} + 31\epsilon^{P_3}_{s_5s_4s_1s_3} - 118\epsilon^{P_3}_{s_5s_2s_4s_3} + 325\epsilon^{P_3}_{s_6s_5s_4s_3}.$$
$$e_5\left(y_3^{L_3}, y_4^{L_3}, y_5^{L_3}, y_6^{L_3}, y_7^{L_3}\right) \mapsto$$
$$35\epsilon^{P_3}_{s_3s_2s_4s_1s_3} - 40\epsilon^{P_3}_{s_5s_2s_4s_1s_3} - 93\epsilon^{P_3}_{s_5s_3s_4s_1s_3}492\epsilon^{P_3}_{s_6s_5s_4s_1s_3} + 78\epsilon^{P_3}_{s_4s_5s_2s_4s_3} - 331\epsilon^{P_3}_{s_6s_5s_2s_4s_3}.$$

(d) (L_4)**:** *For* L_4, *we let*

$$y_1^{L_4} = 6\Theta_1 - \Theta_4,\ y_2^{L_4} = -6\Theta_1 + 6\Theta_3 - \Theta_4,\ y_3^{L_4} = -6\Theta_3 + 5\Theta_4,\ y_4^{L_4} = -\Theta_4 + 6\Theta_6,$$
$$y_5^{L_4} = -\Theta_4 + 6\Theta_5 - 6\Theta_6,\ y_6^{L_4} = 5\Theta_4 - 6\Theta_5,\ y_7^{L_4} = 3\Theta_2 - \Theta_4, y_8^{L_4} = -3\Theta_2 + 2\Theta_4.$$

Then,

$$\mathrm{Rep}^{\mathbb{C}}_{\mathrm{poly}}(L_4) = \mathbb{C}_{\mathrm{sym}}[y_1^{L_4}, y_2^{L_4}, y_3^{L_4}] \otimes_{\mathbb{C}} \mathbb{C}_{\mathrm{sym}}[y_4^{L_4}, y_5^{L_4}, y_6^{L_4}] \otimes_{\mathbb{C}} \mathbb{C}_{\mathrm{sym}}[y_7^{L_4}, y_8^{L_4}]/(R_{6,4}, R'_{6,4}),$$

where $R_{6,4} = y_1^{L_4} + y_2^{L_4} + y_3^{L_4} - 3y_7^{L_4} - 3y_8^{L_4}$, *and* $R'_{6,4} = y_4^{L_4} + y_5^{L_4} + y_6^{L_4} - 3y_7^{L_4} - 3y_8^{L_4}$.
Further, the generators of $\mathrm{Rep}^{\mathbb{C}}_{\mathrm{poly}}(L_4)$ *under* ξ^{P_4} *go to*

$$e_1\left(y_1^{L_4}, y_2^{L_4}, y_3^{L_4}\right) \mapsto 3\epsilon^{P_4}_{s_4},$$
$$e_2\left(y_1^{L_4}, y_2^{L_4}, y_3^{L_4}\right) \to -9\epsilon^{P_4}_{s_2s_4} + 27\epsilon^{P_4}_{s_3s_4} - 9\epsilon^{P_4}_{s_5s_4},$$
$$e_3\left(y_1^{L_4}, y_2^{L_4}, y_3^{L_4}\right) \mapsto -26\epsilon^{P_4}_{s_3s_2s_4} + 10\epsilon^{P_4}_{s_5s_2s_4} + 185\epsilon^{P_4}_{s_1s_3s_4} - 26\epsilon^{P_4}_{s_5s_3s_4} + 5\epsilon^{P_4}_{s_6s_5s_4},$$
$$e_1\left(y_4^{L_4}, y_5^{L_4}, y_6^{L_4}\right) \mapsto 3\epsilon^{P_4}_{s_4},$$
$$e_2\left(y_4^{L_4}, y_5^{L_4}, y_6^{L_4}\right) \mapsto -9\epsilon^{P_4}_{s_2s_4} - 9\epsilon^{P_4}_{s_3s_4} + 27\epsilon^{P_4}_{s_5s_4},$$
$$e_3\left(y_4^{L_4}, y_5^{L_4}, y_6^{L_4}\right) \mapsto 10\epsilon^{P_4}_{s_3s_2s_4} - 26\epsilon^{P_4}_{s_5s_2s_4} + 5\epsilon^{P_4}_{s_1s_3s_4} - 26\epsilon^{P_4}_{s_5s_3s_4} + 185\epsilon^{P_4}_{s_6s_5s_4},$$
$$e_1\left(y_7^{L_4}, y_8^{L_4}\right) \mapsto \epsilon^{P_4}_{s_4},$$
$$e_2\left(y_7^{L_4}, y_8^{L_4}\right) \mapsto 7\epsilon^{P_4}_{s_2s_4} - 2\epsilon^{P_4}_{s_3s_4} - 2\epsilon^{P_4}_{s_5s_4}.$$

(e) (L_5)**:** *For* L_5, *we let*

$$y_1^{L_5} = -\Theta_5 + 5\Theta_6,\ y_2^{L_5} = 4\Theta_5 - 5\Theta_6,\ y_3^{L_5} = -4\Theta_2 + 3\Theta_5,\ y_4^{L_5} = 4\Theta_2 - 4\Theta_4 + 3\Theta_5,$$
$$y_5^{L_5} = -4\Theta_3 + 4\Theta_4 - \Theta_5,\ y_6^{L_5} = -4\Theta_1 + 4\Theta_3 - \Theta_5,\ y_7^{L_5} = 4\Theta_1 - \Theta_5.$$

Then,

$$\mathrm{Rep}^{\mathbb{C}}_{\mathrm{poly}}(L_5) = \mathbb{C}_{\mathrm{sym}}[y_1^{L_5}, y_2^{L_5}] \otimes_{\mathbb{C}} \mathbb{C}_{\mathrm{sym}}[y_3^{L_5}, y_4^{L_5}, y_5^{L_5}, y_6^{L_5}, y_7^{L_5}]/R_{6,5},$$

where $R_{6,5} = y_1^{L_5} + y_2^{L_5} - y_3^{L_5} - y_4^{L_5} - y_5^{L_5} - y_6^{L_5} - y_7^{L_5}$.
Further, the generators of $\mathrm{Rep}^{\mathbb{C}}_{\mathrm{poly}}(L_5)$ *under* ξ^{P_5} *go to*

$$e_1\left(y_1^{L_5}, y_2^{L_5}\right) \mapsto 3\epsilon^{P_5}_{s_5},$$

$$e_2\left(y_1^{L_5},y_2^{L_5}\right)\mapsto -4\epsilon^{P_5}_{s_4s_5}+21\epsilon^{P_5}_{s_6s_5},$$
$$e_1\left(y_3^{L_5},y_4^{L_5},y_5^{L_5},y_6^{L_5},y_7^{L_5}\right)\mapsto 3\epsilon^{P_5}_{s_5},$$
$$e_2\left(y_3^{L_5},y_4^{L_5},y_5^{L_5},y_6^{L_5},y_7^{L_5}\right)\mapsto 10\epsilon^{P_5}_{s_4s_5}-6\epsilon^{P_5}_{s_6s_5},$$
$$e_3\left(y_3^{L_5},y_4^{L_5},y_5^{L_5},y_6^{L_5},y_7^{L_5}\right)\mapsto -58\epsilon^{P_5}_{s_2s_4s_5}+70\epsilon^{P_5}_{s_3s_4s_5}-4\epsilon^{P_5}_{s_6s_4s_5},$$
$$e_4\left(y_3^{L_5},y_4^{L_5},y_5^{L_5},y_6^{L_5},y_7^{L_5}\right)\mapsto$$
$$-118\epsilon^{P_5}_{s_3s_2s_4s_5}+31\epsilon^{P_5}_{s_6s_2s_4s_5}+325\epsilon^{P_5}_{s_1s_3s_4s_5}+31\epsilon^{P_5}_{s_6s_3s_4s_5}-38\epsilon^{P_5}_{s_5s_6s_4s_5},$$
$$e_5\left(y_3^{L_5},y_4^{L_5},y_5^{L_5},y_6^{L_5},y_7^{L_5}\right)\mapsto$$
$$-331\epsilon^{P_5}_{s_1s_3s_2s_4s_5}+78\epsilon^{P_5}_{s_4s_3s_2s_4s_5}-40\epsilon^{P_5}_{s_6s_3s_2s_4s_5}+35\epsilon^{P_5}_{s_5s_6s_2s_4s_5}+492\epsilon^{P_5}_{s_6s_1s_3s_4s_5}-93\epsilon^{P}_{s_5s_6s_3s_4s_5}.$$

(f) (L_6): *For L_6, we let*

$$y_1^{L_6}=3\Theta_6,\ y_2^{L_6}=2\Theta_1-\Theta_6,\ y_3^{L_6}=-2\Theta_1+2\Theta_3-\Theta_6,$$
$$y_4^{L_6}=-2\Theta_3+2\Theta_4-\Theta_6,\ y_5^{L_6}=2\Theta_2-2\Theta_4+2\Theta_5-\Theta_6,\ y_6^{L_6}=-2\Theta_2+2\Theta_5-\Theta_6.$$

Then,

$$\mathrm{Rep}^{\mathbb{C}}_{\mathrm{poly}}(L_6)=\mathbb{C}[y_1^{L_6}]\otimes_{\mathbb{C}}\left[R\oplus(y_2^{L_6}y_3^{L_6}y_4^{L_6}y_5^{L_6}y_6^{L_6})R\right],$$

where $R=\mathbb{C}_{\mathrm{sym}}\left[(y_2^{L_6})^2,(y_3^{L_6})^2,(y_4^{L_6})^2,(y_5^{L_6})^2,(y_6^{L_6})^2\right]$.
Further, the generators of $\mathrm{Rep}^{\mathbb{C}}_{\mathrm{poly}}(L_6)$ *under* ξ^{P_6} *go to*

$$y_1^{L_6}\mapsto 3\epsilon^{P_6}_{s_6},$$
$$e_1\left((y_2^{L_6})^2,(y_3^{L_6})^2,(y_4^{L_6})^2,(y_5^{L_6})^2,(y_6^{L_6})^2\right)\mapsto -5\epsilon^{P_6}_{s_5s_6},$$
$$e_2\left((y_2^{L_6})^2,(y_3^{L_6})^2,(y_4^{L_6})^2,(y_5^{L_6})^2,(y_6^{L_6})^2\right)\mapsto -54\epsilon^{P_6}_{s_2s_4s_5s_6}+4254\epsilon^{P_6}_{s_3s_4s_5s_6},$$
$$e_3\left((y_2^{L_6})^2,(y_3^{L_6})^2,(y_4^{L_6})^2,(y_5^{L_6})^2,(y_6^{L_6})^2\right)\mapsto 174\epsilon^{P_6}_{s_1s_3s_2s_4s_5s_6}-108\epsilon^{P_6}_{s_4s_3s_2s_4s_5s_6},$$
$$e_4\left((y_2^{L_6})^2,(y_3^{L_6})^2,(y_4^{L_6})^2,(y_5^{L_6})^2,(y_6^{L_6})^2\right)\mapsto$$
$$-519\epsilon^{P_6}_{s_3s_4s_1s_3s_2s_4s_5s_6}+291\epsilon^{P_6}_{s_5s_4s_1s_3s_2s_4s_5s_6}+42\epsilon^{P_6}_{s_6s_5s_4s_3s_2s_4s_5s_6},$$
$$e_5\left((y_2^{L_6})^2,(y_3^{L_6})^2,(y_4^{L_6})^2,(y_5^{L_6})^2,(y_6^{L_6})^2\right)\mapsto -180\epsilon^{P_6}_{s_4s_5s_3s_4s_1s_3s_2s_4s_5s_6}+333\epsilon^{P_6}_{s_6s_5s_3s_4s_1s_3s_2s_4s_5s_6},$$

8. E_7

We use $\{\omega_1,\dots,\omega_7\}$ as a basis for $\mathfrak{t}^*$. This gives rise to a coordinate system on T as earlier using the identity (34).

Lemma 8.1.

$$\theta_{\omega_7}(t_1,\dots,t_7)=(\Theta_1,\dots,\Theta_7), \tag{38}$$

where

$$\begin{aligned}\Theta_1(t)=&\frac{1}{24}(2t_7+2t_6t_7^{-1}+2t_5t_6^{-1}+2t_4t_5^{-1}+2t_2t_3t_4^{-1}+2t_2^{-1}t_3+2t_1t_2t_3^{-1}\\&+2t_1t_2^{-1}t_3^{-1}t_4+2t_1t_4^{-1}t_5+2t_1t_5^{-1}t_6+2t_1t_6^{-1}t_7+2t_1t_7^{-1}-2t_1^{-1}t_7-2t_1^{-1}t_6t_7^{-1}\\&-2t_1^{-1}t_5t_6^{-1}-2t_1^{-1}t_4t_5^{-1}-2t_1^{-1}t_2t_3t_4^{-1}-2t_1^{-1}t_2^{-1}t_3-2t_2t_3^{-1}-2t_2^{-1}t_3^{-1}t_4\\&-2t_4^{-1}t_5-2t_5^{-1}t_6-2t_6^{-1}t_7-2t_7^{-1})\end{aligned}$$

$$\Theta_2(t) = \frac{1}{24}(3t_7 + 3t_6t_7^{-1} + 3t_5t_6^{-1} + 3t_4t_5^{-1} + 3t_2t_3t_4^{-1} + t_2^{-1}t_3 + 3t_1t_2t_3^{-1}$$
$$+ t_1t_2^{-1}t_3^{-1}t_4 + 3t_1^{-1}t_2 + t_1^{-1}t_2^{-1}t_4 + t_1t_4^{-1}t_5 + t_1^{-1}t_3t_4^{-1}t_5 + t_1t_5^{-1}t_6 + t_3^{-1}t_5$$
$$+ t_1^{-1}t_3t_5^{-1}t_6 + t_1t_6^{-1}t_7 + t_3^{-1}t_4t_5^{-1}t_6 + t_1^{-1}t_3t_6^{-1}t_7 + t_1t_7^{-1} + t_2t_4^{-1}t_6$$
$$+ t_3^{-1}t_4t_6^{-1}t_7 + t_1^{-1}t_3t_7^{-1} - t_2^{-1}t_6 + t_2t_4^{-1}t_5t_6^{-1}t_7 + t_3^{-1}t_4t_7^{-1} - t_2^{-1}t_5t_6^{-1}t_7$$
$$+ t_2t_5^{-1}t_7 + t_2t_4^{-1}t_5t_7^{-1} - t_2^{-1}t_4t_5^{-1}t_7 - t_2^{-1}t_5t_7^{-1} + t_2t_5^{-1}t_6t_7^{-1} - t_3t_4^{-1}t_7$$
$$- t_2^{-1}t_4t_5^{-1}t_6t_7^{-1} + t_2t_6^{-1} - t_1t_3^{-1}t_7 - t_3t_4^{-1}t_6t_7^{-1} - t_2^{-1}t_4t_6^{-1} - t_1^{-1}t_7 - t_1t_3^{-1}t_6t_7^{-1}$$
$$- t_3t_4^{-1}t_5t_6^{-1} - t_1^{-1}t_6t_7^{-1} - t_1t_3^{-1}t_5t_6^{-1} - t_3t_5^{-1} - t_1^{-1}t_5t_6^{-1} - t_1t_3^{-1}t_4t_5^{-1}$$
$$- t_1^{-1}t_4t_5^{-1} - t_1t_2t_4^{-1} - t_1^{-1}t_2t_3t_4^{-1} - 3t_1t_2^{-1} - 3t_1^{-1}t_2^{-1}t_3 - t_2t_3^{-1}$$
$$- 3t_2^{-1}t_3^{-1}t_4 - 3t_4^{-1}t_5 - 3t_5^{-1}t_6 - 3t_6^{-1}t_7 - 3t_7^{-1})$$

$$\Theta_3(t) = \frac{1}{24}(4t_7 + 4t_6t_7^{-1} + 4t_5t_6^{-1} + 4t_4t_5^{-1} + 4t_2t_3t_4^{-1} + 4t_2^{-1}t_3 + 2t_1t_2t_3^{-1}$$
$$+ 2t_1t_2^{-1}t_3^{-1}t_4 + 2t_1^{-1}t_2 + 2t_1^{-1}t_2^{-1}t_4 + 2t_1t_4^{-1}t_5 + 2t_1^{-1}t_3t_4^{-1}t_5 + 2t_1t_5^{-1}t_6$$
$$+ 2t_1^{-1}t_3t_5^{-1}t_6 + 2t_1t_6^{-1}t_7 + 2t_1^{-1}t_3t_6^{-1}t_7 + 2t_1t_7^{-1} + 2t_1^{-1}t_3t_7^{-1} - 2t_1t_3^{-1}t_7 - 2t_1^{-1}t_7$$
$$- 2t_1t_3^{-1}t_6t_7^{-1} - 2t_1^{-1}t_6t_7^{-1} - 2t_1t_3^{-1}t_5t_6^{-1} - 2t_1^{-1}t_5t_6^{-1} - 2t_1t_3^{-1}t_4t_5^{-1}$$
$$- 2t_1^{-1}t_4t_5^{-1} - 2t_1t_2t_4^{-1} - 2t_1^{-1}t_2t_3t_4^{-1} - 2t_1t_2^{-1} - 2t_1^{-1}t_2^{-1}t_3$$
$$- 4t_2t_3^{-1} - 4t_2^{-1}t_3^{-1}t_4 - 4t_4^{-1}t_5 - 4t_5^{-1}t_6 - 4t_6^{-1}t_7 - 4t_7^{-1})$$

$$\Theta_4(t) = \frac{1}{24}(6t_7 + 6t_6t_7^{-1} + 6t_5t_6^{-1} + 6t_4t_5^{-1} + 4t_2t_3t_4^{-1}$$
$$+ 4t_2^{-1}t_3 + 4t_1t_2t_3^{-1} + 4t_1t_2^{-1}t_3^{-1}t_4 + 4t_1^{-1}t_2 + 4t_1^{-1}t_2^{-1}t_4$$
$$+ 2t_1t_4^{-1}t_5 + 2t_1^{-1}t_3t_4^{-1}t_5 + 2t_1t_5^{-1}t_6 + 2t_3^{-1}t_5 + 2t_1^{-1}t_3t_5^{-1}t_6$$
$$+ 2t_1t_6^{-1}t_7 + 2t_3^{-1}t_4t_5^{-1}t_6 + 2t_1^{-1}t_3t_6^{-1}t_7 + 2t_1t_7^{-1} + 2t_3^{-1}t_4t_6^{-1}t_7$$
$$+ 2t_1^{-1}t_3t_7^{-1} + 2t_3^{-1}t_4t_7^{-1} - 2t_3t_4^{-1}t_7 - 2t_1t_3^{-1}t_7 - 2t_3t_4^{-1}t_6t_7^{-1}$$
$$- 2t_1^{-1}t_7 - 2t_1t_3^{-1}t_6t_7^{-1} - 2t_3t_4^{-1}t_5t_6^{-1} - 2t_1^{-1}t_6t_7^{-1} - 2t_1t_3^{-1}t_5t_6^{-1}$$
$$- 2t_3t_5^{-1} - 2t_1^{-1}t_5t_6^{-1} - 2t_1t_3^{-1}t_4t_5^{-1} - 2t_1^{-1}t_4t_5^{-1} - 4t_1t_2t_4^{-1}$$
$$- 4t_1^{-1}t_2t_3t_4^{-1} - 4t_1t_2^{-1} - 4t_1^{-1}t_2^{-1}t_3 - 4t_2t_3^{-1} - 4t_2^{-1}t_3^{-1}t_4$$
$$- 6t_4^{-1}t_5 - 6t_5^{-1}t_6 - 6t_6^{-1}t_7 - 6t_7^{-1})$$

$$\begin{aligned}\Theta_5(t) =&\frac{1}{24}(5t_7 + 5t_6t_7^{-1} + 5t_5t_6^{-1} + 3t_4t_5^{-1} + 3t_2t_3t_4^{-1}\\ &+ 3t_2^{-1}t_3 + 3t_1t_2t_3^{-1} + 3t_1t_2^{-1}t_3^{-1}t_4 + 3t_1^{-1}t_2 + 3t_1^{-1}t_2^{-1}t_4\\ &+ 3t_1t_4^{-1}t_5 + 3t_1^{-1}t_3t_4^{-1}t_5 + t_1t_5^{-1}t_6 + 3t_3^{-1}t_5 + t_1^{-1}t_3t_5^{-1}t_6\\ &+ t_1t_6^{-1}t_7 + t_3^{-1}t_4t_5^{-1}t_6 + t_1^{-1}t_3t_6^{-1}t_7 + t_1t_7^{-1} + t_2t_4^{-1}t_6\\ &+ t_3^{-1}t_4t_6^{-1}t_7 + t_1^{-1}t_3t_7^{-1} + t_2^{-1}t_6 + t_2t_4^{-1}t_5t_6^{-1}t_7 + t_3^{-1}t_4t_7^{-1}\\ &+ t_2^{-1}t_5t_6^{-1}t_7 - t_2t_5^{-1}t_7 + t_2t_4^{-1}t_5t_7^{-1} - t_2^{-1}t_4t_5^{-1}t_7 + t_2^{-1}t_5t_7^{-1}\\ &- t_2t_5^{-1}t_6t_7^{-1} - t_3t_4^{-1}t_7 - t_2^{-1}t_4t_5^{-1}t_6t_7^{-1} - t_2t_6^{-1} - t_1t_3^{-1}t_7\\ &- t_3t_4^{-1}t_6t_7^{-1} - t_2^{-1}t_4t_6^{-1} - t_1^{-1}t_7 - t_1t_3^{-1}t_6t_7^{-1} - t_3t_4^{-1}t_5t_6^{-1}\\ &- t_1^{-1}t_6t_7^{-1} - t_1t_3^{-1}t_5t_6^{-1} - 3t_3t_5^{-1} - t_1^{-1}t_5t_6^{-1} - 3t_1t_3^{-1}t_4t_5^{-1}\\ &- 3t_1^{-1}t_4t_5^{-1} - 3t_1t_2t_4^{-1} - 3t_1^{-1}t_2t_3t_4^{-1} - 3t_1t_2^{-1} - 3t_1^{-1}t_2^{-1}t_3\\ &- 3t_2t_3^{-1} - 3t_2^{-1}t_3^{-1}t_4 - 3t_4^{-1}t_5 - 5t_5^{-1}t_6 - 5t_6^{-1}t_7 - 5t_7^{-1})\end{aligned}$$

$$\begin{aligned}\Theta_6(t) =&\frac{1}{24}(4t_7 + 4t_6t_7^{-1} + 2t_5t_6^{-1} + 2t_4t_5^{-1} + 2t_2t_3t_4^{-1}\\ &+ 2t_2^{-1}t_3 + 2t_1t_2t_3^{-1} + 2t_1t_2^{-1}t_3^{-1}t_4 + 2t_1^{-1}t_2 + 2t_1^{-1}t_2^{-1}t_4\\ &+ 2t_1t_4^{-1}t_5 + 2t_1^{-1}t_3t_4^{-1}t_5 + 2t_1t_5^{-1}t_6 + 2t_3^{-1}t_5 + 2t_1^{-1}t_3t_5^{-1}t_6\\ &+ 2t_3^{-1}t_4t_5^{-1}t_6 + 2t_2t_4^{-1}t_6 + 2t_2^{-1}t_6 - 2t_2t_6^{-1} - 2t_2^{-1}t_4t_6^{-1}\\ &- 2t_3t_4^{-1}t_5t_6^{-1} - 2t_1t_3^{-1}t_5t_6^{-1} - 2t_3t_5^{-1} - 2t_1^{-1}t_5t_6^{-1} - 2t_1t_3^{-1}t_4t_5^{-1}\\ &- 2t_1^{-1}t_4t_5^{-1} - 2t_1t_2t_4^{-1} - 2t_1^{-1}t_2t_3t_4^{-1} - 2t_1t_2^{-1} - 2t_1^{-1}t_2^{-1}t_3\\ &- 2t_2t_3^{-1} - 2t_2^{-1}t_3^{-1}t_4 - 2t_4^{-1}t_5 - 2t_5^{-1}t_6 - 4t_6^{-1}t_7 - 4t_7^{-1})\end{aligned}$$

$$\begin{aligned}\Theta_7(t) =&\frac{1}{24}(3t_7 + t_6t_7^{-1} + t_5t_6^{-1} + t_4t_5^{-1} + t_2t_3t_4^{-1} + t_2^{-1}t_3 + t_1t_2t_3^{-1}\\ &+ t_1t_2^{-1}t_3^{-1}t_4 + t_1^{-1}t_2 + t_1^{-1}t_2^{-1}t_4 + t_1t_4^{-1}t_5 + t_1^{-1}t_3t_4^{-1}t_5 + t_1t_5^{-1}t_6 + t_3^{-1}t_5\\ &+ t_1^{-1}t_3t_5^{-1}t_6 + t_1t_6^{-1}t_7 + t_3^{-1}t_4t_5^{-1}t_6 + t_1^{-1}t_3t_6^{-1}t_7 - t_1t_7^{-1} + t_2t_4^{-1}t_6\\ &+ t_3^{-1}t_4t_6^{-1}t_7 - t_1^{-1}t_3t_7^{-1} + t_2^{-1}t_6 + t_2t_4^{-1}t_5t_6^{-1}t_7 - t_3^{-1}t_4t_7^{-1} + t_2^{-1}t_5t_6^{-1}t_7\\ &+ t_2t_5^{-1}t_7 - t_2t_4^{-1}t_5t_7^{-1} + t_2^{-1}t_4t_5^{-1}t_7 - t_2^{-1}t_5t_7^{-1} - t_2t_5^{-1}t_6t_7^{-1} + t_3t_4^{-1}t_7\\ &- t_2^{-1}t_4t_5^{-1}t_6t_7^{-1} - t_2t_6^{-1} + t_1t_3^{-1}t_7 - t_3t_4^{-1}t_6t_7^{-1} - t_2^{-1}t_4t_6^{-1} + t_1^{-1}t_7\\ &- t_1t_3^{-1}t_6t_7^{-1} - t_3t_4^{-1}t_5t_6^{-1} - t_1^{-1}t_6t_7^{-1} - t_1t_3^{-1}t_5t_6^{-1} - t_3t_5^{-1} - t_1^{-1}t_5t_6^{-1}\\ &- t_1t_3^{-1}t_4t_5^{-1} - t_1^{-1}t_4t_5^{-1} - t_1t_2t_4^{-1} - t_1^{-1}t_2t_3t_4^{-1} - t_1t_2^{-1} - t_1^{-1}t_2^{-1}t_3\\ &- t_2t_3^{-1} - t_2^{-1}t_3^{-1}t_4 - t_4^{-1}t_5 - t_5^{-1}t_6 - t_6^{-1}t_7 - 3t_7^{-1})\end{aligned}$$

Combining the equation (2) with Lemma 8.1 and using Theorem 4.1, we get the following:

Theorem 8.2. *The generators of* $\mathrm{Rep}^{\mathbb{C}}_{\mathrm{poly}}(L_r)$ *under* ξ^{P_r} *are mapped as follows:*

(a) (L_1)**:** *For* L_1*, we let*

$$y_1^{L_1} = \Theta_1,\ y_2^{L_1} = -\Theta_1 - 2\Theta_2 + 2\Theta_3,\ y_3^{L_1} = -\Theta_1 + 2\Theta_2 + 2\Theta_3 - 2\Theta_4,$$
$$y_4^{L_1} = -\Theta_1 + 2\Theta_4 - 2\Theta_5,\ y_5^{L_1} = -\Theta_1 + 2\Theta_5 - 2\Theta_6,$$

$$y_6^{L_1} = -\Theta_1 + 2\Theta_6 - 2\Theta_7,\ y_7^{L_1} = -\Theta_1 + 2\Theta_7.$$

Then,

$$\mathrm{Rep}^{\mathbb{C}}_{\mathrm{poly}}(L_1) = \mathbb{C}[y_1^{L_1}] \otimes_{\mathbb{C}} \left[R \oplus (y_2^{L_1} y_3^{L_1} y_4^{L_1} y_5^{L_1} y_6^{L_1} y_7^{L_1})R\right],$$

where $R = \mathbb{C}_{\mathrm{sym}}\left[(y_2^{L_1})^2, (y_3^{L_1})^2, (y_4^{L_1})^2, (y_5^{L_1})^2, (y_6^{L_1})^2, (y_7^{L_1})^2\right]$.
Further, the generators of $\mathrm{Rep}^{\mathbb{C}}_{\mathrm{poly}}(L_1)$ *under* ξ^{P_1} *go to*

$$y_1^{L_1} \mapsto \epsilon^{P_1}_{s_1},$$

$$e_1\left((y_2^{L_1})^2, (y_3^{L_1})^2, (y_4^{L_1})^2, (y_5^{L_1})^2, (y_6^{L_1})^2, (y_7^{L_1})^2\right) \mapsto -2\epsilon^{P_1}_{s_3 s_1},$$

$$e_2\left((y_2^{L_1})^2, (y_3^{L_1})^2, (y_4^{L_1})^2, (y_5^{L_1})^2, (y_6^{L_1})^2, (y_7^{L_1})^2\right) \mapsto -57\epsilon^{P_1}_{s_2 s_4 s_3 s_1} + 39\epsilon^{P_1}_{s_5 s_4 s_3 s_1},$$

$$e_3\left((y_2^{L_1})^2, (y_3^{L_1})^2, (y_4^{L_1})^2, (y_5^{L_1})^2, (y_6^{L_1})^2, (y_7^{L_1})^2\right) \mapsto$$
$$-120\epsilon^{P_1}_{s_4 s_5 s_2 s_4 s_3 s_1} + 204\epsilon^{P_1}_{s_6 s_5 s_2 s_4 s_3 s_1} - 316\epsilon^{P_1}_{s_7 s_6 s_5 s_4 s_3 s_1},$$

$$e_4\left((y_2^{L_1})^2, (y_3^{L_1})^2, (y_4^{L_1})^2, (y_5^{L_1})^2, (y_6^{L_1})^2, (y_7^{L_1})^2\right) \mapsto$$
$$-66\epsilon^{P_1}_{s_1 s_3 s_4 s_5 s_2 s_4 s_3 s_1} + 249\epsilon^{P_1}_{s_6 s_3 s_4 s_5 s_2 s_4 s_3 s_1} - 453\epsilon^{P_1}_{s_5 s_6 s_4 s_5 s_2 s_4 s_3 s_1} + 247\epsilon^{P_1}_{s_7 s_6 s_4 s_5 s_2 s_4 s_3 s_1},$$

$$e_5\left((y_2^{L_1})^2, (y_3^{L_1})^2, (y_4^{L_1})^2, (y_5^{L_1})^2, (y_6^{L_1})^2, (y_7^{L_1})^2\right) \mapsto$$
$$438\epsilon^{P_1}_{s_5 s_6 s_1 s_3 s_4 s_5 s_2 s_4 s_3 s_1} - 1618\epsilon^{P_1}_{s_7 s_6 s_1 s_3 s_4 s_5 s_2 s_4 s_3 s_1} - 408\epsilon^{P_1}_{s_4 s_5 s_6 s_3 s_4 s_5 s_2 s_4 s_3 s_1}$$
$$+1132\epsilon^{P_1}_{s_7 s_5 s_6 s_3 s_4 s_5 s_2 s_4 s_3 s_1} - 2140\epsilon^{P_1}_{s_6 s_7 s_5 s_6 s_4 s_5 s_2 s_4 s_3 s_1}.$$

$$e_6\left((y_2^{L_1})^2, (y_3^{L_1})^2, (y_4^{L_1})^2, (y_5^{L_1})^2, (y_6^{L_1})^2, (y_7^{L_1})^2\right) \mapsto$$
$$-27\epsilon^{P_1}_{s_2 s_4 s_5 s_6 s_1 s_3 s_4 s_5 s_2 s_4 s_3 s_1} + 153\epsilon^{P_1}_{s_3 s_4 s_5 s_6 s_1 s_3 s_4 s_5 s_2 s_4 s_3 s_1} - 585\epsilon^{P_1}_{s_7 s_4 s_5 s_6 s_1 s_3 s_4 s_5 s_2 s_4 s_3 s_1}$$
$$+1552\epsilon^{P_1}_{s_6 s_7 s_5 s_6 s_1 s_3 s_4 s_5 s_2 s_4 s_3 s_1} + 498\epsilon^{P_1}_{s_7 s_2 s_4 s_5 s_6 s_3 s_4 s_5 s_2 s_4 s_3 s_1} - 450\epsilon^{P_1}_{s_6 s_7 s_4 s_5 s_6 s_3 s_4 s_5 s_2 s_4 s_3 s_1}.$$

$$e_6\left(y_2^{L_1}, y_3^{L_1}, y_4^{L_1}, y_5^{L_1}, y_6^{L_1}, y_7^{L_1}\right) \mapsto -6\epsilon^{P_1}_{s_4 s_5 s_2 s_4 s_3 s_1} + 15\epsilon^{P_1}_{s_6 s_5 s_2 s_4 s_3 s_1} - 43\epsilon^{P_1}_{s_7 s_6 s_5 s_4 s_3 s_1},$$

(b) (L_2)**:** *For* L_2*, we let*

$$y_1^{L_2} = 4\Theta_1 - 2\Theta_2,\ y_2^{L_2} = -4\Theta_1 - 2\Theta_2 + 4\Theta_3,\ y_3^{L_2} = -2\Theta_2 - 4\Theta_3 + 4\Theta_4,$$

$y_4^{L_2} = 2\Theta_2 - 4\Theta_4 + 4\Theta_5,\ y_5^{L_2} = 2\Theta_2 - 4\Theta_5 + 4\Theta_6,\ y_6^{L_2} = 2\Theta_2 - 4\Theta_6 + 4\Theta_7,\ y_7^{L_2} = 2\Theta_2 - 4\Theta_7.$
Then,

$$\mathrm{Rep}^{\mathbb{C}}_{\mathrm{poly}}(L_2) = \mathbb{C}_{\mathrm{sym}}[y_1^{L_2}, y_2^{L_2}, y_3^{L_2}, y_4^{L_2}, y_5^{L_2}, y_6^{L_2}, y_7^{L_2}].$$

Further, the generators of $\mathrm{Rep}^{\mathbb{C}}_{\mathrm{poly}}(L_2)$ *under* ξ^{P_2} *go to*

$$e_1\left(y_1^{L_2}, y_2^{L_2}, y_3^{L_2}, y_4^{L_2}, y_5^{L_2}, y_6^{L_2}, y_7^{L_2}\right) \mapsto 2\epsilon^{P_2}_{s_2},$$

$$e_2\left(y_1^{L_2}, y_2^{L_2}, y_3^{L_2}, y_4^{L_2}, y_5^{L_2}, y_6^{L_2}, y_7^{L_2}\right) \mapsto 4\epsilon^{P_2}_{s_4 s_2},$$

$$e_3\left(y_1^{L_2}, y_2^{L_2}, y_3^{L_2}, y_4^{L_2}, y_5^{L_2}, y_6^{L_2}, y_7^{L_2}\right) \mapsto 72\epsilon^{P_2}_{s_3 s_4 s_2} - 56\epsilon^{P_2}_{s_5 s_4 s_2},$$

$$e_4\left(y_1^{L_2}, y_2^{L_2}, y_3^{L_2}, y_4^{L_2}, y_5^{L_2}, y_6^{L_2}, y_7^{L_2}\right) \mapsto 432\epsilon^{P_2}_{s_1 s_3 s_4 s_2} - 160\epsilon^{P_2}_{s_5 s_3 s_4 s_2} + 176\epsilon^{P_2}_{s_6 s_5 s_4 s_2},$$

$$e_5\left(y_1^{L_2}, y_2^{L_2}, y_3^{L_2}, y_4^{L_2}, y_5^{L_2}, y_6^{L_2}, y_7^{L_2}\right) \mapsto 32\epsilon^{P_2}_{s_5 s_1 s_3 s_4 s_2} - 320\epsilon^{P_2}_{s_4 s_5 s_3 s_4 s_2} + 544\epsilon^{P_2}_{s_6 s_5 s_3 s_4 s_2} - 1184\epsilon^{P_2}_{s_7 s_6 s_5 s_4 s_2},$$

$$e_6\left(y_1^{L_2}, y_2^{L_2}, y_3^{L_2}, y_4^{L_2}, y_5^{L_2}, y_6^{L_2}, y_7^{L_2}\right) \mapsto$$
$$-1088\epsilon^{P_2}_{s_4 s_5 s_1 s_3 s_4 s_2} + 2176\epsilon^{P_2}_{s_6 s_5 s_1 s_3 s_4 s_2} + 384\epsilon^{P_2}_{s_2 s_4 s_5 s_3 s_4 s_2} - 64\epsilon^{P_2}_{s_6 s_4 s_5 s_3 s_4 s_2} - 1280\epsilon^{P_2}_{s_7 s_6 s_5 s_3 s_4 s_2}.$$

$$e_7\left(y_1^{L_2}, y_2^{L_2}, y_3^{L_2}, y_4^{L_2}, y_5^{L_2}, y_6^{L_2}, y_7^{L_2}\right) \mapsto$$
$$-384\epsilon^{P_2}_{s_2 s_4 s_5 s_1 s_3 s_4 s_2} - 1152\epsilon^{P_2}_{s_3 s_4 s_5 s_1 s_3 s_4 s_2} + 2048\epsilon^{P_2}_{s_6 s_4 s_5 s_1 s_3 s_4 s_2} - 8448\epsilon^{P_2}_{s_7 s_6 s_5 s_1 s_3 s_4 s_2}$$
$$-384\epsilon^{P_2}_{s_6 s_2 s_4 s_5 s_3 s_4 s_2} - 1152\epsilon^{P_2}_{s_5 s_6 s_4 s_5 s_3 s_4 s_2} + 2432\epsilon^{P}_{s_7 s_6 s_4 s_5 s_3 s_4 s_2}.$$

(c) **(L_3):** *For L_3, we let*

$$y_1^{L_3} = 3\Theta_1 - \Theta_3,\ y_2^{L_3} = -3\Theta_1 + 2\Theta_3,\ y_3^{L_3} = -4\Theta_2 + 3\Theta_3,\ y_4^{L_3} = 4\Theta_2 + 3\Theta_3 - 4\Theta_4,$$

$$y_5^{L_3} = -\Theta_3 + 4\Theta_4 - 4\Theta_5,\ y_6^{L_3} = -\Theta_3 + 4\Theta_5 - 4\Theta_6,\ y_7^{L_3} = -\Theta_3 + 4\Theta_6 - 4\Theta_7,\ y_8^{L_3} = -\Theta_3 + 4\Theta_7.$$

Then,

$$\mathrm{Rep}^{\mathbb{C}}_{\mathrm{poly}}(L_3) = \mathbb{C}_{\mathrm{sym}}[y_1^{L_3}, y_2^{L_3}] \otimes_{\mathbb{C}} \mathbb{C}_{\mathrm{sym}}[y_3^{L_3}, y_4^{L_3}, y_5^{L_3}, y_6^{L_3}, y_7^{L_3}, y_8^{L_3}]/R_{7,3},$$

where $R_{7,3} = 2y_1^{L_3} + 2y_2^{L_3} - y_3^{L_3} - y_4^{L_3} - y_5^{L_3} - y_6^{L_3} - y_7^{L_3} - y_8^{L_3}$.
Further, the generators of $\mathrm{Rep}^{\mathbb{C}}_{\mathrm{poly}}(L_3)$ *under* ξ^{P_3} *go to*

$$e_1\left(y_1^{L_3}, y_2^{L_3}\right) \mapsto \epsilon^{P_3}_{s_3},$$

$$e_2\left(y_1^{L_3}, y_2^{L_3}\right) \mapsto 7\epsilon^{P_3}_{s_1s_3} - 2\epsilon^{P_3}_{s_4s_3},$$

$$e_1\left(y_3^{L_3}, y_4^{L_3}, y_5^{L_3}, y_6^{L_3}, y_7^{L_3}, y_8^{L_3}\right) \mapsto 2\epsilon^{P_3}_{s_3},$$

$$e_2\left(y_3^{L_3}, y_4^{L_3}, y_5^{L_3}, y_6^{L_3}, y_7^{L_3}, y_8^{L_3}\right) \mapsto -9\epsilon^{P_3}_{s_1s_3} + 7\epsilon^{P_3}_{s_4s_3},$$

$$e_3\left(y_3^{L_3}, y_4^{L_3}, y_5^{L_3}, y_6^{L_3}, y_7^{L_3}, y_8^{L_3}\right) \to -8\epsilon^{P_3}_{s_4s_1s_3} - 68\epsilon^{P_3}_{s_2s_4s_3} + 60\epsilon^{P_3}_{s_5s_4s_3},$$

$$e_4\left(y_3^{L_3}, y_4^{L_3}, y_5^{L_3}, y_6^{L_3}, y_7^{L_3}, y_8^{L_3}\right) \mapsto$$
$$93\epsilon^{P_3}_{s_2s_4s_1s_3} - 34\epsilon^{P_3}_{s_3s_4s_1s_3} - 35\epsilon^{P_3}_{s_5s_4s_1s_3} - 130\epsilon^{P_3}_{s_5s_2s_4s_3} + 255\epsilon^{P_3}_{s_6s_5s_4s_3}.$$

$$e_5\left(y_3^{L_3}, y_4^{L_3}, y_5^{L_3}, y_6^{L_3}, y_7^{L_3}, y_8^{L_3}\right) \mapsto$$
$$42\epsilon^{P_3}_{s_3s_2s_4s_1s_3} + 16\epsilon^{P_3}_{s_5s_2s_4s_1s_3} - 86\epsilon^{P_3}_{s_5s_3s_4s_1s_3} + 136\epsilon^{P_3}_{s_6s_5s_4s_1s_3} + 196\epsilon^{P_3}_{s_4s_5s_2s_4s_3}$$
$$-538\epsilon^{P_3}_{s_6s_5s_2s_4s_3} + 1314\epsilon^{P_3}_{s_7s_6s_5s_4s_3}.$$

$$e_6\left(y_3^{L_3}, y_4^{L_3}, y_5^{L_3}, y_6^{L_3}, y_7^{L_3}, y_8^{L_3}\right) \mapsto$$
$$-35\epsilon^{P_3}_{s_4s_3s_2s_4s_1s_3} + 98\epsilon^{P_3}_{s_5s_3s_2s_4s_1s_3} - 38\epsilon^{P_3}_{s_4s_5s_2s_4s_1s_3} - 121\epsilon^{P_3}_{s_6s_5s_2s_4s_1s_3}$$
$$+93\epsilon^{P_3}_{s_4s_5s_3s_4s_1s_3} - 399\epsilon^{P_3}_{s_6s_5s_3s_4s_1s_3} + 1965\epsilon^{P_3}_{s_7s_6s_5s_4s_1s_3}$$
$$-78\epsilon^{P_3}_{s_3s_4s_5s_2s_4s_3} + 253\epsilon^{P_3}_{s_6s_4s_5s_2s_4s_3} - 1308\epsilon^{P_3}_{s_7s_6s_5s_2s_4s_3}.$$

(d) **(L_4):** *For L_4, we let*

$$y_1^{L_4} = 4\Theta_1 - \Theta_4,\ y_2^{L_4} = -4\Theta_1 + 4\Theta_3 - \Theta_4,\ y_3^{L_4} = -4\Theta_3 + 3\Theta_4,$$
$$y_4^{L_4} = -\Theta_4 + 6\Theta_7,\ y_5^{L_4} = -\Theta_4 + 6\Theta_6 - 6\Theta_7,\ y_6^{L_4} = -\Theta_4 + 6\Theta_5 - 6\Theta_6,$$
$$y_7^{L_4} = 5\Theta_4 - 6\Theta_5, y_8^{L_4} = 12\Theta_2 - 5\Theta_4, y_9^{L_4} = -12\Theta_2 + 7\Theta_4.$$

Then,

$$\mathrm{Rep}^{\mathbb{C}}_{\mathrm{poly}}(L_4) = \mathbb{C}_{\mathrm{sym}}[y_1^{L_4}, y_2^{L_4}, y_3^{L_4}] \otimes_{\mathbb{C}} \mathbb{C}_{\mathrm{sym}}[y_4^{L_4}, y_5^{L_4}, y_6^{L_4}, y_7^{L_4}] \otimes_{\mathbb{C}} \mathbb{C}_{\mathrm{sym}}[y_8^{L_4}, y_9^{L_4}]/(R_{7,4}, R'_{7,4}),$$

where $R_{7,4} = y_1^{L_4} + y_2^{L_4} + y_3^{L_4} - y_8^{L_4} - y_9^{L_4}$ *and* $R'_{7,4} = y_4^{L_4} + y_5^{L_4} + y_6^{L_4} + y_7^{L_4} - y_8^{L_4} - y_9^{L_4}$.
Further, the generators of $\mathrm{Rep}^{\mathbb{C}}_{\mathrm{poly}}(L_4)$ *under* ξ^{P_4} *go to*

$$e_1\left(y_1^{L_4}, y_2^{L_4}, y_3^{L_4}\right) \mapsto \epsilon^{P_4}_{s_4},$$

$$e_2\left(y_1^{L_4}, y_2^{L_4}, y_3^{L_4}\right) \mapsto -5\epsilon^{P_4}_{s_2s_4} + 11\epsilon^{P_4}_{s_3s_4} - 5\epsilon^{P_4}_{s_5s_4},$$

$$e_3\left(y_1^{L_4}, y_2^{L_4}, y_3^{L_4}\right) \mapsto -10\epsilon^{P_4}_{s_3s_2s_4} + 6\epsilon^{P_4}_{s_5s_2s_4} + 51\epsilon^{P_4}_{s_1s_3s_4} - 10\epsilon^{P_4}_{s_5s_3s_4} + 3\epsilon^{P_4}_{s_6s_5s_4},$$

$$e_1\left(y_4^{L_4}, y_5^{L_4}, y_6^{L_4}, y_7^{L_4}\right) \mapsto 2\epsilon^{P_4}_{s_4},$$

$$e_2\left(y_4^{L_4}, y_5^{L_4}, y_6^{L_4}, y_7^{L_4}\right) \mapsto -12\epsilon^{P_4}_{s_2s_4} - 12\epsilon^{P_4}_{s_3s_4}24\epsilon^{P_4}_{s_5s_4},$$

$$e_3\left(y_4^{L_4}, y_5^{L_4}, y_6^{L_4}, y_7^{L_4}\right) \mapsto 28\epsilon^{P_4}_{s_3s_2s_4} - 44\epsilon^{P_4}_{s_5s_2s_4} + 14\epsilon^{P_4}_{s_1s_3s_4} - 44\epsilon^{P_4}_{s_5s_3s_4} + 158\epsilon^{P_4}_{s_6s_5s_4},$$

$$e_4\left(y_4^{L_4}, y_5^{L_4}, y_6^{L_4}, y_7^{L_4}\right) \mapsto$$
$$-15\epsilon^{P_4}_{s_1s_3s_2s_4} - 10\epsilon^{P_4}_{s_4s_3s_2s_4} + 42\epsilon^{P_4}_{s_5s_3s_2s_4} + 26\epsilon^{P_4}_{s_4s_5s_2s_4} - 159\epsilon^{P_4}_{s_6s_5s_2s_4}$$
$$+21\epsilon^{P_4}_{s_5s_1s_3s_4} + 26\epsilon^{P_4}_{s_4s_5s_3s_4} - 159\epsilon^{P_4}_{s_6s_5s_3s_4} + 1111\epsilon^{P_4}_{s_7s_6s_5s_4},$$
$$e_1\left(y_8^{L_4}, y_9^{L_4}\right) \mapsto 2\epsilon^{P_4}_{s_4},$$
$$e_2\left(y_8^{L_4}, y_9^{L_4}\right) \mapsto 109\epsilon^{P_4}_{s_2s_4} - 35\epsilon^{P_4}_{s_3s_4} - 35\epsilon^{P_4}_{s_5s_4}.$$

(e) **(L_5):** *For L_5, we let*

$$y_1^{L_5} = 6\Theta_1 - 2\Theta_5,\ y_2^{L_5} = -6\Theta_1 + 6\Theta_3 - 2\Theta_5,\ y_3^{L_5} = -6\Theta_3 + 6\Theta_4 - 2\Theta_5,\ y_4^{L_5} = 6\Theta_2 - 6\Theta_4 + 4\Theta_5,$$
$$y_5^{L_5} = -6\Theta_2 + 4\Theta_5,\ y_6^{L_5} = -\Theta_5 + 5\Theta_7,\ y_7^{L_5} = -\Theta_5 + 5\Theta_6 - 5\Theta_7,\ y_8^{L_5} = 4\Theta_5 - 5\Theta_6.$$

Then,

$$\mathrm{Rep}^{\mathbb{C}}_{\mathrm{poly}}(L_5) = \mathbb{C}_{\mathrm{sym}}[y_1^{L_5}, y_2^{L_5}, y_3^{L_5}, y_4^{L_5}, y_5^{L_5}] \otimes_{\mathbb{C}} \mathbb{C}_{\mathrm{sym}}[y_6^{L_5}, y_7^{L_5}, y_8^{L_5}]/R_{7,5},$$

where $R_{7,5} = y_1^{L_5} + y_2^{L_5} + y_3^{L_5} + y_4^{L_5} + y_5^{L_5} - y_6^{L_5} - y_7^{L_5} - y_8^{L_5}$.
Further, the generators of $\mathrm{Rep}^{\mathbb{C}}_{\mathrm{poly}}(L_5)$ *under* ξ^{P_5} *go to*

$$e_1\left(y_1^{L_5}, y_2^{L_5}, y_3^{L_5}, y_4^{L_5}, y_5^{L_5}\right) \mapsto 2\epsilon^{P_5}_{s_5},$$
$$e_2\left(y_1^{L_5}, y_2^{L_5}, y_3^{L_5}, y_4^{L_5}, y_5^{L_5}\right) \mapsto 16\epsilon^{P_5}_{s_4s_5} - 20\epsilon^{P_5}_{s_6s_5},$$
$$e_3\left(y_1^{L_5}, y_2^{L_5}, y_3^{L_5}, y_4^{L_5}, y_5^{L_5}\right) \mapsto -224\epsilon^{P_5}_{s_2s_4s_5} + 208\epsilon^{P_5}_{s_3s_4s_5} - 16\epsilon^{P_5}_{s_6s_4s_5} - 8\epsilon^{P_5}_{s_7s_6s_5},$$
$$e_4\left(y_1^{L_5}, y_2^{L_5}, y_3^{L_5}, y_4^{L_5}, y_5^{L_5}\right) \mapsto$$
$$-608\epsilon^{P_5}_{s_3s_2s_4s_5} + 384\epsilon^{P_5}_{s_6s_2s_4s_5} + 1424\epsilon^{P_5}_{s_1s_3s_4s_5} - 48\epsilon^{P_5}_{s_6s_3s_4s_5} - 176\epsilon^{P_5}_{s_5s_6s_4s_5} - 48\epsilon^{P_5}_{s_7s_6s_4s_5},$$
$$e_5\left(y_1^{L_5}, y_2^{L_5}, y_3^{L_5}, y_4^{L_5}, y_5^{L_5}\right) \mapsto$$
$$-2976\epsilon^{P_5}_{s_1s_3s_2s_4s_5} + 896\epsilon^{P_5}_{s_4s_3s_2s_4s_5} - 160\epsilon^{P_5}_{s_6s_3s_2s_4s_5} + 224\epsilon^{P_5}_{s_5s_6s_2s_4s_5} + 96\epsilon^{P_5}_{s_7s_6s_2s_4s_5}$$
$$+2944\epsilon^{P_5}_{s_6s_1s_3s_4s_5} - 640\epsilon^{P_5}_{s_5s_6s_3s_4s_5} - 768\epsilon^{P_5}_{s_7s_6s_3s_4s_5} + 512\epsilon^{P_5}_{s_7s_5s_6s_4s_5},$$
$$e_1\left(y_6^{L_5}, y_7^{L_5}, y_8^{L_5}\right) \mapsto 2\epsilon^{P_5}_{s_5},$$
$$e_2\left(y_6^{L_5}, y_7^{L_5}, y_8^{L_5}\right) \mapsto -7\epsilon^{P_5}_{s_4s_5} + 18\epsilon^{P_5}_{s_6s_5},$$
$$e_3\left(y_6^{L_5}, y_7^{L_5}, y_8^{L_5}\right) \mapsto 4\epsilon^{P_5}_{s_2s_4s_5} + 4\epsilon^{P_5}_{s_3s_4s_5} - 17\epsilon^{P_5}_{s_6s_4s_5} + 104\epsilon^{P_5}_{s_7s_6s_5},$$

(f) **(L_6):** *For L_6, we let*

$$y_1^{L_6} = 2\Theta_1 - \Theta_6,\ y_2^{L_6} = -2\Theta_1 + 2\Theta_3 - \Theta_6,\ y_3^{L_6} = -2\Theta_3 + 2\Theta_4 - \Theta_6,\ y_4^{L_6} = 2\Theta_2 - 2\Theta_4 + 2\Theta_5 - \Theta_6,$$
$$y_5^{L_6} = -2\Theta_2 + 2\Theta_5 - \Theta_6,\ y_6^{L_6} = -\Theta_6 + 4\Theta_7,\ y_7^{L_6} = 3\Theta_6 - 4\Theta_7.$$

Then,

$$\mathrm{Rep}^{\mathbb{C}}_{\mathrm{poly}}(L_6) = \left[R \oplus (y_1^{L_6}y_2^{L_6}y_3^{L_6}y_4^{L_6}y_5^{L_6})R\right] \otimes_{\mathbb{C}} \mathbb{C}[y_6^{L_6}, y_7^{L_6}],$$

where $R = \mathbb{C}_{\mathrm{sym}}\left[(y_1^{L_6})^2, (y_2^{L_6})^2, (y_3^{L_6})^2, (y_4^{L_6})^2, (y_5^{L_6})^2\right]$.
Further, the generators of $\mathrm{Rep}^{\mathbb{C}}_{\mathrm{poly}}(L_6)$ *under* ξ^{P_6} *go to*

$$e_1\left((y_1^{L_6})^2, (y_2^{L_6})^2, (y_3^{L_6})^2, (y_4^{L_6})^2, (y_5^{L_6})^2\right) \mapsto -3\epsilon^{P_6}_{s_5s_6} + 5\epsilon^{P_6}_{s_7s_6},$$
$$e_2\left((y_1^{L_6})^2, (y_2^{L_6})^2, (y_3^{L_6})^2, (y_4^{L_6})^2, (y_5^{L_6})^2\right) \mapsto -54\epsilon^{P_6}_{s_2s_4s_5s_6} + 42\epsilon^{P_6}_{s_3s_4s_5s_6} - 2\epsilon^{P_6}_{s_7s_4s_5s_6} + 4\epsilon^{P_6}_{s_6s_7s_5s_6},$$
$$e_3\left((y_1^{L_6})^2, (y_2^{L_6})^2, (y_3^{L_6})^2, (y_4^{L_6})^2, (y_5^{L_6})^2\right) \mapsto$$
$$174\epsilon^{P_6}_{s_1s_3s_2s_4s_5s_6} - 108\epsilon^{P_6}_{s_4s_3s_2s_4s_5s_6} + 100\epsilon^{P_6}_{s_7s_3s_2s_4s_5s_6} - 54\epsilon^{P_6}_{s_6s_7s_2s_4s_5s_6}$$
$$-174\epsilon^{P_6}_{s_7s_1s_3s_4s_5s_6} - 22\epsilon^{P_6}_{s_6s_7s_3s_4s_5s_6} + 50\epsilon^{P_6}_{s_5s_6s_7s_4s_5s_6},$$

$$e_4\left((y_1^{L_6})^2,(y_2^{L_6})^2,(y_3^{L_6})^2,(y_4^{L_6})^2,(y_5^{L_6})^2\right)\mapsto$$
$$-519\epsilon^{P_6}_{s_3s_4s_1s_3s_2s_4s_5s_6}+291\epsilon^{P_6}_{s_5s_4s_1s_3s_2s_4s_5s_6}+207\epsilon^{P_6}_{s_7s_4s_1s_3s_2s_4s_5s_6}-164\epsilon^{P_6}_{s_6s_7s_1s_3s_2s_4s_5s_6}$$
$$+42\epsilon^{P_6}_{s_6s_5s_4s_3s_2s_4s_5s_6}-250\epsilon^{P_6}_{s_7s_5s_4s_3s_2s_4s_5s_6}+168\epsilon^{P_6}_{s_6s_7s_4s_3s_2s_4s_5s_6}-136\epsilon^{P_6}_{s_5s_6s_7s_3s_2s_4s_5s_6}$$
$$+70\epsilon^{P_6}_{s_4s_5s_6s_7s_2s_4s_5s_6}-4\epsilon^{P_6}_{s_5s_6s_7s_1s_3s_4s_5s_6}+102\epsilon^{P_6}_{s_4s_5s_6s_7s_3s_4s_5s_6}.$$
$$e_5\left((y_1^{L_6})^2,(y_2^{L_6})^2,(y_3^{L_6})^2,(y_4^{L_6})^2,(y_5^{L_6})^2\right)\mapsto$$
$$-180\epsilon^{P_6}_{s_4s_5s_3s_4s_1s_3s_2s_4s_5s_6}+333\epsilon^{P_6}_{s_6s_5s_3s_4s_1s_3s_2s_4s_5s_6}-84\epsilon^{P_6}_{s_7s_5s_3s_4s_1s_3s_2s_4s_5s_6}+223\epsilon^{P_6}_{s_6s_7s_3s_4s_1s_3s_2s_4s_5s_6}$$
$$-444\epsilon^{P_6}_{s_7s_6s_5s_4s_1s_3s_2s_4s_5s_6}-106\epsilon^{P_6}_{s_6s_7s_5s_4s_1s_3s_2s_4s_5s_6}-76\epsilon^{P_6}_{s_5s_6s_7s_4s_1s_3s_2s_4s_5s_6}-131\epsilon^{P_6}_{s_4s_5s_6s_7s_1s_3s_2s_4s_5s_6}$$
$$+152\epsilon^{P_6}_{s_6s_7s_6s_5s_4s_3s_2s_4s_5s_6}+12\epsilon^{P_6}_{s_5s_6s_7s_5s_4s_3s_2s_4s_5s_6}+26\epsilon^{P_6}_{s_4s_5s_6s_7s_4s_3s_2s_4s_5s_6}+136\epsilon^{P_6}_{s_2s_4s_5s_6s_7s_3s_2s_4s_5s_6}$$
$$-56\epsilon^{P_6}_{s_3s_4s_5s_6s_7s_3s_2s_4s_5s_6}+264\epsilon^{P_6}_{s_1s_3s_4s_5s_6s_7s_2s_4s_5s_6}-504\epsilon^{P_6}_{s_2s_4s_5s_6s_7s_1s_3s_4s_5s_6}+506\epsilon^{P_6}_{s_3s_4s_5s_6s_7s_1s_3s_4s_5s_6}.$$
$$e_5\left(y_1^{L_6},y_2^{L_6},y_3^{L_6},y_4^{L_6},y_5^{L_6}\right)\mapsto$$
$$6\epsilon^{P_6}_{s_3s_2s_4s_5s_6}-8\epsilon^{P_6}_{s_7s_2s_4s_5s_6}-21\epsilon^{P_6}_{s_1s_3s_4s_5s_6}+8\epsilon^{P_6}_{s_7s_3s_4s_5s_6}-1\epsilon^{P_6}_{s_6s_7s_4s_5s_6},$$

and

$$e_1\left(y_6^{L_6},y_7^{L_6}\right)\mapsto 2\epsilon^{P_6}_{s_6},$$
$$e_2\left(y_6^{L_6},y_7^{L_6}\right)\mapsto -3\epsilon^{P_6}_{s_5s_6}+13\epsilon^{P_6}_{s_7s_6}.$$

(g) (L_7): *For E_6, we let*

$$x_1=5\alpha_1+4\alpha_3+3\alpha_4+2\alpha_5+\alpha_6,\ \ x_2=-\alpha_1+4\alpha_3+3\alpha_4+2\alpha_5+\alpha_6,$$
$$x_3=-\alpha_1-2\alpha_3+3\alpha_4+2\alpha_5+\alpha_6,\ \ x_4=-\alpha_1-2\alpha_3-3\alpha_4+2\alpha_5+\alpha_6,$$
$$x_5=-\alpha_1-2\alpha_3-3\alpha_4-4\alpha_5+\alpha_6,\ ,x_6=-\alpha_1-2\alpha_3-3\alpha_4-4\alpha_5-5\alpha_6$$

and

$$x=-3(\alpha_1+2\alpha_2+2\alpha_3+3\alpha_4+2\alpha_5+\alpha_6).$$

Set

$$a_i=x_i+x,\ b_i=x_i-x\ (i=1,\dots,6),$$
$$c_{ij}=-x_i-x_j\ (1\le i<j\le 6).$$

Then, the following provides a set of generators for the Weyl group invariant polynomials in $S(\mathfrak{t}_6^*)=\mathbb{C}[\alpha_1,\dots,\alpha_6]$ for E_6, where $\mathfrak{t}_6$ is the Cartan subalgebra of E_6 (cf. [L, §3], [C]*):*

$$\psi_m=\sum_{i=1}^{6}a_i^m+\sum_{i=1}^{6}b_i^m+\sum_{1\le i<j\le 6}c_{ij}^m,\ m=2,5,6,8,9,12. \tag{39}$$

We view ψ_m as elements of $S(\mathfrak{t}^*)=\mathbb{C}[\alpha_1,\dots,\alpha_6,\alpha_7]\supset\mathbb{C}[\alpha_1,\dots,\alpha_6]$. Then,

$$S(\mathfrak{t}^*)^{W_7}=\mathbb{C}[\psi_2,\psi_5,\psi_6,\psi_8,\psi_9,\psi_{12}]\otimes_{\mathbb{C}}\mathbb{C}[\omega_7],$$

where W_7 is the Weyl group of L_7. Thus,

$$\mathrm{Rep}^{\mathbb{C}}_{\mathrm{poly}}(L_7)=\mathbb{C}[\bar\psi_2,\bar\psi_5,\bar\psi_6,\bar\psi_8,\bar\psi_9,\bar\psi_{12}]\otimes_{\mathbb{C}}\mathbb{C}[\bar\omega_7],$$

where $\bar\psi_m:=\theta^*_{\omega_7}(\psi_m)$ and $\bar\omega_7:=\theta^*_{\omega_7}(\omega_7)$ ($\theta^*_{\omega_7}:S(\mathfrak{t}^*)^{W_7}\to\mathrm{Rep}^{\mathbb{C}}_{\mathrm{poly}}(L_7)$ being the induced map from the Springer morphism $\theta_{\omega_7}:T\to\mathfrak{t}$).

Further, the generators of $\mathrm{Rep}^{\mathbb{C}}_{\mathrm{poly}}(L_7)$ under ξ^{P_7} go to

$$\bar\psi_2\mapsto -144\epsilon^{P_7}_{s_6s_7},$$
$$\bar\psi_5\mapsto 11520\left(-46\epsilon^{P_7}_{s_2s_4s_5s_6s_7}+35\epsilon^{P_7}_{s_3s_4s_5s_6s_7}\right),$$

$$\bar{\psi}_6 \mapsto 6912\left(-254\epsilon^{P_7}_{s_3s_2s_4s_5s_6s_7} + 683\epsilon^{P_7}_{s_1s_3s_4s_5s_6s_7}\right),$$
$$\bar{\psi}_8 \mapsto 9216\left(-20165\epsilon^{P_7}_{s_4s_1s_3s_2s_4s_5s_6s_7} + 23686\epsilon^{P_7}_{s_5s_4s_3s_2s_4s_5s_6s_7}\right),$$
$$\bar{\psi}_9 \mapsto 774144\left(-8816\epsilon^{P_7}_{s_3s_4s_1s_3s_2s_4s_5s_6s_7} + 7292\epsilon^{P_7}_{s_5s_4s_1s_3s_2s_4s_5s_6s_7} - 3575\epsilon^{P_7}_{s_6s_5s_4s_3s_2s_4s_5s_6s_7}\right),$$
$$\bar{\psi}_{12} \mapsto 1327104\Big(- 3256896\epsilon^{P_7}_{s_2s_4s_5s_3s_4s_1s_3s_2s_4s_5s_6s_7} + 1330131\epsilon^{P_7}_{s_6s_4s_5s_3s_4s_1s_3s_2s_4s_5s_6s_7} - 980876\epsilon^{P_7}_{s_7s_6s_5s_3s_4s_1s_3s_2s_4s_5s_6s_7}\Big).$$

and

$$\bar{\omega}_7 \mapsto \epsilon^{P_7}_{s_7}$$

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