CATEGORY $\mathscr{C}_k$ OF MULTI-LOOP ALGEBRA REPRESENTATIONS VERSUS MODULAR REPRESENTATIONS: QUESTIONS OF G. LUSZTIG

SHRAWAN KUMAR

1. Abstract

Lusztig defined an abelian category $\mathscr{C}_k$ of a class of representations of a multi-loop algebra and asked various questions connecting it to the modular representation theory of simple algebraic groups in char. $p > 0$. The aim of this paper is to show that some of these questions have negative answer.

2. Introduction

Let G be a connected simply-connected simple algebraic group over an algebraically closed field of characteristic $p > 0$ with a fixed maximal torus T and a fixed Borel subgroup B containing T. Let X_+ be the set of dominant characters of T. Consider the category $\mathfrak{C} = \mathfrak{C}(G)$ of finite dimensional rational representations of G. The simple objects of $\mathfrak{C}$, up to isomorphism, are indexed by X_+; let L_λ be the simple object indexed by $\lambda \in X_+$. Let E^0_λ be the Weyl module indexed by $\lambda \in X_+$. The Weyl modules form another basis of the Grothendieck group $\mathcal{G}(\mathfrak{C})$ of $\mathfrak{C}$. Hence, for any $\lambda \in X_+$, we can write

$$L_\lambda = \sum_{\mu \in X_+} c_{\mu,\lambda}\, E^0_\mu,$$

where $c_{\mu,\lambda}$ are integers, and they are zero for all but finitely many μ. It is of considerable interest to understand the character of each L_λ or, equivalenty, to understand the integers $c_{\mu,\lambda}$ (since the characters of E^0_μ are given by the Weyl character formula). By a famous conjecture of Lusztig [Lu1], the integers $c_{\mu,\lambda}$ for λ in a finite subset of X_+ containing the restricted weights $X^{\text{res}}_+ := \{\mu \in X_+ : \mu(\alpha_i^\vee) < p \text{ for all the simple coroots } \alpha_i^\vee\}$, are given in terms of the Kazhdan-Lusztig polynomials of the affine Weyl group of the Langlands dual of G, assuming that p is sufficiently large relative to the type of G. (By the Steinberg tensor product theorem, this leads to a formula for $c_{\mu,\lambda}$ for any $\lambda \in X_+$.) This conjecture has been proved for p 'very large' by Andersen-Jantzen-Soergel [AJS]. Fiebig [F] proved the conjecture still for very large but explicit bound on p.

Now, more recently, Lusztig [Lu2] has formulated a conjecture in such a way that the tensor product theorem is not used. Namely, for any $\lambda \in X_+$ and any integer $k \geq 0$, he defined an element $E^k_\lambda \in \mathcal{G}(\mathfrak{C})$ by induction on k. When $k = 0$, E^0_λ is already defined above. Define E^1_λ to be the reduction mod p of the simple module with highest weight λ of the quantum group associated to G at a p-th root of 1. Then,

$$E^1_\lambda = \sum_{\mu \in X_+} \mathcal{P}_{\mu,\lambda} E^0_\mu, \quad \text{for } \mathcal{P}_{\mu,\lambda} \in \mathbb{Z},$$

where $\mathcal{P}_{\mu,\lambda}$ is explicitly known in terms of the Kazhdan-Lusztig polynomials of the affine Weyl group due to the works of Kazhdan-Lusztig [KL0-4] and Kashiwara-Tanisaki [KT]. Express λ in terms of its p-power expansion: $\lambda = \sum_{r\geq 0} p^r\lambda_r$, where $\lambda_r \in X_+^{\text{res}}$. Following Lusztig [Lu2], define

$$E_\lambda^2 = \sum_{\mu\in X_+:\mu-\lambda_0\in pX} \mathcal{P}_{(\mu-\lambda_0)/p,(\lambda-\lambda_0)/p} E_\mu^1, \;\; E_\lambda^3 = \sum_{\mu\in X_+:\mu-\lambda_0-p\lambda_1\in p^2X} \mathcal{P}_{(\mu-\lambda_0-p\lambda_1)/p^2,(\lambda-\lambda_0-p\lambda_1)/p^2} E_\mu^2,$$

where X is the set of all the characters of T, and continue this way to define E_λ^k for any k. Then, $\{E_\lambda^k : \lambda \in X_+\}$ is a $\mathbb{Z}$-basis of $\mathcal{G}(\mathfrak{C})$ for any $k \geq 0$. Moreover, each E_λ^k is, in fact, a G-module (not just a virtual module). Further, for large k, $E_\lambda^k = E_\lambda^{k+1} = E_\lambda^{k+2} = \cdots$. This common value is denoted by E_λ^∞. If p is 'very large' compared to the type of G, then $E_\lambda^\infty = L_\lambda$. Thus, this gives an explicit successive approximation to construct L_λ for large p (cf. [Lu2] for all these results).

Now, Lusztig has defined parallel objects in characteristic zero in terms of the representation theory of multi-loop algebras [Lu3]. Specifically, let $\mathfrak{g}$ be the (complex) Lie algebra of a simple group over the complex numbers of the same type as G and let $\langle\, ,\, \rangle$ be the invariant (symmetric) bilinear form on $\mathfrak{g}$ normalized so that the induced form on the dual $\mathfrak{t}^*$ of the Cartan subalgebra $\mathfrak{t}$ of $\mathfrak{g}$ satisfies: $\langle\theta,\theta\rangle = 2$ for the highest root θ. For any $k \geq 0$, define the Laurent polynomial ring

$$A_k := \mathbb{C}[t_1^{\pm 1}, \ldots, t_k^{\pm 1}]$$

and let

$$\tilde{\mathfrak{g}}_k := (\mathfrak{g}\otimes A_k) \oplus (\mathbb{C}c_1 \oplus \cdots \oplus \mathbb{C}c_k)$$

with bracket defined by

$$[x\otimes \mathbf{t^n}, y\otimes \mathbf{t^m}] = [x,y]\otimes \mathbf{t^{n+m}} + \delta_{\mathbf{n+m},0}\langle x,y\rangle(n_1c_1 + \cdots + n_kc_k), \;\text{ for } x,y\in\mathfrak{g},$$
$$\mathbf{n} = (n_1,\ldots,n_k)\in\mathbb{Z}^k \;\text{ and }\; \mathbf{m} = (m_1,\ldots,m_k)\in\mathbb{Z}^k,$$

where $\mathbf{t^n} := t_1^{n_1}\cdots t_k^{n_k}$ and c_i are central elements.

Let $'\mathcal{C}_k$ be the category of $\tilde{\mathfrak{g}}_k$-modules on which c_i acts by $-p^i - h^\vee$ for $1 \leq i \leq k$, where $h^\vee$ is the dual Coxeter number of $\mathfrak{g}$. (In particular, $'\mathcal{C}_0$ is the category of $\mathfrak{g}$-modules.) For any $M \in {'\mathcal{C}_{k-1}}$ (with $k \geq 1$) define a $\tilde{\mathfrak{g}}_k$-module $\text{Ind}_{k-1}^k(M)$ as follows: Let

$$A_k^+ := \mathbb{C}[t_1^{\pm 1}, \ldots, t_{k-1}^{\pm 1}, t_k^1] \subset A_k$$

and regard M as a $\tilde{\mathfrak{g}}_k^+$ -module by letting $\mathfrak{g}\otimes(t_kA_k^+)$ act by zero and c_i acts by $-p^i - h^\vee$, where

$$\tilde{\mathfrak{g}}_k^+ := (\mathfrak{g}\otimes A_k^+) \oplus (\mathbb{C}c_1 \oplus \cdots \oplus \mathbb{C}c_k).$$

Now, let

$$\text{Ind}_{k-1}^k(M) := \text{Ind}_{U(\tilde{\mathfrak{g}}_k^+)}^{U(\tilde{\mathfrak{g}}_k)}(M).$$

It clearly belongs to the category $'\mathcal{C}_k$.

Define the sequence of objects $\mathcal{E}_{\lambda,k}^0, \mathcal{E}_{\lambda,k}^1, \ldots, \mathcal{E}_{\lambda,k}^k$ in $'\mathcal{C}_k$ by induction on k by setting $\mathcal{E}_{\lambda,0}^0$ as the Weyl (which is the same as irreducible) module $V(\lambda)$ of $\mathfrak{g}$ with highest weight λ. For $k \geq 1$ and $k' \in \{1, 2, \ldots, k-1\}$, set

$$\mathcal{E}_{\lambda,k}^{k'} := \text{Ind}_{k-1}^k(\mathcal{E}_{\lambda,k-1}^{k'}).$$

Lusztig asked the following questions $(Q_1) - (Q_4)$ (cf. [Lu3]). Actually, he termed these questions as his 'expectations'.

Question Q_1 (Lu3): *The $\tilde{\mathfrak{g}}_k$-module $\mathcal{E}_{\lambda,k}^0$ has a unique irreducible quotient. Denote it by $\mathcal{E}_{\lambda,k}^k$. (For $k = 1$ this is proved in [KL1].)*

Assuming the validity of the above question (Q_1), let $\mathcal{C}_k$ be the abelian subcategory of $'\mathcal{C}_k$ consisting of those $\tilde{\mathfrak{g}}_k$-modules which have finite length and have all their composition factors in the $\tilde{\mathfrak{g}}_k$- irreducible modules $\{\mathcal{E}_{\lambda,k}^k : \lambda \in X_+\}$.

Question Q_2 **(Lu3):** *Each of the modules* $\mathscr{E}^{k'}_{\lambda,k}$, *where* $0 \le k' \le k$ *and* $\lambda \in X_+$, *lie in* $\mathscr{C}_k$. *(For* $k = 1$ *this is proved in [KL1].)*

Question Q_3 **(Lu3):** *For* $0 \le k' \le k$ *and* $\lambda \in X_+$, *the matrix expressing* $\mathscr{E}^{k'}_{\lambda,k}$ *in terms of the irreducible objects* $\mathscr{E}^{k}_{\mu,k}$ $(\mu \in X_+)$ *in the Grothendieck group* $\mathcal{G}(\mathscr{C}_k)$ *is the same as the matrix expressing* $(E^{k'}_{\lambda})$ *in terms of* (E^{k}_{μ}).

Question Q_4 **(Lu3):** *The category* $\mathscr{C}_k$ *is a rigid braided monoidal category. In particular, the category* $\mathscr{C}_k$ *is equipped with a duality operator. (For* $k = 1$ *this is proved in [KL1, KL2].)*

We enlarge the Lie algebra $\tilde{\mathfrak{g}}_k$ (resp. $\tilde{\mathfrak{g}}^+_k$) by adding the standard derivations $d_1, \dots, d_k$ to get the Lie algebra $\hat{\mathfrak{g}}_k$ (resp. $\hat{\mathfrak{g}}^+_k$) (cf. Definition 3). Replacing $U(\tilde{\mathfrak{g}}_k)$ (resp. $U(\tilde{\mathfrak{g}}^+_k)$) by $U(\hat{\mathfrak{g}}_k)$ (resp. $U(\hat{\mathfrak{g}}^+_k)$) in the definition of $\operatorname{Ind}^{U(\tilde{\mathfrak{g}}_k)}_{U(\tilde{\mathfrak{g}}^+_k)}(M)$, we get the $\hat{\mathfrak{g}}_k$-modules (cf. Definition 3)

$$\hat{\mathscr{E}}^0_{\lambda,k}, \hat{\mathscr{E}}^1_{\lambda,k}, \dots, \hat{\mathscr{E}}^k_{\lambda,k}.$$

We prove that for any $\lambda \in X_+$, the module $\hat{\mathscr{E}}^0_{\lambda,k}$ (and hence any $\hat{\mathscr{E}}^j_{\lambda,k}$ for $j < k$) has a unique irreducible quotient as a $\hat{\mathfrak{g}}_k$-module denoted by $\hat{\mathscr{E}}^k_{\lambda,k}$ (cf. Lemma 4). This answers the Question Q_1 affirmatively for the enlarged Lie algebra $\hat{\mathfrak{g}}_k$.

Our main technical result of the paper is the following theorem (cf. Theorem 5 and Remark 6):

Theorem 1. *Take* $k \ge 2$. *Then, for any non-constant element* $\mathcal{Z} \in U\left(\hat{\mathfrak{g}}^-_k\right)$, *there exists* $p_o > 0$ *such that* $\left[\mathfrak{g} \otimes t^r_{k-1}, \mathcal{Z}\right] \neq 0$ *for all* $r \ge p_o$, *where* $\hat{\mathfrak{g}}^-_k$ *is defined by the equation* (3.2).

We next prove that each of $\hat{\mathscr{E}}^k_{\lambda,k}$ is an irreducible $\tilde{\mathfrak{g}}_k$-module for any $k \ge 0$ (cf. Proposition 10). Even though we do not know if the $\tilde{\mathfrak{g}}_k$-module $\mathscr{E}^0_{\lambda,k}$ has a unique irreducible quotient, but it has a 'preferred' irreducible $\tilde{\mathfrak{g}}_k$-module quotient $\hat{\mathscr{E}}^k_{\lambda,k}$. Thus, for any $k \ge 0$,

$$\mathscr{E}^{k'}_{\lambda,k} = \hat{\mathscr{E}}^{k'}_{\lambda,k}, \text{ for any } \lambda \in X_+ \text{ and } 0 \le k' \le k.$$

The following result, which is the main result of the paper, asserts that Question Q_2 has a negative answer for any $\mathfrak{g}$ and any $k \ge 2$ (cf. Corollary 13 deduced from Theorem 12). Since Q_2 has negative answer, Question Q_3 does not make sense at least as it is.

Corollary 2. *Lusztig's question* Q_2 *has negative answer for any* $\mathfrak{g}$ *and any* $k \ge 2$. *Specifically,* $\mathscr{E}^{k-1}_{\lambda,k}$ *(and hence any* $\mathscr{E}^{k'}_{\lambda,k}$ *for* $k' < k$*) does not belong to the category* $\mathscr{C}_k$ *unless* $\mathscr{E}^{k-1}_{\lambda,k}$ *is irreducible as a* $\tilde{\mathfrak{g}}_k$*-module.*

In the last section we study the action of the Sugawara operator on any $\hat{\mathfrak{g}}_k$-module V which has its d_k-eigenvalues bounded above (cf. Theorem 14). In fact, our theorem is slightly more general and it extends the corresponding well-known theorem for $k = 1$.

Acknowledgements: I am grateful to G. Lusztig for bringing the above questions to my knowledge and many subsequent conversations.

3. Questions Q_1 and Q_2 for loop algebras with derivations

Definition 3. Let $\mathfrak{g}$ be a simple Lie algebra over $\mathbb{C}$. Fix $k \ge 1$. Just as in the definition of affine Kac-Moody Lie algebras one adds a derivation to make the root spaces as well as weight spaces in integrable highest weight modules finite dimensional, we add derivations $d_1, \dots, d_k$ to $\tilde{\mathfrak{g}}_k$ to get

$$\hat{\mathfrak{g}}_k := \tilde{\mathfrak{g}}_k \oplus \left(\overset{k}{\underset{i=1}{\oplus}} \mathbb{C} d_i \right), \tag{3.1}$$

where the bracket is defined by (for all $1 \le i, j \le k$),

$$[d_i, x \otimes \mathbf{t}^{\mathbf{n}}] = n_i x \otimes \mathbf{t}^{\mathbf{n}}, \text{ for } x \in \mathfrak{g} \text{ and } \mathbf{n} = (n_1, \dots, n_k) \in \mathbb{Z}^k, \text{ and } [d_i, d_j] = [d_i, c_j] = 0.$$

We also define

$$\hat{\mathfrak{g}}_k^+ = \tilde{\mathfrak{g}}_k^+ \oplus \left(\overset{k}{\underset{i=1}{\oplus}} \mathbb{C} d_i \right).$$

Let $'\hat{\mathscr{C}}_k$ be the category of $\hat{\mathfrak{g}}_k$-modules on which c_i acts as $-p^i - h^\vee$. For $\lambda \in X_+$, we define the sequence of objects $\hat{\mathscr{E}}^0_{\lambda,k}, \hat{\mathscr{E}}^1_{\lambda,k}, \dots, \hat{\mathscr{E}}^k_{\lambda,k}$ in $'\hat{\mathscr{C}}_k$ exactly by the definition as in the Introduction replacing $\tilde{\mathfrak{g}}_k$ by $\hat{\mathfrak{g}}_k$. In defining $\hat{\mathscr{E}}^j_{\lambda,k}$, we let d_k act trivially on $\hat{\mathscr{E}}^j_{\lambda,k-1}$ (for any $0 \le j < k$). ($\hat{\mathscr{E}}^0_{\lambda,0}$ is defined to be the Weyl module $\mathscr{E}^0_{\lambda,0} = V(\lambda)$.)

Let $A_k^- := t_k^{-1} \mathbb{C} \left[t_1^{\pm 1}, \dots, t_{k-1}^{\pm 1}, t_k^{-1} \right]$ and

$$\hat{\mathfrak{g}}_k^- = \mathfrak{g} \otimes A_k^-. \tag{3.2}$$

The following lemma answers the Question Q_1 affirmatively for $\hat{\mathscr{E}}^0_{\lambda,k}$ replacing $\mathscr{E}^0_{\lambda,k}$.

Lemma 4. *For any $\lambda \in X_+$, the module $\hat{\mathscr{E}}^0_{\lambda,k}$ (and hence any $\hat{\mathscr{E}}^j_{\lambda,k}$ for $j < k$) has a unique irreducible quotient as a $\hat{\mathfrak{g}}_k$-module. Let us denote it by $\hat{\mathscr{E}}^k_{\lambda,k}$.*

Proof. Let $\{M_\alpha\}$ be the collection of all the proper $\hat{\mathfrak{g}}_k$-submodules of $\hat{\mathscr{E}}^0_{\lambda,k}$ and let

$$M := \sum M_\alpha \subset \hat{\mathscr{E}}^0_{\lambda,k}.$$

Then, M is the maximal proper $\hat{\mathfrak{g}}_k$-submodule of $\hat{\mathscr{E}}^0_{\lambda,k}$. Observe that it is a proper submodule since $\hat{\mathscr{E}}^0_{\lambda,k}$ is diagonalizable as a $D_k := (\mathbb{C}d_1 \oplus \dots, \oplus \mathbb{C}d_k)$-module and any non-zero element of the 0-weight space of $\hat{\mathscr{E}}^0_{\lambda,k}$ with respect to the action of D_k generates $\hat{\mathscr{E}}^0_{\lambda,k}$ as a $\hat{\mathfrak{g}}_k$ -module. Thus, any M_α (and hence M) cannot contain any non-zero element of the 0-weight space. □

Let U denote the universal enveloping algebra.

Theorem 5. *Take $k = 2$. For any non-constant element $\mathcal{Z} \in U\left(\hat{\mathfrak{g}}_2^-\right)$, there exists $p_o > 0$ (depending upon $\mathcal{Z}$) such that the commutator $\left[\mathfrak{g} \otimes t_1^r, \mathcal{Z}\right] \neq 0$ for all $r \ge p_o$.*

Proof. Choose a basis $\{x_i\}_{i \ge 1}$ of $\hat{\mathfrak{g}}_2^-$ of the form

$$x_i = e_{\beta_i} \otimes t_1^{p_i} t_2^{q_i}, \quad q_i < 0,$$

where $\left\{e_\beta\right\}$ is a basis of $\mathfrak{g}$ consisting of root vectors (e_β has root β) and a basis of the Cartan subalgebra $\mathfrak{t} \subset \mathfrak{g}$ dual to the simple roots $\{\alpha_i\}$. Write (by renumbering the basis elements) the top homogeneous component $\mathcal{Z}^0$ of $\mathcal{Z}$ in the PBW-basis:

$$\mathcal{Z}^0 = \sum c_{\mathbf{d}} x_1^{d_1} \dots x_n^{d_n}, \; 0 \neq c_{\mathbf{d}} \in \mathbb{C} \quad \text{where } \mathbf{d} = (d_1, \dots, d_n).$$

In the above expression we only list those x_i such that x_i appears with exponent $d_i \ge 1$ in at least one monomial $\mathbf{x}^{\mathbf{d}} := x_1^{d_1} \dots x_n^{d_n}$ with $c_{\mathbf{d}} \neq 0$. We also index x_i so that

$$x_1, \dots, x_m \in \quad \mathfrak{t} \otimes A_2^- \qquad (m \ge 0)$$

and $x_{m+1}, \dots, x_n \in$ Root spaces $\otimes A_2^-$.

Let the homogeneous component $\mathcal{Z}^o$ be of degree $d_{\mathcal{Z}}$.

Case I : $m < n$, i.e., $\mathcal{Z}^o \notin U\left(\mathfrak{t} \otimes A_2^-\right)$.

Of course, $\mathcal{Z}^o = \sum_{|\mathbf{d}|=d_{\mathcal{Z}}} c_{\mathbf{d}} x_1^{d_1} \ldots x_n^{d_n}$, where $|\mathbf{d}| := d_1+d_2+\cdots+d_n$. Let $\bar{d}_{m+1} := \max\{d_{m+1} : c_{\mathbf{d}} \neq 0\}$. Fix $\left(\bar{d}_1, \ldots, \bar{d}_m\right)$ such that $c_{(\bar{d}_1,\ldots,\bar{d}_m,\bar{d}_{m+1},\ldots)} \neq 0$ for some $d_{m+2}, \ldots, d_n$.
Set

$$\bar{d}_{m+2} := \max\left\{d_{m+2} : c_{(\bar{d}_1,\bar{d}_2,\ldots,\bar{d}_m,\bar{d}_{m+1},d_{m+2},\ldots)} \neq 0 \text{ for some } d_{m+3}, \ldots, d_n\right\}$$
$$\vdots$$
$$\bar{d}_n := \max\left\{d_n : c_{(\bar{d}_1,\ldots,\bar{d}_{n-1},d_n)} \neq 0\right\}.$$

Then, of course, $\bar{d}_n = d_{\mathcal{Z}} - \left(\bar{d}_1 + \cdots + \bar{d}_{n-1}\right)$. Let $m+1 \leq k \leq n$ be the maximum integer such that

$$\bar{d}_k \neq 0.$$

Let $x_k = e_{\beta_k} \otimes t_1^{p_k} t_2^{q_k}$ for a root $\beta_k \neq 0$. Take $h_o \in \mathfrak{t}$ such that $\beta_k(h_o) \neq 0$. Then, we claim that the coefficient of

$$A := x_1^{\bar{d}_1} x_2^{\bar{d}_2} \ldots x_{k-1}^{\bar{d}_{k-1}} x_k^{\bar{d}_k - 1} x_{k+1}^0 \ldots x_n^0 \cdot \left(e_{\beta_k} \otimes \left(t_1^{p_k+r} t_2^{q_k}\right)\right) \text{ in } \left[h_o \otimes t_1^r, \mathcal{Z}^o\right]_{d_{\mathcal{Z}}} \tag{3.3}$$

is nonzero if we take $r >> 0$, $\left[h_o \otimes t_1^r, \mathcal{Z}^o\right]_{d_{\mathcal{Z}}}$ denotes the $d_{\mathcal{Z}}$-graded component in $\mathrm{Gr}\left(U\left(\mathfrak{g} \otimes A_2^-\right)\right)$ under the standard filtration of the enveloping algebra.

To prove the above claim, observe that the coefficient of A in $\left[h_o \otimes t_1^r, \mathcal{Z}^o\right]_{d_{\mathcal{Z}}}$ can only come from (by using the definition of $\bar{d}_i$) the commutator of $h_o \otimes t_1^r$ with the monomial $x_1^{\bar{d}_1} x_2^{\bar{d}_2} \ldots x_{k-1}^{\bar{d}_{k-1}} x_k^{\bar{d}_k} x_{k+1}^0 \ldots x_n^0$ or the monomials:

$$\left\{B_j := x_1^{\bar{d}_1} x_2^{\bar{d}_2} \ldots x_{k-1}^{\bar{d}_{k-1}} x_k^{\bar{d}_k - 1} x_{k+1}^0 \ldots x_j^1 \ldots x_n^0\right\}_{k+1 \leq j \leq n}.$$

Now, the component of A in

$$\left[h_o \otimes t_1^r, x_1^{\bar{d}_1} \ldots x_k^{\bar{d}_k} x_{k+1}^0 \ldots x_n^0\right]_{d_{\mathcal{Z}}} = \beta_k(h_o) \cdot A.$$

Further, for any $k+1 \leq j \leq n$, let $x_j = e_{\beta_j}\left(t_1^{p_j} t_2^{q_j}\right)$. Then, if $\left(p_j, q_j\right) \neq (p_k, q_k)$, then the component of A in $\left[h_o \otimes t_1^r, B_j\right]_{d_{\mathcal{Z}}}$ is clearly zero. But, if $\left(p_j, q_j\right) = (p_k, q_k)$, then the root β_j will have to be different from β_k. Thus, again the component of A in $\left[h_o \otimes t_1^r, B_j\right]_{d_{\mathcal{Z}}}$ is zero. Hence, the coefficient of A in $\left[h_o \otimes t_1^r, \mathcal{Z}^o\right]_{d_{\mathcal{Z}}}$ is nonzero, proving the claim (3.3). Thus, we have proved the theorem in this Case I by showing that $\left[h_o \otimes t_1^r, \mathcal{Z}^o\right]_{d_{\mathcal{Z}}} \neq 0$ which of course implies that $\left[h_o \otimes t_1^r, \mathcal{Z}\right] \neq 0$.

Case II: $m = n$, i.e., $\mathcal{Z}^o \in U\left(\mathfrak{t} \otimes A_2^-\right)$.

Again we write

$$\mathcal{Z}^o = \sum c_{\mathbf{d}} x_1^{d_1} \ldots x_m^{d_m},$$

where we now have each $x_i \in \mathfrak{t} \otimes A_2^-$. Similar to the proof in Case I, define

$$\begin{aligned} \bar{d}_1 &= \max\{d_1 : c_{\mathbf{d}} \neq 0\} \\ \bar{d}_2 &= \max\left\{d_2 : c_{(\bar{d}_1, d_2, \dots)} \neq 0 \quad \text{for some } d_3, \dots, d_m\right\}, \\ &\vdots \\ \bar{d}_m &= \max\left\{d_m : c_{(\bar{d}_1, \dots, \bar{d}_{m-1}, d_m)} \neq 0\right\}. \end{aligned}$$

Of course, $\bar{d}_m = d_{\mathcal{Z}} - \left(\bar{d}_1 + \bar{d}_2 + \cdots + \bar{d}_{m-1}\right)$. Let $1 \leq k \leq m$ be the maximum integer such that $\bar{d}_k \neq 0$. Let $x_k = h_k \otimes t_1^{p_k} t_2^{q_k}$. Take the simple root β_k such that $\beta_k(h_k) \neq 0$. (Observe that we have taken the basis of $\mathfrak{t}$ dual to the simple roots.) Then, we claim that the coefficient of

$$A' := x_1^{\bar{d}_1} \dots x_{k-1}^{\bar{d}_{k-1}} x_k^{\bar{d}_k - 1} x_{k+1}^0 \dots x_m^0 \left(e_{\beta_k} \otimes t_1^{r+p_k} t_2^{q_k}\right) \text{ in } \left[e_{\beta_k} \otimes t_1^r, \mathcal{Z}^o\right]_{d_{\mathcal{Z}}} \tag{3.4}$$

is nonzero for $r >> 0$.

Again by a proof similar to that of Case I, the coefficient of A' in $\left[e_{\beta_k} \otimes t_1^r, \mathcal{Z}^o\right]_{d_{\mathcal{Z}}}$ can only come from the commutator of $e_{\beta_k} \otimes t_1^r$ with the monomial $x_1^{\bar{d}_1} \dots x_k^{\bar{d}_k} x_{k+1}^0 \dots x_m^0$ or the monomials

$$\left\{B'_j := x_1^{\bar{d}_1} \dots x_{k-1}^{\bar{d}_{k-1}} x_k^{\bar{d}_k - 1} x_{k+1}^0 \dots x_j^1 \dots x_m^0\right\}_{k+1 \leq j \leq m}.$$

Now, the component of A' in $\left[e_{\beta_k} \otimes t_1^r, x_1^{\bar{d}_1} \dots x_m^{\bar{d}_m}\right]_{d_{\mathcal{Z}}}$ is easily seen to be equal to $-A'$. Further, for any $k+1 \leq j \leq m$, let

$$x_j = h_j \otimes t_1^{p_j} t_2^{q_j}.$$

If $(p_j, q_j) \neq (p_k, q_k)$, then, of course, the component of A' in $\left[e_{\beta_k} \otimes t_1^r, B'_j\right]_{d_{\mathcal{Z}}}$ is clearly zero. But, if $(p_j, q_j) = (p_k, q_k)$, then $h_j \neq h_k$ and hence $\beta_k(h_j) = 0$. Thus, again the coefficient of A' in $\left[e_{\beta_k} \otimes t_1^r, B'_j\right]_{d_{\mathcal{Z}}}$ is zero. Hence, the coefficient of A' in $\left[e_{\beta_k} \otimes t_1^r, \mathcal{Z}^o\right]_{d_{\mathcal{Z}}}$ is nonzero, proving the claim (3.4). Thus, we have proved the theorem in this case as well. This completes the proof of the theorem. □

Remark 6. The above proof can easily be adapted to prove the following generalization of Theorem 5:

Take $k \geq 2$. Then, for any non-constant element $\mathcal{Z} \in U\left(\hat{\mathfrak{g}}_k^-\right)$, there exists $p_o > 0$ such that $\left[\mathfrak{g} \otimes t_{k-1}^r, \mathcal{Z}\right] \neq 0$ for all $r \geq p_o$.

Let λ be a dominant integral weight of $\mathfrak{g}$ (i.e., $\lambda \in X_+$) and let $V(\lambda)$ be the corresponding Weyl (=irreducible) module of $\mathfrak{g}$ with highest weight λ. Realize $V(\lambda)$ as a $\hat{\mathfrak{g}}_1^+$-module by letting $\mathfrak{g} \otimes t_1^d$ act trivially for any $d > 0$ and c_1 to act by $-p - h^\vee$ (where $h^\vee$ is the dual Coxeter number of $\mathfrak{g}$). We let d_1 to act trivially on $V(\lambda)$. Define

$$\hat{M}(\lambda) := \operatorname{Ind}_{U(\hat{\mathfrak{g}}_1^+)}^{U(\hat{\mathfrak{g}}_1)} (V(\lambda))$$

and $\hat{L}(\lambda)$ its $\hat{\mathfrak{g}}_1$-module (unique) irreducible quotient (cf. Lemma 4). Observe that for any $v_o \in \hat{M}(\lambda)$ (and hence for $\hat{L}(\lambda)$) there exists $p_o > 0$ (depending upon v_o) such that

$$(\mathfrak{g} \otimes t_1^r) \cdot v_o = 0 \quad \text{for all} \quad r \geq p_o.$$

As a corollary of Theorem 5, we get the following.

Corollary 7. *Let $\hat{L}(\lambda)$ be an irreducible $\hat{\mathfrak{g}}_1$-module as above. Then, for any proper $\hat{\mathfrak{g}}_2$-submodule*

$$M \subsetneq Ind_{U(\hat{\mathfrak{g}}_2^+)}^{U(\hat{\mathfrak{g}}_2)}\left(\hat{L}(\lambda)\right)$$

and for any nonzero vector $v \in M$, there exists $p_o = p_o(v) > 0$ such that

$$(\mathfrak{g} \otimes t_1^r) \cdot v \neq 0 \quad \text{for all } r \geq p_o.$$

Proof. Write $v = \sum_{i=1}^N z_i \otimes \omega_i$, for $z_i \in U\left(\hat{\mathfrak{g}}_2^-\right)$ and some linearly independent elements $\omega_i \in \hat{L}(\lambda)$. Then, for any $x \in \mathfrak{g}$,

$$\begin{aligned}(x \otimes t_1^r) \cdot v &= \sum_i (x \otimes t_1^r)\, z_i \otimes \omega_i \\ &= \sum_i [x \otimes t_1^r, z_i] \otimes \omega_i + \sum_i z_i \otimes (x \otimes t_1^r) \cdot \omega_i \\ &= \sum_i [x \otimes t_1^r, z_i] \otimes \omega_i, \quad \text{for } r >> 0.\end{aligned}$$

But, z_i is not a scalar for some i since M is a proper submodule of $\mathrm{Ind}_{U(\hat{\mathfrak{g}}_2^+)}^{U(\hat{\mathfrak{g}}_2)}\left(\hat{L}(\lambda)\right)$ and $\hat{L}(\lambda)$ is an irreducible $\hat{\mathfrak{g}}_1$-module. Thus, the corollary follows from Theorem 5. □

Let $\hat{\mathscr{E}}^2_{\lambda,2}$ be the $\hat{\mathfrak{g}}_2$-module irreducible quotient of $\mathrm{Ind}_{U(\hat{\mathfrak{g}}_2^+)}^{U(\hat{\mathfrak{g}}_2)}\left(\hat{L}(\lambda)\right)$ (cf. Lemma 4). For any integers m, n, let $\hat{\mathscr{E}}^2_{\lambda,2}(m,n)$ be the same module as $\hat{\mathscr{E}}^2_{\lambda,2}$ except that we shift the d_1, d_2 weights of the highest weight vector $v_\lambda \in \hat{L}(\lambda)$ to (m,n) respectively and extend the d_1, d_2 action compatibly to $\hat{\mathscr{E}}^2_{\lambda,2}$.

As a corollary of Corollary 7, we get the following.

Corollary 8. *Let $\hat{L}(\lambda)$ be an irreducible $\hat{\mathfrak{g}}_1$-module as above (for $\lambda \in X_+$). Then, if $M \subset Ind_{U(\hat{\mathfrak{g}}_2^+)}^{U(\hat{\mathfrak{g}}_2)}\left(\hat{L}(\lambda)\right)$ is an irreducible proper $\hat{\mathfrak{g}}_2$-submodule, then M cannot be isomorphic as a $\hat{\mathfrak{g}}_2$-module with the irreducible $\hat{\mathfrak{g}}_2$-module $\hat{\mathscr{E}}^2_{\mu,2}(m,n)$ for any $\mu \in X_+$ and any pair of integers (m,n).*

Thus, if $Ind_{U(\hat{\mathfrak{g}}_2^+)}^{U(\hat{\mathfrak{g}}_2)}\left(\hat{L}(\lambda)\right)$ has finite length but not irreducible; in particular, it has an irreducible proper $\hat{\mathfrak{g}}_2$-submodule M, then M **cannot** *be $\hat{\mathfrak{g}}_2$-module isomorphic with $\hat{\mathscr{E}}^2_{\mu,2}(m,n)$ for any $\mu \in X_+$ and any pair of integers (m,n).*

Proof. Assume, if possible, that there is a $\hat{\mathfrak{g}}_2$-module isomorphism

$$\Phi : M \to \hat{\mathscr{E}}^2_{\mu,2}(m,n) \quad \text{for some } (m,n) \in \mathbb{Z}^2.$$

Let $M^o \subset M$ be the eigenspace of d_2 with the maximum d_2-eigenvalue say $\alpha_o \leq 0$. (Observe that the d_2-eigenvalues of $\mathrm{Ind}_{U(\hat{\mathfrak{g}}_2^+)}^{U(\hat{\mathfrak{g}}_2)}\left(\hat{L}(\lambda)\right)$ are ≤ 0.) Let $m_\mu = \Phi^{-1}(v_\mu)$, where v_μ is the highest weight vector of $\hat{L}(\mu)$. Then, $m_\mu \in M^o$. Now, we assign d_1, d_2-eigenvalues of v_μ to be that of m_μ. Clearly, the d_2-eigenspace of $\hat{\mathscr{E}}^2_{\mu,2}(m,n)$ with maximum eigenvalue is $\hat{L}(\mu)$. Thus, the isomorphism Φ would induce an isomorphism

$$\Phi^o : M^o \xrightarrow{\sim} \hat{L}(\mu) \ \text{ as } \hat{\mathfrak{g}}_2^+\text{-modules}.$$

By Corollary 7, any nonzero vector $v^o \in M^o$ satisfies:

$$(\mathfrak{g} \otimes t_1^r) \cdot v^o \neq 0 \quad \text{for all } r >> 0. \tag{3.5}$$

But, since $\hat{L}(\mu)$ has d_1-eigenvalues bounded above, no vector in $\hat{L}(\mu)$ satisfies (3.5). This contradicts the existence of Φ, proving the corollary. □

Remark 9. The above argument can easily be extended to prove similar negative results for $\hat{\mathscr{E}}^{k'}_{\lambda,k}$ for $k \geq 2$ and $0 \leq k' < k$ as long as $\hat{\mathscr{E}}^{k'}_{\lambda,k}$ is not irreducible $\hat{\mathfrak{g}}_k$-module.

4. Questions Q_1 and Q_2 for multi-loop algebras

Proposition 10. *Let $k \geq 1$. Assume that $\hat{\mathscr{E}}^{k-1}_{\lambda,k-1}$ is an irreducible $\tilde{\mathfrak{g}}_{k-1}$-module. Let $\hat{M}_k := \hat{M}_k(\lambda) \subset \hat{\mathscr{E}}^{0}_{\lambda,k}$ be the unique proper maximal $\hat{\mathfrak{g}}_k$-submodule (cf. Lemma 4). Then, $\hat{M}_k$ is also maximal as a $\tilde{\mathfrak{g}}_k$-submodule.*

Thus, $\hat{\mathscr{E}}^{k}_{\lambda,k}$ is an irreducible $\tilde{\mathfrak{g}}_k$-module as well.

Hence, by induction, each of $\hat{\mathscr{E}}^{k}_{\lambda,k}$ is an irreducible $\tilde{\mathfrak{g}}_k$-module for any $k \geq 0$. Thus, for any $k \geq 0$, taking the preferred choice for $\mathscr{E}^{k}_{\lambda,k}$ to be $\hat{\mathscr{E}}^{k}_{\lambda,k}$,

$$\mathscr{E}^{k'}_{\lambda,k} = \hat{\mathscr{E}}^{k'}_{\lambda,k}, \text{ for any } \lambda \in X_+ \text{ and } 0 \leq k' \leq k.$$

Proof. Let $\hat{M}_k \subset \tilde{M}_k \subsetneq \hat{\mathscr{E}}^{0}_{\lambda,k}$ be a $\tilde{\mathfrak{g}}_k$-submodule. Let $\hat{L}_k = \hat{\mathscr{E}}^{k}_{\lambda,k}$ be the $\hat{\mathfrak{g}}_k$-module irreducible quotient $\hat{\mathscr{E}}^{0}_{\lambda,k}/\hat{M}_k$. Then, $\tilde{N}_k := \frac{\tilde{M}_k}{\hat{M}_k} \subsetneq \hat{L}_k$ is a $\tilde{\mathfrak{g}}_k$-submodule. Assume, if possible, that $\tilde{N}_k$ is nonzero. Observe that d_k acts on $\hat{L}_k$ with eigenvalues bounded above by 0. For nonzero $v \in \tilde{N}_k$, decompose it as a sum of eigenvectors under d_k:

$$v = \sum_{i \geq 0} v_i, \text{ where } d_k v_i = -i v_i. \tag{4.6}$$

(Since $\hat{L}_k$ is a $\hat{\mathfrak{g}}_k$-module, each $v_i \in \hat{L}_k$.) Now, define

$$|v|_k = \{\Sigma i : v_i \neq 0\}. \tag{4.7}$$

Choose a nonzero $\overset{o}{v} \in \tilde{N}_k$ such that $|\overset{o}{v}|_k \leq |v|_k$, for all nonzero $v \in \tilde{N}_k$. If $|\overset{o}{v}|_k > 0$, then at least one of the components $\overset{o}{v}_{i_o} \neq 0$ with $i_o > 0$. For any $x \in \mathfrak{g}$ and $\mathbf{p} = (p_1, \dots, p_k) \in \mathbb{Z}^{k-1} \times \mathbb{Z}_{>0}$, $(x \otimes \mathbf{t}^{\mathbf{p}}) \cdot \overset{o}{v} = 0$, because of the minimality of $|\overset{o}{v}|_k$ in $\tilde{N}_k$. In particular,

$$(x \otimes \mathbf{t}^{\mathbf{p}}) \cdot \overset{o}{v}_{i_o} = 0.$$

Thus, the $\hat{\mathfrak{g}}_k$-submodule $\hat{Q}_k \subset \hat{L}_k$ generated by $\overset{o}{v}_{i_o}$ satisfies:

$$|v|_k \geq |\overset{o}{v}_{i_o}|_k = i_o > 0, \text{ for all } v \in \hat{Q}_k.$$

But, clearly $\hat{L}_k$ contains nonzero elements v' with $|v'|_k = 0$ coming from the image of $1 \otimes \hat{\mathscr{E}}^{0}_{\lambda,k-1}$ under the projection $\hat{\mathscr{E}}^{0}_{\lambda,k} \to \hat{L}_k$. Hence, the $\hat{\mathfrak{g}}_k$-submodule

$$0 \neq \hat{Q}_k \subsetneq \hat{L}_k.$$

This contradicts the $\hat{\mathfrak{g}}_k$-module irreducibility of $\hat{L}_k$ and hence $\tilde{N}_k = 0$ in the case $|\overset{o}{v}|_k > 0$.

So, assume now that $|\overset{o}{v}|_k = 0$. Thus, $\overset{o}{v}$ itself is an d_k-eigenvetor of eigenvalue 0. But, the d_k-eigenspace of eigenvalue 0 in $\hat{L}_k$ clearly is equal to the image of $1 \otimes \hat{\mathscr{E}}^{0}_{\lambda,k-1}$ under the projection $\hat{\mathscr{E}}^{0}_{\lambda,k} \to \hat{L}_k$. In fact, $\hat{L}_k$ being the unique $\hat{\mathfrak{g}}_k$-module irreducible quotient of $\hat{\mathscr{E}}^{0}_{\lambda,k}$, it is also an irreducible quotient:

$$\hat{\mathscr{E}}^{k-1}_{\lambda,k} \twoheadrightarrow \hat{L}_k,$$

and d_k-eigenspace of eigenvalue 0 in $\hat{L}_k$ is equal to the image of $1 \otimes \hat{\mathscr{E}}^{k-1}_{\lambda,k-1}$. Since $\hat{\mathscr{E}}^{k-1}_{\lambda,k-1}$ is an irreducible $\tilde{\mathfrak{g}}_{k-1}$-module by assumption, we get that $\tilde{\mathfrak{g}}_{k-1}$-submodule of $\tilde{N}_k$ generated by $\overset{o}{v}$ must contain the full image $\hat{L}_\mu$ of $1 \otimes \hat{\mathscr{E}}^{k-1}_{\lambda,k-1}$ in $\hat{L}_k$. But, then

$$\tilde{N}_k = \hat{L}_k.$$

This is a contradiction, since $\tilde{N}_k$ was assumed to be properly contained in $\hat{L}_k$. Hence, $\tilde{N}_k = 0$ in this case as well. This complete the proof of the proposition. □

Lemma 11. *Any $\tilde{\mathfrak{g}}_k$-module automorphism φ of $\hat{\mathscr{E}}^k_{\lambda,k}$ is the identity automorphism up to a nonzero scalar (for any $k \geq 0$). In particular, it is a $\hat{\mathfrak{g}}_k$-module automorphism.*

Proof. The lemma is clearly true for $k = 0$ (by Schur's Lemma). By induction, assume the validity of the lemma for $k-1$ and take $k \geq 1$. Let

$$M^o = \left\{ v \in \hat{\mathscr{E}}^k_{\lambda,k} : (\mathfrak{g} \otimes \mathbf{t}^{\mathbf{p}}) \cdot v = 0 \quad \text{for all } \mathbf{p} \in \mathbb{Z}^{k-1} \times \mathbb{Z}_{>0} \right\}.$$

Clearly, M^o is stable under the action of d_k (in fact, M^o is stable under the action of each d_i, $1 \leq i \leq k$). Assume, if possible, that M^o contains a nonzero d_k-eignevector $\overset{o}{v}$ of eigenvalue < 0. Then, the $\hat{\mathfrak{g}}_k$-submodule $\overset{o}{V}$ of $\hat{\mathscr{E}}^k_{\lambda,k}$ generated by $\overset{o}{v}$ clearly satisfies:

$$\overset{o}{V} \underset{\neq}{\subset} \hat{\mathscr{E}}^k_{\lambda,k}.$$

This is a contradiction since $\hat{\mathscr{E}}^k_{\lambda,k}$ is an irreducible $\hat{\mathfrak{g}}_k$-module. Thus, M^o = Image of $1 \otimes \hat{\mathscr{E}}^{k-1}_{\lambda,k-1}$ under the projection $\hat{\mathscr{E}}^{k-1}_{\lambda,k} \twoheadrightarrow \hat{\mathscr{E}}^k_{\lambda,k}$. Hence, the automorphism φ of $\hat{\mathscr{E}}^k_{\lambda,k}$ restricts to a $\tilde{\mathfrak{g}}_{k-1}$-module automorphism of $1 \otimes \hat{\mathscr{E}}^{k-1}_{\lambda,k-1}$ (which is the identity automorphism up to a scalar by induction). This proves the lemma since $\hat{\mathscr{E}}^k_{\lambda,k}$ is generated by the image of $1 \otimes \hat{\mathscr{E}}^{k-1}_{\lambda,k-1}$ as a $\tilde{\mathfrak{g}}_k$-module. □

Theorem 12. *Let $k \geq 1$ and let $M \subset \hat{\mathscr{E}}^{k-1}_{\lambda,k}$ be an irreducble $\hat{\mathfrak{g}}^{(k)}_k$-submodule, where $\hat{\mathfrak{g}}^{(k)}_k \subset \hat{\mathfrak{g}}_k$ is the Lie subalgebra $\tilde{\mathfrak{g}}_k \oplus \mathbb{C}d_k$. Then, M is an irreducible $\tilde{\mathfrak{g}}_k$-module.*

Moreover, for $k \geq 2$, if $\hat{\mathfrak{g}}^{(k)}_k$-irredicible $M \underset{\neq}{\subset} \hat{\mathscr{E}}^{k-1}_{\lambda,k}$, then M cannot be isomorphic with $\hat{\mathscr{E}}^k_{\mu,k}$ as $\tilde{\mathfrak{g}}_k$-modules for any $\mu \in X_+$.

Proof. We first prove that M is an irreducible $\tilde{\mathfrak{g}}_k$-module:

Let $d\,(\leq 0)$ be the largest d_k-eignevalue of d_k-eigenvectors in M. Let us renormalize the d_k-eigenvalues by subtracting d so that the largest d_k-eigenvalue of M becomes 0. Let $N \subset M$ be a nonzero $\tilde{\mathfrak{g}}_k$-submodule. Recall the definition of $|v|_k$ for nonzero vectors from the identity (4.7) of the proof of Proposition 10. Choose nonzero $\overset{o}{v} \in N$ such that

$$|\overset{o}{v}|_k \leq |v|_k \quad \text{for all nonzero } v \in N. \tag{4.8}$$

Then, for any $x \in \mathfrak{g}$ and $\mathbf{p} = (p_1, \ldots, p_k) \in \mathbb{Z}^{k-1} \times \mathbb{Z}_{>0}$, by (4.8),

$$(x \otimes \mathbf{t}^{\mathbf{p}}) \cdot \overset{o}{v} = 0. \tag{4.9}$$

If $|\overset{o}{v}|_k > 0$, choose a nonzero d_k-eigen component $\overset{o}{v}_{i_0}$ with $|\overset{o}{v}_{i_0}|_k = i_o > 0$. Then, by the equation (4.9), for any $x \in \mathfrak{g}$ and $\mathbf{p} \in \mathbb{Z}^{k-1} \times \mathbb{Z}_{>0}$,

$$(x \otimes \mathbf{t}^{\mathbf{p}}) \cdot \overset{o}{v}_{i_o} = 0.$$

Thus, the $\hat{\mathfrak{g}}_k^{(k)}$-submodule M' of M generated by $\overset{o}{v}_{i_o}$ is properly contained in M. This contradicts the irreducibility of M as a $\hat{\mathfrak{g}}_k^{(k)}$-module. Thus, we conclude that $|\overset{o}{v}|_k = 0$. In particular, $\overset{o}{v}$ is a d_k-eigenvector. Thus, the $\tilde{\mathfrak{g}}_k$-submodule of N generated by $\overset{o}{v}$ is d_k-stable. This forces

$$N = M$$

from the irreducibility of M as a $\hat{\mathfrak{g}}_k^{(k)}$-module. This proves that M is irreducible as a $\tilde{\mathfrak{g}}_k$-module as well.

We now prove the second part of the proposition:

If possible, assume that there exists a $\tilde{\mathfrak{g}}_k$-module isomorphism (for some $\mu \in X_+$):

$$\varphi : M \xrightarrow{\sim} \hat{\mathscr{E}}^k_{\mu,k}.$$

Let

$$M^o := \left\{ v \in M : (x \otimes \mathbf{t}^{\mathbf{p}}) \cdot v = 0 \quad \text{for all } x \in \mathfrak{g} \text{ and } \mathbf{p} \in \mathbb{Z}^{k-1} \times \mathbb{Z}_{>0} \right\},$$

and similarly define $\left(\hat{\mathscr{E}}^k_{\mu,k}\right)^o$. Then, φ induces an isomorphism:

$$M^o \simeq \left(\hat{\mathscr{E}}^k_{\mu,k}\right)^o.$$

Irreducibility of M as a $\hat{\mathfrak{g}}_k^{(k)}$-module forces $M^o = M_0$, and similarly for $\hat{\mathscr{E}}^k_{\mu,k}$, where M_0 is the d_k-eigenspace of M of eigenvalue 0. (Observe that $\hat{\mathscr{E}}^k_{\mu,k}$ is irreducible even as a $\tilde{\mathfrak{g}}_k$-module by Proposition 10.) Thus, we get an isomorphism of $\tilde{\mathfrak{g}}_{k-1}$-modules (by restricting φ):

$$\varphi_0 : M_0 \simeq 1 \otimes \hat{\mathscr{E}}^{k-1}_{\mu,k-1}.$$

For any nonzero $v \in M_0$, there exists $p_o = p_o(v) > 0$ such that

$$(\mathfrak{g} \otimes t^r_{k-1}) \cdot v \neq 0 \ \text{ for all } r \geq p_o. \tag{4.10}$$

This is proved for $k = 2$ in Corollary 7. (The proof of Corollary 7 works equally well if we only assume that M is a proper $\hat{\mathfrak{g}}_2^{(2)}$-submodule.) The proof for any $k \geq 2$ is the same by using Remark 6 and the proof of Corollary 7. However, for any vector w in $\hat{\mathscr{E}}^{k-1}_{\mu,k-1}$ (by the definition)

$$(\mathfrak{g} \otimes t^r_{k-1}) \cdot w = 0 \ \text{ for all } r >> 0. \tag{4.11}$$

The identities (4.10) and (4.11) contradict the existence of the $\tilde{\mathfrak{g}}_{k-1}$-module isomorphism φ_0. This concludes the proof that M cannot be $\tilde{\mathfrak{g}}_k$-module isomorphic with $\hat{\mathscr{E}}^k_{\mu,k}$ (for any $\mu \in X_+$). □

Corollary 13. *Lusztig's question Q_2 (cf. Introduction) has negative answer for any $\mathfrak{g}$ and any $k \geq 2$. Specifically, $\mathscr{E}^{k-1}_{\lambda,k}$ (and hence any $\mathscr{E}^{k'}_{\lambda,k}$ for $k' < k$) does not belong to the category $\mathscr{C}_k$ unless $\mathscr{E}^{k-1}_{\lambda,k}$ is irreducible as a $\tilde{\mathfrak{g}}_k$-module. (Observe that $\mathscr{E}^{k'}_{\lambda,k} = \hat{\mathscr{E}}^{k'}_{\lambda,k}$ as in Proposition 10.)*

Proof. Assume that $\hat{\mathscr{E}}^{k-1}_{\lambda,k}$ has finite length as a $\tilde{\mathfrak{g}}_k$-module; in particular, it would have finite length as $\hat{\mathfrak{g}}_k^{(k)}$-module. This would insure the existence of an irreducible $\hat{\mathfrak{g}}_k^{(k)}$-submodule $M \subset \hat{\mathscr{E}}^{k-1}_{\lambda,k}$. By Theorem 12, M would be an irreducible $\tilde{\mathfrak{g}}_k$-submodule of $\hat{\mathscr{E}}^{k-1}_{\lambda,k}$; in particular, a $\tilde{\mathfrak{g}}_k$-module composition factor of $\hat{\mathscr{E}}^{k-1}_{\lambda,k}$. Thus, if $\hat{\mathscr{E}}^{k-1}_{\lambda,k}$ is not irreducible as $\tilde{\mathfrak{g}}_k$-module, then $M \subsetneq \hat{\mathscr{E}}^{k-1}_{\lambda,k}$. But, then by Theorem 12, M is not $\tilde{\mathfrak{g}}_k$-module isomorphic with $\hat{\mathscr{E}}^k_{\mu,k}$ (for any $\mu \in X_+$). This shows that Question Q_2 has negative answer. □

5. Sugawara operator for multiloop algebras

For $\mathbf{n} = (n_1, \dots n_k) \in \mathbb{Z}^k$ also written as $\mathbf{n} = n_1\delta_1 + \cdots + n_k\delta_k$, denote

$$\mathbf{n}' = \mathbf{n} - n_k\delta_k.$$

For $x \in \mathfrak{g}$ and $\mathbf{n} \in \mathbb{Z}^k$, denote

$$x(\mathbf{n}) = x \otimes \mathbf{t}^{\mathbf{n}} \in \tilde{\mathfrak{g}}_k.$$

We abbreviate $d\delta_k$ simply by d. Thus, $x(\mathbf{n} + d) := x \otimes (\mathbf{t}^{\mathbf{n}} \cdot t_k^d)$.

Define the *Sugawara Operator*

$$L_0^{(k)} := \frac{1}{2}\sum_j \sum_{d\in\mathbb{Z}} : e_j(-d) \cdot e^j(d) : \in \hat{U}\left((\mathfrak{g} \otimes t_k^{\pm 1}) \oplus \mathbb{C}c_k\right) \subset \hat{U}(\tilde{\mathfrak{g}}_k),$$

where $\{e_j\}$ is a basis of $\mathfrak{g}$, $\left\{e^j\right\}$ is the dual basis with respect to the normalized invariant form $\langle\, ,\rangle$ on $\mathfrak{g}$ (as in the Introduction), and $\hat{U}$ is the standard completion along the positive root spaces as in, e.g., [K, Definition 1.5.8]. Even though we have not used the following theorem in rest of the paper, we include it for its future use. This extends the corresponding well-known theorem for $k = 1$ (cf. [KRR, Proposition 10.1]).

Theorem 14. *Let V be a $\hat{\mathfrak{g}}_k^{(k)}$-module which has d_k-eigenvalues bounded above (cf. Theorem 12 for the definition of the Lie algebra $\hat{\mathfrak{g}}_k^{(k)}$). Then, for $x \in \mathfrak{g}$ and $\mathbf{n} \in \mathbb{Z}^k$, such that $\mathbf{n}' \neq \mathbf{0}$, $\left[x(\mathbf{n}), L_0^{(k)}\right]$ as an operator on V is given by :*

$$\begin{aligned}\left[x(\mathbf{n}), L_0^{(k)}\right] \;=\; & \sum_j \left(-\sum_{d<\frac{n_k}{2}} e_j\left(\mathbf{n}' + d\right) \cdot \left[x, e^j\right](n_k - d) + \sum_{d<\frac{n_k}{2}} e_j(d) \cdot \left[x, e^j\right](\mathbf{n} - d)\right) \\ & \sum_j \left(-\frac{1}{2} e_j\left(\mathbf{n}' + \frac{n_k}{2}\right) \cdot \left[x, e^j\right]\left(\frac{n_k}{2}\right) + \frac{1}{2} e_j\left(\frac{n_k}{2}\right) \cdot \left[x, e^j\right]\left(\mathbf{n} - \frac{n_k}{2}\right)\right) + h^\vee n_k x(\mathbf{n}),\end{aligned}$$

where we follow the convention that $x\left(\frac{n}{2}\right) = 0$ for odd n.

Proof. Let $\psi : \mathbb{R} \to \{0, 1\}$ be the cut-off function defined by

$$\begin{aligned}\psi(t) &= 1 \qquad \text{for} \quad |t| \leq 1 \\ &= 0 \qquad \text{otherwise.}\end{aligned}$$

Define the element

$$L_0^{(k)}(\epsilon) = \frac{1}{2}\sum_j \sum_{d\in\mathbb{Z}} e_j(-d) \cdot e^j(d)\psi(\epsilon d) \in U\left((\mathfrak{g} \otimes t_k^{\pm 1}) \oplus \mathbb{C}c_k\right).$$

Assume that $\mathbf{n}' \neq \mathbf{0}$ (which is an assumption in the theorem). Then,

$$\begin{aligned}
2\left[x(\mathbf{n}), L_0^{(k)}(\epsilon)\right] &= \sum_j \left(\sum_{\frac{n_k}{2} \leq d} \left[x, e_j\right](\mathbf{n} - d) \cdot e^j(d)\psi(\epsilon d) + \sum_{\frac{n_k}{2} > d} e^j(d) \cdot \left[x, e_j\right](\mathbf{n} - d)\, \psi(\epsilon d) \right) \\
&+ \sum_j \left(\sum_{\frac{n_k}{2} > d} \left[\left[x, e_j\right](\mathbf{n} - d), e^j(d)\right] \psi(\epsilon d) + \sum_{-\frac{n_k}{2} \leq d} e_j(-d) \cdot \left[x, e^j\right](\mathbf{n} + d)\, \psi(\epsilon d) \right) \\
&+ \sum_j \left(\sum_{-\frac{n_k}{2} > d} \left[x, e^j\right](\mathbf{n} + d) \cdot e_j(-d)\psi(\epsilon d) + \sum_{-\frac{n_k}{2} > d} \left[e_j(-d), \left[x, e^j\right](\mathbf{n} + d)\right] \psi(\epsilon d) \right) \\
&= \sum_j \left(- \sum_{-\frac{n_k}{2} \leq d} e_j(\mathbf{n}' - d) \cdot \left[x, e^j\right](n_k + d)\, \psi(\epsilon(d + n_k)) \right) \\
&+ \sum_j \left(\sum_{-\frac{n_k}{2} \leq d} e_j(-d) \cdot \left[x, e^j\right](\mathbf{n} + d)\, \psi(\epsilon d) + \sum_{\frac{n_k}{2} > d} e^j(d) \cdot \left[x, e_j\right](\mathbf{n} - d)\, \psi(\epsilon d) \right) \\
&\sum_j \left(- \sum_{\frac{n_k}{2} > d} e^j(\mathbf{n}' + d) \cdot \left[x, e_j\right](-d + n_k)\, \psi(\epsilon(d - n_k)) \right) + 2h^\vee x(\mathbf{n}) \sum_{-\frac{n_k}{2} \leq d < \frac{n_k}{2}} \psi(\epsilon d),
\end{aligned}$$

since $\sum_j [[x, e_j], e^j] = 2h^\vee x$, cf. [GW, Lemma 3.3.8] as $h^\vee := 1 + \langle \rho, \theta^\vee \rangle$

$$\begin{aligned}
&= \sum_j \left(- \sum_{d \leq \frac{n_k}{2}} e_j(\mathbf{n}' + d) \cdot \left[x, e^j\right](n_k - d)\, \psi(\epsilon(-d + n_k)) \right) \\
&+ \sum_j \left(\sum_{d < \frac{n_k}{2}} e_j(d) \cdot \left[x, e^j\right](\mathbf{n} - d)\, \psi(\epsilon d) + \sum_{d \leq \frac{n_k}{2}} e_j(d) \cdot \left[x, e^j\right](\mathbf{n} - d)\, \psi(\epsilon d) \right) \\
&\sum_j \left(- \sum_{d < \frac{n_k}{2}} e_j(\mathbf{n}' + d) \cdot \left[x, e^j\right](-d + n_k)\, \psi(\epsilon(d - n_k)) \right) + 2h^\vee x(\mathbf{n}) \sum_{-\frac{n_k}{2} \leq d < \frac{n_k}{2}} \psi(\epsilon d)
\end{aligned}$$

$$\begin{aligned}
&= \sum_j \left(-2 \sum_{d < \frac{n_k}{2}} e_j(\mathbf{n}' + d) \cdot \left[x, e^j\right](n_k - d)\, \psi(\epsilon(-d + n_k)) \right) \\
&\sum_j \left(2 \sum_{d < \frac{n_k}{2}} e_j(d) \cdot \left[x, e^j\right](\mathbf{n} - d)\, \psi(\epsilon d) - e_j\left(\mathbf{n}' + \frac{n_k}{2}\right) \cdot \left[x, e^j\right]\left(\frac{n_k}{2}\right) \psi\left(\epsilon\left(\frac{n_k}{2}\right)\right) \right) \\
&\sum_j \left(e_j\left(\frac{n_k}{2}\right) \cdot \left[x, e^j\right]\left(\mathbf{n} - \frac{n_k}{2}\right) \psi\left(\epsilon \frac{n_k}{2}\right) \right) + 2h^\vee x(\mathbf{n}) \sum_{-\frac{n_k}{2} \leq d < \frac{n_k}{2}} \psi(\epsilon d),
\end{aligned} \tag{5.12}$$

since ψ is symmetric. We have used the following relation in the above equation.

$$[x, \sum_j e_j \otimes e^j] = \sum_j \left[x, e_j\right] \otimes e^j + \sum_j e_j \otimes \left[x, e^j\right] = 0 \quad \text{in} \quad \mathfrak{g} \otimes \mathfrak{g}.$$

Taking limit as $\epsilon \to 0$ in the equation (5.12), we get

$$\begin{aligned}\left[x(\mathbf{n}), \varinjlim_{\epsilon\to 0} L_0^{(k)}(\epsilon)\right] &= \sum_j \left(-\sum_{d<\frac{n_k}{2}} e_j(\mathbf{n}'+d)\cdot\left[x, e^j\right](n_k - d) + \sum_{d<\frac{n_k}{2}} e_j(d)\cdot\left[x, e^j\right](\mathbf{n}-d)\right) \\ &\quad \sum_j \left(-\frac{1}{2}e_j\left(\mathbf{n}'+\frac{n_k}{2}\right)\cdot\left[x, e^j\right]\left(\frac{n_k}{2}\right) + \frac{1}{2}e_j\left(\frac{n_k}{2}\right)\cdot\left[x, e^j\right]\left(\mathbf{n}-\frac{n_k}{2}\right)\right) \\ &\quad + h^\vee n_k x(\mathbf{n}), \end{aligned} \tag{5.13}$$

since, in all the above summands, the d_k-eigenvalues of the first term is not more than the d_k-eigenvalues of the second term. Now,

$$\begin{aligned} 2L_0^{(k)}(\epsilon) &= \sum_j \sum_{d\in\mathbb{Z}} e_j(-d)\cdot e^j(d)\psi(\epsilon d) \\ &= \sum_j \left(\sum_{d\geq 0} e_j(-d)\cdot e^j(d)\psi(\epsilon d) + \sum_{d<0} e^j(d)\cdot e_j(-d)\psi(\epsilon d)\right) \\ &\quad \sum_j \left(\sum_{d<0}\left(\left[e_j, e^j\right] - dc_k\right)\psi(\epsilon d)\right) \\ &= \sum_j \left(\sum_{d\geq 0} e_j(-d)\cdot e^j(d)\psi(\epsilon d) + \sum_{d<0} e^j(d)\cdot e_j(-d)\psi(\epsilon d)\right) \\ &\quad -\sum_j \left(\sum_{d<0} dc_k\psi(\epsilon d)\right) \end{aligned} \tag{5.14}$$

But, c_k commutes with $x(\mathbf{n})$. Hence,

$$\left[x(\mathbf{n}), \varinjlim_{\epsilon\to 0} L_0^{(k)}(\epsilon)\right] = \left[x(\mathbf{n}), \frac{1}{2}\sum_j \sum_{d\in\mathbb{Z}} : e_j(-d)\cdot e^j(d) :\right]. \tag{5.15}$$

Combining the equations (5.13) and (5.15), we get the theorem. □

Remark 15. If $\mathbf{n}' = \mathbf{0}$ in the above theorem, then $\left[x(\mathbf{n}), L_0^{(k)}\right]$ as an operator on V is given by the classical formula given in [KRR, Proposition 10.1].

References

[AJS] H. Andersen, J. Jantzen and W. Soergel: Representations of quantum groups at a p-th root of unity and of semisimple groups in characteristic p : independence of p, *Astérisque* **220**, 1-323 (1994).

[F] P. Fiebig, An upper bound on the exceptional characteristics for Lusztig's character formula, *Journal für die reine und angewandte Mathematik (Crelles Journal)* **673**, 1–31 (2012).

[GW] R. Goodman and N. Wallach, *Symmetry, Representations, and Invariants*. Graduate Texts in Mathematics Vol. **255**, Springer (2009).

[KRR] V. Kac, A. Raina and N. Rezhkouskaya: *Bombay Lectures on highest weight representation of infinite dimensional Lie algebra (second edition)*, World Scientific (2013).

[KT] M. Kashiwara and Tanisaki: Kazhdan-Lusztig conjecture for affine Lie algebras with negative level, *Duke Math. J.* **77**, 21– 62 (1995).

[KL0] D. Kazhdan and G. Lusztig: Affine Lie algebras and quantum groups, *International Math. Res. Notices* **2**, 21–29 (1991).

[KL1] D. Kazhdan and G. Lusztig: Tensor structures arising from affine Lie algebras I, *J. Amer. Math. Soc.* **6**, 905–947 (1993).

[KL2] D. Kazhdan and G. Lusztig: Tensor structures arising from affine Lie algebras II, *J. Amer. Math. Soc.* **6**, 949–1011 (1993).

[KL3] D. Kazhdan and G. Lusztig: Tensor structures arising from affine Lie algebras III, *J. Amer. Math. Soc.* **7**, 335–381 (1994).

[KL4] D. Kazhdan and G. Lusztig: Tensor structures arising from affine Lie algebras IV, *J. Amer. Math. Soc.* **7**, 383–453 (1994).

[K] S. Kumar, *Kac-Moody Groups, their Flag Varieties and Representation Theory*. Progress in Mathematics Vol. **204**, Birkhäuser (2002).

[Lu1] G. Lusztig: Some problems in the representation theory of finite Chevalley groups, *Proc. Symp. Pure Math.* **37**, Amer. Math. Soc., 313–317 (1980).

[Lu2] G. Lusztig: On the character of certain irreducible modular representations, *Representation Theory, AMS* **19**, 3–8 (2015).

[Lu3] G. Lusztig: Comments on my papers, paper [229], arXiv:1707.09368 (v7) (2021).

S. Kumar: Department of Mathematics, University of North Carolina, Chapel Hill, NC 27599-3250, USA
E-mail address: `shrawan@email.unc.edu`