Subalgebra generated by ad-locally nilpotent elements of Borcherds Generalized Kac-Moody Lie algebras

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ABSTRACT

We determine the Lie subalgebra $\mathfrak{g}_{nil}$ of a Borcherds symmetrizable generalized Kac-Moody Lie algebra $\mathfrak{g}$ generated by ad-locally nilpotent elements and show that it is ‘essentially’ the same as the Levi subalgebra of $\mathfrak{g}$ with its simple roots precisely the real simple roots of $\mathfrak{g}$.

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1. Introduction

Let $\mathfrak{g} = \mathfrak{g}(A)$ be the symmetrizable Generalized Kac-Moody (GKM) algebra associated to a $\ell \times \ell$ matrix $A$ (cf. Section 2). Let

$$\mathfrak{g}_{nil}^o := \{ x \in \mathfrak{g} : \text{ad } x \text{ acts locally nilpotently on } \mathfrak{g} \},$$

and let $\mathfrak{g}_{nil} \subset \mathfrak{g}$ be the Lie subalgebra generated by $\mathfrak{g}_{nil}^o$. Then, we prove the following theorem (cf. Theorem 3.1):

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Theorem. Let $g = g(A)$ be as above, where $\ell \geq 2$ and $A$ is indecomposable, i.e., the corresponding Dynkin diagram is connected. Then,

$$g'(B) \subset g_{nil} \subset g'(B) + h,$$

where $B \subset A$ is the submatrix parameterized by those $i$ such that $a_{i,i} = 2$, $h$ is the Cartan subalgebra and $g'(B)$ is the derived subalgebra of $g(B)$.

As shown in Remark 3.2, the assumption $\ell \geq 2$ in the above theorem is necessary in general.

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2. Basic definition

In this section, we recall the definition of Borcherds Generalized Kac-Moody Lie algebras (for short GKM algebras). For a more extensive treatment of $g$ and its properties, see Chapters 1, 11 of [3] and the papers [1] and [2].

Definition 2.1. Let $A = (a_{i,j})$ be a $\ell \times \ell$ matrix (for $\ell \geq 1$) with real entries, satisfying the following properties:

(P1) either $a_{i,i} = 2$ or $a_{i,i} \leq 0$,

(P2) $a_{i,j} \leq 0$ if $i \neq j$, and $a_{i,j} \in \mathbb{Z}$ if $a_{i,i} = 2$,

(P3) $a_{i,j} = 0$ if and only if $a_{j,i} = 0$.

Fix a realization of $A$, which is a triple $(h, \Pi, \Pi^\vee)$ consisting of a complex vector space $h$, $\Pi = \{\alpha_1, \ldots, \alpha_\ell\} \subset h^*$ and $\Pi^\vee = \{\alpha_1^\vee, \ldots, \alpha_\ell^\vee\} \subset h$ are indexed subsets, satisfying the following three conditions:

(Q1) both sets $\Pi$ and $\Pi^\vee$ are linearly independent,

(Q2) $\alpha_j(\alpha_i^\vee) = a_{i,j}$, for all $i, j$,

(Q3) $\ell - \text{rank } A = \dim h - \ell$.

By [3], Proposition 1.1, such a realization is unique up to an isomorphism of the triple.

Now, the Borcherds Generalized Kac-Moody Lie algebra (for short GKM algebra) $g(A)$ is defined as the Lie algebra generated by $\{e_i, f_i, h\}_{1 \leq i \leq \ell}$ subject to the following relations:

(R1) $[e_i, f_j] = \delta_{ij} \alpha_i^\vee$, for all $i$,

(R2) $[h, h'] = 0$, for all $h, h' \in h$,

(R3) $[h, e_i] = \alpha_i(h)e_i$; $[h, f_i] = -\alpha_i(h)f_i$, for all $1 \leq i \leq \ell$ and $h \in h$,.
(R4) \((\text{ad} e_i)^{1-a_{i,j}} e_j = (\text{ad} f_i)^{1-a_{i,j}} f_j = 0\), if \(a_{i,i} = 2\) and \(i \neq j\),
(R5) \([e_i, e_j] = [f_i, f_j] = 0\), if \(a_{i,j} = 0\).

The matrix \(A\) (or the Lie algebra \(\mathfrak{g}(A)\)) is called symmetrizable if there exists an invertible diagonal matrix \(D = \text{diag}(\epsilon_1, \ldots, \epsilon_\ell)\) such that the matrix \(DA\) is symmetric.

3. Main theorem and its proof

**Theorem 3.1.** Let \(\mathfrak{g} = \mathfrak{g}(A)\) be the symmetrizable GKM algebra associated to a \(\ell \times \ell\) matrix \(A\) as in the last section. Assume further that \(\ell \geq 2\) and \(A\) is indecomposable, i.e., the corresponding Dynkin diagram is connected. Let

\[
\mathfrak{g}_\text{nil}^o := \{x \in \mathfrak{g} : \text{ad} x \text{ acts locally nilpotently on } \mathfrak{g}\},
\]

and let \(\mathfrak{g}_\text{nil} \subset \mathfrak{g}\) be the Lie subalgebra generated by \(\mathfrak{g}_\text{nil}^o\). Then,

\[
\mathfrak{g}'(B) \subset \mathfrak{g}_\text{nil} \subset \mathfrak{g}'(B) + \mathfrak{h},
\]

where \(B \subset A\) is the submatrix parameterized by those \(i\) such that \(a_{i,i} = 2\), i.e., \(\alpha_i\) is a real root and \(\mathfrak{g}'(B)\) is the derived subalgebra of \(\mathfrak{g}(B)\).

**Proof.** Consider the \(\mathbb{Z}\)-gradation of \(\mathfrak{g}\) induced from a homomorphism \(\theta : Q := \bigoplus_i \mathbb{Z} \alpha_i \rightarrow \mathbb{Z}\). Then, for any \(x \in \mathfrak{g}_\text{nil}^o, x_+ (\theta) \in \mathfrak{g}_\text{nil}^o\), where \(x_+(\theta)\) is the top degree component of \(x\) in the \(\mathbb{Z}\)-gradation of \(\mathfrak{g}\) induced by \(\theta\). To prove this, observe that for any \(y \in \mathfrak{g}_\alpha\) (where \(\mathfrak{g}_\alpha\) is the root space corresponding to the root \(\alpha\) or 0),

\[
(\text{ad} x)^n(y) = (\text{ad} x_+(\theta))^n(y) + \text{lower degree terms}.
\]

Similarly, for \(x \in \mathfrak{g}_\text{nil}^o, x_- (\theta) \in \mathfrak{g}_\text{nil}^o\), where \(x_- (\theta)\) is the lowest degree component of \(x\).

Further, given any nonzero \(x \in \mathfrak{g}\), we can get a gradation \(\theta_x : \mathcal{Q} \rightarrow \mathbb{Z}\) as above (depending upon \(x\)) such that all the homogeneous degree components of \(x\) (under \(\theta_x\)) belong to root spaces \(\mathfrak{g}_\beta\). To prove this, write \(x = \sum_j x_{\beta_j}\), where \(\beta_j\) are distinct roots or zero, \(x_{\beta_j} \in \mathfrak{g}_{\beta_j}\) and each \(x_{\beta_j} \neq 0\). Consider the finite collection of weights: \(\{\beta_j - \beta_k\}_{j \neq k} \subset \mathfrak{h}^*\). Now, we can find a vector \(\gamma = \gamma_x \in \mathcal{Q}^\ell = \mathcal{Q} \otimes \mathbb{Z} \mathcal{Q}\) such that for the standard dot product \((\cdot, \cdot)\) in \(\mathcal{Q}^\ell\),

\[
\theta_x(\beta_j - \beta_k) = (\beta_j - \beta_k, \gamma) \neq 0, \text{ for any } j \neq k.
\]

To prove the above equation, consider the \((\ell - 1)\)-dimensional subspace \(V_{j,k} \subset \mathcal{Q}^\ell\) (for any \(j \neq k\)) perpendicular to \(\beta_j - \beta_k\). Since the collection \(\{\beta_j - \beta_k\}_{j \neq k}\) is finite, we can find a vector \(\gamma\) such that the equation (1) is satisfied. We can further take \(\gamma \in \mathcal{Q} \simeq \mathbb{Z}^\ell\) by clearing the denominators.
So, if $x \in g_{\text{nil}}^{\circ}$, then either $x$ belongs to the center $Z(g)$ of $g$ or the root component $x_{\beta} \in g_{\text{nil}}^{\circ}$ for some root $\beta$ ($\beta \neq 0$). (To prove this: if $x$ belongs to the Cartan subalgebra $h$, then it will have to lie in $Z(g)$ of $g$ by [3], Proposition 1.6. But, if it does not lie in $h$, then, as observed in the beginning of the proof by making a choice of $\theta_x$ as above, $x_{+}(\theta_x) \in g_{\text{nil}}^{\circ}$ for the top degree component $x_{+}(\theta_x)$ of $x$ in $Z$-gradation $\theta_x$ of $g$.) Moreover, if some nonzero root component of $x$ belongs to the root space $g_{\beta}$ such that $\delta$ contains an imaginary simple root $\delta_p$ (i.e., with $a_{p,p} \leq 0$) with nonzero coefficient, we can assume that $x_{\delta} \in g_{\text{nil}}^{\circ}$ (possibly with a different nonzero root component of $x$ corresponding to a root containing an imaginary simple root with nonzero coefficient). This is achieved by taking $\gamma$ as above but requiring $\theta_x(\alpha_p)$ to be much larger for all the imaginary simple roots $\alpha_p$ as compared to the values $\theta_x(\alpha_q)$ for all the real simple roots $\alpha_q$ (i.e., those with $a_{q,q} = 2$).

By using the Cartan involution $\omega$ of $g$ (i.e., $\omega(e_i) = -f_i, \omega(f_i) = -e_i, \omega(h) = -h \forall h \in h$), if needed, we can further assume that $\delta$ is a positive root. Write

$$\delta = \sum_p (m_p \alpha_p) + \sum_q (n_q \alpha_q), \text{ for } m_p, n_q \geq 0,$$

where $\alpha_p$ (resp. $\alpha_q$) run over all the imaginary (resp. real) simple roots. In particular, some $m_p > 0$. By [3], Exercise 11.21, the support $\text{supp}(\delta)$ is connected. Assume first that $\delta$ is not an imaginary simple root. Further, taking some $W$-translate (where $W$ is the Weyl group of $g$, cf. [3], §11.13), we can assume that $\delta(\alpha_q^\vee) \leq 0$ for all the real simple coroots $\alpha_q^\vee$ (cf. [3], Identity 11.13.3). Now, with respect to the $W$-invariant symmetric bilinear form $\langle \cdot, \cdot \rangle$ on $h^*$ (cf. [3], §2.1),

$$\langle \delta, \delta \rangle = \sum_p m_p \langle \delta, \alpha_p \rangle + \sum_q n_q \langle \delta, \alpha_q \rangle$$

$$= \sum_q n_q \langle \delta, \alpha_q \rangle + \sum_{p,q} m_p n_q \langle \alpha_q, \alpha_p \rangle + \sum_{p,p'} m_p m_{p'} \langle \alpha_{p'}, \alpha_p \rangle,$$

where $\alpha_{p'}$ also runs over imaginary simple roots. Now, by assumption,

$$\langle \delta, \alpha_q \rangle \leq 0, \text{ for all the real simple roots.}$$

For any imaginary simple root $\alpha_p$ and any real simple root $\alpha_q$, we have

$$\langle \alpha_q, \alpha_p \rangle \leq 0, \text{ since } a_{p,q} \leq 0.$$

Further, for imaginary simple roots $\alpha_p, \alpha_{p'}$,

$$\langle \alpha_{p'}, \alpha_p \rangle \leq 0, \text{ by [3], Identity 2.1.6.}$$

Observe that we can take the normalizing factor $\epsilon_i > 0$ for each $1 \leq i \leq \ell$ as can be seen from the identity:
\[ \epsilon_i a_{i,j} = \epsilon_j a_{j,i}, \text{ for all } 1 \leq i, j \leq \ell, \]

where the diagonal matrix \( D = \text{diag}(\epsilon_1, \cdots, \epsilon_\ell) \) is such that \( DA \) is a symmetric matrix. Moreover, since there exists \( p \) with \( m_p \neq 0 \) and since \( \text{supp} \delta \) is connected and \( \delta \) is not a simple root, by [3], Identity 2.1.6,

\[
\langle \alpha_p, \alpha_p \rangle < 0, \text{ for some } p' \neq p \text{ with } m_{p'} \neq 0 \text{ and } \alpha_{p'} \text{ an imaginary simple root or } \langle \alpha_q, \alpha_p \rangle < 0 \text{ for some } q \text{ with } n_q \neq 0 \text{ and } \alpha_q \text{ a real simple root.} \tag{6} \]

Thus, combining the equations (2) - (6), we get:

\[ \langle \delta, \delta \rangle < 0. \]

By [3], Corollary 9.12, \( \oplus_{k>0} g_{k\delta} \) is a free Lie algebra on a basis of the form \( \oplus_{k>0} g_{k\delta}^0 \), where

\[ g_{k\delta}^0 := \{ x \in g_{k\delta} : \langle x, y \rangle = 0 \forall y \text{ in the Lie subalgebra generated by } \]

\[ g_{-\delta}, g_{-2\delta}, \ldots, g_{-(k-1)\delta} \} \]

Observe next that \( g_{k\delta} \neq 0 \) for any \( k > 0 \) by [3], Identity 11.13.3. If \( g_\delta \) is one dimensional, then so is \( g_{-\delta} \) and hence \( g_{2\delta}^0 \neq 0 \). (To prove \( \dim g_{-\delta} = 1 \), observe that, due to the existence of the Cartan involution, \( \dim g_\beta = \dim g_{-\beta} \) for any root \( \beta \), cf. [3], Identity 1.3.5 and Theorem 11.13.1. Moreover, \( g_{-\delta} \) being one dimensional, the Lie subalgebra of \( g \) generated by \( g_{-\delta} \) is \( g_{-\delta} \) itself. Thus, \( g_{2\delta}^0 \neq 0 \) by the definition.) Thus, \( \oplus_{k>0} g_{k\delta} \) is a free Lie algebra on at least 2 generators. If \( \dim g_\delta \geq 2 \), then \( \oplus_{k>0} g_{k\delta} \) is again a free Lie algebra on at least two generators (since \( g_\delta^0 = g_\delta \)). Thus, \( \text{ad}(x_\delta) \) can not act locally nilpotently on \( \oplus_{k>0} g_{k\delta} \) and hence on \( g \) (since the enveloping algebra of a free Lie algebra is the tensor algebra on the same generators and now use [4], Identity (3) of Definition 1.3.2).

Now, let \( \delta = \alpha_p \) be an imaginary simple root. Then, again \( x_\delta = e_p \) can not act nilpotently on any \( e_i, i \neq p \) such that \( a_{i,p} \neq 0 \). (This is where we have used the assumption that \( A \) is indecomposable and \( \ell \geq 2 \).) To prove this, use [3], Identity 11.13.3 by observing that \( (n\alpha_p + \alpha_i) \in K \) for all \( n \geq 2 \) in the notation of [3].

Thus, we conclude that any \( x \in g_{\text{nil}}^0 \) must be of the form \( x \in g(B) + h \). Hence,

\[ g_{\text{nil}} \subset g'(B) + h. \]

Further, by [4], Lemma 1.3.3(a) and the defining relations of \( g(A) \), \( e_i, f_i \in g_{\text{nil}}^0 \) for any real simple root \( \alpha_i \). Thus,

\[ g'(B) \subset g_{\text{nil}}. \]

This proves the theorem. \( \Box \)
Remark 3.2. (a) Define

\[ g'_\text{nil} := \{ x \in g' : \text{ad} x \text{ acts locally nilpotently on } g' \} \]

and let \( g'_\text{nil} \subset g' \) be the Lie subalgebra generated by \( g'_\text{nil} \). Then, by the same proof as above,

\[ g'(B) \subset g'_\text{nil} \subset (g'(B) + h) \cap g'. \]

(b) It is easy to see that the above theorem remains true in the case \( A \) is parameterized by \( \mathbb{N} \times \mathbb{N} \).

(c) For the \( 1 \times 1 \)-matrix \( A = (0) \), following [3], §2.9, \( g(A) = h \oplus \mathbb{C}e_1 \oplus \mathbb{C}f_1 \), where \( h = \mathbb{C}\alpha_1^\vee \oplus \mathbb{C}d \) and \([e_1, f_1] = \alpha_1^\vee, [\alpha_1^\vee, g] = 0, [d, e_1] = e_1, [d, f_1] = -f_1 \). Thus, in this case, \( g_{\text{nil}} = g' \). Hence, the assumption \( \ell \geq 2 \) in the above theorem is necessary in general.

(d) One interesting consequence of the above theorem is that the only connected ‘reasonable’ group attached to a GKM algebra \( g(A) \) is the one coming from its subalgebra \( g(B) \) (up to an \( h \)-factor).

References