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Representation ring of Levi subgroups versus cohomology ring of flag varieties II

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ABSTRACT

For any reductive group G and a parabolic subgroup P with its Levi subgroup L , the first author in [5] introduced a ring homomorphism $\xi_\lambda^P : \text{Rep}_{\lambda\text{-poly}}^{\mathbb{C}}(L) \rightarrow H^*(G/P, \mathbb{C})$, where $\text{Rep}_{\lambda\text{-poly}}^{\mathbb{C}}(L)$ is a certain subring of the complexified representation ring of L (depending upon the choice of an irreducible representation $V(\lambda)$ of G with highest weight λ). In this paper we study this homomorphism for $G = \text{Sp}(2n)$ and its maximal parabolic subgroups P_{n-k} for any $1 \leq k \leq n-1$ (with the choice of $V(\lambda)$ to be the defining representation $V(\omega_1)$ in \mathbb{C}^{2n}). Thus, we obtain a \mathbb{C} -algebra homomorphism $\xi_{n,k} : \text{Rep}_{\omega_1\text{-poly}}^{\mathbb{C}}(\text{Sp}(2k)) \rightarrow H^*(IG(n-k, 2n), \mathbb{C})$. Our main result asserts that $\xi_{n,k}$ is injective when n tends to ∞ keeping k fixed. Similar results are obtained for the odd orthogonal groups.

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1. Introduction

This is a follow-up of first author’s work [5].

Let G be a connected reductive group over \mathbb{C} with a Borel subgroup B and maximal torus $T \subset B$. Let P be a standard parabolic subgroup with the Levi subgroup L containing T . Let $V(\lambda)$ be an irreducible almost faithful representation of G with highest weight λ (i.e., the corresponding map $\rho_\lambda : G \rightarrow \text{Aut}(V(\lambda))$ has finite kernel). Then, Springer defined an adjoint-equivariant regular map with Zariski dense image $\theta_\lambda : G \rightarrow \mathfrak{g}$ (depending upon λ) (cf. Definition 1). Using this the first author defined in [5] a certain subring $\text{Rep}_{\lambda\text{-poly}}^{\mathbb{C}}(L)$ of the complexified representation ring $\text{Rep}^{\mathbb{C}}(L)$ (cf. Definition 3). For $G = \text{GL}(n)$ and $V(\lambda)$ the defining representation \mathbb{C}^n , the ring $\text{Rep}_{\lambda\text{-poly}}(G) := \text{Rep}_{\lambda\text{-poly}}^{\mathbb{C}}(G) \cap \text{Rep}(G)$ coincides with the standard notion of polynomial representation ring of $\text{GL}(n)$ (cf. the equation (4)).

Coming back to the general case, the first author [5] defined a surjective \mathbb{C} -algebra homomorphism

$$\xi_\lambda^P : \text{Rep}_{\lambda\text{-poly}}^{\mathbb{C}}(L) \rightarrow H^*(G/P, \mathbb{C}) \tag{1}$$

(cf. Theorem 4).

Specializing the above result to the case when $G = \text{GL}(n)$, $V(\lambda)$ is the standard defining representation \mathbb{C}^n and $P = P_r$ (for any $1 \leq r \leq n - 1$) is the maximal parabolic subgroup so that the flag variety G/P_r is the Grassmannian $\text{Gr}(r, n)$ and $L_r = \text{GL}(r) \times \text{GL}(n - r)$ and restricting ξ_λ^P to the component $\text{GL}(r)$, one recovers the classical ring homomorphism

$$\phi_n : \text{Rep}_{\text{poly}}(\text{GL}(r)) \rightarrow H^*(\text{Gr}(r, n))$$

as shown in [5, §5].

Fix $r \geq 1$ and define the stable cohomology ring

$$\mathbb{H}^*(\text{Gr}_r, \mathbb{Z}) := \varprojlim H^*(\text{Gr}(r, n), \mathbb{Z})$$

as the inverse limit. Then, the homomorphisms ϕ_n combine to give a ring homomorphism

$$\phi_\infty : \text{Rep}_{\text{poly}}(\text{GL}(r)) \rightarrow \mathbb{H}^*(\text{Gr}_r, \mathbb{Z}).$$

Moreover, by the explicit description of ϕ_n (cf. [3, §9.4] and also [5, §5]) it is immediately seen that ϕ_∞ is a ring *isomorphism*.

The aim of this paper is to analyze the corresponding question for the Symplectic groups $\text{Sp}(2k)$ as well as the odd orthogonal groups $\text{SO}(2k + 1)$.

Let us fix a positive integer $1 \leq k \leq n - 1$ and consider the isotropic Grassmannian $\text{IG}(n - k, 2n)$ consisting of $n - k$ -dimensional isotropic subspaces of $V = \mathbb{C}^{2n}$ with respect to a non-degenerate symplectic form. Then, $\text{IG}(n - k, 2n)$ is the quotient $\text{Sp}(2n)/P_{n-k}^C$ of

$\mathrm{Sp}(2n)$ by the standard maximal parabolic subgroup P_{n-k}^C corresponding to the $n - k$ -th node of the Dynkin diagram of $\mathrm{Sp}(2n)$ (following the indexing convention as in [2]). Let L_{n-k}^C denote the Levi subgroup of P_{n-k}^C . Then,

$$L_{n-k}^C \simeq \mathrm{GL}(n - k) \times \mathrm{Sp}(2k).$$

We take the standard representation of $\mathrm{Sp}(2n)$ in \mathbb{C}^{2n} and abbreviate the corresponding $\mathrm{Rep}_{\lambda\text{-poly}}^{\mathbb{C}}(L_{n-k}^C)$ by $\mathrm{Rep}_{\mathrm{poly}}^{\mathbb{C}}(L_{n-k}^C)$. Thus, following (1), we get a ring homomorphism

$$\xi^{P_{n-k}^C} : \mathrm{Rep}_{\mathrm{poly}}^{\mathbb{C}}(L_{n-k}^C) \rightarrow H^*(\mathrm{IG}(n - k, 2n), \mathbb{C}).$$

Restricting $\xi^{P_{n-k}^C}$ to the component $\mathrm{Sp}(2k)$, we get a ring homomorphism

$$\xi_{n,k} : \mathrm{Rep}_{\mathrm{poly}}^{\mathbb{C}}(\mathrm{Sp}(2k)) \rightarrow H^*(\mathrm{IG}(n - k, 2n), \mathbb{C}).$$

Define the *stable cohomology ring* (cf. Definition 15)

$$\mathbb{H}^*(\mathrm{IG}_k, \mathbb{Z}) := \varprojlim H^*(\mathrm{IG}(n - k, 2n), \mathbb{Z})$$

as the inverse limit. Then, the homomorphisms $\xi_{n,k}$ combine to give a ring homomorphism

$$\xi_k : \mathrm{Rep}_{\mathrm{poly}}^{\mathbb{C}}(\mathrm{Sp}(2k)) \rightarrow \mathbb{H}^*(\mathrm{IG}_k, \mathbb{C}).$$

Following is our first main result of the paper (cf. Theorem 16 for a more precise assertion).

Theorem A. *The above ring homomorphism $\xi_k : \mathrm{Rep}_{\mathrm{poly}}^{\mathbb{C}}(\mathrm{Sp}(2k)) \rightarrow \mathbb{H}^*(\mathrm{IG}_k, \mathbb{C})$ is injective.*

However, it is not surjective (cf. Remark 17).

There are parallel results for the odd orthogonal groups $\mathrm{SO}(2k + 1)$. Specifically, consider the isotropic Grassmannian $\mathrm{OG}(n - k, 2n + 1)$ consisting of $n - k$ -dimensional isotropic subspaces of $V = \mathbb{C}^{2n+1}$ with respect to a non-degenerate symmetric form (for $1 \leq k \leq n - 1$). Then, $\mathrm{OG}(n - k, 2n + 1)$ is the quotient $\mathrm{SO}(2n)/P_{n-k}^B$ of $\mathrm{SO}(2n + 1)$ by the standard maximal parabolic subgroup P_{n-k}^B corresponding to the $n - k$ -th node of the Dynkin diagram of $\mathrm{SO}(2n + 1)$. Let L_{n-k}^B denote the Levi subgroup of P_{n-k}^B . Then,

$$L_{n-k}^B \simeq \mathrm{GL}(n - k) \times \mathrm{SO}(2k + 1).$$

We take the standard representation of $\mathrm{SO}(2n + 1)$ in \mathbb{C}^{2n+1} and abbreviate the corresponding $\mathrm{Rep}_{\lambda\text{-poly}}^{\mathbb{C}}(L_{n-k}^B)$ by $\mathrm{Rep}_{\mathrm{poly}}^{\mathbb{C}}(L_{n-k}^B)$. Thus, following (1), we get a ring homomorphism

$$\xi^{P_{n-k}^B} : \text{Rep}_{\text{poly}}^{\mathbb{C}}(L_{n-k}^B) \rightarrow H^*(\text{OG}(n - k, 2n + 1), \mathbb{C}).$$

Restricting $\xi^{P_{n-k}^B}$ to the component $\text{SO}(2k + 1)$, we get a ring homomorphism

$$\bar{\xi}_{n,k} : \text{Rep}_{\text{poly}}^{\mathbb{C}}(\text{SO}(2k + 1)) \rightarrow H^*(\text{OG}(n - k, 2n + 1), \mathbb{C}).$$

Similar to $\mathbb{H}^*(\text{IG}_k, \mathbb{Z})$, define the *stable cohomology ring* (cf. Definition 28)

$$\mathbb{H}^*(\text{OG}_k, \mathbb{Z}) := \varprojlim H^*(\text{OG}(n - k, 2n + 1), \mathbb{Z})$$

as the inverse limit. Then, the homomorphisms $\bar{\xi}_{n,k}$ combine to give a ring homomorphism

$$\bar{\xi}_k : \text{Rep}_{\text{poly}}^{\mathbb{C}}(\text{SO}(2k + 1)) \rightarrow \mathbb{H}^*(\text{OG}_k, \mathbb{C}).$$

Following is our second main result of the paper (cf. Theorem 29 for a more precise assertion).

Theorem B. *The above ring homomorphism $\bar{\xi}_k : \text{Rep}_{\text{poly}}^{\mathbb{C}}(\text{SO}(2k + 1)) \rightarrow \mathbb{H}^*(\text{OG}_k, \mathbb{C})$ is injective.*

However, it is not surjective (cf. Remark 30).

The proofs rely on some results of Buch-Kresch-Tamvakis from [6] and [7] and earlier results of the first author [5].

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2. Preliminaries and notation

We recall some notation and results from [5].

Let G be a connected reductive group over \mathbb{C} with a Borel subgroup B and maximal torus $T \subset B$. Let P be a standard parabolic subgroup with the Levi subgroup L containing T . We denote their Lie algebras by the corresponding Gothic characters: $\mathfrak{g}, \mathfrak{b}, \mathfrak{t}, \mathfrak{p}, \mathfrak{l}$ respectively. We denote by $\Delta = \{\alpha_1, \dots, \alpha_\ell\} \subset \mathfrak{t}^*$ the set of simple roots. The fundamental weights of \mathfrak{g} are denoted by $\{\omega_1, \dots, \omega_\ell\} \subset \mathfrak{t}^*$. Let W (resp. W_L) be the Weyl group of G (resp. L). Then, W is generated by the simple reflections $\{s_i\}_{1 \leq i \leq \ell}$. Let W^P denote the set of smallest coset representatives in the cosets in W/W_L . *Throughout the paper we follow the indexing convention as in [2, Planche I - IX].*

Let $X(T)$ be the group of characters of T and let $D \subset X(T)$ be the set of dominant characters (with respect to the given choice of B and hence positive roots, which are the roots of \mathfrak{b}). Then, the isomorphism classes of finite dimensional irreducible representations of G are bijectively parameterized by D under the correspondence $\lambda \in D \rightsquigarrow V(\lambda)$,

where $V(\lambda)$ is the irreducible representation of G with highest weight λ . We call $V(\lambda)$ *almost faithful* if the corresponding map $\rho_\lambda : G \rightarrow \text{Aut}(V(\lambda))$ has finite kernel.

Recall the Bruhat decomposition for the flag variety:

$$G/P = \sqcup_{w \in W^P} \Lambda_w^P, \text{ where } \Lambda_w^P := BwP/P.$$

Let $\bar{\Lambda}_w^P$ denote the closure of Λ_w^P in G/P . We denote by $[\bar{\Lambda}_w^P] \in H_{2\ell(w)}(G/P, \mathbb{Z})$ its fundamental class. Let $\{\epsilon_w^P\}_{w \in W^P}$ denote the Kronecker dual basis of the cohomology, i.e.,

$$\epsilon_w^P([\bar{\Lambda}_v^P]) = \delta_{w,v}, \text{ for any } v, w \in W^P.$$

Thus, ϵ_w^P belongs to the singular cohomology:

$$\epsilon_w^P \in H^{2\ell(w)}(G/P, \mathbb{Z}).$$

We abbreviate ϵ_w^B by ϵ_w . Then, for any $w \in W^P$, $\epsilon_w^P = \pi^*(\epsilon_w)$, where $\pi : G/B \rightarrow G/P$ is the standard projection.

We will often abbreviate ϵ_w^P by ϵ_w when the reference to P is clear from the context.

Definition 1. Let $V(\lambda)$ be any almost faithful irreducible representation of G . Following Springer (cf. [1, §9]), define the map

$$\theta_\lambda : G \rightarrow \mathfrak{g} \quad (\text{depending upon } \lambda)$$

as follows:

$$\begin{array}{ccc} G & \xrightarrow{\rho_\lambda} & \text{Aut}(V(\lambda)) \subset \text{End}(V(\lambda)) = \mathfrak{g} \oplus \mathfrak{g}^\perp \\ & \searrow \theta_\lambda & \downarrow \pi \\ & & \mathfrak{g} \end{array}$$

where \mathfrak{g} sits canonically inside $\text{End}(V(\lambda))$ via the derivative $d\rho_\lambda$, the orthogonal complement \mathfrak{g}^\perp is taken with respect to the standard conjugate $\text{Aut}(V(\lambda))$ -invariant form on $\text{End}(V(\lambda))$: $\langle A, B \rangle := \text{tr}(AB)$, and π is the projection to the \mathfrak{g} -factor. (By considering a compact form K of G , it is easy to see that $\mathfrak{g} \cap \mathfrak{g}^\perp = \{0\}$.)

Since $\pi \circ d\rho_\lambda$ is the identity map, θ_λ is a local diffeomorphism at 1 (and hence with Zariski dense image). Of course, by construction, θ_λ is an algebraic morphism. Moreover, since the decomposition $\text{End}(V(\lambda)) = \mathfrak{g} \oplus \mathfrak{g}^\perp$ is G -stable, it is easy to see that θ_λ is G -equivariant under conjugation.

We recall the following lemma from [5, Lemma 2].

Lemma 2. *The above morphism restricts to $\theta_{\lambda|_T} : T \rightarrow \mathfrak{t}$.*

For any $\mu \in X(T)$, we have a G -equivariant line bundle $\mathcal{L}(\mu)$ on G/B associated to the principal B -bundle $G \rightarrow G/B$ via the one dimensional B -module μ^{-1} . (Any $\mu \in X(T)$ extends uniquely to a character of B .) The one dimensional B -module μ is also denoted by \mathbb{C}_μ . Recall the surjective Borel homomorphism

$$\beta : S(\mathfrak{t}^*) \rightarrow H^*(G/B, \mathbb{C}),$$

which takes a character $\mu \in X(T)$ to the first Chern class of the line bundle $\mathcal{L}(\mu)$. (We realize $X(T)$ as a lattice in \mathfrak{t}^* via taking derivative.) We then extend this map linearly over \mathbb{C} to \mathfrak{t}^* and extend further as a graded algebra homomorphism from $S(\mathfrak{t}^*)$ (doubling the degree). Under the Borel homomorphism,

$$\beta(\omega_i) = \epsilon_{s_i}, \text{ for any fundamental weight } \omega_i. \tag{2}$$

Fix a compact form K of G such that $T_o := K \cap T$ is a (compact) maximal torus of K . Then, $W \simeq N(T_o)/T_o$, where $N(T_o)$ is the normalizer of T_o in K . Recall that β is W -equivariant under the standard action of W on $S(\mathfrak{t}^*)$ and the W -action on $H^*(G/B, \mathbb{C})$ induced from the W -action on $G/B \simeq K/T_o$ via

$$(nT_o) \cdot (kT_o) := kn^{-1}T_o, \text{ for } n \in N(T_o) \text{ and } k \in K.$$

Thus, for any standard parabolic subgroup P with the Levi subgroup L containing T , restricting β , we get a surjective graded algebra homomorphism:

$$\beta^P : S(\mathfrak{t}^*)^{W_L} \rightarrow H^*(G/B, \mathbb{C})^{W_L} \simeq H^*(G/P, \mathbb{C}),$$

where the last isomorphism, which is induced from the projection $G/B \rightarrow G/P$, can be found, e.g., in [4, Corollary 11.3.14].

Now, the Springer morphism $\theta_{\lambda|_T} : T \rightarrow \mathfrak{t}$ (restricted to T) gives rise to the corresponding W -equivariant injective algebra homomorphism on the affine coordinate rings:

$$(\theta_{\lambda|_T})^* : \mathbb{C}[\mathfrak{t}] = S(\mathfrak{t}^*) \rightarrow \mathbb{C}[T].$$

Thus, on restriction to W_L -invariants, we get an injective algebra homomorphism

$$\theta_{\lambda}(P)^* : \mathbb{C}[\mathfrak{t}]^{W_L} = S(\mathfrak{t}^*)^{W_L} \rightarrow \mathbb{C}[T]^{W_L}.$$

(Since W_L -invariants depend upon the choice of the parabolic subgroup P , we have included P in the notation of $\theta_{\lambda}(P)^*$.) Now, let $\text{Rep}(L)$ be the representation ring of L and let $\text{Rep}^{\mathbb{C}}(L) := \text{Rep}(L) \otimes_{\mathbb{Z}} \mathbb{C}$ be its complexification. Then, as it is well known,

$$\text{Rep}^{\mathbb{C}}(L) \simeq \mathbb{C}[T]^{W_L} \tag{3}$$

obtained from taking the character of an L -module restricted to T .

We will often identify a virtual representation of L with its character restricted to T (which is automatically W_L -invariant).

Definition 3. We call a virtual character $\chi \in \text{Rep}^{\mathbb{C}}(L)$ of L a λ -polynomial character if the corresponding function in $\mathbb{C}[T]^{W_L}$ is in the image of $\theta_\lambda(P)^*$. The set of all λ -polynomial characters of L , which is, by definition, a subalgebra of $\text{Rep}^{\mathbb{C}}(L)$ isomorphic to the algebra $S(\mathfrak{t}^*)^{W_L}$, is denoted by $\text{Rep}_{\lambda\text{-poly}}^{\mathbb{C}}(L)$. Of course, the map $\theta_\lambda(P)^*$ induces an algebra isomorphism (still denoted by)

$$\theta_\lambda(P)^* : S(\mathfrak{t}^*)^{W_L} \simeq \text{Rep}_{\lambda\text{-poly}}^{\mathbb{C}}(L),$$

under the identification (3).

It is easy to see that

$$\text{Rep}_{\omega_1\text{-poly}}(\text{GL}(n)) = \text{Rep}_{\text{poly}}(\text{GL}(n)), \tag{4}$$

where $\text{Rep}_{\text{poly}}(\text{GL}(n))$ denotes the subring of the representation ring $\text{Rep}(\text{GL}(n))$ spanned by the irreducible polynomial representations of $\text{GL}(n)$.

We recall the following result from [5, Theorem 5].

Theorem 4. Let $V(\lambda)$ be an almost faithful irreducible G -module and let P be any standard parabolic subgroup. Then, the above maps (specifically $\beta^P \circ (\theta_\lambda(P)^*)^{-1}$) give rise to a surjective \mathbb{C} -algebra homomorphism

$$\xi_\lambda^P : \text{Rep}_{\lambda\text{-poly}}^{\mathbb{C}}(L) \rightarrow H^*(G/P, \mathbb{C}).$$

Moreover, let Q be another standard parabolic subgroup with Levi subgroup R containing T such that $P \subset Q$ (and hence $L \subset R$). Then, we have the following commutative diagram:

$$\begin{CD} \text{Rep}_{\lambda\text{-poly}}^{\mathbb{C}}(R) @>\xi_\lambda^Q>> H^*(G/Q, \mathbb{C}) \\ @V\gamma VV @VV\pi^*V \\ \text{Rep}_{\lambda\text{-poly}}^{\mathbb{C}}(L) @>\xi_\lambda^P>> H^*(G/P, \mathbb{C}), \end{CD}$$

where π^* is induced from the standard projection $\pi : G/P \rightarrow G/Q$ and γ is induced from the restriction of representations.

3. Injectivity result for the symplectic group

In this section, we consider the symplectic group $G = \text{Sp}(2n)$ ($n \geq 2$). We take the Springer morphism for $\text{Sp}(2n)$ with respect to the first fundamental weight $\lambda = \omega_1$. We will abbreviate the Springer morphism θ_{ω_1} by θ , ξ_λ^P by ξ^P and $\text{Rep}_{\omega_1\text{-poly}}^{\mathbb{C}}(G)$ by $\text{Rep}_{\text{poly}}^{\mathbb{C}}(G)$.

Let $V = \mathbb{C}^{2n}$ be equipped with the nondegenerate symplectic form \langle , \rangle so that its matrix $(\langle e_i, e_j \rangle)_{1 \leq i, j \leq 2n}$ in the standard basis $\{e_1, \dots, e_{2n}\}$ is given by

$$E_C = \begin{pmatrix} 0 & J \\ -J & 0 \end{pmatrix},$$

where J is the anti-diagonal matrix $(1, \dots, 1)$ of size n . Let

$$\text{Sp}(2n) := \{g \in \text{SL}(2n) : g \text{ leaves the form } \langle , \rangle \text{ invariant}\}$$

be the associated symplectic group. Clearly, $\text{Sp}(2n)$ can be realized as the fixed point subgroup $\text{SL}(2n)^\sigma$ under the involution $\sigma : \text{SL}(2n) \rightarrow \text{SL}(2n)$ defined by $\sigma(A) = E_C(A^t)^{-1}E_C^{-1}$.

The involution σ keeps both of B and T stable, where B and T are the standard Borel and maximal torus respectively of $\text{SL}(2n)$. Moreover, B^σ (respectively, T^σ) is a Borel subgroup (respectively, a maximal torus) of $\text{Sp}(2n)$. We denote B^σ, T^σ by $B_C = B_{C_n}, T_C = T_{C_n}$ respectively. Then, T_C is given as follows:

$$T_C = \{ \mathfrak{t} = \text{diag}(t_1, \dots, t_n, t_n^{-1}, \dots, t_1^{-1}) : t_i \in \mathbb{C}^* \}. \tag{5}$$

Its Lie algebra is given by

$$\mathfrak{t}_C = \{ \mathfrak{t} = \text{diag}(x_1, \dots, x_n, -x_n, \dots, -x_1) : x_i \in \mathbb{C} \}. \tag{6}$$

We recall the following lemma from [5, Lemma 10].

Lemma 5. *The Springer morphism $\theta : G \rightarrow \mathfrak{g}$ for $G = \text{Sp}(2n)$ is given by*

$$g \mapsto \frac{g - E_C^{-1}g^tE_C}{2}, \text{ for } g \in G.$$

(Observe that this is the Cayley transform.)

From the description of the Springer morphism given above, we immediately get the following (cf. [5, Corollary 11]):

Corollary 6. *Restricted to the maximal torus as above, we get the following description of the Springer map θ :*

$$\theta(\mathbf{t}) = \text{diag}(\bar{t}_1, \dots, \bar{t}_n, -\bar{t}_n, \dots, -\bar{t}_1), \text{ where } \bar{t}_i := \frac{t_i - t_i^{-1}}{2}.$$

The following result follows easily from Corollary 6 together with the description of the Weyl group (cf. [5, Proposition 12]).

Proposition 7. *Let $f : T \rightarrow \mathbb{C}$ be a regular map. Then, $f \in \text{Rep}_{\text{poly}}^{\mathbb{C}}(G)$ if and only if the following is satisfied:*

There exists a symmetric polynomial $P_f(x_1, \dots, x_n)$ such that

$$f(\mathbf{t}) = P_f((\bar{t}_1)^2, \dots, (\bar{t}_n)^2), \text{ for } \mathbf{t} \in T_{\mathbb{C}} \text{ given by (5).}$$

We recall the following result from [5, Proposition 24].

Lemma 8. *Under the homomorphism $\xi^B : \text{Rep}_{\text{poly}}^{\mathbb{C}}(T) \rightarrow H^*(G/B, \mathbb{C})$ of Theorem 4,*

$$\bar{t}_i \mapsto (\epsilon_{s_i} - \epsilon_{s_{i-1}}), \text{ for any } 1 \leq i \leq n,$$

where ϵ_{s_0} is to be interpreted as 0.

Definition 9. For $1 \leq r \leq n$, we let $\text{IG}(r, 2n)$ to be the set of r -dimensional isotropic subspaces of V with respect to the form $\langle \cdot, \cdot \rangle$, i.e.,

$$\text{IG}(r, 2n) := \{M \in \text{Gr}(r, 2n) : \langle v, v' \rangle = 0, \forall v, v' \in M\}.$$

Then, $\text{IG}(r, 2n)$ is the quotient $\text{Sp}(2n)/P_r^{\mathbb{C}}$ of $\text{Sp}(2n)$ by the standard maximal parabolic subgroup $P_r^{\mathbb{C}}$ with $\Delta \setminus \{\alpha_r\}$ as the set of simple roots of its Levi component $L_r^{\mathbb{C}}$. (Again we take $L_r^{\mathbb{C}}$ to be the unique Levi subgroup of $P_r^{\mathbb{C}}$ containing $T_{\mathbb{C}}$.) Then,

$$L_r^{\mathbb{C}} \simeq \text{GL}(r) \times \text{Sp}(2(n-r)).$$

In this case, by the identity (4), Corollary 6 and Proposition 7,

$$\text{Rep}_{\text{poly}}^{\mathbb{C}}(L_r^{\mathbb{C}}) \simeq \mathbb{C}_{\text{sym}}[\bar{t}_1, \dots, \bar{t}_r] \otimes_{\mathbb{C}} \mathbb{C}_{\text{sym}}[(\bar{t}_{r+1})^2, \dots, (\bar{t}_n)^2], \tag{7}$$

where \mathbb{C}_{sym} denotes the subalgebra of the polynomial ring consisting of symmetric polynomials.

From now on we fix $0 \leq k \leq n - 1$ and consider $\text{IG}(n - k, 2n)$.

Following [6, Definition 1.1], a partition $\lambda : \lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_d > 0$ is said to be k -strict if no part greater than k is repeated (i.e., $\lambda_j > k \Rightarrow \lambda_{j+1} < \lambda_j$). The

Schubert varieties in $\text{IG}(n - k, 2n)$ are parametrized by k -strict partitions contained in the $(n - k) \times (n + k)$ rectangle. The codimension of this variety is equal to $|\lambda| := \sum \lambda_i$. Let $\sigma_\lambda \in H^{2|\lambda|}(\text{IG}(n - k, 2n), \mathbb{Z})$ denote the cohomology class Poincaré dual to the fundamental class $[X_\lambda]$ of the Schubert variety associated to λ . Let $\mathcal{P}(k, n)$ denote the set of k -strict partitions contained in the $(n - k) \times (n + k)$ rectangle. Thus, $\{\sigma_\lambda\}_{\lambda \in \mathcal{P}(k, n)}$ gives the Schubert basis of $H^*(\text{IG}(n - k, 2n), \mathbb{Z})$.

We have the following short exact sequence of vector bundles over $\text{IG}(n - k, 2n)$:

$$0 \rightarrow \mathcal{S} \rightarrow \bar{\mathcal{E}} \rightarrow \mathcal{Q} \rightarrow 0,$$

where $\bar{\mathcal{E}}$ is the trivial bundle of rank $2n$, \mathcal{S} is the tautological subbundle of rank $n - k$ and \mathcal{Q} is the quotient bundle of rank $n + k$. Let $c_i = c_i(\mathcal{Q})$ ($1 \leq i \leq n + k$) denote the i th Chern class of the quotient bundle \mathcal{Q} . Then, these classes are so called the *special Schubert classes*. Then, by [6, §1.2],

$$c_i = \sigma_i, \tag{8}$$

where $\sigma_i := \sigma_{(i)}$ and (i) is the partition with single term i .

We have the following presentation of the cohomology ring due to [6, Theorem 1.2]. In the following we follow the convention that $c_0 = 1$ and $c_p = 0$ if $p < 0$ or $p > n + k$.

Theorem 10. *The cohomology ring $H^*(\text{IG}(n - k, 2n), \mathbb{Z})$ is presented as a quotient of the polynomial ring $\mathbb{Z}[c_1, \dots, c_{n+k}]$ modulo the relations:*

$$(R_{n,k}^p) \quad (n - k + 1 \leq p \leq n + k) : \quad \det(c_{1+j-i})_{1 \leq i, j \leq p} = 0,$$

and

$$(S_{n,k}^s) \quad (k + 1 \leq s \leq n) : \quad c_s^2 + 2 \sum_{i=1}^{n+k-s} (-1)^i c_{s+i} c_{s-i} = 0.$$

Our original proof of the following result was longer. The following shorter proof is due to L. Mihalcea.

Proposition 11. *Take $1 \leq k \leq n - 1$. The map $\xi^{P_{n-k}} : \text{Rep}_{\text{poly}}^{\mathbb{C}}(L_{n-k}^{\mathbb{C}}) \rightarrow H^*(\text{IG}(n - k, 2n), \mathbb{C})$ of Theorem 4 under the decomposition (7) for $r = n - k$ takes, for $1 \leq i \leq k$,*

$$e_i \left((\bar{t}_{n-k+1})^2, \dots, (\bar{t}_n)^2 \right) \mapsto c_i^2 + 2 \sum_{j=1}^i (-1)^j c_{i+j} c_{i-j},$$

where $c_i = c_i(\mathcal{Q})$ is as defined before Theorem 10 and e_i is the i -th elementary symmetric function.

Before we come to the proof of the proposition, we need the following two lemmas:
 Let $\text{Fl} = G/B$ be the full flag variety for $G = \text{Sp}(2n)$. It consists of partial flags

$$F_\bullet : F_1 \subset F_2 \subset \dots \subset F_n \subset E := \mathbb{C}^{2n}, \text{ such that each } F_j \text{ is isotropic and } \dim F_j = j.$$

We can complete the partial flag to a full flag by taking $F_{n+j} := F_{n-j}^\perp$. The flags F_\bullet give rise to a sequence of tautological vector bundles over Fl :

$$\mathcal{F}_1 \subset \mathcal{F}_2 \subset \dots \subset \mathcal{F}_n \subset \mathcal{E}, \text{ with rank } \mathcal{F}_j = j,$$

where $\mathcal{E} : \text{Fl} \times \mathbb{C}^{2n} \rightarrow \text{Fl}$ is the trivial rank $2n$ vector bundle. For $1 \leq j \leq n$, define

$$x_j := -c_1(\mathcal{F}_j/\mathcal{F}_{j-1}),$$

where \mathcal{F}_0 is taken to be the vector bundle of rank 0.

Lemma 12. For $1 \leq j \leq n$, the Schubert divisor $\epsilon_{s_j} \in H^2(\text{Fl}, \mathbb{Z})$ is given by

$$\epsilon_{s_j} = -c_1(\mathcal{F}_j) = x_1 + \dots + x_j.$$

In particular, under ξ^B for $G = \text{Sp}(2n)$, $\bar{t}_j \mapsto x_j$ for any $1 \leq j \leq n$.

Proof. The first part follows from the identity (8).

The ‘In particular’ statement follows from Lemma 8. \square

For $0 \leq j \leq n$, let

$$\mathcal{Q}_j := \mathcal{E}/\mathcal{F}_j.$$

Observe that the symplectic form gives an isomorphism of vector bundles:

$$\mathcal{Q}_j \simeq (\mathcal{F}_j^\perp)^*. \tag{9}$$

Lemma 13. For $0 \leq j \leq n$, the following holds:

$$c(\mathcal{Q}_j)c(\mathcal{Q}_j^*) = \prod_{p=j+1}^n (1 - x_p)(1 + x_p),$$

where c is the total Chern class.

Proof. By definition,

$$\prod_{p=j+1}^n (1 - x_p)(1 + x_p) = \prod_{p=j+1}^n c(\mathcal{F}_p/\mathcal{F}_{p-1}) \cdot c((\mathcal{F}_p/\mathcal{F}_{p-1})^*) = \frac{c(\mathcal{F}_n)}{c(\mathcal{F}_j)} \cdot \frac{c(\mathcal{F}_n^*)}{c(\mathcal{F}_j^*)}. \tag{10}$$

From the exact sequence $0 \rightarrow \mathcal{F}_j \rightarrow \mathcal{E} \rightarrow \mathcal{Q}_j \rightarrow 0$, we get

$$c(\mathcal{Q}_j) \cdot c(\mathcal{F}_j) = 1 \text{ and } c(\mathcal{Q}_j^*)c(\mathcal{F}_j^*) = 1, \tag{11}$$

and hence taking $j = n$ in the above equation and using the equation (9), we get

$$c(\mathcal{F}_n)c(\mathcal{F}_n^*) = 1, \text{ since } \mathcal{F}_n^\perp = \mathcal{F}_n. \tag{12}$$

Combining the equations (10), (11) and (12), we get the lemma. \square

Proof of Proposition 11. By taking terms of degree $2i$ and $j = n - k$ in Lemma 13, we obtain in $H^*(\text{Fl}, \mathbb{Z})$:

$$c_i(\mathcal{Q}_j)^2 + 2 \sum_{p=1}^i (-1)^p c_{i+p}(\mathcal{Q}_j) \cdot c_{i-p}(\mathcal{Q}_j) = e_i(x_{j+1}^2, \dots, x_n^2).$$

By the definition, the bundle \mathcal{S} pulls back to the bundle \mathcal{F}_{n-k} over Fl under the projection $\text{Fl} \rightarrow \text{IG}(n - k, 2n)$. Thus, the proposition follows from Lemma 12. \square

Remark 14. Even though we do not need, the map $\xi^{P_{n-k}} : \text{Rep}_{\text{poly}}^{\mathbb{C}}(L_{n-k}^{\mathbb{C}}) \rightarrow H^*(\text{IG}(n - k, 2n), \mathbb{C})$ of Theorem 4 under the decomposition (7) takes for $1 \leq i \leq n - k$,

$$e_i(\bar{t}_1, \dots, \bar{t}_{n-k}) \mapsto c_i(\mathcal{S}) = \epsilon_{s_{n-k-i+1} \dots s_{n-k}}.$$

This follows from [6, §1.2].

Definition 15 (Inverse Limit). For any $k \geq 0$, define the *stable cohomology ring* [7, §1.3] as

$$\mathbb{H}^*(\text{IG}_k, \mathbb{Z}) = \varprojlim H^*(\text{IG}(n - k, 2n), \mathbb{Z})$$

as the inverse limit (in the category of graded rings) of the inverse system

$$\dots \leftarrow H^*(\text{IG}(n - k, 2n), \mathbb{Z}) \xleftarrow{\pi_n^*} H^*(\text{IG}(n - k + 1, 2n + 2), \mathbb{Z}) \leftarrow \dots,$$

where $\pi_n : \text{IG}(n - k, 2n) \hookrightarrow \text{IG}(n - k + 1, 2n + 2)$ is given by $V \mapsto T_n(V) \oplus \mathbb{C}e_{n+1}$ and $T_n : \mathbb{C}^{2n} \rightarrow \mathbb{C}^{2n+2}$ is the linear embedding taking $e_i \mapsto e_i$ for $1 \leq i \leq n$ and taking $e_i \mapsto e_{i+2}$ for $n + 1 \leq i \leq 2n$.

This ring has an additive basis consisting of Schubert classes σ_λ for each k -strict partition λ . The natural ring homomorphism $\varphi_{k,n} : \mathbb{H}^*(\text{IG}_k, \mathbb{Z}) \rightarrow H^*(\text{IG}(n - k, 2n), \mathbb{Z})$ takes σ_λ to σ_λ whenever λ fits in a $(n - k) \times (n + k)$ rectangle and to zero otherwise. In particular, $\varphi_{k,n}$ is surjective. From the definition of the Chern classes $c_j = c_j^n(\mathcal{Q})$,

it is easy to see that under the restriction map $\pi_n^* : H^*(\text{IG}(n - k + 1, 2n + 2), \mathbb{Z}) \rightarrow H^*(\text{IG}(n - k, 2n), \mathbb{Z})$, $c_j^{n+1} \mapsto c_j^n$ for $1 \leq j \leq n + k$ and $c_{n+k+1}^{n+1} \mapsto 0$.

From the presentation of the ring $H^*(\text{IG}(n - k, 2n), \mathbb{Z})$ (Theorem 10), none of the determinantal relations hold in the inverse limit. So, $\mathbb{H}^*(\text{IG}_k, \mathbb{Z})$ is isomorphic to the polynomial ring $\mathbb{Z}[c_1, c_2, \dots]$ modulo the relations:

$$(S^s) \ (s > k) : \quad c_s^2 + 2 \sum_{i=1}^s (-1)^i c_{s+i} c_{s-i} = 0. \tag{13}$$

Take $k \geq 1$. Recall from Proposition 7 (which remains valid for $k = 1$ as well) that

$$\text{Rep}_{\text{poly}}^{\mathbb{C}}(\text{Sp}(2k)) \simeq \mathbb{C}_{\text{sym}} [(\bar{h}_1)^2, \dots, (\bar{h}_k)^2].$$

Define a ring homomorphism (for any $1 \leq k \leq n - 1$)

$$\iota_k^n : \text{Rep}_{\text{poly}}^{\mathbb{C}}(\text{Sp}(2k)) \rightarrow \text{Rep}_{\text{poly}}^{\mathbb{C}}(L_{n-k}^{\mathbb{C}})$$

by taking $f(\bar{\mathbf{h}}) \mapsto 1 \otimes f(\bar{\mathbf{t}})$, where $\bar{\mathbf{h}} := (\bar{h}_1, \dots, \bar{h}_k)$, $\bar{\mathbf{t}} := (\bar{t}_{n-k+1}, \dots, \bar{t}_n)$ and $f(\bar{\mathbf{t}})$ is the same polynomial written in the $\bar{\mathbf{t}}$ -variables under the transformation $\bar{h}_p \mapsto \bar{t}_{n-k+p}$. This gives rise to the map $\xi_{n,k} := \xi^{P_{n-k}} \circ \iota_k^n : \text{Rep}_{\text{poly}}^{\mathbb{C}}(\text{Sp}(2k)) \rightarrow H^*(\text{IG}(n - k, 2n), \mathbb{C})$. Consider the following diagram, which is commutative because of Proposition 11.

$$\begin{array}{ccc}
 & & \uparrow \pi_{n-1}^* \\
 \text{Rep}_{\text{poly}}^{\mathbb{C}}(\text{Sp}(2k)) & \xrightarrow{\xi_{n,k}} & H^*(\text{IG}(n - k, 2n), \mathbb{C}) \\
 & \searrow \xi_{n+1,k} & \uparrow \pi_n^* \\
 & & H^*(\text{IG}(n - k + 1, 2n + 2), \mathbb{C}) \\
 & & \uparrow \pi_{n+1}^*
 \end{array}$$

The compatible ring homomorphisms $\xi_{n,k} : \text{Rep}_{\text{poly}}^{\mathbb{C}}(\text{Sp}(2k)) \rightarrow H^*(\text{IG}(n - k, 2n), \mathbb{C})$ combine to give a ring homomorphism

$$\xi_k : \text{Rep}_{\text{poly}}^{\mathbb{C}}(\text{Sp}(2k)) \rightarrow \mathbb{H}^*(\text{IG}_k, \mathbb{C}).$$

The following theorem is one of our main results of the paper.

Theorem 16. *Let $k \geq 1$ be an integer. The above ring homomorphism $\xi_k : \text{Rep}_{\text{poly}}^{\mathbb{C}}(\text{Sp}(2k)) \rightarrow \mathbb{H}^*(\text{IG}_k, \mathbb{C})$ takes the generators*

$$e_i((\bar{h}_1)^2, \dots, (\bar{h}_k)^2) \mapsto c_i^2 + 2 \sum_{j=1}^i (-1)^j c_{i+j} c_{i-j}, \text{ for any } 1 \leq i \leq k, \tag{14}$$

where $c_i = c_i(\mathcal{Q})$.

Moreover, ξ_k is injective.

Proof. The first part follows from Proposition 11.

We next prove the injectivity of ξ_k :

By Proposition 7, $\text{Rep}_{\text{poly}}^{\mathbb{C}}(\text{Sp}(2k))$ is a polynomial ring over \mathbb{C} generated by $\{e_1(\bar{\mathbf{z}}), \dots, e_k(\bar{\mathbf{z}})\}$, where $\bar{\mathbf{z}} = (z_1, \dots, z_k)$ and $z_i := (\bar{h}_i)^2$. Let $\text{Rep}_{\text{poly}}^{\mathbb{Z}}(\text{Sp}(2k))$ be the polynomial subring over \mathbb{Z} generated by $\{e_1(\bar{\mathbf{z}}), \dots, e_k(\bar{\mathbf{z}})\}$. Then, by the equation (14),

$$\xi_k(\text{Rep}_{\text{poly}}^{\mathbb{Z}}(\text{Sp}(2k))) \subset \mathbb{H}^*(\text{IG}_k, \mathbb{Z}).$$

Thus, on restriction, we get the ring homomorphism

$$\xi_k^{\mathbb{Z}} : \text{Rep}_{\text{poly}}^{\mathbb{Z}}(\text{Sp}(2k)) \rightarrow \mathbb{H}^*(\text{IG}_k, \mathbb{Z}).$$

Observe further that $\xi_k^{\mathbb{Z}}$ is a homomorphism of graded rings if we assign degree $4i$ to each e_i (and the standard cohomological degree to $\mathbb{H}^*(\text{IG}_k, \mathbb{Z})$). Let K be the Kernel of $\xi_k^{\mathbb{Z}}$. Since $\mathbb{H}^*(\text{IG}_k, \mathbb{Z})$ is a free \mathbb{Z} -module of finite rank in each degree, the induced homomorphism

$$\mathbb{Z}/(2) \otimes_{\mathbb{Z}} K \rightarrow \mathbb{Z}/(2) \otimes_{\mathbb{Z}} \text{Rep}_{\text{poly}}^{\mathbb{Z}}(\text{Sp}(2k)) \text{ is injective.} \tag{15}$$

We next observe that the induced homomorphism

$$\mathbb{Z}/(2) \otimes_{\mathbb{Z}} \text{Rep}_{\text{poly}}^{\mathbb{Z}}(\text{Sp}(2k)) \rightarrow \mathbb{Z}/(2) \otimes_{\mathbb{Z}} \mathbb{H}^*(\text{IG}_k, \mathbb{Z}) \text{ is injective.} \tag{16}$$

To prove this, observe that

$$\mathbb{Z}/(2) \otimes_{\mathbb{Z}} \text{Rep}_{\text{poly}}^{\mathbb{Z}}(\text{Sp}(2k)) \simeq \mathbb{Z}/(2)[e_1, \dots, e_k], \tag{17}$$

and, by the defining relations $(S^s) (s > k)$ of $\mathbb{H}^*(\text{IG}_k, \mathbb{Z})$ as in equation (13),

$$\mathbb{Z}/(2) \otimes_{\mathbb{Z}} \mathbb{H}^*(\text{IG}_k, \mathbb{Z}) \simeq \mathbb{Z}/(2)[c_1, \dots, c_k] \otimes \frac{\mathbb{Z}/(2)[c_{k+1}, c_{k+2} \dots]}{\langle c_{k+1}^2, c_{k+2}^2 \dots \rangle}. \tag{18}$$

Moreover, under the above identifications (17) and (18), by the first part of the theorem, the ring homomorphism $\xi_k^{\mathbb{Z}}$ modulo 2 is given by

$$e_i \mapsto c_i^2, \text{ for any } 1 \leq i \leq k.$$

In particular, it is injective. From this we obtain that

$$\mathbb{Z}/(2) \otimes_{\mathbb{Z}} K = 0.$$

But, since K is a finitely generated torsionfree \mathbb{Z} -module in each graded degree (thus free) we get that

$$K = 0.$$

Since \mathbb{C} is a torsionfree \mathbb{Z} -module, this clearly gives the injectivity of ξ_k (cf. [8, Chap. 5, §2, Lemma 5]). This proves the theorem. \square

Remark 17. The ring homomorphism $\xi_k : \text{Rep}_{\text{poly}}^{\mathbb{C}}(\text{Sp}(2k)) \rightarrow \mathbb{H}^*(\text{IG}_k, \mathbb{C})$ of the above Theorem 16 is *not* surjective, as can be easily seen since the domain is a finitely generated \mathbb{C} -algebra (by Proposition 7) whereas the range is not (for otherwise for each n , $H^*(\text{IG}(n - k, 2n), \mathbb{C})$ would be generated by a fixed finite number of generators independent of n).

4. Injectivity result for the odd orthogonal group

The treatment in this section is parallel to that of the last section dealing with $\text{Sp}(2n)$. But, we include some details for completeness.

In this section, we consider the special orthogonal group $G = \text{SO}(2n + 1)$ ($n \geq 2$). We take the Springer morphism for $\text{SO}(2n + 1)$ with respect to the first fundamental weight $\lambda = \omega_1$. We will abbreviate θ_{ω_1} by θ , ξ_{λ}^P by ξ^P and $\text{Rep}_{\omega_1 - \text{poly}}^{\mathbb{C}}(G)$ by $\text{Rep}_{\text{poly}}^{\mathbb{C}}(G)$.

Let $V' = \mathbb{C}^{2n+1}$ be equipped with the nondegenerate symmetric form $\langle \cdot, \cdot \rangle$ so that its matrix $E_B = (\langle e_i, e_j \rangle)_{1 \leq i, j \leq 2n+1}$ (in the standard basis $\{e_1, \dots, e_{2n+1}\}$) is the $(2n+1) \times (2n+1)$ antidiagonal matrix with 1's all along the antidiagonal except at the $(n+1, n+1)$ -th place where the entry is 2. Note that the associated quadratic form on V' is given by

$$Q(\sum t_i e_i) = t_{n+1}^2 + \sum_{i=1}^n t_i t_{2n+2-i}.$$

Let

$$\text{SO}(2n + 1) := \{g \in \text{SL}(2n + 1) : g \text{ leaves the quadratic form } Q \text{ invariant}\}$$

be the associated special orthogonal group. Clearly, $\text{SO}(2n + 1)$ can be realized as the fixed point subgroup $\text{SL}(2n + 1)^{\delta}$ under the involution $\delta : \text{SL}(2n + 1) \rightarrow \text{SL}(2n + 1)$ defined by $\delta(A) = E_B^{-1}(A^t)^{-1}E_B$. The involution δ keeps both of B and T stable, where B (resp. T) is the standard Borel (resp. maximal torus) of $\text{SL}(2n + 1)$. Moreover, B^{δ} (respectively, T^{δ}) is a Borel subgroup (respectively, a maximal torus) of $\text{SO}(2n + 1)$. We denote B^{δ}, T^{δ} by $B_B = B_{B_n}, T_B = T_{B_n}$ respectively. Then, T_B is given by:

$$T_B = \{ \mathbf{t} = \text{diag}(t_1, \dots, t_n, 1, t_n^{-1}, \dots, t_1^{-1}) : t_i \in \mathbb{C}^* \}. \tag{19}$$

Its Lie algebra is given by

$$\mathfrak{t}_B = \{ \dot{\mathbf{t}} = \text{diag}(x_1, \dots, x_n, 0, -x_n, \dots, -x_1) : x_i \in \mathbb{C} \}. \tag{20}$$

We recall the following lemma from [5, Lemma 10].

Lemma 18. *The Springer morphism $\theta : G \rightarrow \mathfrak{g}$ for $G = \text{SO}(2n + 1)$ is given by*

$$g \mapsto \frac{g - E_B^{-1} g^t E_B}{2}, \text{ for } g \in G.$$

(Observe that this is the Cayley transform.)

From the description of the Springer morphism given above, we immediately get the following (cf. [5, Corollary 11]):

Corollary 19. *Restricted to the maximal torus T_B as above, we get the following description of the Springer map θ :*

$$\theta(\mathbf{t}) = \text{diag}(\bar{t}_1, \dots, \bar{t}_n, 0, -\bar{t}_n, \dots, -\bar{t}_1), \text{ where } \bar{t}_i := \frac{t_i - t_i^{-1}}{2}.$$

The following result follows easily from Corollary 19 together with the description of the Weyl group (cf. [5, Proposition 12]).

Proposition 20. *Let $f : T_B \rightarrow \mathbb{C}$ be a regular map. Then, $f \in \text{Rep}_{\text{poly}}^{\mathbb{C}}(G)$ if and only if the following is satisfied:*

There exists a symmetric polynomial $P_f(x_1, \dots, x_n)$ such that

$$f(\mathbf{t}) = P_f((\bar{t}_1)^2, \dots, (\bar{t}_n)^2), \text{ for } \mathbf{t} \in T_B \text{ given by (19)}.$$

We recall the following result from [5, Proposition 24].

Lemma 21. *Under the homomorphism $\xi^B : \text{Rep}_{\text{poly}}^{\mathbb{C}}(T_B) \rightarrow H^*(G/B, \mathbb{C})$ of Theorem 4 for $G = \text{SO}(2n + 1)$,*

$$\bar{t}_i \mapsto (\epsilon_{s_i} - \epsilon_{s_{i-1}}), \text{ for any } 1 \leq i < n,$$

and

$$\bar{t}_n \mapsto 2\epsilon_{s_n} - \epsilon_{s_{n-1}},$$

where ϵ_{s_0} is to be interpreted as 0.

Definition 22. For $1 \leq r \leq n$, let $\text{OG}(r, 2n + 1)$ be the set of r -dimensional isotropic subspaces of V' with respect to the quadratic form Q , i.e.,

$$\text{OG}(r, 2n + 1) := \{M \in \text{Gr}(r, V') : Q(v) = 0, \forall v \in M\}.$$

Then, $\text{OG}(r, 2n + 1)$ is the quotient $\text{SO}(2n + 1)/P_r^B$ of $\text{SO}(2n + 1)$ by the standard maximal parabolic subgroup P_r^B with $\Delta \setminus \{\alpha_r\}$ as the set of simple roots of its Levi component L_r^B . (Again we take L_r^B to be the unique Levi subgroup of P_r^B containing T_B .) Then,

$$L_r^B \simeq \text{GL}(r) \times \text{SO}(2(n - r) + 1). \tag{21}$$

In this case, by the identity (4) and Proposition 20,

$$\text{Rep}_{\text{poly}}^{\mathbb{C}}(L_r^B) \simeq \mathbb{C}_{\text{sym}}[\bar{t}_1, \dots, \bar{t}_r] \otimes_{\mathbb{C}} \mathbb{C}_{\text{sym}}[(\bar{t}_{r+1})^2, \dots, (\bar{t}_n)^2].$$

From now on we fix $0 < k \leq n - 1$ and consider $\text{OG}(n - k, 2n + 1)$.

The Schubert varieties in $\text{OG}(n - k, 2n + 1)$ are again parametrized by $\mathcal{P}(k, n)$ consisting of k -strict partitions contained in the $(n - k) \times (n + k)$ rectangle. The codimension of this variety is equal to $|\lambda|$. Let $\tau_{\lambda} \in H^{2|\lambda|}(\text{OG}(n - k, 2n + 1), \mathbb{Z})$ denote the cohomology class Poincaré dual to the corresponding fundamental class $[X_{\lambda}^B]$ of the Schubert variety associated to λ . Thus, $\{\tau_{\lambda}\}_{\lambda \in \mathcal{P}(k, n)}$ gives the Schubert basis of $H^*(\text{OG}(n - k, 2n + 1), \mathbb{Z})$ (cf. [6, §2.1]).

We have the following short exact sequence of vector bundles over $\text{OG}(n - k, 2n + 1)$:

$$0 \rightarrow \mathcal{S}_B \rightarrow \bar{\mathcal{E}}' \rightarrow \mathcal{Q}_B \rightarrow 0,$$

where $\bar{\mathcal{E}}'$ is the trivial bundle of rank $2n + 1$, \mathcal{S}_B is the tautological subbundle of rank $n - k$ and \mathcal{Q}_B is the quotient bundle of rank $n + k + 1$. Let $c_i = c_i(\mathcal{Q}_B)$ ($1 \leq i \leq n + k$) denote the i th Chern class of the quotient bundle \mathcal{Q} . (Observe that $c_{n+k+1} = 0$ as can be seen by pulling \mathcal{Q}_B to $\text{SO}(2n + 1)/T_B$, where it admits a nowhere vanishing section given by the vector e_{n+1} .) Then, by [6, §2.3],

$$c_i(\mathcal{Q}_B) = \begin{cases} \tau_i & \text{if } 1 \leq i \leq k \\ 2\tau_i & \text{if } k < i \leq n + k, \end{cases} \tag{22}$$

where $\tau_i := \tau_{(i)}$ and (i) is the partition with single term i .

We have the following presentation of the cohomology ring due to [6, Theorem 2.2(a)]. In the following we follow the convention that $\tau_0 = 1$ and $\tau_p = 0$ if $p < 0$ or $p > n + k$.

Theorem 23. *The cohomology ring $H^*(\text{OG}(n - k, 2n + 1), \mathbb{Z})$ is presented as a quotient of the polynomial ring $\mathbb{Z}[\tau_1, \dots, \tau_{n+k}]$ modulo the relations:*

$$\begin{aligned}
 (\bar{R}_{n,k}^p) \ (n - k + 1 \leq p \leq n) : \quad & \det(\delta_{1+j-i}\tau_{1+j-i})_{1 \leq i, j \leq p} = 0, \\
 (\bar{R}_{n,k}^p) \ (n + 1 \leq p \leq n + k) : \quad & \sum_{r=k+1}^p (-1)^r \tau_r \det(\delta_{1+j-i}\tau_{1+j-i})_{1 \leq i, j \leq p-r} = 0,
 \end{aligned}$$

and

$$(\bar{S}_{n,k}^s) \ (k + 1 \leq s \leq n) : \quad \tau_s^2 + \sum_{i=1}^s (-1)^i \delta_{s-i} \tau_{s+i} \tau_{s-i} = 0,$$

where $\delta_p = 1$ if $p \leq k$ and $\delta_p = 2$ otherwise.

Proposition 24. Take $1 \leq k \leq n - 1$. The map $\xi^{P_{n-k}^B} : \text{Rep}_{\text{poly}}^{\mathbb{C}}(L_{n-k}^B) \rightarrow H^*(\text{OG}(n - k, 2n + 1), \mathbb{C})$ of Theorem 4 under the decomposition (21) takes, for $1 \leq i \leq k$,

$$e_i \left((\bar{t}_{n-k+1})^2, \dots, (\bar{t}_n)^2 \right) \mapsto c_i^2 + 2 \sum_{j=1}^i (-1)^j c_{i+j} c_{i-j},$$

where $c_i = c_i(\mathcal{Q}_B)$ and e_i is the i -th elementary symmetric function.

Proof. It follows by the same proof as that of the corresponding Proposition 11 once we use the following two lemmas. \square

Let $\text{Fl}_B = G/B_B$ be the full flag variety for $G = \text{SO}(2n + 1)$. It consists of partial flags

$$\bar{F}_{\bullet} : \bar{F}_1 \subset \bar{F}_2 \subset \dots \subset \bar{F}_n \subset E' := \mathbb{C}^{2n+1}, \text{ such that each } \bar{F}_j \text{ is isotropic and } \dim \bar{F}_j = j.$$

We can complete the partial flag to a full flag by taking $\bar{F}_{n+j} := \bar{F}_{n+1-j}^{\perp}$. The flags \bar{F}_{\bullet} give rise to a sequence of tautological vector bundles over Fl_B :

$$\bar{\mathcal{F}}_1 \subset \bar{\mathcal{F}}_2 \subset \dots \subset \bar{\mathcal{F}}_n \subset \mathcal{E}', \text{ with rank } \bar{\mathcal{F}}_j = j,$$

where $\mathcal{E}' : \text{Fl}_B \times \mathbb{C}^{2n+1} \rightarrow \text{Fl}_B$ is the trivial rank $2n + 1$ vector bundle. For $1 \leq j \leq n$, define

$$\bar{x}_j := -c_1(\bar{\mathcal{F}}_j / \bar{\mathcal{F}}_{j-1}),$$

where $\bar{\mathcal{F}}_0$ is taken to be the vector bundle of rank 0.

The first part of the following lemma follows from equation (22). The ‘In particular’ statement follows from Lemma 21.

Lemma 25. For $1 \leq j \leq n$, the Schubert divisor $\epsilon_{s_j} \in H^2(\mathbb{F}l_B, \mathbb{Z})$ is given by

$$\begin{aligned} \epsilon_{s_j} &= -c_1(\bar{\mathcal{F}}_j) = \bar{x}_1 + \cdots + \bar{x}_j, \text{ for } j < n, \text{ and} \\ 2\epsilon_{s_n} &= -c_1(\bar{\mathcal{F}}_n) = \bar{x}_1 + \cdots + \bar{x}_n. \end{aligned}$$

In particular, under ξ^B for $G = \text{SO}(2n + 1)$, $\bar{t}_j \mapsto \bar{x}_j$ for any $1 \leq j \leq n$.

For $0 \leq j \leq n$, let

$$\bar{Q}_j := \mathcal{E}' / \bar{\mathcal{F}}_j.$$

Observe that the orthogonal form gives an isomorphism of vector bundles:

$$\bar{Q}_j \simeq (\bar{\mathcal{F}}_j^\perp)^*. \tag{23}$$

Lemma 26. For $0 \leq j \leq n$, the following holds:

$$c(\bar{Q}_j)c(\bar{Q}_j^*) = \prod_{p=j+1}^n (1 - \bar{x}_p)(1 + \bar{x}_p),$$

where c is the total Chern class.

Proof. The lemma follows by the same proof as that of the corresponding Lemma 13 once we observe that

$$c(\bar{\mathcal{F}}_n) = c(\bar{\mathcal{F}}_{n+1}),$$

which follows from the fact that $\bar{\mathcal{F}}_{n+1} / \bar{\mathcal{F}}_n$ pulled back to $\text{SO}(2n+1) / T_B$ admits a nowhere vanishing section since the vector e_{n+1} is held fixed by T_B . \square

Remark 27. Even though we do not need, the map $\xi^{P_{n-k}^B} : \text{Rep}_{\text{poly}}^{\mathbb{C}}(L_{n-k}^B) \rightarrow H^*(\text{OG}(n-k, 2n+1), \mathbb{C})$ of Theorem 4 under the decomposition (21) takes for $1 \leq i \leq n-k$,

$$\begin{aligned} e_i(\bar{t}_1, \dots, \bar{t}_{n-k}) &\mapsto c_i(\mathcal{S}_B) = \epsilon_{s_{n-k-i+1} \cdots s_{n-k}}, \text{ if } k > 0, \\ e_i(\bar{t}_1, \dots, \bar{t}_{n-k}) &\mapsto c_i(\mathcal{S}_B) = 2\epsilon_{s_{n-k-i+1} \cdots s_{n-k}}, \text{ if } k = 0. \end{aligned}$$

Definition 28 (Inverse Limit). Analogous to Definition 15, for any $k \geq 1$, define the stable cohomology ring [7, §3.2] as

$$\mathbb{H}^*(\text{OG}_k, \mathbb{Z}) = \varprojlim H^*(\text{OG}(n-k, 2n+1), \mathbb{Z})$$

as the inverse limit (in the category of graded rings) of the inverse system

$$\cdots \leftarrow H^*(\text{OG}(n - k, 2n + 1), \mathbb{Z}) \xleftarrow{\bar{\pi}_n^*} H^*(\text{OG}(n - k + 1, 2n + 3), \mathbb{Z}) \leftarrow \cdots ,$$

where $\bar{\pi}_n : \text{OG}(n - k, 2n + 1) \hookrightarrow \text{OG}(n - k + 1, 2n + 3)$ is given by $V \mapsto \bar{T}_n(V) \oplus \mathbb{C}e_{n+1}$ and $\bar{T}_n : \mathbb{C}^{2n+1} \rightarrow \mathbb{C}^{2n+3}$ is the linear embedding taking $e_i \mapsto e_i$ for $1 \leq i \leq n$, taking $e_{n+1} \mapsto e_{n+2}$ and taking $e_i \mapsto e_{i+2}$ for $n + 2 \leq i \leq 2n + 1$.

This ring has an additive basis consisting of Schubert classes τ_λ for each k -strict partition λ . The natural ring homomorphism $\bar{\varphi}_{k,n} : \mathbb{H}^*(\text{OG}_k, \mathbb{Z}) \rightarrow H^*(\text{OG}(n - k, 2n + 1), \mathbb{Z})$ takes τ_λ to τ_λ whenever λ fits in a $(n - k) \times (n + k)$ rectangle and to zero otherwise. In particular, $\bar{\varphi}_{k,n}$ is surjective. From the definition of the Chern classes $c_j^n = c_j^n(\mathcal{Q}_B)$, it is easy to see that under the restriction map $\bar{\pi}_n^* : H^*(\text{OG}(n - k + 1, 2n + 3), \mathbb{Z}) \rightarrow H^*(\text{OG}(n - k, 2n + 1), \mathbb{Z})$, $c_j^{n+1} \mapsto c_j^n$ for $1 \leq j \leq n + k$ and $c_{n+k+1}^{n+1} \mapsto 0$.

From the presentation of the ring $H^*(\text{OG}(n - k, 2n + 1), \mathbb{Z})$ (Theorem 23), $\mathbb{H}^*(\text{OG}_k, \mathbb{Z})$ is isomorphic with the polynomial ring $\mathbb{Z}[\tau_1, \tau_2, \dots]$ modulo the relations:

$$(\bar{S}^s) \ (s > k) : \quad \tau_s^2 + \sum_{i=1}^s (-1)^i \delta_{s-i} \tau_{s+i} \tau_{s-i} = 0. \tag{24}$$

Take $k \geq 1$. Recall from Proposition 20 (which remains valid for $k = 1$ as well) that

$$\text{Rep}_{\text{poly}}^{\mathbb{C}}(\text{SO}(2k + 1)) \simeq \mathbb{C}_{\text{sym}} [(\bar{h}_1)^2, \dots, (\bar{h}_k)^2].$$

Define a ring homomorphism (for any $1 \leq k \leq n - 1$)

$$\bar{l}_k^n : \text{Rep}_{\text{poly}}^{\mathbb{C}}(\text{SO}(2k + 1)) \rightarrow \text{Rep}_{\text{poly}}^{\mathbb{C}}(L_{n-k}^B)$$

by taking $f(\bar{\mathbf{h}}) \mapsto 1 \otimes f(\bar{\mathbf{t}})$, where $\bar{\mathbf{h}} := (\bar{h}_1, \dots, \bar{h}_k)$, $\bar{\mathbf{t}} := (\bar{t}_{n-k+1}, \dots, \bar{t}_n)$ and $f(\bar{\mathbf{t}})$ is the same polynomial written in the $\bar{\mathbf{t}}$ -variables under the transformation $\bar{h}_p \mapsto \bar{t}_{n-k+p}$. This gives rise to the map $\bar{\xi}_{n,k} := \xi_{n-k}^{P^B} \circ \bar{l}_k^n : \text{Rep}_{\text{poly}}^{\mathbb{C}}(\text{SO}(2k + 1)) \rightarrow H^*(\text{OG}(n - k, 2n + 1), \mathbb{C})$. Consider the following diagram, which is commutative because of Proposition 24.

$$\begin{array}{ccc}
 & & \uparrow \bar{\pi}_{n-1}^* \\
 \text{Rep}_{\text{poly}}^{\mathbb{C}}(\text{SO}(2k + 1)) & \xrightarrow{\bar{\xi}_{n,k}} & H^*(\text{OG}(n - k, 2n + 1), \mathbb{C}) \\
 & \searrow \bar{\xi}_{n+1,k} & \uparrow \bar{\pi}_n^* \\
 & & H^*(\text{OG}(n - k + 1, 2n + 3), \mathbb{C}) \\
 & & \uparrow \bar{\pi}_{n+1}^*
 \end{array}$$

The compatible ring homomorphisms $\bar{\xi}_{n,k} : \text{Rep}_{\text{poly}}^{\mathbb{C}}(\text{SO}(2k + 1)) \rightarrow H^*(\text{OG}(n - k, 2n + 1), \mathbb{C})$ combine to give a ring homomorphism

$$\bar{\xi}_k : \text{Rep}_{\text{poly}}^{\mathbb{C}}(\text{SO}(2k + 1)) \rightarrow \mathbb{H}^*(\text{OG}_k, \mathbb{C}).$$

The following theorem is our second main result of the paper, which is analogous to Theorem 16.

Theorem 29. *Let $k \geq 1$ be an integer. The above ring homomorphism $\bar{\xi}_k : \text{Rep}_{\text{poly}}^{\mathbb{C}}(\text{SO}(2k + 1)) \rightarrow \mathbb{H}^*(\text{OG}_k, \mathbb{C})$ takes the generators*

$$e_i((\bar{h}_1)^2, \dots, (\bar{h}_k)^2) \mapsto c_i^2 + 2 \sum_{j=1}^i (-1)^j c_{i+j} c_{i-j}, \text{ for any } 1 \leq i \leq k, \tag{25}$$

where $c_i := c_i(\mathcal{Q}_B)$.

Moreover, $\bar{\xi}_k$ is injective.

Proof. The first part follows from Proposition 24.

We next prove the injectivity of $\bar{\xi}_k$:

By Proposition 20, $\text{Rep}_{\text{poly}}^{\mathbb{C}}(\text{SO}(2k + 1))$ is a polynomial ring over \mathbb{C} generated by $\{e_1(\bar{\mathbf{z}}), \dots, e_k(\bar{\mathbf{z}})\}$, where $\bar{\mathbf{z}} = (z_1, \dots, z_k)$ and $z_i := (\bar{h}_i)^2$. Let $\text{Rep}_{\text{poly}}^{\mathbb{Z}}(\text{SO}(2k + 1))$ be the polynomial subring over \mathbb{Z} generated by $\{e_1(\bar{\mathbf{z}}), \dots, e_k(\bar{\mathbf{z}})\}$.

Let $\bar{\mathbb{H}}^*(\text{OG}_k, \mathbb{Z}) \subset \mathbb{H}^*(\text{OG}_k, \mathbb{Z})$ be the subring generated by $\{c_i\}_{i \geq 1}$. Then, by the identity (22),

$$\mathbb{C} \otimes_{\mathbb{Z}} \bar{\mathbb{H}}^*(\text{OG}_k, \mathbb{Z}) = \mathbb{C} \otimes_{\mathbb{Z}} \mathbb{H}^*(\text{OG}_k, \mathbb{Z}) = \mathbb{H}^*(\text{OG}_k, \mathbb{C}). \tag{26}$$

Then, by the equation (25),

$$\bar{\xi}_k(\text{Rep}_{\text{poly}}^{\mathbb{Z}}(\text{SO}(2k + 1))) \subset \bar{\mathbb{H}}^*(\text{OG}_k, \mathbb{Z}).$$

Thus, on restriction, we get the ring homomorphism

$$\bar{\xi}_k^{\mathbb{Z}} : \text{Rep}_{\text{poly}}^{\mathbb{Z}}(\text{SO}(2k + 1)) \rightarrow \bar{\mathbb{H}}^*(\text{OG}_k, \mathbb{Z}).$$

Observe further that $\bar{\xi}_k^{\mathbb{Z}}$ is a homomorphism of graded rings if we assign degree $4i$ to each e_i (and the standard cohomological degree to $\bar{\mathbb{H}}^*(\text{OG}_k, \mathbb{Z})$). Let \bar{K} be the Kernel of $\bar{\xi}_k^{\mathbb{Z}}$. Since $\bar{\mathbb{H}}^*(\text{OG}_k, \mathbb{Z})$ is a free \mathbb{Z} -module of finite rank in each degree and hence so is $\bar{\mathbb{H}}^*(\text{OG}_k, \mathbb{Z})$, the induced homomorphism

$$\mathbb{Z}/(2) \otimes_{\mathbb{Z}} \bar{K} \rightarrow \mathbb{Z}/(2) \otimes_{\mathbb{Z}} \text{Rep}_{\text{poly}}^{\mathbb{Z}}(\text{SO}(2k + 1)) \text{ is injective.} \tag{27}$$

We next observe that the induced homomorphism

$$\mathbb{Z}/(2) \otimes_{\mathbb{Z}} \text{Rep}_{\text{poly}}^{\mathbb{Z}}(\text{SO}(2k + 1)) \rightarrow \mathbb{Z}/(2) \otimes_{\mathbb{Z}} \bar{\mathbb{H}}^*(\text{OG}_k, \mathbb{Z}) \text{ is injective.} \tag{28}$$

To prove this, observe that

$$\mathbb{Z}/(2) \otimes_{\mathbb{Z}} \text{Rep}_{\text{poly}}^{\mathbb{Z}}(\text{SO}(2k + 1)) \simeq \mathbb{Z}/(2)[e_1, \dots, e_k]. \tag{29}$$

Moreover, by the defining relations (\bar{S}^s) ($s > k$) of $\mathbb{H}^*(\text{OG}_k, \mathbb{Z})$ as in equation (24) together with the identity (22), we can rewrite the equation (24) as:

$$(\hat{S}^s) \ (s > k) : \quad c_s^2 + 2 \sum_{i=1}^s (-1)^i c_{s+i} c_{s-i} = 0.$$

Thus,

$$\mathbb{Z}/(2) \otimes_{\mathbb{Z}} \bar{\mathbb{H}}^*(\text{OG}_k, \mathbb{Z}) \simeq \mathbb{Z}/(2)[c_1, \dots, c_k] \otimes \frac{\mathbb{Z}/(2)[c_{k+1}, c_{k+2} \dots]}{\langle c_{k+1}^2, c_{k+2}^2 \dots \rangle}. \tag{30}$$

Moreover, under the above identifications (29) and (30), by the first part of the theorem, the ring homomorphism $\bar{\xi}_k^{\mathbb{Z}}$ modulo 2 is given by

$$e_i \mapsto c_i^2, \text{ for any } 1 \leq i \leq k.$$

In particular, it is injective. From this we obtain that

$$\mathbb{Z}/(2) \otimes_{\mathbb{Z}} \bar{K} = 0.$$

But, since \bar{K} is a finitely generated torsionfree \mathbb{Z} -module in each graded degree (thus free) we get that

$$\bar{K} = 0.$$

Since \mathbb{C} is a torsionfree \mathbb{Z} -module, by the equation (26), this clearly gives the injectivity of $\bar{\xi}_k$ proving the theorem. \square

Remark 30. The ring homomorphism $\bar{\xi}_k : \text{Rep}_{\text{poly}}^{\mathbb{C}}(\text{SO}(2k + 1)) \rightarrow \mathbb{H}^*(\text{OG}_k, \mathbb{C})$ of the above Theorem 29 is *not* surjective, as can be easily seen since the domain is a finitely generated \mathbb{C} -algebra (by Proposition 20) whereas the range is not (for otherwise for each n , $H^*(\text{OG}(n - k, 2n + 1), \mathbb{C})$ would be generated by a fixed finite number of generators independent of n).

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