CONFORMAL BLOCKS FOR GALOIS COVERS OF ALGEBRAIC CURVES

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Abstract. We study the spaces of twisted conformal blocks attached to a $\Gamma$-curve $\Sigma$ with marked $\Gamma$-orbits and an action of $\Gamma$ on a simple Lie algebra $g$, where $\Gamma$ is a finite group. We prove that if $\Gamma$ stabilizes a Borel subalgebra of $g$, then Propagation Theorem and Factorization Theorem hold. We endow a projectively flat connection on the sheaf of twisted conformal blocks attached to a smooth family of pointed $\Gamma$-curves; in particular, it is locally free. We also prove that the sheaf of twisted conformal blocks on the stable compactification of Hurwitz stack is locally free.

Let $\mathcal{G}$ be the parahoric Bruhat-Tits group scheme on the quotient curve $\Sigma/\Gamma$ obtained via the $\Gamma$-invariance of Weil restriction associated to $\Sigma$ and the simply-connected simple algebraic group $G$ with Lie algebra $g$. We prove that the space of twisted conformal blocks can be identified with the space of generalized theta functions on the moduli stack of quasi-parabolic $\mathcal{G}$-torsors on $\Sigma/\Gamma$ when the level $c$ is divisible by $|\Gamma|$ (establishing a conjecture due to Pappas-Rapoport).

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1. Introduction

Wess-Zumino-Witten model is a type of two dimensional conformal field theory, which associates to an algebraic curve with marked points and integrable highest weight
modules of an affine Kac-Moody Lie algebra associated to the points, a finite dimensional vector space consisting of conformal blocks. The space of conformal blocks has many important properties including Propagation of Vacua and Factorization. Deforming the pointed algebraic curves in a family, we get a sheaf of conformal blocks. This sheaf admits a projectively flat connection when the family of pointed curves is a smooth family. The mathematical theory of conformal blocks was first established in a pioneering work by Tsuchiya-Ueno-Yamada [TUY] where all these properties were obtained. All the above properties are important ingredients in the proof of the celebrated Verlinde formula for the dimension of the space of conformal blocks (cf. [Be, Fa, So1, V]). This theory has a geometric counterpart in the theory of moduli spaces of principal bundles over algebraic curves and also the moduli of curves and its stable compactification.

In this paper we study a twisted theory of conformal blocks on Galois covers of algebraic curves. More precisely, we consider an algebraic curve \( \Sigma \) with an action of a finite group \( \Gamma \). Moreover, we take a group homomorphism \( \phi : \Gamma \to \text{Aut}(g) \) of \( \Gamma \) acting on a simple Lie algebra \( g \). Given any smooth point \( q \in \Sigma \), we attach an affine Lie algebra \( \hat{L}(g, \Gamma_q) \) (in general a twisted affine Lie algebra), where \( \Gamma_q \) is the stabilizer group of \( q \) at \( q \). The integrable highest weight representations of \( \hat{L}(g, \Gamma_q) \) of level \( c \) (where \( c \) is a positive integer) are parametrized by certain finite set \( D_{c,q} \) of dominant weights of the reductive Lie algebra \( \hat{g}^F \), i.e., for any \( \lambda \in D_{c,q} \) we attach an integrable highest weight representation \( \mathcal{H}(\lambda) \) of \( \hat{L}(g, \Gamma_q) \) of level \( c \) and conversely (cf. Section 2). Given a collection \( \vec{q} := (q_1, \ldots, q_s) \) of smooth points in \( \Sigma \) such that their \( \Gamma \)-orbits are disjoint and a collection of weights \( \vec{\lambda} = (\lambda_1, \ldots, \lambda_s) \) with \( \lambda_i \in D_{c,q_i} \), we consider the representation \( \mathcal{H}(\vec{\lambda}) := \mathcal{H}(\lambda_1) \otimes \cdots \otimes \mathcal{H}(\lambda_s) \). Now, define the associated space of twisted covacua (or twisted dual conformal blocks) as follows:

\[
y_{\Sigma, \Gamma, \phi}(\vec{q}, \vec{\lambda}) := \frac{\mathcal{H}(\vec{\lambda})}{\mathcal{H}(\vec{\lambda})},
\]

where \( g[\Sigma \setminus \Gamma \cdot \vec{q}]^\Gamma \) is the Lie algebra of \( \Gamma \)-equivariant regular functions from \( \Sigma \setminus \Gamma \cdot \vec{q} \) to \( g \) acting on the \( i \)-th factor \( \mathcal{H}(\lambda_i) \) of \( \mathcal{H}(\vec{\lambda}) \) via its Laurent series expansion at \( q_i \). In this paper we often work with a more intrinsic but equivalent definition of the space of twisted covacua (see Definition 3.5), where we work with marked \( \Gamma \)-orbits.

The following Propagation of Vacua is the first main result of the paper (cf. Corollary 4.5 (a)).

**Theorem A.** Assume that \( \Gamma \) stabilizes a Borel subalgebra of \( g \). Let \( q \) be a smooth point of \( \Sigma \) such that \( q \) is not \( \Gamma \)-conjugate to any point \( \vec{q} \). Assume further that \( 0 \in D_{c,q} \) (cf. Corollary 2.2). Then, we have the following isomorphism of spaces of twisted covacua:

\[
y_{\Sigma, \Gamma, \phi}(\vec{q}, \vec{\lambda}) \cong y_{\Sigma, \Gamma, \phi}(0, \vec{\lambda}).
\]

In fact, a stronger version of Propagation Theorem is proved (cf. Theorem 4.3 and Corollary 4.5 (b)). Even though, we generally follow the argument given in [Be, Proposition 2.3], in our equivariant setting we need to generalize some important ingredients. For example, the fact that

"The endormorphism \( X_q \otimes f \) of \( \mathcal{H} \) is locally nilpotent for all \( f \in \mathcal{O}(U) \)"
in the proof of Proposition 2.3 of [Be], can not easily be generalized to the twisted case. To prove an analogous result, we need to assume that $\Gamma$ stabilizes a Borel subalgebra of $g$, and use Lemma 2.5 crucially. It will be interesting to see if this assumption can be removed.

Let $q$ be a nodal point in $\Sigma$. Assume that the action of $\Gamma$ at $q$ is stable (see Definition 5.1) and the stabilizer group $\Gamma_q$ does not exchange the two formal branches around $q$. Let $\Sigma'$ be the normalization of $\Sigma$ at the points $\Gamma \cdot q$, and let $q', q''$ be the two smooth points in $\Sigma'$ over $q$. The following Factorization Theorem is our second main result (cf. Theorem 5.4).

**Theorem B.** Assume that $\Gamma$ stabilizes a Borel subalgebra of $g$. Then, there exists a natural isomorphism:

$$\mathcal{V}_{\Sigma, \Gamma, \phi}(\vec{q}, \vec{\lambda}) \cong \bigoplus_{\mu \in D_c, q''} \mathcal{V}_{\Sigma', \Gamma, \phi}(\vec{q}, \vec{q}', \vec{q}'', \vec{\lambda}, \vec{\mu}^*, \mu),$$

where $\mu^*$ is the dominant weight of $g^{F \sigma}$ such that $V(\mu^*)$ is the dual representation $V(\mu)$ of $g^{F \sigma} = g^{F \cdot \sigma} = g^{F \sigma}$. The formulation of the Factorization Theorem in the twisted case is a bit more delicate, since the parameter sets $D_c, q'$ and $D_c, q''$ attached to the points $q', q''$ are different in general; nevertheless they are related by the dual of representations under the assumption that the action of $\Gamma$ at the node $q$ is stable and the stabilizer group $\Gamma_q$ does not exchange the branches (cf. Lemma 5.3). Its proof requires additional care (from that of the untwisted case) at several places. The assumption that $\Gamma$ stabilizes a Borel subalgebra of $g$ also appears in this theorem as we use the Propagation Theorem in its proof.

Given a family $(\Sigma_T, \vec{q})$ of $s$-pointed $\Gamma$-curves over a connected scheme $T$ and weights $\vec{\lambda} = (\lambda_1, \ldots, \lambda_s)$ with $\lambda_i \in D_c, q_i$ as above, one can attach a functorial coherent sheaf $\mathcal{V}_{\Sigma_T, \Gamma, \phi}(\vec{q}, \vec{\lambda})$ of twisted covacua over the base $T$ (cf. Definition 7.7 and Theorem 7.8). We prove the following stronger theorem (cf. Theorems 7.10 and 7.12).

**Theorem C.** Assume that the family $\Sigma_T \to T$ is a smooth family over a smooth base $T$. Then, the sheaf $\mathcal{V}_{\Sigma_T, \Gamma, \phi}(\vec{q}, \vec{\lambda})$ of twisted covacua is locally free of finite rank over $T$. In fact, there exists a projectively flat connection on $\mathcal{V}_{\Sigma_T, \Gamma, \phi}(\vec{q}, \vec{\lambda})$.

This theorem relies mainly on the Sugawara construction for the twisted affine Kac-Moody algebras. In the untwisted case, this construction is quite well-known (cf. [Ka, §12.8]). In the twisted case, the construction can be found in [KW, W], where the formulae are written in terms of the abstract Kac-Moody presentation of $\hat{L}(g, \sigma)$, where $\sigma$ is a finite order automorphism of $g$. For our application, we require the formulae in terms of the affine realization of $\hat{L}(g, \sigma)$ as a central extension of the twisted loop algebra $\mathfrak{g}((t))^{F \sigma}$. We present such a formula in (79) in Section 6, which might be new (to our knowledge).

Let $\mathcal{H}M_{g, \Gamma, \eta}$ be the Hurwitz stack of $\Gamma$-stable $s$-pointed $\Gamma$-curves of genus $g$ with marking data $\eta$ at the marked points such that the set of $\Gamma$-orbits of the marked points contains the full ramification divisor (cf. Definition 8.7). It was proved by Bertin-Romagny [BR] that $\mathcal{H}M_{g, \Gamma, \eta}$ is a smooth and proper Deligne-Mumford stack (cf. Theorem 8.8).
We can attach a collection \( \hat{\lambda} \) of dominant weights to the marking data \( \eta \), and associate a coherent sheaf \( \mathcal{V}_{g, \Gamma, \phi}(\eta, \hat{\lambda}) \) of twisted covacua over the Hurwitz stack \( \mathcal{M}_{g, \Gamma, \eta} \). The presence of the Hurwitz stack is a new phenomenon in the twisted theory. We prove the following theorem (cf. Theorem 8.9).

**Theorem D.** Assume that \( \Gamma \) stabilizes a Borel subalgebra of \( \mathfrak{g} \). Then, the sheaf \( \mathcal{V}_{g, \Gamma, \phi}(\eta, \hat{\lambda}) \) is locally free over the stack \( \mathcal{M}_{g, \Gamma, \eta} \).

Our proof of this theorem follows closely the work of Looijenga [L] in the non-equivariant setting; in particular, we use the canonical smoothing deformation of nodal curves (Lemma 8.3) and gluing tensor elements (Lemma 8.5 and the construction before that). The Factorization Theorem also plays a crucial role in the proof. In the case \( \Gamma \) is cyclic, this theorem together with the Factorization Theorem allows us to reduce the computation of the dimension of the space of twisted covacua to the case of Galois covers of projective line with three marked points (see Remark 8.11 (1)).

There were some earlier works related to the twisted theory of conformal blocks. For example, Frenkel-Szczesny [FS] studied the twisted modules over Vertex algebras on algebraic curves, and Kuroki-Takebe [KT] studied a twisted Wess-Zumino-Witten model on elliptic curves. When \( \Gamma \) is of prime order and the marked points are unramified, the space \( \mathcal{V}_{g, \Gamma, \phi}(\tilde{q}, \tilde{\lambda}) \) has been studied recently by Damiolini [D], where she proved similar results as ours described above. Our work is a vast generalization of her work, since we do not need to put any restrictions on the \( \Gamma \)-orbits, and the only restriction on \( \Gamma \) is that \( \Gamma \) stabilizes a Borel subalgebra of \( \mathfrak{g} \) (when \( \Gamma \) is a cyclic group it automatically holds). In particular, when \( \Gamma \) has nontrivial stabilizers at the marked points \( \tilde{q} \), general twisted affine Kac-Moody Lie algebras and their representations occur naturally in this twisted theory of conformal blocks, which was not the case in Damiolini’s work (since marked points being unramified in her work, only untwisted affine Lie algebras appeared). Nor did she consider the identification of \( \mathcal{V}_{g, \Gamma, \phi}(\tilde{q}, \tilde{\lambda})^\dagger \) with the space of global sections of certain line bundles over \( \mathcal{P}_{\text{arbun}}(\tilde{P}) \) (see Theorem E). We also learnt from S. Mukhopadhyay that he has obtained some results (unpublished) in this direction.

In the usual (untwisted) theory of conformal blocks, the space of conformal blocks has a beautiful geometric interpretation in that it can be identified with the space of generalized theta functions on the moduli space of parabolic \( G \)-bundles over the algebraic curve, where \( G \) is the simply-connected simple algebraic group associated to \( \mathfrak{g} \) (cf. Beauville-Laszlo [BL], Faltings [Fa], Kumar-Narasimhan-Ramanathan [KNR], Laszlo-Sorger [LS] and Pauly [P]).

From a \( \Gamma \)-curve \( \Sigma \) and an action of \( \Gamma \) on \( G \), the \( \Gamma \)-invariants of Weil restriction produces a parahoric Bruhat-Tits group scheme \( \mathcal{G} \) on \( \tilde{\Sigma} = \Sigma/\Gamma \). Recently, the geometry of the moduli stack \( \mathcal{Brun}_G \) of \( \mathcal{G} \)-torsors over \( \tilde{\Sigma} \) has extensively been studied by Pappas-Rapoport [PR1, PR2], Heinloth [He], Balaji-Seshadri [BS] and Zhu [Zh]. A connection between generalized theta functions on \( \mathcal{Bun}_G \) and twisted conformal blocks associated to the Lie algebra of \( \mathcal{G} \) was conjectured by Pappas-Rapoport [PR2]. Along this direction, some results have recently been obtained by Zelaci [Z] when \( \Gamma \) is of order 2 acting on \( \mathfrak{g} = \mathfrak{sl}_n \) by certain involutions and very special weights.
We study this connection in full generality in the setting of \( \Gamma \)-curves \( \Sigma \). Let \( G \) be the simply-connected simple algebraic group with the action of \( \Gamma \) corresponding to \( \phi : \Gamma \to \text{Aut}(\mathfrak{g}) \). We assume that \( \Sigma \) is a smooth irreducible projective curve with a collection \( \vec{q} = (q_1, \cdots, q_s) \) of marked points such that their \( \Gamma \)-orbits are disjoint. To this, we attach a collection \( \vec{\lambda} = (\lambda_1, \cdots, \lambda_s) \) of weights with \( \lambda_i \in D_{\ell, q_i} \) as before. Assume that \( c \) is divisible by \( |\Gamma| \). Then, the irreducible representation \( V(\lambda_i) \) of \( g^{\Gamma_{\lambda_i}} \) of highest weight \( \lambda_i \) integrates to an algebraic representation of \( G^{\Gamma_{\lambda_i}} \) (cf. Proposition 10.9), where \( G^{\Gamma_{\lambda_i}} \) is the fixed subgroup of \( \Gamma_{q_i} \) in \( G \). Let \( P_i^{\phi} \) be the stabilizer in \( G^{\Gamma_{\lambda_i}} \) of the highest weight line \( \ell_{\lambda_i} \subset V(\lambda_i) \). Let \( \mathcal{G} \) be the parahoric Bruhat-Tits group scheme over \( \Sigma := \Sigma/\Gamma \) obtained from the \( \Gamma \)-invariants of the Weil restriction via \( \pi : \Sigma \to \bar{\Sigma} \) from the constant group scheme \( G \times \Sigma \rightarrow \Sigma \) over \( \Sigma \) (cf. Definition 11.1). One can attach the moduli stack \( \mathcal{P}arbun_{\vec{q}}(\bar{\mathcal{P}}) \) of quasi-parabolic \( \mathcal{G} \)-torsors with parabolic subgroups \( \bar{\mathcal{P}} = (P_i^{\phi}) \) attached to \( q_i \) for each \( i \) (cf. Definition 11.2). With the assumption that \( c \) is divisible by \( |\Gamma| \), we can define a line bundle \( \mathcal{L}(c, \vec{\lambda}) \) on \( \mathcal{P}arbun_{\vec{q}}(\bar{\mathcal{P}}) \) (cf. Definition 11.6). The following is our last main theorem (cf. Theorem 12.1) confirming a conjecture of Pappas-Rapoport for \( \mathcal{G} \).

**Theorem E.** Assume that \( \Gamma \) stabilizes a Borel subalgebra of \( \mathfrak{g} \) and that \( c \) is divisible by \( |\Gamma| \). Then, there exists a canonical isomorphism:

\[
H^0(\mathcal{P}arbun_{\vec{q}}(\bar{\mathcal{P}}), \mathcal{L}(c, \vec{\lambda})) \cong \mathcal{V}_{\Sigma, \Gamma, \phi}(\vec{q}, \vec{\lambda})^\dagger,
\]

where \( H^0(\mathcal{P}arbun_{\vec{q}}(\bar{\mathcal{P}}), \mathcal{L}(c, \vec{\lambda})) \) denotes the space of global sections of the line bundle \( \mathcal{L}(c, \vec{\lambda}) \) and \( \mathcal{V}_{\Sigma, \Gamma, \phi}(\vec{q}, \vec{\lambda})^\dagger \) denotes the space of twisted conformal blocks, i.e., the dual space of \( \mathcal{V}_{\Sigma, \Gamma, \phi}(\vec{q}, \vec{\lambda}) \).

One of the main ingredients in the proof of this theorem is the connectedness of the ind-group \( \text{Mor}_\Gamma(\Sigma^*, G) \) consisting of \( \Gamma \)-equivariant morphisms from \( \Sigma^* \) to \( G \) (cf. Theorem 9.5), where \( \Sigma^* \) is a \( \Gamma \)-stable affine open subset of \( \Sigma \). Another important ingredient is the Uniformization Theorem for the stack of \( \mathcal{G} \)-torsors on the parahoric Bruhat-Tits group scheme \( \mathcal{G} \) due to Heinloth [He]; in fact, its parabolic analogue (cf. Theorem 11.3). Finally, yet another ingredient is the splitting of the central extension of the twisted loop group \( G(D_q^\times)^{\Gamma_{\lambda_i}} \) over \( \Xi = \text{Mor}_\Gamma(\Sigma\setminus q, G) \) and the reducedness and the irreducibility of \( \Xi \) (cf. Theorem 10.7 and Corollary 11.5), where \( q \) is a point in \( \Sigma \) and \( D_q^\times \) (resp. \( D_q \)) is the punctured formal disc (resp. formal disc) around \( q \) in \( \Sigma \).

In spite of the parallels with the classical case, there are some important essential differences in the twisted case. First of all the constant group scheme is to be replaced by the parahoric Bruhat-Tits group scheme \( \mathcal{G} \). Further, the group \( \Xi \) could have nontrivial characters resulting in the splitting over \( \Xi \) non-unique. (It might be mentioned that in the special case considered by Zelaci [Z, Proposition 5.1] mentioned above, \( \Xi \) has only trivial character.) To overcome this difficulty, we need to introduce a *canonical* splitting over \( \Xi \) of the central extension of the twisted loop group \( G(D_q^\times)^{\Gamma_{\lambda_i}} \) (cf. Theorem 10.7). We are able to do it when \( c \) is divisible by \( |\Gamma| \) (cf. Remark 12.2 (b)).

It is interesting to remark that Zhu [Zh] proved that for any line bundle on the moduli stack \( \mathcal{B}un_{\vec{q}} \) for a ‘reasonably good’ parahoric Bruhat-Tits group scheme \( \mathcal{G} \) over a curve \( \bar{\Sigma} \), the pull-back of the line bundle to the twisted affine Grassmannian at every point of
\( \Sigma \) is of the same central charge. It matches the way we define the space of covacua, i.e., we attach integrable highest weight representations of twisted affine Lie algebras of the same central charge at every point.

Our work was initially motivated by a conjectural connection predicted by Fuchs-Schweigert [FSc] between the trace of diagram automorphism on the space of conformal blocks and certain conformal field theory related to twisted affine Lie algebras. A Verlinde type formula for the trace of diagram automorphism on the space of conformal blocks has been proved recently by the first author [Ho1, Ho2], where the formula involves the twisted affine Kac-Moody algebras mysteriously.

In the following we recall the structure of this paper.

In Section 2, we introduce the twisted affine Lie algebra \( \hat{L}(\mathfrak{g}, \sigma) \) attached to a finite order automorphism \( \sigma \) of \( \mathfrak{g} \) following [Ka, §8]. We prove some preparatory lemmas which will be used later in Section 4.

In Section 3, we define the space of twisted covacua attached to a Galois cover of an algebraic curve. We prove that this space is finite dimensional.

Section 4 is devoted to proving the Propagation Theorem.

Section 5 is devoted to proving the Factorization Theorem.

In Section 6, we prove the independence of parameters for integrable highest weight representations of twisted affine Kac-Moody algebras over a base. We also prove that the Sugawara operators acting on the integrable highest weight representations of twisted affine Kac-Moody algebras are independent of the parameters up to scalars. This section is preparatory for Section 7.

In Section 7, we define the sheaf of twisted covacua for a family \( \Sigma_T \) of \( s \)-pointed \( \Gamma \)-curves. We further show that this sheaf is locally free of finite rank for a smooth family \( \Sigma_T \) over a smooth base \( T \). In fact, it admits a projectively flat connection.

In Section 8, we consider families of \( \Gamma \)-stable \( s \)-pointed \( \Gamma \)-curves and we show that the sheaf of twisted covacua over the stable compactification of Hurwitz stack is locally free.

In Section 9, we prove the connectedness of the ind-group \( \text{Mor}_\Gamma(\Sigma^*, G) \), following an argument by Drinfeld in the non-equivariant case. In particular, we show that the twisted Grassmannian \( X^\gamma = G(\mathbb{D}_q)^{\Gamma_q}/G(\mathbb{D}_q)^{\Gamma_q} \) is irreducible.

In Section 10, we construct the central extensions of the twisted loop group \( G(\mathbb{D}_q)^{\Gamma_q} \) and prove the existence of its canonical splitting over \( \Xi := \text{Mor}_\Gamma(\Sigma \setminus \Gamma \cdot q, G) \).

In Section 11, we recall the uniformization theorem due to Heinloth for the parahoric Bruhat-Tits group schemes \( \mathcal{G} \) in our setting, and introduce the moduli stack \( \mathcal{P}arbun_{\mathcal{G}} \) of quasi-parabolic \( \mathcal{G} \)-torsors over \( \Sigma \) and construct the line bundles over \( \mathcal{P}arbun_{\mathcal{G}} \).

In Section 12, we establish the identification of twisted conformal blocks and generalized theta functions on the moduli stack \( \mathcal{P}arbun_{\mathcal{G}} \).

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After the appearance of our paper, a preprint [DM] by Deshpande-Mukhopadhyay on Verlinde formula for twisted conformal blocks appeared recently.

2. Twisted affine Kac-Moody algebras

This section is devoted to recalling the definition of twisted affine Kac-Moody Lie algebras and their basic properties (we need).

Let \( \sigma \) be an operator of finite order \( m \) acting on two vector spaces \( V \) and \( W \) over \( \mathbb{C} \). Consider the diagonal action of \( \sigma \) on \( V \otimes W \). We have the following decomposition of the \( \sigma \)-invariant subspace in \( V \otimes W \),

\[
(V \otimes W)^\sigma = \oplus_V V_\xi \otimes W_{\xi^{-1}},
\]

where the summation is over \( m \)-th roots of unity and \( V_\xi \) (resp. \( W_{\xi^{-1}} \)) is the \( \xi \)-eigenspace of \( V \) (resp. \( \xi^{-1} \)-eigenspace of \( W \)). We say \( v \otimes w \) is pure or more precisely \( \xi \)-pure if \( v \otimes w \in V_\xi \otimes W_{\xi^{-1}} \). Throughout this paper, if we write \( v \otimes w \in (V \otimes W)^\sigma \), we mean \( v \otimes w \) is pure.

Let \( l \) be a Lie algebra over \( \mathbb{C} \) and let \( A \) be an algebra over \( \mathbb{C} \). Let \( \sigma \) act on \( l \) (resp. \( A \)) as Lie algebra (resp. algebra) automorphism of finite orders. For any \( x \otimes a \in l \otimes A \), we denote it by \( x[a] \) for brevity. There is a Lie algebra structure on \( l \otimes A \) with the Lie bracket given by

\[
[x[a], y[b]] := [x, y][ab], \quad \text{for any elements } x[a], y[b] \in l \otimes A.
\]

Then, \((l \otimes A)^\sigma\) is a Lie subalgebra.

Let \( \mathfrak{g} \) be a simple Lie algebra over \( \mathbb{C} \) and let \( \sigma \) be an automorphism of \( \mathfrak{g} \) such that \( \sigma^m = 1 \) (\( \sigma \) is not necessarily of order \( m \)). Let \( \langle \cdot, \cdot \rangle \) be the invariant (symmetric, nondegenerate) bilinear form on \( \mathfrak{g} \) normalized so that the induced form on the dual space \( \mathfrak{g}^* \) satisfies \( \langle \theta, \theta \rangle = 2 \) for the highest root \( \theta \) of \( \mathfrak{g} \). The bilinear form \( \langle \cdot, \cdot \rangle \) is \( \sigma \)-invariant since \( \sigma \) is a Lie algebra automorphism of \( \mathfrak{g} \).

Let \( \mathcal{K} = \mathbb{C}(t) := \mathbb{C}[[t]][t^{-1}] \) be the field of Laurent power series, and let \( O \) be the ring of formal power seires \( \mathbb{C}[[t]] \) with the maximal ideal \( m = tO \). We fix a primitive \( m \)-th root of unity \( \epsilon = e^{\frac{2\pi i}{m}} \) throughout the paper. We define an action of \( \sigma \) on \( \mathcal{K} \) as field automorphism by setting

\[
\sigma(t) = e^{-1}t \quad \text{and } \sigma \text{ acting trivially on } \mathbb{C}.
\]

It gives rise to an action of \( \sigma \) on the loop algebra \( L(\mathfrak{g}) := \mathfrak{g} \otimes \mathcal{K} \). Under this action,

\[
L(\mathfrak{g})^\sigma = \oplus_{j=0}^{m-1} \left( \mathfrak{g}_j \otimes \mathcal{K}_j \right),
\]

where

\[
\mathfrak{g}_j := \{ x \in \mathfrak{g} : \sigma(x) = e^{j}x \}, \quad \text{and } \mathcal{K}_j = \{ P \in \mathcal{K} : \sigma(P) = e^{-j}P \}.
\]

We now define a central extension \( \hat{L}(\mathfrak{g}, \sigma) := L(\mathfrak{g})^\sigma \oplus \mathbb{C}C \) of \( L(\mathfrak{g})^\sigma \) under the bracket

\[
[x[P] + zC, x'[P'] + z'C] = [x, x'][PP'] + m^{-1} \text{Res}_{s=0} ((dP)P') \langle x, x' \rangle C,
\]
for $x[P], x'[P'] \in L(g)^\nu$, $z, z' \in \mathbb{C}$; where $\text{Res}_{t=0}$ denotes the coefficient of $t^{-1}dt$. Let $\hat{L}(g, \sigma)^{\geq 0}$ denote the subalgebra

$$\hat{L}(g, \sigma)^{\geq 0} := \oplus_{j=0}^{m-1} g_j \otimes O_j \oplus \mathbb{C}C,$$

where $O_j = \mathcal{K}_j \cap O$. We also denote

$$\hat{L}(g, \sigma)^{\leq 0} := \oplus_{j=0}^{m-1} g_j \otimes m_j, \quad \text{and} \quad \hat{L}(g, \sigma)^{-} := \oplus_{j=0}^{m-1} g_j \otimes t^j,$$

where $m_j = m \cap O_j$. Then, $\hat{L}(g, \sigma)^{\leq 0}$ is an ideal of $\hat{L}(g, \sigma)^{\geq 0}$ and the quotient $\hat{L}(g, \sigma)^{\geq 0}/\hat{L}(g, \sigma)^{\leq 0}$ is isomorphic to $g_0 \oplus \mathbb{C}C$. Note that $g_0$ is the Lie algebra $g^\sigma$ of $\sigma$-fixed points in $g$. As vector spaces we have

$$\hat{L}(g, \sigma) = \hat{L}(g, \sigma)^{\geq 0} \oplus \hat{L}(g, \sigma)^{-}.$$

By the classification theorem of finite order automorphisms of simple Lie algebras (cf. [Ka, Proposition 8.1, Theorems 8.5, 8.6]), there exists a ‘compatible’ Cartan subalgebra $h$ and a ‘compatible’ Borel subalgebra $b \supset h$ of $g$ both stable under the action of $\sigma$ such that

$$(3) \quad \sigma = \tau e^{ad h},$$

where $\tau$ is a diagram automorphism of $g$ of order $r$ preserving $h$ and $b$, and $e^{ad h}$ is the inner automorphism of $g$ such that for any root $\alpha$ of $g$, $e^{ad h}$ acts on the root space $g_\alpha$ by the multiplication $e^{\alpha(h)}$, and $e^{ad h}$ acts on $h$ by the identity. We consider $\tau = \text{Id}$ also as a diagram automorphism. Here $h$ is an element in $h^\tau$. In particular, $\tau$ and $e^{ad h}$ commute. Moreover, $\alpha(h) \in \mathbb{Z}^{\geq 0}$ for any simple root $\alpha$ of $g^\tau$, $\beta(h) \in \mathbb{Z}$ for any simple root $\beta$ of $g$ and $\theta_0(h) \leq \frac{m}{r}$ where $\theta_0 \in (b^\tau)^\tau$ denotes the following weight of $g^\tau$

$$\theta_0 = \begin{cases} 
\text{highest root of } g_0, & \text{if } r = 1 \\
\text{highest short root of } g^\tau, & \text{if } r > 1 \text{ and } (g, r) \neq (A_{2n}, 2) \\
2 \cdot \text{highest short root of } g^\tau, & \text{if } (g, r) = (A_{2n}, 2).
\end{cases}$$

Observe that $r$ divides $m$, and $r$ can only be $1, 2, 3$. Note that $g^\sigma$ and $g^\tau$ share the common Cartan subalgebra $h^\tau = b^\tau$.

Let $I(g^\tau)$ denote the set of vertices of Dynkin diagram of $g^\tau$. Let $\alpha_i$ denote the simple root associated to $i \in I(g^\tau)$. Let $\hat{I}(g, \sigma)$ denote the set $I(g^\tau) \cup \{o\}$, where $o$ is just a symbol. (Observe that $\tau$ is determined from $\sigma$.) Set

$$s_i = \begin{cases} 
\alpha_i(h) & \text{if } i \in I(g^\tau) \\
\frac{m}{r} - \theta_0(h) & \text{if } i = o.
\end{cases}$$

Then, $s = \{s_i | i \in \hat{I}(g, \sigma)\}$ is a tuple of non-negative integers. Let $\hat{L}(g, \tau)$ denote the Lie algebra with the construction similar to $\hat{L}(g, \sigma)$ where $\sigma$ is replaced by $\tau$, $m$ is replaced by $r$ and $\epsilon$ is replaced by $\epsilon^\tau$. There exists an isomorphism of Lie algebras (cf. [Ka, Theorem 8.5]):

$$(4) \quad \phi_{\sigma} : \hat{L}(g, \tau) \cong \hat{L}(g, \sigma)$$

given by $C \mapsto C$ and $x[t^j] \mapsto x[t^{\frac{m}{r} + k}]$, for any $x$ an $e^{\frac{m}{r} e^\tau}$-eigenvector of $\tau$, and $x$ also a $k$-eigenvector of $ad h$. We remark that in the case $(g, r) = (A_{2n}, 2)$, our labelling for $i = o$
is the same as \( i = n \) in [Ka, Chapter 8]. It is well-known that \( \hat{L}(g, \tau) \) is an affine Lie algebra, more precisely \( \hat{L}(g, \tau) \) is untwisted if \( r = 1 \) and twisted if \( r > 1 \).

By Theorem 8.7 in [Ka], there exists a \( sl_2 \)-triple \( x_i, y_i, h_i \in g \) for each \( i \in \hat{I}(g, \sigma) \) where

- \( x_i \in (g^+)_{\alpha_i}, y_i \in (g^-)_{-\alpha_i} \) when \( i \in I(g^+) \);
- \( x_o \) (resp. \( y_o \)) is a \((-\theta_0)\) (resp. \( \theta_0 \))-weight vector with respect to the adjoint action of \( \mathfrak{h}^* \) on \( g \), and is also an \( e^{\tau} \) (resp. \( e^{-\tau} \))-eigenvector of \( \tau \);
- \( x_i \in n^+ \) for \( i \in I(g^+) \) and \( x_o \in n^- \), where \( n^+ \) (resp. \( n^- \)) is the nil-radical of \( b \) (resp. the opposite Borel subalgebra \( b^- \)). Similarly, \( y_i \in n^- \) for \( i \in I(g^-) \) and \( y_o \in n^+ \),

(see explicit construction of \( x_i, y_i, i \in \hat{I}(g, \sigma) \) in [Ka, §7.4, §8.3]), such that

\[
x_i[I^+] \cdot y_i[I^-], [x_i[I^+], y_i[I^-]], \quad i \in \hat{I}(g, \sigma),
\]

are Chevalley generators of \( \hat{L}(g, \sigma) \). We set

\[
\tilde{x}_i := x_i[I^+] \cdot \tilde{y}_i := y_i[I^-], \quad \text{and} \quad \tilde{h}_i := [\tilde{x}_i, \tilde{y}_i], \quad \text{for any} \ i \in \hat{I}(g, \sigma).
\]

Via the isomorphism \( \phi_{\sigma} \), we have

\[
\phi_{\sigma}(x_i) = \tilde{x}_i, \quad \phi_{\sigma}(y_i) = \tilde{y}_i, \quad \text{for any} \ i \in I(g^+),
\]

and

\[
\phi_{\sigma}(x_o[I]) = \tilde{x}_o, \quad \phi_{\sigma}(y_o[I^{-1}]) = \tilde{y}_o.
\]

Thus, \( \deg \tilde{x}_i = s_i \) and \( \deg \tilde{y}_i = -s_i \). The Lie algebra \( \hat{L}(g, \sigma) \) is called an \((s, r)\)-realization of the associated affine Lie algebra.

From the above discussion, for any \( i \in \hat{I}(g, \sigma) \), we have

\[
\sigma(x_i) = e^{s_i} x_i, \quad \text{and} \quad \sigma(y_i) = e^{-s_i} y_i.
\]

We fix a positive integer \( c \) called the level or central charge. Let \( \text{Rep}_c \) be the set of isomorphism classes of integrable highest weight (in particular, irreducible) \( \hat{L}(g, \sigma) \)-modules with central charge \( c \), where in our realization \( C \) acts by \( c \), the standard Borel subalgebra of \( \hat{L}(g, \sigma) \) is generated by \( \{\tilde{x}_i, \tilde{h}_i\}_{i \in \hat{I}(g, \sigma)} \) and \( \hat{L}(g, \sigma^-) \) is generated by \( \{\tilde{y}_i\}_{i \in \hat{I}(g, \sigma), s_i > 0} \), (cf. [Ka, Theorem 8.7]).

Thus, \( \hat{L}(g, \sigma)^{\geq 0} \) is a standard parabolic subalgebra of \( \hat{L}(g, \sigma) \). For any \( \mathcal{H} \in \text{Rep}_c \), let \( \mathcal{H}^0 \) be the subspace of \( \mathcal{H} \) annihilated by \( \hat{L}(g, \sigma)^+ \). Then, \( \mathcal{H}^0 \) is an irreducible finite dimensional \( g^r \)-submodule of \( \mathcal{H} \) with highest weight (say) \( \lambda(\mathcal{H}^0) = (\mathfrak{h}^r)^* \) for the choice of the Borel subalgebra of \( g^r \) generated by \( \mathfrak{h}^r \) and \( \{x_i : s_i = 0\} \). The correspondence \( \mathcal{H} \mapsto \lambda(\mathcal{H}) \) sets up an injective map \( \text{Rep}_c \rightarrow (\mathfrak{h}^r)^* \). Let \( D_c \) be its image. For \( \lambda \in D_c \), let \( \mathcal{H}(\lambda) \) be the corresponding irreducible highest weight \( \hat{L}(g, \sigma) \)-module with central charge \( c \).

For any \( \lambda \in D_c \) and \( i \in \hat{I}(g, \sigma) = I(g^+) \sqcup \{o\} \), we associate an integer \( n_{i,\lambda} \) as follows. Set

\[
n_{i,\lambda} = \lambda([x_i, y_i]) + \langle x_i, y_i \rangle \frac{s_i c}{m}.
\]

For \( \sigma = \tau \) a diagram automorphism of \( g \) (including \( \tau = 1d \)), by definition \( s_i = 0 \) for \( i \in I(g^-) \) and \( s_o = 1 \).
For any diagram automorphism \( \tau \) of order \( r \) (including \( r = 1 \)), from the concrete realization of \( x_i, y_i, i \in \hat{I}(g^r) \cup \{0\} \) in [Ka, §8.3], when \((\varrho, r) \neq (A_{2n}, 2)\), we have

\[
\langle x_i, y_i \rangle = \begin{cases} r & \text{if } i = 0, \text{ or } \alpha_i \text{ is a short root for } i \in I(g^r) \\ 1 & \text{if } \alpha_i \text{ is a long root for } i \in I(g^r) \end{cases}
\]

and when \((\varrho, r) = (A_{2n}, 2)\),

\[
\langle x_i, y_i \rangle = \begin{cases} 1 & \text{if } i = 0 \\ 2 & \text{if } \alpha_i \text{ is a long root for } i \in I(g^r) \\ 4 & \text{if } \alpha_i \text{ is a short root for } i \in I(g^r) \end{cases}
\]

**Lemma 2.1.** The set \( D_c \) can be described as follows:

\[
D_c = \{ \lambda \in (b^r)^* | n_{\lambda, i} \in \mathbb{Z}_{\geq 0} \text{ for any } i \in \hat{I}(\varrho, \sigma) \}.
\]

**Proof.** The lemma follows from the fact that the irreducible highest weight \( \hat{L}(\varrho, \sigma) \)-module \( \mathcal{H}(\lambda) \) with highest weight \( \lambda \) is integrable if and only the eigenvalues of \( \hat{h}_i, i \in I(g^r) \) and \([\tilde{x}_o, \tilde{y}_o]\) on the highest weight vector in \( \mathcal{H}(\lambda) \) are non-negative integers. \( \square \)

Define

\[
\tilde{s}_i = \langle x_i, y_i \rangle s_i, \text{ for any } i \in \hat{I}(\varrho, \sigma)
\]

and let

\[
\tilde{s} := \gcd \{ \tilde{s}_i : i \in \hat{I}(\varrho, \sigma) \}.
\]

As an immediate consequence of Lemma 2.1, we get the following:

**Corollary 2.2.** For any integer \( c \geq 1 \), \( 0 \in D_c \) if and only if \( m \) divides \( \tilde{s}c \).

In particular, \( 0 \in D_c \) if \( m \) divides \( c \).

Also, for a diagram automorphism \( \sigma = \tau \), \( 0 \in D_c \) for all \( c \) if \((\varrho, r) \neq (A_{2n}, 2)\). If \((\varrho, r) = (A_{2n}, 2)\), \( 0 \in D_c \) if and only if \( c \) is even.

We recall the following well known result.

**Lemma 2.3.** For any automorphism \( \sigma \) and any \( c \geq 1 \), \( D_c \neq \emptyset \).

**Proof.** By the isomorphism \( \phi_{\sigma} \) (as in equation (4)), it suffices to prove the lemma for the diagram automorphisms \( \tau \) (including \( \tau = \Id \)). By Corollary 2.2, \( 0 \in D_c \) if \((\varrho, r) \neq (A_{2n}, 2)\). If \((\varrho, r) = (A_{2n}, 2)\), take \( \lambda = \omega_n \): the \( n \)-th fundamental weight of type \( B_n \) (following the Bourbaki convention of indexing as in [Bo, Planche II]). Then, \( \omega_n \in D_c \) for odd values of \( c \) and \( 0 \in D_c \) for even values of \( c \). \( \square \)

Let \( V(\lambda) \) be the irreducible \( g^r \)-module with highest weight \( \lambda \) and highest weight vector \( v_+ \). Let \( \hat{M}(V(\lambda), c) \) be the generalized Verma module \( U(\hat{L}(\varrho, \sigma)) \otimes_{U(\hat{L}(\varrho, \sigma)^{\geq 0})} V(\lambda) \) with highest weight vector \( v_+ = 1 \otimes v_+ \), where the action of \( \hat{L}(\varrho, \sigma)^{\geq 0} \) on \( V(\lambda) \) factors through the projection map \( \hat{L}(\varrho, \sigma)^{\geq 0} \rightarrow g^r \otimes \mathbb{C}C \) and the center \( C \) acts by \( c \). If \( \lambda \in D_c \), then the irreducible quotient of \( \hat{M}(V(\lambda), c) \) is the integrable representation \( \mathcal{H}(\lambda) \). Let \( K_1 \) be the kernel of \( \hat{M}(V(\lambda), c) \rightarrow \mathcal{H}(\lambda) \). Set

\[
\hat{I}(\varrho, \sigma)^+ := \{ i \in \hat{I}(\varrho, \sigma) | s_i > 0 \}.
\]
Then, as $U(\hat{L}(\mathfrak{g}, \sigma))$-module, $K_I$ is generated by
\begin{equation}
\{yz_{\alpha_{i}+1} \cdot v_+ | i \in I(\mathfrak{g}, \sigma^+)\} \quad \text{(cf. [Ku, § 2.1]).}
\end{equation}

**Lemma 2.4.** Fix $i \in \hat{I}(\mathfrak{g}, \sigma)$. Let $f \in \mathcal{K}$ be such that $\sigma(f) = e^{-\gamma_i} f$ and $f \equiv t_i$ mod $t_i^{n+1}$. Put $X = x_i[f]$ and $Y = \hat{y}_i = y_i[r^{-\gamma_i}]$. For any $p > n_{\lambda,i}$ and $q > 0$, there exists $\alpha \neq 0$ such that
\[ Y^p \cdot v_+ = \alpha X^q Y^{p+q} \cdot v_+ \]
in the generalized Verma module $\hat{M}(V(\lambda), c)$.

**Proof.** Let $H := \{X, Y\} = h_i[t^{-\gamma_i}f] + \frac{\hat{y}_i}{m} C$. Then, $[H, Y] = -2y_i[t^{-2\gamma_i}f]$ commutes with $Y$. Note that $H \cdot v_+ = n_{\lambda,i}v_+$. Then, one can check that, for $d \geq 0$,
\[ HY^d \cdot v_+ = (n_{\lambda,i} - 2d) Y^d \cdot v_+ , \]
and, for $p \geq 0$,
\[ X Y^{p+1} \cdot v_+ = (p + 1)(n_{\lambda,i} - p) Y^p \cdot v_+ . \]
By induction on $q$, the lemma follows. \qed

**Lemma 2.5.** Let $\mathfrak{g}$ and $\sigma$ be as above and let $\mathfrak{b}$ be a $\sigma$-stable Borel subalgebra with $\sigma$-stable Cartan subalgebra $\mathfrak{h} \subset \mathfrak{b}$ of $\mathfrak{g}$. Then, any element $x$ of $(\mathfrak{n} \otimes \mathcal{K})^\sigma$ acts locally nilpotently on any integrable highest weight $\hat{L}(\mathfrak{g}, \sigma)$-module $\mathcal{H}(\lambda)$, where $\mathfrak{n}$ is the nilradical of $\mathfrak{h}$.

Replacing the Borel subalgebra $\mathfrak{b}$ by the opposite Borel subalgebra $\mathfrak{b}^-$, the lemma holds for any $x \in (\mathfrak{n}^- \otimes \mathcal{K})^\sigma$ as well, where $\mathfrak{n}^-$ is the nil-radical of $\mathfrak{b}^-$. 

**Proof.** Take a basis $\{y_\beta\}_\beta$ of $\mathfrak{n}$ consisting of common eigenvectors under the action of $\sigma$ as well as $\mathfrak{h}^\sigma$ (which is possible since the adjoint action of $\mathfrak{h}^\sigma$ on $\mathfrak{g}$ commutes with the action of $\sigma$) and write $x = \sum_\beta y_\beta P_\beta$ for some $P_\beta \in \mathcal{K}$. Since $x$ is $\sigma$-invariant and each $y_\beta$ is an eigenvector, each $y_\beta[P_\beta]$ is $\sigma$-invariant. Let $\hat{L}(\mathfrak{g}, \sigma)_x$ be the Lie subalgebra of $\hat{L}(\mathfrak{g}, \sigma)$ generated by the elements $\{y_\beta[P_\beta]\}_\beta$. Then, since $\mathfrak{n}$ is nilpotent (in particular, $N$-bracket of elements from $\mathfrak{n}$ is zero, for some large enough $N$), $\hat{L}(\mathfrak{g}, \sigma)_x$ is a finite dimensional nilpotent Lie algebra. (Observe that $\mathfrak{n}$ being nilpotent, for any two elements $s_1, s_2 \in \mathfrak{n}$, $(s_1, s_2) = 0$.) Take any element $v \in \mathcal{H}(\lambda)$ and let $\mathcal{H}(x, v)$ be the $\hat{L}(\mathfrak{g}, \sigma)_x$ -submodule of $\mathcal{H}(\lambda)$ generated by $v$. Since $y_\beta$ is an eigenvector for the adjoint action of $\mathfrak{h}^\sigma$ with non-trivial action (cf. [Ka, Lemma 8.1(b)]), any $y_\beta[P_\beta]$ can be written as a finite sum of commuting real root vectors for $\hat{L}(\mathfrak{g}, \sigma)$ and a $\sigma$-invariant element of the form $y_\beta[P_\beta]$ with $P_\beta \in \mathfrak{c}^\sigma([t])$ (cf. [Ka, Exercises 8.1 and 8.2, §8.8]). Thus, $y_\beta[P_\beta]$ acts locally nilpotently on $\mathcal{H}(\lambda)$ (in particular, on $\mathcal{H}(x, v)$). Now, using [Ku, Lemma 1.3.3 (c2)], we get that $\mathcal{H}(x, v)$ is finite dimensional. Using Lie’s theorem, the lemma follows. \qed

### 3. Twisted Analogue of Conformal Blocks

In this section we define the space of twisted covacua attached to a Galois cover of an algebraic curve. We prove that this space is finite dimensional.

For a smooth point $p$ in an algebraic curve $\Sigma$ over $\mathbb{C}$, let $\mathcal{K}_p$ denote the quotient field of the completed local ring $\hat{\Theta}_p$ of $\Sigma$ at $p$. We denote by $\mathbb{D}_p$ (resp. $\mathbb{D}_p^\circ$) the formal disc Spec $\hat{\Theta}_p$ (resp. the punctured formal disc Spec $\mathcal{K}_p$).
Definition 3.1. A morphism \( \pi : \Sigma \to \hat{\Sigma} \) of projective curves is said to be a \textit{Galois cover with finite Galois group} \( \Gamma \) (for short \( \Gamma \)-cover) if the group \( \Gamma \) acts on \( \Sigma \) as algebraic automorphisms and \( \Sigma/\Gamma \cong \hat{\Sigma} \) and no nontrivial element of \( \Gamma \) fixes pointwise any irreducible component of \( \Sigma \).

For any smooth point \( q \in \Sigma \), the stabilizer group \( \Gamma_q \) of \( \Gamma \) at \( q \) is always cyclic. The order \( e_q := |\Gamma_q| \) is called the \textit{ramification index of} \( q \). Thus, \( q \) is unramified if and only if \( e_q = 1 \). Denote \( p = \pi(q) \). We can also write \( e_p = e_q \), since \( e_q = e_{q'} \) for any \( q, q' \in \pi^{-1}(p) \).

We also say that \( e_p \) is the ramification index of \( p \). Denote by \( d_p \), the cardinality of the fiber \( \pi^{-1}(p) \). Then \( |\Gamma| = e_p \cdot d_p \).

The action of \( \Gamma_q \) on the tangent space \( T_q \Sigma \) induces a primitive character \( \chi_q : \Gamma_q \to \mathbb{C}^\times \), i.e., \( \chi_q(\sigma q) \) is a primitive \( e_p \)-th root of unity for any generator \( \sigma q \) in \( \Gamma_q \). From now on we shall fix \( \sigma q \in \Gamma_q \) so that

\[
\chi_q(\sigma q) = e^{2\pi i/e_p}.
\]

For any two smooth points \( q, q' \in \Sigma \), if \( \pi(q) = \pi(q') \) then

\[
\Gamma_{q'} = \gamma \Gamma_q \gamma^{-1}, \quad \text{for any element } \gamma \in \Gamma \text{ such that } q' = \gamma \cdot q.
\]

Moreover,

\[
\chi_{q'}(\gamma \sigma q \gamma^{-1}) = \chi_q(\sigma), \quad \text{for any } \sigma \in \Gamma_q.
\]

Given a smooth point \( p \in \hat{\Sigma} \) such that \( \pi^{-1}(p) \) consists of smooth points in \( \Sigma \), let \( \pi^{-1}(\mathcal{D}_p) \) (resp. \( \pi^{-1}(\mathcal{D}_p^\times) \)) denote the fiber product of \( \Sigma \) and \( \mathcal{D}_p \) (resp. \( \mathcal{D}_p^\times \)) over \( \hat{\Sigma} \). Then,

\[
\pi^{-1}(\mathcal{D}_p) \cong \sqcup_{q \in \pi^{-1}(p)} \mathcal{D}_q, \quad \text{and} \quad \pi^{-1}(\mathcal{D}_p^\times) \cong \sqcup_{q \in \pi^{-1}(p)} \mathcal{D}_q^\times,
\]

where \( \mathcal{D}_q \) (resp. \( \mathcal{D}_q^\times \)) denotes the formal disc (resp. formal punctured disc) in \( \Sigma \) around \( q \).

Let the finite group \( \Gamma \) also act on \( g \) as Lie algebra automorphisms.

Let \( g[\pi^{-1}(\mathcal{D}_p^\times)]^\Gamma \) be the Lie algebra consisting of \( \Gamma \)-equivariant regular maps from \( \pi^{-1}(\mathcal{D}_p) \) to \( g \). There is a natural isomorphism \( g[\pi^{-1}(\mathcal{D}_p^\times)]^\Gamma \cong (g \otimes \mathbb{C}[\pi^{-1}(\mathcal{D}_p^\times)])^\Gamma \). Let

\[
\hat{g}_p := g[\pi^{-1}(\mathcal{D}_p^\times)]^\Gamma \oplus \mathbb{C}C
\]

be the central extension of \( g[\pi^{-1}(\mathcal{D}_p^\times)]^\Gamma \) defined as follows:

\[
[X, Y] = [X, Y]_0 + \frac{1}{|\Gamma|} \sum_{q \in \pi^{-1}(p)} \text{Res}_q(dX, Y)_0 C,
\]

for any \( X, Y \in g[\pi^{-1}(\mathcal{D}_p^\times)]^\Gamma \), where \([,]_0 \) denotes the point-wise Lie bracket induced from the bracket on \( g \). We set the subalgebra

\[
\hat{g}_p := g[\pi^{-1}(\mathcal{D}_p)]^\Gamma \oplus \mathbb{C}C
\]

and

\[
\hat{g}_p^\times := \text{Ker}(g[\pi^{-1}(\mathcal{D}_p)]^\Gamma \to g[\pi^{-1}(p)]^\Gamma)
\]

obtained by the restriction map \( \mathbb{C}[\pi^{-1}(\mathcal{D}_p)] \to \mathbb{C}[\pi^{-1}(p)] \), where \( g[\pi^{-1}(p)]^\Gamma \) denote the Lie algebra consisting of \( \Gamma \)-equivariant maps \( x : \pi^{-1}(p) \to g \). Let \( g_p \) denote \( g[\pi^{-1}(p)]^\Gamma \). The following lemma is obvious.
Lemma 3.2. The evaluation map $\text{ev}_q : \mathfrak{g}_p \to \mathfrak{g}^\Gamma$ given by

$$x \mapsto x(q)$$

for any $x \in \mathfrak{g}_p$ and $q \in \pi^{-1}(p)$ is an isomorphism of Lie algebras.

Let $\sigma_q$ be the generator of $\Gamma_q$ such that $x_q(\sigma_q) = e^{2\pi i q}$. Let $\hat{L}(\mathfrak{g}, \sigma_q)$ denote the affine Lie algebra associated to $\mathfrak{g}$ and $\sigma_q$ as defined in Section 2. We denote this algebra by $\hat{L}(\mathfrak{g}, \Gamma_q, x_q)$ or $\hat{L}(\mathfrak{g}, \Gamma_q)$ in short.

Lemma 3.3. The restriction map $\text{res}_q : \hat{L}_p \to \hat{L}(\mathfrak{g}, \Gamma_q)$ given by

$$X \mapsto X_q, \text{ and } C \mapsto C,$$

is an isomorphism of Lie algebras, where $X \in \mathfrak{g}[\pi^{-1}(\mathbb{D}_p^\mathfrak{g})]^\Gamma$ and $X_q$ is the restriction of $X$ to $\mathbb{D}_q^\mathfrak{g}$. Moreover,

$$\text{res}_q(\hat{L}_p) = \hat{L}(\mathfrak{g}, \Gamma_q)^{0,0}, \text{ and } \text{res}_q(\hat{L}_p^+) = \hat{L}(\mathfrak{g}, \Gamma_q)^+.$$

Proof. For any $X, Y \in \mathfrak{g}[\pi^{-1}(\mathbb{D}_p^\mathfrak{g})]^\Gamma$, the restriction of $[X, Y]$ to $\mathbb{D}_q^\mathfrak{g}$ is equal to $[X_q, Y_q]_0$. Note that for any $\gamma \in \Gamma$ and $x, y \in \mathfrak{g}$, we have $\langle \gamma(x), y \rangle = \langle x, \gamma(y) \rangle$, which follows from the Killing form realization of $\langle, \rangle$ on $\mathfrak{g}$. Since $X, Y$ are $\Gamma$-equivariant, for any $q, q' \in \pi^{-1}(p)$ we have

$$\text{Res}_q(dX, Y) = \text{Res}_{q'}(dX, Y).$$

It is now easy to see that $\text{res}_q : \hat{L}_p \to \hat{L}(\mathfrak{g}, \Gamma_q)$ is an isomorphism of Lie algebras, and

$$\text{res}_q(\hat{L}_p) = \hat{L}(\mathfrak{g}, \Gamma_q)^{0,0}, \text{ and } \text{res}_q(\hat{L}_p^+) = \hat{L}(\mathfrak{g}, \Gamma_q)^+.$$

By the above lemma, we have a faithful functor $\text{Rep}_c(\hat{L}_p) \to \text{Rep}(\mathfrak{g}_p)$ from the category of integrable highest weight representations of $\hat{L}_p$ of level $c$ to the category of finite dimensional representations of $\mathfrak{g}_p$. We denote by $D_{c,p}$ the parameter set of (irreducible) integrable highest weight representations of $\hat{L}_p$ of level $c$ obtained as the subset of the set of dominant integral weights of $\mathfrak{g}_p$ under the above faithful functor. Let $D_{c,q}$ denote the parameter set of (irreducible) integrable highest weight representations of $\hat{L}(\mathfrak{g}, \Gamma_q)$ as in Section 2. Then, we can identify $D_{c,p}$ and $D_{c,q}$ via the restriction isomorphism $\text{res}_q : \hat{L}_p \to \hat{L}(\mathfrak{g}, \Gamma_q)$ as in Lemma 3.3.

Definition 3.4. For any $s \geq 1$, by an $s$-pointed curve, we mean the pair $(\Sigma, \vec{p} = (p_1, \ldots, p_s))$ consisting of distinct and smooth points $\{p_1, \ldots, p_s\}$ of $\Sigma$, such that the following condition is satisfied.

(*) Each irreducible component of $\Sigma$ contains at least one point $p_i$.

Similarly, by an $s$-pointed $\Gamma$-curve, we mean the pair $(\Sigma, \vec{q} = (q_1, \ldots, q_s))$ consisting of smooth points $\{q_1, \ldots, q_s\}$ of $\Sigma$ such that $\langle \pi(q_i), \pi(q_j) \rangle$ is an $s$-pointed curve.

From now on we fix an $s$-pointed curve $(\Sigma, \vec{p})$ (for any $s \geq 1$), where $\vec{p} = (p_1, \ldots, p_s)$, and a Galois cover $\pi : \Sigma \to \tilde{\Sigma}$ with the finite Galois group $\Gamma$ such that the fiber $\pi^{-1}(p_i)$ consists of smooth points for any $i = 1, 2, \ldots, s$. We also fix a simple Lie algebra $\mathfrak{g}$ and
a group homomorphism $\phi : \Gamma \to \text{Aut}(g)$, where $\text{Aut}(g)$ is the group of automorphisms of $g$.

We now fix an $s$-tuple $\mathbf{\lambda} = (\lambda_1, \ldots, \lambda_s)$ of weights with $\lambda_i \in D_{c, p_i}$ ‘attached’ to the point $p_i$. To this data, there is associated the space of (twisted) vacua $\mathcal{V}_{\Sigma, \Gamma, \phi}(\mathbf{p}, \mathbf{\lambda})$ (or the space of (twisted) conformal blocks) and its dual space $\mathcal{V}_{\Sigma, \Gamma, \phi}^\dagger(\mathbf{p}, \mathbf{\lambda})$ called the space of (twisted) covacua (or the space of (twisted) dual conformal blocks) defined as follows:

**Definition 3.5.** Let $g[\Sigma \setminus \pi^{-1}(\mathbf{p})]^\Gamma$ denote the space of $\Gamma$-equivariant regular maps $f : \Sigma \setminus \pi^{-1}(\mathbf{p}) \to g$. Then, $g[\Sigma \setminus \pi^{-1}(\mathbf{p})]^\Gamma$ is a Lie algebra under the pointwise bracket.

Set

$$\mathcal{H}(\mathbf{\lambda}) := \mathcal{H}(\lambda_1) \otimes \cdots \otimes \mathcal{H}(\lambda_s),$$

where $\mathcal{H}(\lambda_i)$ is the integrable highest weight representation of $\hat{\mathfrak{g}}_{p_i}$ of level $c$ with highest weight $\lambda_i \in D_{c, p_i}$.

Define an action of the Lie algebra $g[\Sigma \setminus \pi^{-1}(\mathbf{p})]^\Gamma$ on $\mathcal{H}(\mathbf{\lambda})$ as follows:

$$X \cdot (v_1 \otimes \cdots \otimes v_s) = \sum_{i=1}^s v_1 \otimes \cdots \otimes X_{p_i} v_i \otimes \cdots \otimes v_s, \text{ for } X \in g[\Sigma \setminus \pi^{-1}(\mathbf{p})]^\Gamma, \text{ and } v_i \in \mathcal{H}(\lambda_i),$$

where $X_{p_i}$ denotes the restriction of $X$ to $\pi^{-1}(\mathcal{D}_{p_i}^\times)$, hence $X_{p_i}$ is an element in $\hat{\mathfrak{g}}_{p_i}$.

By the residue theorem [H, Theorem 7.14.2, Chap. III],

$$\sum_{g \in \pi^{-1}(\mathbf{p})} \text{Res}_g(dX, Y) = 0, \text{ for any } X, Y \in g[\Sigma \setminus \pi^{-1}(\mathbf{p})]^\Gamma.$$  

Thus, the action (15) indeed is an action of the Lie algebra $g[\Sigma \setminus \pi^{-1}(\mathbf{p})]^\Gamma$ on $\mathcal{H}(\mathbf{\lambda})$.

Finally, we are ready to define the space of (twisted) vacua

$$\mathcal{V}_{\Sigma, \Gamma, \phi}(\mathbf{p}, \mathbf{\lambda}) := \text{Hom}_{g[\Sigma \setminus \pi^{-1}(\mathbf{p})]^\Gamma}(\mathcal{H}(\mathbf{\lambda}), \mathbb{C}),$$

and the space of (twisted) covacua

$$\mathcal{V}_{\Sigma, \Gamma, \phi}^\dagger(\mathbf{p}, \mathbf{\lambda}) := [\mathcal{H}(\mathbf{\lambda})]_{g[\Sigma \setminus \pi^{-1}(\mathbf{p})]^\Gamma},$$

where $\mathbb{C}$ is considered as the trivial module under the action of $g[\Sigma \setminus \pi^{-1}(\mathbf{p})]^\Gamma$, and $[\mathcal{H}(\mathbf{\lambda})]_{g[\Sigma \setminus \pi^{-1}(\mathbf{p})]^\Gamma}$ denotes the space of covariants $\mathcal{H}(\mathbf{\lambda})/(g[\Sigma \setminus \pi^{-1}(\mathbf{p})]^\Gamma \cdot \mathcal{H}(\mathbf{\lambda}))$. Clearly,

$$\mathcal{V}_{\Sigma, \Gamma, \phi}^\dagger(\mathbf{p}, \mathbf{\lambda}) \cong \mathcal{V}_{\Sigma, \Gamma, \phi}(\mathbf{p}, \mathbf{\lambda})^\ast.$$  

**Remark 3.6.** Fix any $q_i \in \pi^{-1}(p_i)$. If we choose $\mathbf{\lambda} = (\lambda_1, \ldots, \lambda_s)$ to be a set of weights, where for each $i$, $\lambda_i$ is a dominant weight of $g^{F_{q_i}}$ in $D_{c, q_i}$, we can transfer each $\lambda_i$ to an element in $D_{c, p_i}$ through the restriction isomorphism $g_{p_i} \approx g_{q_i}^{F_{q_i}}$ via Lemma 3.2. Accordingly, we denote the associated space of covacua by $\mathcal{V}_{\Sigma, \Gamma, \phi}(q, \mathbf{\lambda})$. This terminology will often be used interchangeably.

**Lemma 3.7.** With the notation and assumptions as in Definition 3.5, the space of covacua $\mathcal{V}_{\Sigma, \Gamma, \phi}(\mathbf{p}, \mathbf{\lambda})$ is finite dimensional and hence by equation (19), so is the space of vacua $\mathcal{V}_{\Sigma, \Gamma, \phi}(\mathbf{p}, \mathbf{\lambda})^\dagger$. 

Proof. Let \( g[\pi^{-1}(\mathbb{D}_p^\gamma)]^\Gamma \) be the space of \( \Gamma \)-equivariant maps from the disjoint union of formal punctured discs \( \sqcup_{q \in \mathbb{D}_p^\gamma} \mathbb{D}_q^\gamma \) to \( g \). Define a Lie algebra bracket on
\[
(20) \quad \delta_{\tilde{\beta}} := g[\pi^{-1}(\mathbb{D}_p^\gamma)]^\Gamma \oplus \mathbb{C},
\]
by declaring \( C \) to be the central element and the Lie bracket is defined in the similar way as in (11).

Now, define an embedding of Lie algebras:
\[
(21) \quad \beta : g[\Sigma \setminus \pi^{-1}(\tilde{\beta})]^\Gamma \to \delta_{\tilde{\beta}}, \quad X \mapsto X_{\tilde{\beta}}
\]
where \( X_{\tilde{\beta}} \) is the restriction of \( X \) to \( \pi^{-1}(\mathbb{D}_p^\gamma) \).

By the Residue Theorem, \( \beta \) is indeed a Lie algebra homomorphism. Moreover, by Riemann-Roch theorem \( \text{Im} \beta + g[\pi^{-1}(\mathbb{D}_p^\gamma)]^\Gamma \) has finite codimension in \( \delta_{\tilde{\beta}} \), where \( \pi^{-1}(\mathbb{D}_p^\gamma) \) is the disjoint union \( \sqcup_{q \in \mathbb{D}_p^\gamma} \mathbb{D}_q^\gamma \). Further, define the following surjective Lie algebra homomorphism from the direct sum Lie algebra
\[
\bigoplus_{i=1}^s \delta_{\tilde{\beta}_i} \to \delta_{\tilde{\beta}}, \quad \sum_{i=1}^s X_i \mapsto \sum_{i=1}^s \tilde{X}_i, \quad C_i \to C,
\]
here \( C_i \) is the center of \( \delta_{\tilde{\beta}_i} \), and the map \( X_i \in g[\pi^{-1}(\mathbb{D}_p^\gamma)]^\Gamma \) naturally extends to \( \tilde{X}_i \in g[\pi^{-1}(\mathbb{D}_p^\gamma)]^\Gamma \) by taking \( \pi^{-1}(\mathbb{D}_p^\gamma) \) to 0 if \( j \neq i \).

Now, the lemma follows from [Ku, Lemma 10.2.2].

\[ \square \]

4. Propagation of Twisted Vacua

We prove the Propagation Theorem in this section, which asserts that adding marked points and attaching weight 0 to those points does not alter the space of twisted vacua.

Let \( \Sigma \to \tilde{\Sigma} \) be a \( \Gamma \)-cover (cf. Definition 3.1). Moreover, \( \phi : \Gamma \to \text{Aut}(g) \) is a group homomorphism.

**Definition 4.1.** Let \( \tilde{\delta} = (o_1, \ldots, o_s) \) and \( \tilde{\beta} = (p_1, \ldots, p_a) \) be two disjoint non-empty sets of smooth and distinct points in \( \tilde{\Sigma} \) such that \( (\tilde{\Sigma}, \tilde{\delta}) \) is a \( s \)-pointed curve and let \( \tilde{\lambda} = (\lambda_1, \ldots, \lambda_s) \), \( \tilde{\mu} = (\mu_1, \ldots, \mu_a) \) be tuples of dominant weights such that \( \lambda_i \in D_{c,\alpha_i} \) and \( \mu_j \in D_{c,p_j} \) for each \( 1 \leq i \leq s, 1 \leq j \leq a \).

We assume that \( \pi^{-1}(o_i) \) and \( \pi^{-1}(p_j) \) consist of smooth points.

Denote the tensor product
\[
(22) \quad V(\tilde{\mu}) := V(\mu_1) \otimes \cdots \otimes V(\mu_a),
\]
where \( V(\mu_k) \) is the irreducible \( g_{p_k} \)-module with highest weight \( \mu_k \).

Define a \( g[\Sigma \setminus \pi^{-1}(\tilde{\delta})]^\Gamma \)-module structure on \( V(\tilde{\mu}) \) as follows:
\[
X \cdot (v_1 \otimes \cdots \otimes v_a) = \sum_{k=1}^a v_1 \otimes \cdots \otimes X|_{p_k} \cdot v_k \otimes \cdots \otimes v_a,
\]
for \( v_k \in V(\mu_k), X \in g[\Sigma \setminus \pi^{-1}(\tilde{\delta})]^\Gamma \), and \( X|_{p_k} \) denotes the restriction \( X|_{\pi^{-1}(p_k)} \in g_{p_k} \). This gives rise to the tensor product \( g[\Sigma \setminus \pi^{-1}(\tilde{\delta})]^\Gamma \)-module structure on \( \mathcal{H}(\tilde{\lambda}) \otimes V(\tilde{\mu}) \).

The proof of the following lemma was communicated to us by J. Bernstein.
Lemma 4.2. Assume that $\Gamma$ stabilizes a Borel subalgebra $b \subset g$. Then, there exist a Cartan subalgebra $h \subset b$ such that $\Gamma$ stabilizes $h$.

Proof. Let $G$ be the simply-connected simple algebraic group associated to $g$, and let $B$ be the Borel subgroup associated to $b$. Let $N$ be the unipotent radical of $B$. Then, $\Gamma$ acts on $N$. It is known that the space of all Cartan subalgebras in $b$ is a $N$-torsor (it follows easily from the conjugacy theorem of Cartan subalgebras). Let $h_o$ be any fixed Cartan subalgebra in $b$. It defines a function $\psi : \Gamma \rightarrow N$ given by $\gamma \mapsto u_\gamma$, where $u_\gamma$ is the unique element in $N$ such that $\text{Ad} u_\gamma(h_o) = \gamma(h_o)$. It is easy to check that $\psi$ is a 1-cocycle of $\Gamma$ with values in $N$. Note that the group cohomology $H^1(\Gamma, N) = 0$ since $\Gamma$ is a finite group and $N$ is unipotent. It follows that $\psi$ is a 1-coboundary, i.e., there exists $u_\gamma \in N$ such that $\psi(\gamma) = \gamma(u_\gamma)^{-1}u_\gamma$ for any $\gamma \in \Gamma$. Set $h = \text{Ad} u(h_o)$. It is now easy to verify that $h$ is $\Gamma$-stable. \hfill \Box

Theorem 4.3. With the notation and assumptions as in Definition 4.1, assume further that $\Gamma$ stabilizes a Borel subalgebra of $g$. Then, the canonical map

$$\theta : \left[ \mathcal{H}(\tilde{\Lambda}) \otimes V(\tilde{\mu}) \right]_{\mathfrak{sl}(\Sigma/\pi^{-1}(\partial))^\Gamma} \rightarrow \mathcal{Y}_{\Sigma, \Gamma, \phi}(\tilde{\partial}, \tilde{\mu}, \tilde{\Lambda}, \tilde{\mu})$$

is an isomorphism, where $\mathcal{Y}_{\Sigma, \Gamma, \phi}$ is the space of covacua and the map $\theta$ is induced from the $g[\Sigma/\pi^{-1}(\partial)]^\Gamma$-module embedding

$$\mathcal{H}(\tilde{\Lambda}) \otimes V(\tilde{\mu}) \hookrightarrow \mathcal{H}(\tilde{\Lambda}, \tilde{\mu}),$$

with $V(\mu_j)$ identified as a $\mathfrak{g}_\mu$-submodule of $\mathcal{H}(\mu_j)$ annihilated by $\tilde{\mathfrak{h}}^+_p$. (Observe that since the subspace $V(\mu_j) \subset \mathcal{H}(\mu_j)$ is annihilated by $\tilde{\mathfrak{h}}^+_p$, the embedding $V(\mu_j) \subset \mathcal{H}(\mu_j)$ is indeed a $g[\Sigma/\pi^{-1}(\partial)]^\Gamma$-module embedding.)

Proof. By Lemma 4.2, we may assume that $\Gamma$ stabilizes a Borel subalgebra $b$ and a Cartan subalgebra $h$ contained in $b$. From now on we fix such a $b$ and $h$.

Let $\mathcal{H} := \mathcal{H}(\tilde{\Lambda}) \otimes V(\mu_1) \otimes \cdots \otimes V(\mu_a)$. By induction on $a$, it suffices to show that the inclusion $V(\mu_a) \hookrightarrow \mathcal{H}(\mu_a)$ induces an isomorphism (abbreviating $\mu_a$ by $\mu$ and $p_a$ by $p$)

$$\left[ \mathcal{H} \otimes V(\mu) \right]_{\mathfrak{sl}(\Sigma/\pi^{-1}(\partial))^\Gamma} \cong \left[ \mathcal{H} \otimes \mathcal{H}(\mu) \right]_{\mathfrak{sl}(\Sigma/\pi^{-1}(\partial))^\Gamma},$$

where $\Sigma^\mu := \Sigma/\pi^{-1}(\partial)$.

We first prove (23) replacing $\mathcal{H}(\mu)$ by the generalized Verma module $\hat{M}(V(\mu), c)$ for $\hat{\mathfrak{h}}_p$ and the parabolic subalgebra $\hat{\mathfrak{p}}_p$, i.e.,

$$\left[ \mathcal{H} \otimes V(\mu) \right]_{\mathfrak{sl}(\Sigma/\pi^{-1}(\partial))^\Gamma} \cong \left[ \mathcal{H} \otimes \hat{M}(V(\mu), c) \right]_{\mathfrak{sl}(\Sigma/\pi^{-1}(\partial))^\Gamma}.$$ 

Consider the Lie algebra

$$\mathfrak{s}_p := g[\Sigma^\mu/\pi^{-1}(p)]^\Gamma \oplus \mathbb{C}C,$$

where $C$ is central in $\mathfrak{s}_p$ and

$$[X, Y] = [X, Y]_0 + \frac{1}{|\mu|} \sum_{q \in \pi^{-1}(p)} \text{Res}_q(dX, Y) C,$$

where $[X, Y]_0$ is the pointwise Lie bracket.
Let $s_p^{≥0}$ be the subalgebra of $s_p$:

$$s_p^{≥0} := g[Σ^o]Γ ⊕ CC.$$ 

Fix a point $q ∈ π^{-1}(p)$ and a generator $σ_q$ of $Γ_q$ such that $σ_q$ acts on $T_qΣ$ by $ε_q := e^{2πi}$ (which is a primitive $e_p$-th root of unity). By the Riemann-Roch theorem there exists a formal parameter $z_q$ around $q$ such that $z_q^{-1}$ is a regular function on $Σ^o\{q\}$. Moreover, we require $z_q^{-1}$ to vanish at any other point $q'$ in $π^{-1}(p)$. Replacing $z_q^{-1}$ by

$$\sum_{j=1}^{ε_q} ε_q^{-j}σ_q^j(z_q^{-1}),$$

we can (and will) assume that

$$σ_q · z_q^{-1} = ε_qz_q^{-1}.$$ 

Recall the Lie algebra $\hat{L}(g, Γ_q)$ and $\hat{L}(g, Γ_q)^r = (z_q^{-1}g[z_q^{-1}])^{Γ_q}$ from §2. Since $z_q$ is a local parameter at $q$ with $σ_q · z_q = ε_q^{-1}z_q$, we have

$$\hat{L}(g, Γ_q) = \hat{L}(g, Γ_q)^{≥0} ⊕ (z_q^{-1}g[z_q^{-1}])^{Γ_q}.$$ 

Define, for any $x ∈ g$ and $k ≥ 1$,

$$A(x[z_q^{-k}]) := \frac{1}{|Γ_q|} ∑_{γ ∈ Γ} γ · x[z_q^{-k}] ∈ s_p,$$

and let $V ⊂ s_p$ be the span of $\{A(x[z_q^{-k}])\}_{x ∈ g, k ≥ 1}$. It is easy to check that

$$s_p = s_p^{≥0} ⊕ V.$$ 

By Lemmas 3.2 and 3.3, we can view $\hat{M}(V(μ), c)$ as a generalized Verma module over $\hat{L}(g, Γ_q)$ induced from $V(μ)$ as $\hat{L}(g, Γ_q)^{≥0}$-module.

Consider the embedding of the Lie algebra

$$s_p ↪ \hat{L}(g, Γ_q)$$

by taking $C ↪ C$ and any $X ↪ X_q$. We assert that the above embedding $s_p ↪ \hat{L}(g, Γ_q)$ induces a vector space isomorphism

$$γ : s_p^{≥0}/s_p^{≥0} ≃ \hat{L}(g, Γ_q)/\hat{L}(g, Γ_q)^{≥0}.$$ 

To prove the above isomorphism, observe first that $γ$ is injective: For $α ∈ g[Σ^o \setminus π^{-1}(p)]^{Γ}$, if $γ(α) ∈ \hat{L}(g, Γ_q)^{≥0}$, then $α ∈ g[(Σ^o \setminus π^{-1}(p)) ∪ \{q\}]$. The $Γ$-invariance of $α$ forces $α ∈ g[Σ^o]$, proving the injectivity of $γ$. To prove the surjectivity of $γ$, take a $Γ_q$-invariant $α = x[z_q^{-k}]$ for $k ≥ 1$. Thus, $σ_q(x) = ε_q^kx$. By the definition, since $z_q^{-1}$ vanishes at any point $q' ∈ π^{-1}(p)$ different from $q$,

$$γ(A(α)) = α + \hat{L}(g, Γ_q)^{≥0}.$$ 

This proves the surjectivity of $γ$. Thus, by the PBW theorem, as $s_p$-module

$$\hat{M}(V(μ), c) ≃ U(s_p) ⊗_{U(s_p^{≥0})} V(μ).$$
Let \( g[\Sigma^0|\pi^{-1}(p)]^\Gamma \) act on \( \mathcal{H} \) as follows:
\[
X \cdot (v_1 \otimes \cdots \otimes v_s \otimes w_1 \otimes \cdots \otimes w_{a-1})
\]
\[
= \sum_{j=1}^{a-1} v_1 \otimes \cdots \otimes X_{a_j} \cdot v_i \otimes \cdots \otimes v_s \otimes w_1 \otimes \cdots \otimes w_{a-1}
\]
\[
+ \sum_{j=1}^{a-1} v_1 \otimes \cdots \otimes v_s \otimes w_1 \otimes \cdots \otimes X_j \cdot w_j \otimes \cdots \otimes w_{a-1}
\]
for \( X \in g[\Sigma^0|\pi^{-1}(p)]^\Gamma, v_i \in \mathcal{H}(\lambda_i) \) and \( w_j \in V(\mu_j) \),
and let \( C \) act on \( \mathcal{H} \) by the scalar \(-c\). By the Residue Theorem, these actions combine to make \( \mathcal{H} \) into an \( s_p \)-module. Thus, the action of \( C \) on the tensor product \( \mathcal{H} \otimes \hat{M}(V(\mu), c) \) is trivial.

Now, by the isomorphism (31) (in the following, \( g[\Sigma^0]^\Gamma \) acts on \( V(\mu) \) via its restriction on \( \pi^{-1}(p) \) and \( C \) acts via the scalar \( c \))
\[
\left[ \mathcal{H} \otimes \hat{M}(V(\mu), c) \right]_{g[\Sigma^0|\pi^{-1}(p)]^\Gamma} = \left[ \mathcal{H} \otimes \hat{M}(V(\mu), c) \right]_{s_p}, \text{ since } C \text{ acts trivially}
\]
\[
= \mathcal{H} \otimes U(s_p) \hat{M}(V(\mu), c)
\]
\[
= \mathcal{H} \otimes U(s_p) \left( U(s_p) \otimes U(\gamma[\Sigma^0]^\Gamma \otimes \mathbb{C}) \right) V(\mu)
\]
\[
= \mathcal{H} \otimes U(s_p) \otimes U(\gamma[\Sigma^0]^\Gamma \otimes \mathbb{C}) V(\mu)
\]
\[
= \mathcal{H} \otimes V(\mu)_{g[\Sigma^0]^\Gamma}.
\]
This proves (24).

Now, we come to the proof of (23):

Let \( K(\mu) \) be the kernel of the canonical projection \( \hat{M}(V(\mu), c) \rightarrow \mathcal{H}(\mu) \). In view of (24), to prove (23), it suffices to show that the image of
\[
\iota : \left[ \mathcal{H} \otimes K(\mu) \right]_{g[\Sigma^0|\pi^{-1}(p)]^\Gamma} \rightarrow \left[ \mathcal{H} \otimes \hat{M}(V(\mu), c) \right]_{g[\Sigma^0|\pi^{-1}(p)]^\Gamma}
\]
is zero: From the isomorphism (30), we get
\[
\hat{L}(\gamma, \Gamma_q) = s_p + \hat{L}(\gamma, \Gamma_q)^{>0}.
\]
Moreover, write
\[
\hat{L}(\gamma, \Gamma_q)^{>0} = \hat{L}(\gamma, \Gamma_q)^+ + g[\Sigma^0] \otimes \mathbb{C},
\]
and observe that any element of \( g[\Sigma^0] \) can be (uniquely) extended to an element of \( g_p := g[\pi^{-1}(p)]^\Gamma \) (cf. Lemma 3.2). Further, \( \Sigma^0 \) being affine, the restriction map \( g[\Sigma^0]^\Gamma \rightarrow g_p \) is surjective, and, of course, \( g[\Sigma^0]^\Gamma \subset s_p^{>0} \). Thus, we get the decomposition:
\[
\hat{L}(\gamma, \Gamma_q) = s_p + \hat{L}(\gamma, \Gamma_q)^+,
\]
and hence, by the Poincaré-Birkhoff-Witt theorem, \( U(\hat{L}(\gamma, \Gamma_q)) \) is the span of elements of the form
\[
Y_1 \ldots Y_m \cdot X_1 \ldots X_n, \text{ for } Y_i \in s_p, X_j \in \hat{L}(\gamma, \Gamma_q)^+ \text{ and } m, n \geq 0.
\]
Consider the decomposition (3) for $\omega_q$: $\omega_q = \tau_q e^{\omega_q}$, under a choice of $\omega_q$-stable Borel subalgebra $h_q$ containing the Cartan subalgebra $h$ in the sense of Section 2. Under such a choice, there exist $s_{2j}$-triples $x_i, y_i, h_i \in g$ for each $i \in \hat{I}(\omega_q)$ such that $\tilde{x}_i = x_i e^{\omega_q}, \tilde{y}_i = y_i e^{\omega_q}$, $i \in \hat{I}(\omega_q)$ are Chevalley generators of $\hat{L}(\omega_q, \sigma_q)$. Moreover, $x_i, y_i$ satisfy

$$\sigma_q(x_i) = e_q x_i^\vee, \quad \text{and} \quad \sigma_q(y_i) = e_q y_i^\vee.$$  

Let $v_+$ be the highest weight vector of $\hat{M}(\nu(q), c)$. Recall (cf. (9)) that $K(q)$ is generated by $\tilde{y}_i e_q^{n_q + 1} \cdot v_+$, for all $i \in \hat{I}(\omega_q)$. Thus, to prove the vanishing of the map $\iota$, it suffices to show that for any $i \in \hat{I}(\omega_q)$

$$\iota(h \otimes (X_1 \cdots X_n \cdot \tilde{y}_i e_q^{n_q + 1} \cdot v_+)) = 0,$$

for $h \in \mathcal{H}$, any $n \geq 0$ and $X_j \in \hat{L}(\omega_q, \sigma_q)^+$. But, $\tilde{y}_i e_q^{n_q + 1} \cdot v_+$ being a highest weight vector,

$$\hat{L}(\omega_q, \sigma_q)^+ \cdot (\tilde{y}_i e_q^{n_q + 1} \cdot v_+) = 0.$$

Thus, to prove (33), it suffices to show that for any $i \in \hat{I}(\omega_q)^+$

$$\iota(h \otimes (y_i e_q^{n_q + 1} \cdot v_+)) = 0, \quad \text{for any} \quad h \in \mathcal{H}.$$  

Fix $i \in \hat{I}(\omega_q)^+$. Take $f \in \mathbb{C}[\Sigma^+ \gamma]$ such that

$$f_q \equiv z_q^{s_i} \mod z_q^{s_i + 1},$$

and the order of vanishing of $f$ at any $q' \neq q \in \pi^{-1}(p)$ is at least $(n_{\mu,i} + 3)s_i$. Moreover, replacing $f$ by $\frac{1}{|I_q|} \sum |I_j| \varepsilon_{\gamma_j, \gamma} \sigma_{\gamma_j} f$, we can (and will) assume that

$$\sigma_q f = e_q^{-s_i} f.$$  

Now, take

$$Z = \sum_{\gamma \in I \mid I_q} \gamma \cdot (x_i[f]).$$

Then, writing $Y = \tilde{y}_i$,

$$Z^N Y^{n_{\mu,i} + N + 1} \cdot v_+ = \left( \sum_{\gamma \in I \mid I_q} (\gamma \cdot (x_i[f]))_q \right)^N Y^{n_{\mu,i} + N + 1} \cdot v_+ = (x_i[f])^N Y^{n_{\mu,i} + N + 1} \cdot v_+.$$  

To prove the last equality, observe that $(\gamma_1 \cdot (x_i[f]))_q \cdots (\gamma_N \cdot (x_i[f]))_q$ has zero of order at least $(n_{\mu,i} + 3)s_i + (N - 1)s_i$ unless each $s_{\gamma_j} \cdot I_q = I_q$. But, $Y^{n_{\mu,i} + N + 1}$ has order of pole equal to $(n_{\mu,i} + N + 1)s_i$. Since $(n_{\mu,i} + 3)s_i + (N - 1)s_i > (n_{\mu,i} + N + 1)s_i$, we get the last equality. Thus, by Lemma 2.4 for $X = x_i[f]$ and $Y = \tilde{y}_i$, for any $N \geq 1$, there exists $\alpha \neq 0$ such that

$$\iota (h \otimes (Y^{n_{\mu,i} + 1} \cdot v_+)) = \alpha \iota (h \otimes X^N Y^{n_{\mu,i} + N + 1} \cdot v_+),$$

$$\iota (h \otimes Z^N Y^{n_{\mu,i} + N + 1} \cdot v_+), \quad \text{by (35)}$$

$$= (-1)^N \alpha \iota (Z^N \cdot h \otimes Y^{n_{\mu,i} + N + 1} \cdot v_+),$$

$$= 0, \quad \text{by Lemma 2.5 for large} \ N \ (\text{see the argument below}).$$
This proves (34) and hence completes the proof of the theorem.

We now explain more precisely how Lemma 2.5 implies $\mathcal{Z}^N \cdot h = 0$. With respect to the pair $(b, h)$ stable under $\sigma$ (note that $b$ might not be the same as $b_q$), since $\Gamma$ preserves the pair $(b, h)$, the group $\Gamma$ acts on the root system $\Phi(g, b)$ of $g$ by factoring through the group of outer automorphisms. Moreover, $\Gamma$ preserves positive roots and also negative roots. From the construction of $x_i$ in §2, $x_i$ is either a linear combination of positive root vectors or a linear combination of negative root vectors. From this consideration, one can see that either $\gamma \cdot x_i \in b$ for any $\gamma \in \Gamma$, or $\gamma \cdot x_i \in b^-$ for any $\gamma \in \Gamma$, where $b^-$ is the negative Borel of $b$. Therefore, we may apply Lemma 2.5 to show $\mathcal{Z}^N \cdot h = 0$. □

**Remark 4.4.** Observe that the condition that $\Gamma$ stabilizes a Borel subalgebra $b$ and hence also a Cartan subalgebra $h \subset b$ is equivalent to the condition that the image of $\Gamma$ in $\text{Aut} g$ is contained in $D \ltimes \text{Int} H$, where $D$ is the group of diagram automorphisms of $g$ and $H$ is the maximal torus of $G$ with Lie algebra $h$ ($G$ being the adjoint group with Lie algebra $g$).

The following result is the twisted analogue of "Propagation of Vacua" due to Tsuchiya-Ueno-Yamada [TUY].

**Corollary 4.5.** With the notation and assumptions as in Theorem 4.3 (in particular, $(\bar{\Sigma}, \vec{o})$ is a $s$-pointed curve), for any smooth point $q \in \Sigma^0 := \Sigma \setminus \pi^{-1}(\vec{\sigma})$ with $p = \pi(q)$ a smooth point of $\bar{\Sigma}$ and $0 \in D_{c,q}$ (cf. Corollary 2.2), there are canonical isomorphisms:

(a) $\mathcal{Y}_{\Sigma, \Gamma, \vec{\sigma}, \vec{\lambda}, \vec{\mu}}(\vec{\sigma}, \vec{\lambda}) \cong \mathcal{Y}_{\Sigma, \Gamma, \vec{\sigma}, \vec{\mu}}((\vec{\sigma}, p), (\vec{\lambda}, 0))$, and

(b) For $\bar{\Sigma}$ an irreducible curve, $\mathcal{Y}_{\Sigma, \Gamma, \vec{\sigma}, \vec{\lambda}} \cong [\mathcal{H}(0) \otimes V(\vec{\lambda})]_{\mathcal{H}[\Sigma \setminus \pi^{-1}(p)]^\Gamma}$, where the point $p$ is assigned weight 0.

*Proof.* (a): Apply Theorem 4.3 for the case $\vec{\rho} = (p)$ and $\vec{\mu} = (0)$.

(b): It follows from Theorem 4.3 and the (a)-part. (In Theorem 4.3 replace $\vec{\sigma}$ by the singleton $(p)$, $\vec{\lambda}$ by $(0)$, $\vec{\rho}$ by $\vec{\sigma}$ and $\vec{\mu}$ by $\vec{\lambda}$.) □

**Remark 4.6.** (a) A much weaker form of the above Corollary part (a) (where $\Gamma$ is of order 2 and $\vec{\sigma}$ consists of all the ramification points) is proved in [FS, Lemma 7.1]. It should be mentioned that they use the more general setting of twisted Vertex Operator Algebras.

(b) When all the marked points are unramified and $|\Gamma|$ is a prime, the Propagation of Vacua is proved in [D].

### 5. Factorization Theorem

The aim of this section is to prove the Factorization Theorem which identifies the space of covacua for a genus $g$ nodal curve $\Sigma$ with a direct sum of the spaces of covacua for its normalization $\tilde{\Sigma}$ (which is a genus $g - 1$ curve).

Let $\pi : \Sigma \to \tilde{\Sigma}$ be a $\Gamma$-cover of a $s$-pointed curve $(\tilde{\Sigma}, \vec{\sigma})$. We do not assume that $\tilde{\Sigma}$ is irreducible. Moreover, $\phi : \Gamma \to \text{Aut}(g)$ is a group homomorphism.

**Definition 5.1.** [BR, Définition 4.1.4] Let $\Sigma$ be a reduced (but not necessarily connected) projective curve with at worst only nodal singularity. (Recall that a point $P \in \Sigma$ is
called a \textit{node} if analytically a neighborhood of $P$ in $\Sigma$ is isomorphic with an analytic neighborhood of $(0,0)$ in the curve $xy = 0$ in $\mathbb{A}^2$.) Then, the action of $\Gamma$ on $\Sigma$ at any node $q \in \Sigma$ is called \textit{stable} if the derivative $\dot{\sigma}$ of any element $\sigma \in \Gamma_q$ acting on the Zariski tangent space $T_q(\Sigma)$ satisfies the following:

$$\det(\dot{\sigma}) = 1 \quad \text{if } \sigma \text{ fixes the two branches at } q,$$

$$= -1 \quad \text{if } \sigma \text{ exchanges the two branches.} 
\best{\text{(36)}}$$

Assume that $p \in \tilde{\Sigma}$ is a node (possibly among other nodes) and also assume that the fiber $\pi^{-1}(p)$ consists of nodal points. Assume further that the action of $\Gamma$ at the points $q \in \pi^{-1}(p)$ is stable. Observe that, in this case, since $p$ is assumed to be a node, any $\sigma \in \Gamma_q$ can not exchange the branches at $q$ for otherwise the point $p$ would be smooth.

We fix a level $c \geq 1$.

Let $\tilde{\Sigma}'$ be the curve obtained from $\tilde{\Sigma}$ by the normalization $\tilde{v} : \tilde{\Sigma}' \to \tilde{\Sigma}$ at only the point $p$. Thus, $\tilde{v}^{-1}(p)$ consists of two smooth points $p', p''$ in $\tilde{\Sigma}'$ and

$$\tilde{v}|_{\tilde{\Sigma}'\setminus\{p', p''\}} : \tilde{\Sigma}'\setminus\{p', p''\} \to \tilde{\Sigma}\setminus\{p\}$$

is a birational isomorphism. We denote the preimage of any point of $\tilde{\Sigma}\setminus\{p\}$ in $\tilde{\Sigma}'\setminus\{p', p''\}$ by the same symbol. Let $\pi' : \Sigma' \to \Sigma'$ be the pull-back of the Galois cover $\pi$ via $\tilde{v}$. In particular, $\pi'$ is a Galois cover with Galois group $\Gamma$. Thus, we have the fiber diagram:

\[
\begin{array}{ccc}
\Sigma' & \xrightarrow{\pi'} & \Sigma \\
\downarrow & \vline & \downarrow \\
\Sigma' & \xrightarrow{\tilde{v}} & \tilde{\Sigma}.
\end{array}
\]

\textbf{Lemma 5.2.} With the same notation and assumptions as in Definition 5.1,

1. the map $\nu$ is a normalization of $\Sigma$ at every point $q \in \pi^{-1}(p)$;
2. there exists a natural $\Gamma$-equivariant bijection $\kappa : \pi'^{-1}(p') \simeq \pi'^{-1}(p'')$;
3. for any $q \in \pi^{-1}(p)$, we have $\Gamma_q = \Gamma_{q'} = \Gamma_{q''}$,

where $\nu^{-1}(q)$ consists of two smooth points $q', q''$, and $\Gamma_q, \Gamma_{q'}$ and $\Gamma_{q''}$ are stabilizer groups of $\Gamma$ at $q, q'$ and $q''$ respectively. Moreover, $\Gamma_q = \Gamma_{q'} = \Gamma_{q''}$ is a cyclic group.

\textbf{Proof.} Let $q$ be any point in $\pi^{-1}(p)$ of ramification index $e$. Since $\pi^{-1}(p)$ consists of nodal points by assumption, there are two branches in the formal neighborhood of $q$. If any $\sigma \in \Gamma_q$ exchanges two branches then the point $p = \pi(q)$ is smooth in $\tilde{\Sigma}$, which contradicts the assumption that $p$ is a nodal point. Thus, $\Gamma_q$ must preserve branches. In particular, since no nontrivial element of $\Gamma$ fixes pointwise any irreducible component of $\Sigma$, $\Gamma_q$ is cyclic. Therefore, by the condition (36), we can choose a formal coordinate system $z', z''$ around the nodal point $q$ such that $\hat{\otimes}_{\Sigma_q} = \mathbb{C}[[z', z'']] / (z'z'')$, and a generator $\sigma_q$ of $\Gamma_q$ such that

$$\sigma_q(z') = e^{-1}z', \quad \text{and} \quad \sigma_q(z'') = ez''.$$
where $\epsilon := e^{2\pi i}$ is the standard primitive $e$-th root of unity. (Observe that $\epsilon$ must be a primitive $e$-th root of unity, since $\Gamma_q$ acts faithfully on each of the two formal branches through $q$.)

We can choose a formal coordinate around $p$ in $\tilde{\Sigma}$ such that $\partial_{\Sigma,p} \simeq \mathbb{C}[x', x'']/(x'x'')$ and the embedding $\partial_{\Sigma,p} \hookrightarrow \partial_{\Sigma,q}$ is given by $x' \mapsto (z')^e$, $x'' \mapsto (z'')^e$.

The node $p$ splits into two smooth points $p'$, $p''$ via $\tilde{v}$. Without loss of generality, we can assume $x'$ (resp. $x''$) is a formal coordinate around $p'$ (resp. $p''$) in $\tilde{\Sigma}$. Then, $q$ will also split into two smooth points $q'$, $q''$ via the map $\nu$, where $z'$ (resp. $z''$) is a formal coordinate around $q'$ (resp. $q''$). It shows that the map $\nu$ is a normalization at every point $q \in \pi^{-1}(p)$.

The pullback gives a decomposition

$$(\pi \circ \nu)^{-1}(p) = \nu^{-1}(\pi^{-1}(p)) = \pi^{-1}(p') \sqcup \pi^{-1}(p'').$$

From the definition of the fiber product, there exist $\Gamma$-equivariant canonical bijections:

$$(37) \quad \pi^{-1}(p') \simeq \pi^{-1}(p) \text{ and } \pi^{-1}(p'') \simeq \pi^{-1}(p).$$

Hence, we get a $\Gamma$-equivariant canonical bijection $\kappa : \pi^{-1}(p') \simeq \pi^{-1}(p'')$. For any $q \in \pi^{-1}(p)$, $\nu^{-1}(q) = \{q', q''\}$. By the choice of $q'$, $q''$ as above, $\pi'(q') = p'$ and $\pi'(q'') = p''$. Therefore, $\kappa$ maps $q'$ to $q''$. Moreover, from (37), the stabilizer groups $\Gamma_q$, $\Gamma_{q'}$ and $\Gamma_{q''}$ are all the same (and of order $e$). Since $q'$ (resp. $q''$) is a smooth point of $\Sigma'$, $\Gamma_{q'}$ (resp. $\Gamma_{q''}$) is cyclic.

Let $g_p$ denote the Lie algebra $g[\pi^{-1}(p)]^{\Gamma}$ (observe that we can attach a Lie algebra $g_p$ regardless of the smoothness of $p$). Then, the $\Gamma$-equivariant bijections $\nu : \pi^{-1}(p') \simeq \pi^{-1}(p)$ and $\nu : \pi^{-1}(p'') \simeq \pi^{-1}(p)$ (cf. equation (37)) induce isomorphisms of Lie algebras $\kappa' : g_{p'} \simeq g_p$ and $\kappa'' : g_{p''} \simeq g_p$ respectively. Recall that $p'$, $p''$ are smooth points of $\tilde{\Sigma}$. Let $D_{c,p'}$ (resp. $D_{c,p''}$) denote the finite set of highest weights of irreducible representations of $g_p$ induced via the isomorphism $\kappa'$ (resp. $\kappa''$) which give rise to integrable highest weight $\hat{g}_{p'}$-modules (resp. $\hat{g}_{p''}$-modules) with central charge $c$.

Set

$$\Sigma' = \Sigma \setminus \pi^{-1}(\partial), \text{ and } \Sigma'' = \Sigma' \setminus \pi^{-1}(\partial).$$

The map $\nu$ on restriction gives rise to an isomorphism

$$\nu : \Sigma'' \setminus \pi^{-1}(p', p'') \simeq \Sigma' \setminus \pi^{-1}(p) \hookrightarrow \Sigma'$$

which, in turn, gives rise to a Lie algebra homomorphism

$$\nu^* : g[\Sigma'']^{\Gamma} \to g[\Sigma'' \setminus \pi^{-1}(p', p'')]^{\Gamma}.$$ 

Let $\tilde{\lambda} = (\lambda_1, \ldots, \lambda_s)$ be an $s$-tuple of weights with $\lambda_i \in D_{c,0i}$, ‘attached’ to $0_i$. We denote the highest weight of the dual representation $V(\mu)^*$ of $g_p$ by $\mu^*$, thus $V(\mu)^* \simeq V(\mu^*)$.

By Lemma 5.2, there exists a canonical bijection $\kappa : \pi^{-1}(p') \simeq \pi^{-1}(p'')$ compatible with the action of $\Gamma$. Thus, it induces an isomorphism of Lie algebras $g_{p'} \simeq g_{p''}$.

**Lemma 5.3.** In the same setting as in the beginning of this section, we have

1. there exists an isomorphism $\hat{g}_{p'} \simeq \hat{g}_{p''}$ which restricts to the isomorphisms:

   $$\hat{g}_{p'} \simeq \hat{g}_{p''}, \, \hat{g}_p^x \simeq \hat{g}_p^x, \, \text{and } g_{p'} \simeq g_{p''}.$$
See the relevant notation in §3.

(2) $\mu \in D_{c,p'}$ if and only if $\mu^* \in D_{c,p''}$.

Proof. For any $q \in \pi^{-1}(p)$, in view of Lemma 3.3, the restriction gives isomorphisms $\text{res}_{q'} : \hat{\mathcal{L}}(g, \Gamma_{q'}) \cong \hat{\mathcal{L}}(g, \Gamma_{q''})$. By Lemma 5.2, $\Gamma_{q'} = \Gamma_{q''}$. Let $\mathfrak{b}$ (resp. $\mathfrak{h}$) be a suitable Borel (resp. Cartan) subalgebra of $\mathfrak{g}$ stable under $\Gamma_{q'}$. This gives rise to Chevalley generators $e_i \in \mathfrak{n}$ and $f_i \in \mathfrak{n}^-$, where $\mathfrak{n}$ (resp. $\mathfrak{n}^-$) is the nilradical of $\mathfrak{b}$ (resp. the opposite Borel subalgebra $\mathfrak{b}^\vee$). Let $\omega : \mathfrak{g} \to \mathfrak{g}$ be the Cartan involution taking the Chevalley generators of $\mathfrak{g}$: $e_j \mapsto -f_j, f_j \mapsto -e_j$ and $h \mapsto -h$ for any $h \in \mathfrak{h}$.

Write as in Section 2,

$$\sigma_{q'} = \tau e^{ad h}$$

for a diagram automorphism $\tau$ (possibly identity) and $h \in \mathfrak{h}^\vee$.

Thus,

$$\omega^{-1} \sigma_{q'} \omega = \omega^{-1} \tau \omega e^{ad \omega^{-1}(h)} = \omega^{-1} \tau \omega e^{ad(-h)}.$$}

But, by the definition of (any diagram automorphism) $\tau$ and $\omega$, it is easy to see that

$$\omega^{-1} \tau \omega = \tau.$$

We now need to cosider two cases:

Case I: $\tau$ is of order 1 or 2. In this case,

$$\omega^{-1} \sigma_{q'} \omega = \tau e^{-ad h}, \quad \text{by (38) and (39)}$$

$$= \tau^{-1} e^{-ad h}, \quad \text{since } \tau \text{ is assumed to be of order 1 or 2}$$

$$= \sigma_{q'}^{-1}.$$}

Case II: $\tau$ is of order 3, i.e., $g$ is of type $D_4$ with labelled nodes:

and $\tau$ is the diagram automorphism induced from taking the nodes $1 \leftrightarrow 3, 2 \leftrightarrow 2, 3 \leftrightarrow 4, 4 \leftrightarrow 1$. Let $\tau_1$ be the diagram automorphism induced from taking the nodes $1 \leftrightarrow 1, 2 \leftrightarrow 2, 3 \leftrightarrow 4, 4 \leftrightarrow 3$. Then,

$$\tau_1^{-1} \tau \tau_1 = \tau^{-1}.$$

In this case, we have

$$(\omega \tau_1)^{-1} \sigma_{q'} \omega \tau_1 = \tau_1^{-1} \tau e^{-ad h} \tau_1, \quad \text{by (38) and (39)}$$

$$= \tau_1^{-1} \tau_1^{-1} e^{-ad h} \tau_1, \quad \text{by (41)}$$

$$= \tau_1^{-1} e^{-ad h}, \quad \text{since } (\tau_1)_{\mathfrak{h}'} = \text{Id by [Ka, §8.3, Case 4]}$$

$$= \sigma_{q'}^{-1}. $$
Let $\omega_o$ be the Cartan involution $\omega$ in the first case and $\omega_1$ in the second case. Extend $\omega_o$ to an isomorphism of twisted affine Lie algebras:

$$\hat{\omega}_o : \hat{L}(g, \sigma_{q'}) \to \hat{L}(g, \sigma_{q''})$$

for any $x \in \mathfrak{g}$ and $P \in \mathcal{K}$, where $\sigma_{q'}$ and $\sigma_{q''} = \sigma_{q'}^{-1}$ are the preferred generators of $\Gamma_q = \Gamma_{q'} = \Gamma_{q''}$ acting on a formal coordinate $z', z''$ around $q', q''$ respectively via $\epsilon^{-1}$ (see the proof of Lemma 5.2). Indeed, $\hat{\omega}_o$ is an isomorphism by the identities (40) and (42).

Observe that $\hat{\omega}_o$ restricted to $\mathfrak{g}^{r'} = \mathfrak{g}^{r''}$ is nothing but the Cartan involution.

Clearly, $\hat{\omega}_o$ restricts to an isomorphism $\hat{\beta}_{r'} \simeq \hat{\beta}_{r''}, \hat{\delta}^+_{r'} \simeq \hat{\delta}^+_{r''}$ and $\hat{\gamma}_{r'} \simeq \hat{\gamma}_{r''}$ (see (12) and (13) for relevant notation). This proves the first part of the lemma.

From the isomorphism $\hat{\omega}_o$, the second part of the lemma follows immediately. \hfill \Box

We also give another proof of the second part of the above lemma.

**Another proof of Lemma 5.3 Part (2):** Let $\sigma_{q'}$ (resp. $\sigma_{q''}$) be the canonical generator of $\Gamma_q$ (resp. $\Gamma_{q''}$). We can choose formal parameter $z'$ (resp. $z''$) around $q'$ (resp. $q''$), such that

$$\sigma_{q'}(z') = \epsilon^{-1} z', \quad \sigma_{q''}(z'') = \epsilon^{-1} z'',$$

where $\epsilon = e^{\rho_{r'} w_0}$. As in Section 2, we can write $\sigma_{q'} = \tau' \cdot \epsilon \ad \theta'$. Let $x'_i, y'_i, h'_i = [x'_i, y'_i], i \in \hat{I}(g, \sigma_{q'})$ be chosen as in [Ka, §8.3], where

$$x'_i \in (\mathfrak{g}^{r'})_{\alpha'_i}, y'_i \in (\mathfrak{g}^{r'})_{-\alpha'_i}, \quad \text{for any } i \in I(\mathfrak{g}^{r'})$$

where $\alpha'_i$ is the simple root of $\mathfrak{g}^{r'}$ associated to $i \in I(\mathfrak{g}^{r'})$, and

$$x'_i \in \mathfrak{g}^{r'}_{-\theta'_i}, y'_i \in \mathfrak{g}^{r'}_{\theta'_i}.$$

Let $s_i, i \in \hat{I}(g, \sigma_{q'})$ be the integers as in Section 2. We have

$$\sigma_{q'}(x'_i) = e^{s_i} x'_i, \quad \text{and } \sigma_{q'}(y'_i) = e^{-s_i} y'_i,$$

for any $i \in \hat{I}(g, \sigma_{q'})$. Moreover, the elements $x'_i [z'^{s_i}], y'_i [z'^{-s_i}], h'_i + (x'_i, y'_i) s_i C$ in $\hat{L}(g, \sigma_{q'})$, are a set of Chevalley generators generating the non-completed Kac-Moody algebra $\hat{L}(g, \sigma_{q'}) \subset \hat{L}(g, \sigma_{q''})$. It is well-known that there is a natural bijection between the set of integrable highest weight representations of $\hat{L}(g, \sigma_{q'})$ and $\hat{L}(g, \sigma_{q''})$.

We now introduce the following notation:

$$x''_i := -y'_i, \quad y''_i := -x'_i, \quad \text{and } h''_i := -h'_i,$$

for any $i \in \hat{I}(g, \sigma_{q'})$. Note that $\sigma_{q''} = (\sigma_{q'})^{-1}$. We can identify $\hat{I}(g, \sigma_{q'})$ and $\hat{I}(g, \sigma_{q''})$, since $\mathfrak{g}^{r''} = \mathfrak{g}^{r'}$ where $\tau'' = \tau'\epsilon$ is the diagram automorphism part of $\sigma_{q''}$.

Set $\alpha''_i = -\alpha'_i$ for any $i \in I(\mathfrak{g}^{r''})$, and $\theta''_0 = -\theta'_0$. We can choose $\alpha''_i, i \in I(\mathfrak{g}^{r''})$ as a set of simple roots for $\mathfrak{g}^{r''}$. Then, $\theta''_0$ is the weight of $\mathfrak{g}^{r''}$ as in Section 2 with respect to this choice. Moreover, $x''_i, y''_i, i \in I(\mathfrak{g}^{r''})$ is a set of Chevalley generators of $\mathfrak{g}^{r''}$, and $x''_i \in (\mathfrak{g}^{r''})_{-\theta''_i}, y''_i \in (\mathfrak{g}^{r''})_{\theta''_i}$ also satisfies the choice as in [Ka, §8.3]. We also notice that

$$\sigma_{q''}(x''_i) = e^{s_i} x''_i, \quad \text{and } \sigma_{q''}(y''_i) = e^{-s_i} y''_i,$$

for any $i \in \hat{I}(g, \sigma_{q'})$. By [Ka, Theorem 8.7], we see that the elements $x''_i [z'^{s_i}], y''_i [z'^{-s_i}], h''_i + (x''_i, y''_i) s_i C$ as elements in $\hat{L}(g, \sigma_{q''})$ are Chevalley generators generating the non-completed
Kac-Moody algebra $\tilde{L}(g, \sigma_{q'}) \subset \hat{L}(g, \sigma_{q'})$. Again, there is a natural bijection between the set of integrable highest weight representations of $\hat{L}(g, \sigma_{q'})$ and $\hat{L}(g, \sigma_{q'})$.

We now get an isomorphism of Lie algebras:

$$\hat{\omega} : \hat{L}(g, \sigma_{q'}) \cong \hat{L}(g, \sigma_{q'})$$

given by

$$x_i'[z'^n] \mapsto x_i'[z'^n], \quad y_i'[z'^{-n}] \mapsto y''_i[z'^{-n}],$$

and

$$h_i' + \langle x_i', y_i' \rangle s_i C \mapsto h_i'' + \langle x_i'', y_i'' \rangle s_i C,$$

for any $i \in \hat{I}(g, \sigma_{q'})$. Note that $\langle x_i', y_i' \rangle = \langle x_i'', y_i'' \rangle$ for any $i$. The map $\hat{\omega}$ is indeed an isomorphism, since these Chevalley generators correspond to the same vertices of the affine Dynkin diagram.

Set

$$\tilde{L}(g, \sigma_{q'})^+ = \tilde{L}(g, \sigma_{q'}) \cap \hat{L}(g, \sigma_{q'})$$

and

$$\tilde{L}(g, \sigma_{q'})^{\geq 0} = \tilde{L}(g, \sigma_{q'}) \cap \hat{L}(g, \sigma_{q'})^{\geq 0}.$$

Similarly, we can introduce the Lie algebras $\tilde{L}(g, \sigma_{q'})^+$ and $\tilde{L}(g, \sigma_{q'})^{\geq 0}$. We can see easily that

$$\hat{\omega}(\tilde{L}(g, \sigma_{q'})^+) = \tilde{L}(g, \sigma_{q'})^+,$$

and

$$\hat{\omega}(\tilde{L}(g, \sigma_{q'})^{\geq 0}) = \tilde{L}(g, \sigma_{q'})^{\geq 0}.$$

Define the linear map

$$\tilde{F} : \mathcal{H}(\tilde{\Lambda}) \to \mathcal{H}(\Lambda) \otimes \bigoplus_{\mu \in D_{c,q'}} V(\mu) \otimes V(\mu), \quad h \mapsto h \otimes \sum_{\mu \in D_{c,q'}} I_\mu,$$

where $I_\mu$ is the identity map thought of as an element of $V(\mu^*) \otimes V(\mu) \cong \text{End}_c(V(\mu))$.

We view $V(\mu^*)$ (resp. $V(\mu)$) as an irreducible representation of $\mathfrak{g}_{\mu'}$ (resp. $\mathfrak{g}_{\mu''}$) via the isomorphism $\kappa'$ (resp. $\kappa''$). Let $\mathcal{H}(\mu')$ (resp. $\mathcal{H}(\mu)$) denote the highest weight integrable representation of $\mathfrak{g}_{\mu'}$ (resp. $\mathfrak{g}_{\mu''}$) associated to $\mu'$ (resp. $\mu$) of level $c$. Realize

$$\mathcal{H}(\tilde{\Lambda}) \otimes \mathcal{H}(\mu') \otimes \mathcal{H}(\mu)$$

which contains $\mathcal{H}(\tilde{\Lambda}) \otimes V(\mu^*) \otimes V(\mu)$ as a $\mathfrak{g}[^\sigma_{q'}]$-module at the points $\partial'$, $\mu'$, $\nu''$ respectively. Then, $I_\mu$ being a $\mathfrak{g}_{\mu'}$-invariant, $\tilde{F}$ is a $\mathfrak{g}[\Sigma]$-$\tilde{\mathfrak{g}}$-module map, where we realize the range as a $\mathfrak{g}[\Sigma]$-$\tilde{\mathfrak{g}}$-module via the Lie algebra homomorphism $\kappa'$. Hence, $\tilde{F}$ induces a linear map

$$F : \mathcal{Y}_{\Sigma, \Lambda, \phi}(\partial, \tilde{\Lambda}) \to \bigoplus_{\mu \in D_{c,q'}} \mathcal{Y}_{\Sigma, \Lambda, \phi}(\partial, \mu', \mu).$$

The following theorem is the twisted analogue of the Factorization Theorem due to Tsuchiya-Ueno-Yamada [TUY].
Theorem 5.4. With the notation and assumptions as in the beginning of this section, we further assume that \( \Gamma \) stabilizes a Borel subalgebra \( \mathfrak{b} \) and \( \pi^{-1}(\tilde{\sigma}) \) consists of smooth points of \( \Sigma \). Then, the map

\[
F : \mathcal{V}_{\Sigma, \Gamma, \phi}(\tilde{\sigma}, \tilde{\lambda}) \to \bigoplus_{\mu \in D_{c, p'}} \mathcal{V}_{\Sigma, \Gamma, \phi}((\tilde{\sigma}, p', p''), (\tilde{\lambda}, \mu', \mu))
\]

is an isomorphism.

Dualizing the map \( F \), we get an isomorphism

\[
F^* : \bigoplus_{\mu \in D_{c, p'}} \mathcal{V}_{\Sigma, \Gamma, \phi}^*(((\tilde{\sigma}, p', p''), (\tilde{\lambda}, \mu', \mu))) \to \mathcal{V}_{\Sigma, \Gamma, \phi}^*(\tilde{\sigma}, \tilde{\lambda}).
\]

Proof. As discussed above, the map \( \hat{F} \) defined in (43) is \( g[\Sigma^o]^{\Gamma} \)-equivariant. By Propagation theorem (Theorem 4.3) at points \( p' \) and \( p'' \), taking covariants on both sides of \( \hat{F} \) with respect to the action of \( g[\Sigma^o]^{\Gamma} \) on the left side and with respect to the action of \( g[\Sigma^o]^{\Gamma} \) on the right side, we also obtain the map \( F \).

We first prove the surjectivity of \( F \). Fix a point \( q \in \pi^{-1}(p) \), we may view \( V(\mu') \) and \( V(\mu) \) as representations of the Lie algebra \( g^{\Gamma} \) via the evaluation map \( \text{ev}_g : g_{\mu} \to g^{\Gamma} \) (cf. Lemma 3.2). Correspondingly, we may view \( D_{c, p'^o} \) as certain set of highest weights of \( g^{\Gamma} \). Observe first that \( \Gamma \cdot q' \cap \Gamma \cdot q'' = \emptyset \), since \( q' \) is \( \Gamma \)-invariant and \( q' \cdot (\Gamma \cdot q') = p' \) and \( q' \cdot (\Gamma \cdot q'') = p'' \). Choose a function \( f \in \mathbb{C}[\Sigma^o] \) such that

\[
f(q') = 1 \text{ and } f_{\Gamma \cdot q'^o \setminus (\Gamma \cdot q')} = 0.
\]

For any \( x \in g^{\Gamma} \), let

\[
A(x[f]) := \frac{1}{|\Gamma_q|} \sum_{\gamma \in \Gamma_q} \gamma \cdot (x[f]) \in g[\Sigma^o]^{\Gamma}.
\]

For any \( h \in \mathcal{H}(\tilde{\lambda}) \) and \( v \in \bigoplus_{\mu \in D_{c, p'^o}} (V(\mu') \otimes V(\mu)) \), as elements of

\[
Q := \mathcal{H}(\tilde{\lambda}) \otimes \left( \bigoplus_{\mu \in D_{c, p'^o}} V(\mu') \otimes V(\mu) \right),
\]

we have the following equality (for any \( x \in g^{\Gamma} \))

\[
A(x[f]) \cdot (h \otimes v) - h \otimes (x \otimes v) = (A(x[f]) \cdot h) \otimes v,
\]

where the action \( \otimes \) of \( g^{\Gamma} \) on \( V(\mu') \otimes V(\mu) \) is via its action on the first factor only. In particular, as elements of \( Q \),

\[
A(x[f]) \cdot (h \otimes \sum_{\mu \in D_{c, p'^o}} I_\mu) - h \otimes \beta(x) = (A(x[f]) \cdot h) \otimes \sum_{\mu \in D_{c, p'^o}} I_\mu,
\]

where \( \beta \) is the map defined by

\[
\beta : U(g^{\Gamma}) \to \bigoplus_{\mu \in D_{c, p'^o}} V(\mu') \otimes V(\mu), \quad \beta(a) = a \otimes \sum_{\mu} I_\mu.
\]

Observe that \( \text{Im}(\beta) \) is \( g^{\Gamma} \oplus g^{\Gamma} \)-stable under the component wise action of \( g^{\Gamma} \oplus g^{\Gamma} \) on \( V(\mu') \otimes V(\mu) \) since \( I_\mu \) is \( g^{\Gamma} \)-invariant under the diagonal action of \( g^{\Gamma} \). Moreover,
V(μ') ⊗ V(μ) is an irreducible \( g^\Gamma \otimes g^\gamma \)-module with highest weight \((μ', μ)\); and \( \text{Im}(β) \) has a nonzero component in each \( V(μ') ⊗ V(μ) \). Thus, \( β \) is surjective.

From the surjectivity of \( β \), we get that the map \( F \) is surjective by combining the equation (47) and the Propagation theorem (Theorem 4.3).

We next show that \( F \) is injective. Equivalently, we show that the dual map

\[
F^* : \bigoplus_{\mu \in D_{\gamma p''}} \Gamma_{\Sigma, \Gamma, \phi}^p \left( (\delta, p', p''), (\hat{λ}, μ', μ) \right) \to \Gamma_{\Sigma, \Gamma, \phi}^p (\delta, \hat{λ})
\]

is surjective.

From the definition of \( \Gamma_{\Sigma, \Gamma, \phi}^p \) and identifying the domain of \( F^* \) via Theorem 4.3, we think of \( F^* \) as the map

\[
F^* : \text{Hom}_{\text{nl}[\Sigma^\alpha]} \left( \mathcal{H}(\hat{λ}) \otimes (\oplus_{\mu \in D_{\gamma p''}} V(μ') \otimes V(μ)) \right) \to \text{Hom}_{\text{nl}[\Sigma^\alpha]} (\mathcal{H}(\hat{λ}), \mathbb{C})
\]

induced from the inclusion

\[
\check{F} : \mathcal{H}(\hat{λ}) \to \mathcal{H}(\hat{λ}) \otimes (\oplus_{\mu \in D_{\gamma p''}} V(μ') \otimes V(μ)), h \mapsto h \otimes \sum_{\mu \in D_{\gamma p''}} I_μ, \text{ for } h \in \mathcal{H}(\hat{λ}).
\]

Let \( \mathbb{C}_p[\Sigma^\alpha] \subset \mathbb{C}_p[\Sigma^\alpha'] \subset \mathbb{C}[\Sigma^\alpha] \) be the ideals of \( \mathbb{C}[\Sigma^\alpha] \):

\[
\mathbb{C}_p[\Sigma^\alpha] := \{ f \in \mathbb{C}[\Sigma^\alpha] : f|_{(\mathfrak{π}^{-1}(p))} = 0 \},
\]

and

\[
\mathbb{C}_p[\Sigma^\alpha'] := \{ f \in \mathbb{C}[\Sigma^\alpha] : f|_{(\mathfrak{π}^{-1}(p'))} = 0 \}.
\]

(Observe that, under the canonical inclusion \( \mathbb{C}[\Sigma^\alpha] \subset \mathbb{C}[\Sigma^\alpha'], \mathbb{C}_p[\Sigma^\alpha] \) is an ideal of \( \mathbb{C}[\Sigma^\alpha] \) consisting of those functions vanishing at \( \mathfrak{π}^{-1}(p', p'') \).) Now, define the Lie ideals of \( g[\Sigma^\alpha]^\Gamma \):

\[
\mathfrak{g}_p[\Sigma^\alpha]^\Gamma := \left( \mathfrak{g} \otimes \mathbb{C}_p[\Sigma^\alpha] \right)^\Gamma, \text{ and } \mathfrak{g}_p[\Sigma^\alpha']^\Gamma := \left( \mathfrak{g} \otimes \mathbb{C}_p[\Sigma^\alpha'] \right)^\Gamma.
\]

Define the linear map

\[
\mathfrak{g}^\Gamma \to \mathfrak{g}_p[\Sigma^\alpha]^\Gamma / \mathfrak{g}_p[\Sigma^\alpha]^\Gamma, \ x \mapsto A(x(f)) + \mathfrak{g}_p[\Sigma^\alpha]^\Gamma,
\]

where \( x \in \mathfrak{g}^\Gamma \), \( f \in \mathbb{C}_p[\Sigma^\alpha'] \) is any function satisfying (44) and \( A(x(f)) \) is defined by (45).

Clearly, the above map is independent of the choice of \( f \) satisfying (44). Moreover, it is a Lie algebra homomorphism:

For \( x, y \in \mathfrak{g}^\Gamma \),

\[
[A(x(f)), A(y(f))] = \frac{1}{|\mathfrak{g}|^2} \sum_{\gamma, \gamma' \in \Gamma} \left[ γ \cdot (x(f)), γ' \cdot (y(f)) \right]
= \frac{1}{|\mathfrak{g}|^2} \sum_{\sigma \in \Gamma_q} \sum_{\gamma, \gamma' \in \Gamma} \left[ \gamma' \sigma \cdot (x(f)), γ' \cdot (y(f)) \right] + \frac{1}{|\mathfrak{g}|^2} \sum_{\gamma \in \Gamma_q} \sum_{\gamma' \in \Gamma} \left[ γ \cdot (x(f)), γ' \cdot (y(f)) \right]
= \frac{1}{|\mathfrak{g}|} \sum_{\gamma' \in \Gamma} \gamma' \cdot ([x, y][f]) \mod \mathfrak{g}_p[\Sigma^\alpha]^\Gamma.
\]

To prove the last equality, observe that, for \( γ \not\in γ', (γ \cdot f) \cdot (γ' \cdot f) \in \mathbb{C}_p[\Sigma^\alpha]. \) Also, for \( σ \in \Gamma_q, σ \cdot f - f \in \mathbb{C}_p[\Sigma^\alpha] \) and \( f^2 - f \in \mathbb{C}_p[\Sigma^\alpha]. \)
Let
\[ \varphi : U(\mathfrak{g}_F) \to U\left( \mathfrak{g}_p[\Sigma^\omega]^{I}/\mathfrak{g}_p[\Sigma^\omega]^{I} \right) \]
be the induced homomorphism of the enveloping algebras.

To prove the surjectivity of \( F^* \), take \( \Phi \in \text{Hom}_{\mathfrak{g}[\Sigma^\omega]}(\mathcal{H}(\mathcal{L}), \mathbb{C}) \) and define the linear map
\[
\tilde{\Phi} : \mathcal{H}(\mathcal{L}) \otimes \left( \bigoplus_{\mu \in D_{\omega}^{\omega'}} V(\mu)^* \otimes V(\mu) \right) \to \mathbb{C}
\]
via
\[
\tilde{\Phi}(h \otimes \beta(a)) = \Phi(\varphi(a') \cdot h), \text{ for } h \in \mathcal{H}(\mathcal{L}) \text{ and } a \in U(\mathfrak{g}_F),
\]
where \( t : U(\mathfrak{g}_F) \to U(\mathfrak{g}_F) \) is the anti-automorphism taking \( x \mapsto -x \) for \( x \in \mathfrak{g}_F \), \( \beta \) is the map defined by (48) and \( \varphi \) is defined above. (Observe that even though \( \varphi(a) \cdot h \) is not well defined, but \( \Phi(\varphi(a) \cdot h) \) is well defined, i.e., it does not depend upon the choice of the coset representatives in \( \mathfrak{g}_p[\Sigma^\omega]^{I}/\mathfrak{g}_p[\Sigma^\omega]^{I} \).)

To show that \( \tilde{\Phi} \) is well defined, we need to show that for any \( a \in \text{Ker} \beta \) and \( h \in \mathcal{H}(\mathcal{L}) \),
\[
(50) \quad \Phi(\varphi(a') \cdot h) = 0.
\]
This will be proved in the next Lemma 5.6.

We next show that \( \tilde{\Phi} \) is a gr[\Sigma^\omega]^I-module map. For any element \( X = \sum x_i[g_i] \in \mathfrak{g}[\Sigma^\omega]^I \)
where \( x_i \in \mathfrak{g} \) and \( g_i \in \mathbb{C}[\Sigma^\omega] \), we need to check that for any \( h \in \mathcal{H}(\mathcal{L}) \) and \( a \in U(\mathfrak{g}_F) \),
\[
(51) \quad \tilde{\Phi}(X \cdot (h \otimes \beta(a))) = 0.
\]
Take any \( \Gamma_q \)-invariant \( f' \in \mathbb{C}[\Sigma^\omega] \) (resp. \( f'' \in \mathbb{C}[\Sigma^\omega] \)) satisfying (44) (resp. \( f''(q'') = 1 \)
and \( f''_{|\Gamma_{q''}\cup(\Gamma_{q''}\backslash q'')} = 0 \)). Then,
\[
\mathbb{C}[\Sigma^\omega] = \mathbb{C}_p[\Sigma^\omega] + S_{f'} + S_{f''}, \text{ where } S_{f'} := \sum_{\gamma \in \Gamma_q} \mathbb{C}(\gamma \cdot f'), S_{f''} := \sum_{\gamma \in \Gamma_q} \mathbb{C}(\gamma \cdot f'').
\]
Thus,
\[
\mathfrak{g}[\Sigma^\omega]^I = \mathfrak{g}_p[\Sigma^\omega]^I + \left( \mathfrak{g} \otimes S_{f'} \right)^I + \left( \mathfrak{g} \otimes S_{f''} \right)^I.
\]
It suffices to prove the equation (51) in the following three cases of \( X \):

**Case 1.** \( X \in \mathfrak{g}_p[\Sigma^\omega]^I \): In this case
\[
\tilde{\Phi}(X \cdot (h \otimes \beta(a))) = \tilde{\Phi}((X \cdot h) \otimes \beta(a)) + \tilde{\Phi}(h \otimes X \cdot \beta(a))
= \Phi(\varphi(a') \cdot X \cdot h), \quad \text{since } X \in \mathfrak{g}[\Sigma^\omega]^I
= \Phi(X \cdot \varphi(a') \cdot h) + \Phi([\varphi(a'), X] \cdot h)
= 0, \quad \text{since } \Phi \text{ is a } \mathfrak{g}[\Sigma^\omega]^I\text{-module map and } [\varphi(a'), X] \in \mathfrak{g}[\Sigma^\omega]^I.
\]

**Case 2.** \( X \in \left( \mathfrak{g} \otimes S_{f'} \right)^I \): Write
\[
X = \sum_{\gamma \in \Gamma_q} x_{\gamma}[\gamma \cdot f'], \text{ for some } x_{\gamma} \in \mathfrak{g}.
\]
Observe first that since \( \{ \gamma \cdot f' \}_{\gamma \in \Gamma_{q'}} \) are linearly independent, \( x_1 \in g^\Gamma_{q'} \). Moreover, we claim that

\[
X - \varphi(x_1) \in g_p[\Sigma^o]_{\Gamma}, \quad \text{i.e.,} \quad X - \sum_{\gamma \in \Gamma_{q'}} \gamma \cdot (x_1[f']) \in g_p[\Sigma^o]_{\Gamma}.
\]

To prove (52), since \( X \) and \( \sum_{\gamma \in \Gamma_{q'}} \gamma \cdot (x_1[f']) \) both are \( \Gamma \)-invariant, it suffices to observe that their difference vanishes both at \( q' \) and \( q'' \). Now,

\[
\tilde{\Phi}(X \cdot (h \otimes \beta(a))) = \tilde{\Phi}((X \cdot h) \otimes \beta(a)) + \tilde{\Phi}(h \otimes X \cdot \beta(a))
\]

\[
= \Phi(\varphi(a') \cdot X \cdot h) - \Phi(\varphi(a'x_1) \cdot h)
\]

\[
= \Phi(\varphi(a')(X - \varphi(x_1)) \cdot h) + \Phi([\varphi(a'), X - \varphi(x_1)] \cdot h)
\]

\[
= 0,
\]

by (52) and since \( g_p[\Sigma^o]_{\Gamma} \) is an ideal in \( g[\Sigma^o]_{\Gamma} \).

Case 3. \( X \in \left( g \otimes S \right)^\Gamma \): Write

\[
X = \sum_{\gamma \in \Gamma_{q'}} x_{\gamma} \gamma \cdot f'', \quad \text{for some} \quad x_{\gamma} \in g.
\]

Same as in Case 2, we have \( x_1 \in g^\Gamma_{q''} \). Moreover, we claim that

\[
X + \varphi(x_1) \in g[\Sigma^o]_{\Gamma}, \quad \text{i.e.,} \quad X + \sum_{\gamma \in \Gamma_{q'}} \gamma \cdot (x_1[f']) \in g[\Sigma^o]_{\Gamma}.
\]

To prove (53), it suffices to observe (from the \( \Gamma \)-invariance) that \( X + \sum_{\gamma \in \Gamma_{q'}} \gamma \cdot (x_1[f']) \) takes the same value at both \( q' \) and \( q'' \). Now,

\[
\tilde{\Phi}(X \cdot (h \otimes \beta(a))) = \tilde{\Phi}((X \cdot h) \otimes \beta(a)) + \tilde{\Phi}(h \otimes X \cdot \beta(a))
\]

\[
= \tilde{\Phi}((X \cdot h) \otimes \beta(a)) + \Phi(h \otimes x_1 \otimes' \beta(a)), \quad \text{where} \quad \otimes' \text{ denotes the action of} \ g^\Gamma_{q''} \text{ on} \ V(\mu') \otimes V(\mu) \text{ on the second factor only}
\]

\[
= \tilde{\Phi}((X \cdot h) \otimes \beta(a)) + \Phi(h \otimes \beta(ax_1)), \quad \text{since the actions} \ \otimes \text{ and} \ \otimes' \text{ commute}
\]

\[
= \Phi(\varphi(a') \cdot X \cdot h) + \Phi(\varphi(ax_1) \gamma \cdot f' \cdot h)
\]

\[
= 0, \quad \text{since} \ [\varphi(a'), X] \in g[\Sigma^o]_{\Gamma} \text{ and using (53)}.
\]

This completes the proof of (51) and hence \( \tilde{\Phi} \) is a \( g[\Sigma^o]_{\Gamma} \)-module map.

From the definition of \( \tilde{\Phi} \), it is clear that \( F^\ast(\tilde{\Phi}) = \Phi \). This proves the surjectivity of \( F^\ast \) (and hence the injectivity of \( F \)) modulo the next lemma. Thus, the theorem is proved (modulo the next lemma).

\section*{Definition 5.5.}

For any \( \mu \in D \) (where \( D \) is the set of dominant integral weights of \( g^\Gamma_{q''} \)), consider the algebra homomorphism

\[
\tilde{\beta}_\mu : U(g^\Gamma_{q''}) \to \text{End}_\mathbb{C}(V(\mu)),
\]
Let $K_\mu$ be the kernel of $\bar{\beta}_\mu$, which is a two sided ideal of $U(g^{F_v})$ (called a primitive ideal). From the definition of $\beta$ (cf. equation (48)), it is easy to see that, under the identification of $V(\mu') \otimes V(\mu)$ with $\text{End}_C(V(\mu))$,

\begin{equation}
\beta(a)(\bar{v}) = a' \cdot \bar{v}, \text{ for any } a \in U(g^{F_v}) \text{ and } \bar{v} \in V(\mu).
\end{equation}

Thus,

\begin{equation}
\text{Ker} \beta = \bigcap_{\mu \in D_{g^{F_v}}} K^i_{\mu}.
\end{equation}

From the definition of $\bar{\beta}_\mu$, it follows immediately that for any left ideal $K \subset U(g^{F_v})$ such that $U(g^{F_v})/K$ is an integrable $g^{F_v}$-module, if the $g^{F_v}$-module $U(g^{F_v})/K$ has isotypic components of highest weights $\{\mu_i\}_{i \in \Lambda} \subset D$, then

\begin{equation}
K \supset \bigcap_{i \in \Lambda} K_{\mu_i}.
\end{equation}

We are now ready to prove the following lemma.

**Lemma 5.6.** With the notation as in the proof of Theorem 5.4 (cf. identity (50)), for any $a \in \text{Ker} \beta$, $\Phi \in \text{Hom}_{\mathfrak{g}[\Sigma]}(\mathcal{H}(\bar{\lambda}), \mathbb{C})$, and $h \in \mathcal{H}(\bar{\lambda})$,

\begin{equation}
\Phi(\varphi(a') \cdot h) = 0.
\end{equation}

**Proof.** Let $s_{p'}$ be the Lie algebra

\[ s_{p'} := \left( g \otimes \mathbb{C}_{p'}[\Sigma^{\alpha} \setminus \pi^{-1}(p')] \right)^{\Gamma} \oplus C, \]

where $\mathbb{C}_{p'}[\Sigma^{\alpha} \setminus \pi^{-1}(p')] \subset \mathbb{C}[\Sigma^{\alpha} \setminus \pi^{-1}(p')]$ is the ideal consisting of functions vanishing at $\pi^{-1}(p')$, with the Lie bracket defined as in Formula (26) and $C$ is central in $s_{p'}$. There is a Lie algebra embedding

\begin{equation}
\sigma_{p'} \hookrightarrow \delta_{p'}, \quad x_i[f_i] \mapsto \sum_i x_i([f_i]_{p'}) \text{ and } C \mapsto C.
\end{equation}

Let $\mathcal{H}(\bar{\lambda})^\ast$ be the full vector space dual of $\mathcal{H}(\bar{\lambda})$. The Lie algebra $s_{p'}$ acts on $\mathcal{H}(\bar{\lambda})^\ast$ where $x[f]$ acts on $\mathcal{H}(\bar{\lambda})$ as in (15) and the center $C$ acts by the scalar $-c$. By the residue theorem, it is indeed a Lie algebra action.

This gives rise to the (dual) action of $s_{p'}$ on $\mathcal{H}(\bar{\lambda})^\ast$. Let $M \subset \mathcal{H}(\bar{\lambda})^\ast$ be the $s_{p'}$-submodule generated by $\Phi \in \mathcal{H}(\bar{\lambda})^\ast$. We claim that the action of $s_{p'}$ on $M$ extends to a $\delta_{p'}$-module structure on $M$ via the embedding (58). Let $s^+_p \subset s_{p'}$ be the subalgebra $g_p[\Sigma^{\alpha}]^{\Gamma}$ defined by the equation (49) of Theorem 5.4. Then, by the definition of $\Phi$,

\begin{equation}
s^+_p \cdot \Phi = 0.
\end{equation}

For any element $X = \sum_i x_i[f_i] \in s_{p'}$ with a basis $\{x_i\}$ of $g$ and $f_i \in \mathbb{C}_{p'}[\Sigma^{\alpha} \setminus \pi^{-1}(p')]$, we define

\[ o(X) := \max_i o(f_i), \]

defined by

\[ \bar{\beta}_\mu(a)(\bar{v}) = a \cdot \bar{v}, \text{ for any } a \in U(g^{F_v}) \text{ and } \bar{v} \in V(\mu). \]
where $o(f_i)$ is the sum of orders of pole of $f_i$ at the points of $\pi^{-1}(p')$. (If $f_i$ is regular at a point in $\pi^{-1}(p')$, we say that the order of pole at that point is 0.)

Define an increasing filtration $\{\mathcal{F}_d(M)\}_{d \geq 0}$ of $M$ by

$$\mathcal{F}_d(M) = \text{span of } \{(X_1 \ldots X_k) \cdot \Phi : X_i \in s_{p'} \text{ and } \sum_{i=1}^k o(X_i) \leq d\}.$$ 

From (59), it is easy to see that for any $\Psi \in \mathcal{F}_d(M)$, and any $Y = \sum x_i[g_i] \in g_{p'}[\Sigma^d]\Gamma$ such that each $g_i$ vanishes at every point of $\pi^{-1}(p')$ of order at least $d + 1$,

$$Y \cdot \Psi = 0. \tag{60}$$

Now, for any $y \in \hat{g}_{p'}$, pick $\hat{y} \in s_{p'}$ such that

$$\hat{y}_{p'} - y \in \hat{g}_{p'}^{d+1}, \tag{61}$$

where $\hat{y}_{p'}$ denotes the restriction of $\hat{y}$ on $\pi^{-1}(\mathbb{D}_{p'})$ and $\hat{g}_{p'}^{d+1}$ denotes elements of $g[\pi^{-1}(\mathbb{D}_{p'})]\Gamma$ that vanish at $\pi^{-1}(p')$ of order at least $d + 1$ (note that $\hat{y}_{p'}^1 = \hat{y}_{p'}$). In fact, if $y \in t[\pi^{-1}(\mathbb{D}_{p'})]\Gamma$ for some $\Gamma$-stable subspace $t$ of $g$, then we can take $\hat{y} \in (t \otimes \mathbb{C}_{p'}[\Sigma^d \pi^{-1}(p')])\Gamma$.

Define, for any $\Psi \in \mathcal{F}_d(M)$,

$$y \cdot \Psi := \hat{y} \cdot \Psi, \quad \text{and} \quad C \cdot \Psi := c\Psi. \tag{62}$$

From (60), it follows that (62) gives a well defined action $y \cdot \Psi$ (i.e., it does not depend upon the choice of $\hat{y}$ satisfying (61)). Observe that, taking $\hat{y} = 0$,

$$y \cdot \Psi = 0, \quad \text{for } y \in \hat{g}_{p'}^{d+1}. \tag{63}$$

Of course, the action of $\hat{y}_{p'}$ on $M$ defined by (62) extends the action of $s_{p'}$ on $M$.

We next show that this action indeed makes $M$ into a module for the Lie algebra $\hat{g}_{p'}$. To show this, it suffices to show that, for $y_1, y_2 \in \hat{g}_{p'}$ and $\Psi \in \mathcal{F}_d(M)$,

$$y_1 \cdot (y_2 \cdot \Psi) - y_2 \cdot (y_1 \cdot \Psi) = [y_1, y_2] \cdot \Psi \tag{64}$$

Take $\hat{y}_1, \hat{y}_2 \in s_{p'}$ such that

$$(\hat{y}_1)_{p'} - y_1 \quad \text{and} \quad (\hat{y}_2)_{p'} - y_2 \in \hat{g}_{p'}^{d+1 + o(y_1) + o(y_2)},$$

where, for $y = \sum x_i[f_i] \in \hat{g}_{p'}$, $o(y) := \max_i o(f_i)$, and $o(f_i)$ being the sum of the orders of poles at the points of $\pi^{-1}(p')$. Using the definition (62) and observing that $o(\hat{y}_1) = o(y_1)$, it is easy to see that (64) is equivalent to the same identity with $y_1$ replaced by $\hat{y}_1$ and $y_2$ by $\hat{y}_2$. The latter of course follows since $M$ is a representation of $s_{p'}$. As a special case of (63), we get

$$\hat{y}_{p'}^+ \cdot \Phi = 0. \tag{65}$$

We next show that $M$ is an integrable $\hat{g}_{p'}$-module. To prove this, it suffices to show that for any vector $y \in \left(\mathfrak{n}^+ \otimes \mathbb{C}[\pi^{-1}(\mathbb{D}_{p'})]\right)^\Gamma$ ($\mathfrak{n}^+ := \mathfrak{n}$) $y$ acts locally nilpotently on $M$, where $\mathfrak{n}$ (resp. $\mathfrak{n}^-$) is the nilradical of the Borel subalgebra $b$ (resp. of the opposite Borel subalgebra $b^-$) (cf. §2). Since $M$ is generated by $\Phi$ as a $\hat{g}_{p'}$-module, by [Ku, Lemma 1.3.3 and Corollary 1.3.4], it suffices to show that $y$ acts nilpotently on $\Phi$. 

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Choose \( N_o > 0 \) such that

\[
\text{(66)} \quad (\text{ad } n)^{N_o}(g) = 0, \text{ and also } (\text{ad } n^-)^{N_o}(g) = 0.
\]

For any \( y \in \left( n^+ \otimes \mathbb{C}[\pi^{-1}(D^\times_p)] \right)^G \subset \mathfrak{g}_{p'} \), pick \( \hat{y} \in \left( n^+ \otimes \mathbb{C}_p[\Sigma^\omega \setminus \pi^{-1}(p')] \right)^G \) such that (cf. equation (61))

\[
\text{(67)} \quad \hat{y}_{p'} - y \in \mathfrak{g}(\Sigma(N_o - 1) + 1).
\]

For any associative algebra \( A \) and element \( y \in A \), define the operators \( L_y(x) = yx, R_y(x) = xy \), and \( \text{ad}(y) = L_y - R_y \). Considering the operator \( R^p_0 = (L_y - \text{ad}(y))^n \) (for any \( n \geq 1 \)) applied to \( \hat{y}_{p'} - y \) in the algebra \( U(\mathfrak{g}_{p'}) \) and using the Binomial Theorem (since \( L_y \) and \( \text{ad}(y) \) commute), we get

\[
\text{(68)} \quad (\hat{y}_{p'} - y)y^n = \sum_{j=0}^{k} \binom{n}{j} (-1)^j y^{n-j} \left( (\text{ad}(y))^j (\hat{y}_{p'} - y) \right),
\]

where the summation runs only up to \( k = \min{n, N_o - 1} \) because of the choice of \( N_o \) satisfying (66). Then, for any \( d \geq 1 \), by induction on \( d \) using (65) we get

\[
\text{(69)} \quad y^d \cdot \Phi = \hat{y}^d \cdot \Phi.
\]

To prove the above, observe that \( (\hat{y} - y)y^d \cdot \Phi = 0 \) by the choice of \( \hat{y} \) satisfying (67) and the identities (68) and (65).

For any positive integer \( N \), let \( \mathbb{C}_p[\Sigma^\omega]^N \subset \mathbb{C}[\Sigma^\omega] \) be the ideal consisting of those \( g \in \mathbb{C}[\Sigma^\omega] \) such that \( g \) has a zero of order \( \geq N \) at any point of \( \pi^{-1}(p) \). Let \( \mathfrak{g}_p[\Sigma^\omega]^N \subset \mathfrak{g}[\Sigma^\omega]^G \) be the Lie subalgebra defined as \( [\mathfrak{g} \otimes \mathbb{C}_p[\Sigma^\omega]^N]^G \). By the same proof as that of Lemma 3.7, \( \mathfrak{g}_p[\Sigma^\omega]^N \cdot \mathcal{H}(\mathcal{A}) \) is of finite codimension in \( \mathcal{H}(\mathcal{A}) \).

Let \( V \) be a finite dimensional complement of \( \mathfrak{g}_p[\Sigma^\omega]^N, \mathcal{H}(\mathcal{A}) \subset \mathcal{H}(\mathcal{A}) \). Since \( \hat{y} \) acts locally nilpotently on \( \mathcal{H}(\mathcal{A}) \) (cf. Lemma 2.5) and \( V \) is finite dimensional, there exists \( N \) (which we take \( \geq N_o \)) such that

\[
\text{(70)} \quad \hat{y}^N \cdot V = 0.
\]

Considering now the Binomial Theorem for the operator \( L^p_y = (\text{ad} y + R_y)^n \), we get (in any associative algebra)

\[
y^n x = \sum_{j=0}^{n} \binom{n}{j} ((\text{ad} y)^j x) y^{n-j}.
\]

Take any \( \hat{z} \in \mathfrak{g}_p[\Sigma^\omega]^N, \mathcal{H}(\mathcal{A}) \). By the above identity in the enveloping algebra

\[
U \left( (\mathfrak{g} \otimes \mathbb{C}_p[\Sigma^\omega \setminus \pi^{-1}(p')]^G) \right),
\]

\[
\hat{y}^N \cdot \hat{z} = \sum_{j=0}^{N_o - 1} \binom{N_o}{j} ((\text{ad} \hat{y})^j (\hat{z})) \hat{y}^{N-o}.
\]

Thus,

\[
\text{(71)} \quad \hat{y}^N \cdot \left( \mathfrak{g}_p[\Sigma^\omega]^N, \mathcal{H}(\mathcal{A}) \right) \subset \mathfrak{g}[\Sigma^\omega]^G \cdot \mathcal{H}(\mathcal{A}).
\]
Combining (69) - (71), we get that

\[ y^N \cdot \Phi = \tilde{y}^N \cdot \Phi = 0. \]

This proves that \( M \subset \mathcal{H}(\tilde{\lambda})^* \) is an integrable \( \mathfrak{g}_{\sigma'} \)-module (generated by \( \Phi \)). Let \( M_\sigma \subset M \) be the \( \mathfrak{g}_{\sigma'} \)-submodule generated by \( \Phi \). Decompose \( M_\sigma \) into irreducible components:

\[ M_\sigma = \bigoplus_{\mu \in D} V(\mu)^{\otimes n_\mu}, \]

where \( D \) is the set of dominant integral weights of \( \mathfrak{g}_{\sigma'} \). Take any highest weight vector \( v_\sigma \) in any irreducible \( \mathfrak{g}_{\sigma'} \)-submodule \( V(\mu) \) of \( M_\sigma \). Since \( \mathfrak{g}_{\sigma'} \) annihilates \( M_\sigma \) (cf. equation (65)), \( v_\sigma \) generates an integrable highest weight \( \mathfrak{g}_{\sigma'} \)-submodule of \( M \) of highest weight \( \mu \) with central charge \( c \). In particular, any \( V(\mu) \) appearing in \( M_\sigma \) satisfies \( \mu \in D_{c,p'} \) (by the definition of \( D_{c,p'} \)), i.e., \( \mu^* \in D_{c,p'}^{\ast} \) by Lemma 5.3.

By the evaluation map \( ev_{\sigma'} : \mathfrak{g}_{\sigma'} \rightarrow \mathfrak{g}^{\ast} \), we may view \( M_\sigma \) as a module over \( \mathfrak{g}^{\ast} \) and each \( V(\mu) \) the irreducible representation of \( \mathfrak{g}^{\ast} \). Thus, from equation (56) of Definition 5.5, applied to the map

\[ U(\mathfrak{g}^{\ast}) \rightarrow M_\sigma, \quad a \mapsto a \cdot \Phi, \]

we get that for any \( a \in \text{Ker} \beta, a \cdot \Phi = 0 \), i.e., \( \Phi(\varphi(a') \cdot h) = 0 \), for any \( h \in \mathcal{H}(\tilde{\lambda}) \). This proves the lemma and hence Theorem 5.4 is fully established. \( \square \)

6. Twisted Kac-Moody algebras and Sugawara construction over a base

We define twisted Kac-Moody Lie algebras, their Verma modules and integrable highest weight modules with parameters and prove the independence of parameters for the integrable highest weight modules. We also prove that the Sugawara operators acting on the integrable highest weight modules (of twisted affine Kac-Moody algebras) are independent of the parameters up to scalars.

Let \( R \) be a commutative algebra over \( \mathbb{C} \). In this section, all commutative algebras are over \( \mathbb{C} \), and we fix a root of unity \( \epsilon = e^{2\pi i} \) of order \( m \) and a central charge \( c > 0 \). Also, as earlier, \( \mathfrak{g} \) is a simple Lie algebra over \( \mathbb{C} \) and \( \sigma \) is a Lie algebra automorphism such that \( \sigma^m = \text{Id} \).

**Definition 6.1.** (a) We say that an \( R \)-algebra \( O_R \) is a complete local \( R \)-algebra if there exists \( t \in O_R \) such that \( O_R \cong R[[t]] \) as an \( R \)-algebra, where \( R[[t]] \) denotes the \( R \)-algebra of formal power series over \( R \). We say such a \( t \) is an \( R \)-parameter of \( O_R \). Let \( \mathcal{K}_R \) be the \( R \)-algebra containing \( O_R \) by inverting \( t \). Thus, \( \mathcal{K}_R \cong R((t)) \). Note that \( \mathcal{K}_R \) does not depend on the choice of the \( R \)-parameters.

(b) An \( R \)-rotation of \( O_R \) of order \( m \) is an \( R \)-automorphism \( \sigma \) of \( O_R \) (of order \( m \)) such that \( \sigma(t) = \epsilon^{-1}t \) for some \( R \)-parameter \( t \). Such an \( R \)-parameter \( t \) is called a \( \sigma \)-equivariant \( R \)-parameter. Observe that any \( R \)-automorphism of \( O_R \) of order \( m \) may not be an \( R \)-rotation. Clearly, an \( R \)-algebra automorphism of \( O_R \) extends uniquely as an automorphism of \( \mathcal{K}_R \), which we still denote by \( \sigma \).
Given a pair $(O_R, \sigma)$ of a complete local $R$-algebra $O_R$ and an $R$-rotation of order $m$, we can attach an $R$-linear Kac-Moody algebra $\hat{L}(\mathfrak{g}, \sigma)_R$,

$$\hat{L}(\mathfrak{g}, \sigma)_R := (\mathfrak{g} \otimes \mathbb{C} \mathcal{K}_R)^{\sigma} \oplus R \cdot C,$$

where $C$ is a central element of $\hat{L}(\mathfrak{g}, \sigma)_R$, and for any $x[g], y[h] \in (\mathfrak{g} \otimes \mathbb{C} \mathcal{K}_R)^{\sigma}$,

$$(72) \quad [x[g], y[h]] = [x, y][gh] + \frac{1}{m} \text{Res}_{t=0} ((dg)h) \langle x, y \rangle C.$$

Here the residue $\text{Res}(df)g$ is well-defined and independent of the choice of $R$-parameter (cf. [H, Chap. III, Theorem 7.14.1]). We denote by $\hat{L}(\mathfrak{g}, \sigma)_R^{\geq 0}$ the $R$-Lie subalgebra $(\mathfrak{g} \otimes \mathbb{C} \mathcal{K}_R)^{\sigma} \oplus R \cdot C$.

Given a complete local $R$-algebra $O_R$, let $m_R$ denote the ideal of $O_R$ generated by a local parameter $t$. Note that $m_R$ does not depend on the choice of $t$. Then, $O_R/m_R \simeq R$. This allows us to give a natural map

$$(\mathfrak{g} \otimes \mathbb{C} O_R)^{\sigma} \rightarrow (\mathfrak{g} \otimes R)^{\sigma},$$

which is independent of the choice of the parameter $t$. Given any morphism of commutative $\mathbb{C}$-algebras $f : R \rightarrow R'$, we define

$$O_{R}^{\hat{R}}R' := \lim_{\rightarrow k} \left( (O_R/m_R^k) \otimes_R R' \right).$$

Then, $O_{R}^{\hat{R}}R'$ is a complete local $R'$-algebra. For any $R$-parameter $t \in O_R$, $t' := t \otimes 1$ is an $R'$-parameter of $O_{R}^{\hat{R}}R'$. Let $\sigma$ be any $R$-rotation of $O_R$ of order $m$. Then, it induces an $R'$-rotation of $O_{R}^{\hat{R}}R'$. We still denote it by $\sigma$.

**Lemma 6.2.** Let $O_R$ be a complete local $R$-algebra with an $R$-rotation $\sigma$ of order $m$. Given any finite morphism of commutative $\mathbb{C}$-algebras $f : R \rightarrow R'$, there exists a natural isomorphism of Lie algebras $\hat{L}(\mathfrak{g}, \sigma)_R \otimes_R R' \simeq \hat{L}(\mathfrak{g}, \sigma)_{R'}$, where $\hat{L}(\mathfrak{g}, \sigma)_{R'}$ is the $R'$-Kac-Moody algebra attached to $O_{R'} := O_{R}^{\hat{R}}R'$ and the induced rotation $\sigma$.

**Proof.** It suffices to check that $\mathcal{K}_R \otimes_R R' \simeq \mathcal{K}_{R'}$, which is well-known (since $f$ is a finite morphism). \(\square\)

Let $V$ be an irreducible representation of $\mathfrak{g}^{\sigma}$ with highest weight $\lambda \in D_c$, where $D_c$ is defined in Section 2. Then $V_R := V \otimes_R R$ is naturally a representation of $\mathfrak{g}^{\sigma} \otimes \mathbb{C} R$. Define the generalized Verma module

$$\hat{M}(V, c)_R := U_R(\hat{L}(\mathfrak{g}, \sigma)_R) \otimes_{U_R(\hat{L}(\mathfrak{g}, \sigma)_{R})^{\geq 0}} V_R,$$

where $U_R(\cdot)$ denotes the universal enveloping algebra of $R$-Lie algebra, and $V_R$ is a module over $U_R(\hat{L}(\mathfrak{g}, \sigma)_{R})^{\geq 0}$ via the projection map $\hat{L}(\mathfrak{g}, \sigma)_{R}^{\geq 0} \rightarrow \mathfrak{g}^{\sigma} \otimes \mathbb{C} R \oplus R \cdot C$ and such that $C$ acts on $V_R$ by $c$.

**Lemma 6.3.** The Verma module $\hat{M}(V, c)_R$ is a free $R$-module. Given any morphism $R \rightarrow R'$ of $\mathbb{C}$-algebras, there exists a natural isomorphism $\hat{M}(V, c)_R \otimes_R R' \simeq \hat{M}(V, c)_{R'}$ as $\hat{L}(\mathfrak{g}, \sigma)_R \otimes R'$-modules, where $\hat{M}(V, c)_{R'}$ is the generalized Verma module attached to $O_{R'} := O_{R}^{\hat{R}}R'$ and the action of $\hat{L}(\mathfrak{g}, \sigma)_R \otimes_R R'$ on $\hat{M}(V, c)_{R'}$ is via the canonical morphism $\hat{L}(\mathfrak{g}, \sigma)_{R} \otimes_R R' \rightarrow \hat{L}(\mathfrak{g}, \sigma)_{R'}$. 
Proof. Let $t$ be a $\sigma$-equivariant $R$-parameter. There exists a decomposition as $R$-module:

$$(g \otimes_C R((t)))^\sigma = (g \otimes_C R[[t]])^\sigma \oplus (g \otimes_C t^{-1}R[t^{-1}])^\sigma.$$  

Hence, $\hat{M}(V,c)_R = U_R((g \otimes t^{-1}R[t^{-1}])^\sigma) \otimes_C V$. Note that $(g \otimes t^{-1}R[t^{-1}])^\sigma$ is a Lie algebra which is a free module over $R$. By Poincaré-Birkhoff-Witt theorem for any $R$-Lie algebra that is free as an $R$-module (cf. [CE, Theorem 3.1, Chapter XIII]), $\hat{M}(V,c)_R$ is a free $R$-module.

Note that there is a natural morphism $\hat{L}(g,\sigma)_R \otimes_R R' \to \hat{L}(g,\sigma)_R$. It induces a natural morphism $\kappa : \hat{M}(V,c)_R \otimes_R R' \to \hat{M}(V,c)_R$. The map $\kappa$ is an isomorphism since it induces the following natural isomorphism

$$R' \otimes_R (U_R((g \otimes t^{-1}R[t^{-1}])^\sigma) \otimes_C V) \cong U_R((g \otimes t^{-1}R[t^{-1}])^\sigma) \otimes_C V,$$

where $t' = t \hat{\otimes} 1$. □

We can choose a $\sigma$-stable Borel subalgebra $b \subset g$, a $\sigma$-stable Cartan subalgebra $h \subset b$, the elements $\{x_i, y_i\}_{i \in \hat{I}(g,\sigma)}$ and the set of non-negative integers $\{s_i\}_{i \in \hat{I}(g,\sigma)}$ as in Section 2, such that $\hat{L}(g,\sigma)_R$ contains the elements $x_i[t^0], y_i[t^{-s_i}]$. Let $V = V(\lambda)$ be the irreducible $g^\sigma$-module with highest weight $\lambda \in D$, (cf. Lemma 2.1 for the description of $D$). Let $\hat{N}(V,c)_R$ be the $\hat{L}(g,\sigma)_R$-submodule of $\hat{M}(V,c)_R$ generated by $\{y_i[t^{-s_i}]x_j^{\alpha_j}, v_j\}_{i \in \hat{I}(g,\sigma)}$, where $v_j$ is the highest weight vector of $V(\lambda)$ and (as in Section 2) $\hat{I}(g,\sigma)^\times := \{i \in \hat{I}(g,\sigma) : s_i > 0\}$.

**Lemma 6.4.** The module $\hat{N}(V,c)_R$ does not depend on the choice of the $\sigma$-equivariant $R$-parameter $t$.

**Proof.** Let $t'$ be another $\sigma$-equivariant $R$-parameter. It suffices to show that for each $i \in \hat{I}(g,\sigma)^+ = [y_i[t^{-s_i}]x_j^{\alpha_j}, v_j]$ for some constant $c \in R^\times$ (where $R^\times$ denotes the set of units in $R$). By the $\sigma$-equivariance of $t$ and $t'$, we can write $t^{-s_i} = ct^{-s_i} + \sum_{k > s_i, m} a_k t^k$, for some $c \in R^\times$ and $a_k \in R$.

**Case 1:** $i \in \hat{I}(g,\sigma)^+$ and $0 < s_i < m$, then $t^{-s_i} = ct^{-s_i} + g$, where $g = \sum_{k > s_i, m} a_k t^k$ with $a_k \in R$ (since $0 < s_i < m$ and $m(s_i + k)$). Since $y_i[g] \cdot v_j = 0$, it is clear that $y_i[t^{-s_i}]x_j^{\alpha_j}, v_j = (cy_i[t^{-s_i}]x_j^{\alpha_j}, v_j).$

**Case 2:** $s_o = m$, then $t^{-m} = ct^{-m} + g$, where $g = \sum_{k > 0} a_k t^k$ with $a_k \in R$. Since $s_o = m$, by [Ka, Identity 8.5.6], each $s_j = 0$ for $j \neq 0$. Thus, the simple root vectors of $g^\sigma$ are $\{x_j\}_{j \neq 0}$ with (simple) roots $\{\alpha_j\}_{j \neq 0}$. Since $y_o$ is a root vector of the root $\theta_0$ and $\theta_0$ is a positive linear combination $\sum_{j \neq 0} a_j \alpha_j$, $y_o$ is a positive root vector of $g^\sigma$. Hence, $y_o \cdot v_j = 0$. Thus, it follows that $y_o(g) = 0$. Hence, $y_o[t^{-m}]x_j, v_j = (cy_o[t^{-m}]x_j, v_j).$

**Case 3:** If $s_i = m$ for $i \neq 0$, then again by [Ka, Identity 8.5.6], $r = 1$ and each $s_j = 0$ for $j \neq i$. Thus, the simple root vectors of $g^\sigma$ are $\{x_j\}_{j \neq i}$ with (simple) roots $\{\alpha_j\}_{j \neq i} \cup \{-\theta_0\}$. Hence, $-\alpha_i + \sum_{j \neq i, 0} a_j \alpha_j$ giving that $-\alpha_i$ is a positive root of $g^\sigma$ and hence $y_i \cdot v_j = 0$. Rest of the argument is the same as in Case 2. This proves the lemma. □

We now define the following $R$-linear representation of $\hat{L}(g,\sigma)_R$:

$$\mathcal{H}(V)_R := \hat{M}(V,c)_R / \hat{N}(V,c)_R.$$
Lemma 6.5. (1) \( \hat{N}(V,c)_R \) and \( \mathcal{H}(V)_R \) are free over \( R \). The module \( \hat{N}(V,c)_R \) is a \( R \)-module direct summand of \( \hat{M}(V,c)_R \).

(2) For any morphism \( f : R \to R' \) of commutative \( \mathbb{C} \)-algebras, there exists a natural isomorphism \( \hat{N}(V,c)_R \otimes_R R' \cong \hat{N}(V,c)_{R'} \) and \( \mathcal{H}(V)_R \otimes_R R' \cong \mathcal{H}(V)_{R'} \) as modules over \( \hat{L}(\mathfrak{g},\sigma)_R \otimes_R R' \), where the action of \( \hat{L}(\mathfrak{g},\sigma)_R \otimes_R R' \) on \( \hat{N}(V,c)_{R'} \) and \( \mathcal{H}(V)_{R'} \) is via the canonical morphism \( \hat{L}(\mathfrak{g},\sigma)_R \otimes_R R' \to \hat{L}(\mathfrak{g},\sigma)_{R'} \).

(3) Choose any \( \sigma \)-equivariant \( R \)-parameter \( t \). Then, \( \hat{N}(V,c)_R \subset \hat{M}(V,c)_R^+ \). Moreover, for any other \( \hat{L}(\mathfrak{g},\sigma)_R \)-submodule \( A \) of \( \hat{M}(V,c)_R \) such that \( A \cap V_R = (0) \), \( A \) is contained in \( \hat{N}(V,c)_R \). Here \( \hat{M}(V,c)_R^+ := \oplus_{t \geq 1} \hat{M}(V,c)_R(d) \) and (for \( d \geq 1 \))

\[
\hat{M}(V,c)_R(d) := \sum_{n_i \geq 0, \sum n_i = d} X_1^{[r^{-n_1}]} \cdots X_k^{[r^{-n_k}]} \cdot V \subset \hat{M}(V,c)_R, \quad \text{where} \quad X_i^{[r^{-n_i}]} \in \hat{L}(\mathfrak{g},\sigma)_R.
\]

Observe that \( \hat{M}(V,c)_R(d) \) does depend upon the choice of the parameter \( t \).

We set \( \hat{M}(V,c)_R(0) = V_R := V \otimes_{\mathbb{C}} R \).

Hence, \( \hat{N}(V,c)_R \) and \( \mathcal{H}(V)_R \) do not depend on the choice of \( b, h \), and \( x_i, y_i, i \in \hat{I}(\mathfrak{g},\sigma) \).

Proof. Fix a \( \sigma \)-equivariant \( R \)-parameter \( t \). For each \( i \in \hat{I}(\mathfrak{g},\sigma)^+ \), the element \( y_i^{[r^{-\lambda}]} \cdot v_\lambda \) is a highest weight vector. Hence,

\[
\hat{N}(V,c)_R = \sum_{i \in \hat{I}(\mathfrak{g},\sigma)^+} U(i \otimes_{\mathbb{C}} R^{[r^{-1}]})y_i^{[r^{-\lambda}]} \cdot v_\lambda.
\]

Note that \( U(i \otimes_{\mathbb{C}} R^{[r^{-1}]}) \cong U([\mathfrak{g} \otimes_{\mathbb{C}} \mathbb{C}[r^{-1}]]^\sigma) \otimes_{\mathbb{C}} R \) as \( R \)-algebras. Since \( R \) is flat over \( \mathbb{C} \), it is easy to see that (as a submodule of \( \hat{M}(V,c)_C = \hat{M}(V,c)_R \otimes_{\mathbb{C}} R \), where \( \hat{M}(V,c)_C := U(\hat{L}(\mathfrak{g},\sigma)_C) \otimes_{U(\hat{L}(\mathfrak{g},\sigma)_C)} V \) is a \( \mathbb{C} \)-lattice in \( \hat{M}(V,c)_R \), which depends on the choice of \( t \), where \( \hat{L}(\mathfrak{g},\sigma)_C := [\mathfrak{g} \otimes_{\mathbb{C}} \mathbb{C}[[t]]]^{\sigma} \otimes_{\mathbb{C}} \mathbb{C} \) and \( \hat{L}(\mathfrak{g},\sigma)_C^0 := [\mathfrak{g} \otimes_{\mathbb{C}} \mathbb{C}[[t]]]^{\sigma} \otimes_{\mathbb{C}} \mathbb{C} \).

\[
\hat{N}(V,c)_R = \hat{\mathcal{N}}(V,c)_C \otimes_{\mathbb{C}} R,
\]

where \( \hat{\mathcal{N}}(V,c)_C := \sum_{i \in \hat{I}(\mathfrak{g},\sigma)^+} U(i \otimes_{\mathbb{C}} \mathbb{C}[r^{-1}])y_i^{[r^{-\lambda}]} \cdot v_\lambda \) is a \( \mathbb{C} \)-lattice of \( \hat{N}(V,c)_R \).

Hence, \( \hat{N}(V,c)_R \) is free over \( R \) and it is a direct summand (as an \( R \)-module) of \( \hat{M}(V,c)_R \). From this we readily see that

\[
\mathcal{H}(V)_R \cong \hat{\mathcal{N}}(V,c)_C \otimes_{\mathbb{C}} R,
\]

where \( \mathcal{H}(V)_C := \hat{M}(V,c)_C / \hat{N}(V,c)_C \) is a \( \mathbb{C} \)-lattice in \( \mathcal{H}(V)_R \) (depending on the choice of \( t \)), and hence it is also free over \( R \). It finishes the proof of part (1) of the lemma.

By the above equation (74) and the associativity of the tensor product: \( (M \otimes_R S) \otimes_R T = M \otimes_R (S \otimes_R T) \), we also have \( \hat{N}(V,c)_R \otimes_R R' \cong \hat{N}(V,c)_{R'} \). Similarly, by the above equation (75), \( \hat{\mathcal{N}}(V,c)_R \otimes_R R' \cong \hat{\mathcal{N}}(V,c)_{R'} \). It concludes part (2) of the lemma.

We now proceed to prove part (3) of the lemma. By the equation (73), \( \hat{N}(V,c)_R \subset \hat{M}(V,c)_R^+ \). Observe first that

\[
\left\{ v \in \mathcal{H}(V)_C : X[R^t] \cdot v = 0 \forall n > 0 \text{ and } X[R^t] \in \hat{L}(\mathfrak{g},\sigma)_C \right\} = V.
\]

This is easy to see since \( \mathcal{H}(V)_C \) is an irreducible \( \hat{L}(\mathfrak{g},\sigma)_C \)-module. Choosing a basis of \( R \) over \( \mathbb{C} \), from this we easily conclude that

\[
\left\{ v \in \mathcal{H}(V)_R : X[R^t] \cdot v = 0 \forall n > 0 \text{ and } X[R^t] \in \hat{L}(\mathfrak{g},\sigma)_C \right\} = V_R.
\]
For any nonzero \( v \in A' := A/(A \cap \hat{N}(V,c)_R) \hookrightarrow \mathcal{H}(V)_R \), \( v = \sum \nu_d \) with \( \nu_d \in \mathcal{H}(V)_R(d) \), set \( |v| = \sum d : \nu_d \neq 0 \), where the gradation \( \mathcal{H}(V)_R(d) \) is induced from that of \( \hat{M}(V,c)_R \). Choose a nonzero \( \nu^o \in A' \) such that \( |\nu^o| \leq |v| \) for all nonzero \( v \in A' \). Then,

\[
(78) \quad X[t^n] \cdot \nu^o = 0 \quad \text{for all} \ n \geq 1 \ \text{and} \ X[t^n] \in \hat{L}(g,\sigma)_{C}.
\]

For, otherwise, \( |X[t^n]| \cdot |\nu^o| < |\nu^o| \), which contradicts the choice of \( \nu^o \).

By the equation (77), we get that \( \nu^o \in V_R \), which contradicts the choice of \( A \) since \( A \cap V_R = (0) \). Thus, \( A' = 0 \), i.e., \( A \subset \hat{N}(V,c)_R \). This proves the third part of the lemma. \( \square \)

We now begin with the definition of Sugawara operators \( \{ \Xi_n | n \in \mathbb{Z} \} \) for the Kac-Moody algebra \( \hat{L}(g,\sigma)_R \) attached to a complete local \( R \)-algebra \( O_R \) with an \( R \)-rotation of order \( m \), and an automorphism \( \sigma \) of \( g \) such that \( \sigma^m = \text{Id} \). We fix a \( \sigma \)-equivariant \( R \)-parameter \( t \).

Recall the eigenspace decomposition \( g = \bigoplus_{n \in \mathbb{Z}/m\mathbb{Z}} g_n \) of \( \sigma \), where

\[
g_n := \{ x \in g | \sigma(x) = e^n x \}.
\]

Note that \( \sigma \) preserves the normalized invariant form \( \langle , \rangle \) on \( g \), i.e., for any \( x, y \in g \) we have \( \langle \sigma(x), \sigma(y) \rangle = \langle x, y \rangle \). For each \( h \in \mathbb{Z}/m\mathbb{Z} \) it induces a non-degenerate bilinear form \( \langle , \rangle_h : g_h \times g_{-h} \to \mathbb{C} \). We choose a basis \( \{ u_a | a \in A_\sigma \} \) of \( g_n \) indexed by a set \( A_\sigma \). Let \( \{ u^a | a \in A_\sigma \} \) be the basis of \( g_{-\sigma} \) dual to the basis \( \{ u_a | a \in A_\sigma \} \) of \( g_\sigma \).

The normalized \( R \)-bilinear form on \( \hat{L}(g,\sigma)_R \) is given as follows (cf. [Ka, Theorem 8.7]),

\[
(\langle x[f], y[g] \rangle = \frac{1}{r} \left( \text{Res}_{t=0} t^{-1} f(t) g(t) \right) \langle x, y \rangle, \ \langle x[f], C \rangle = 0, \ \text{and} \ \langle C, C \rangle = 0,
\]

where \( x[f], y[g] \in \hat{L}(g,\sigma)_R \) and \( r \) is the order of the diagram automorphism associated to \( \sigma \). Then, the following relation is satisfied:

\[
\langle u_a[t^n], u^b[t^{-k}] \rangle = \frac{1}{r^n} \delta_{a,b} \delta_{a,k} \quad \text{for any} \ a \in A_\sigma \text{and} \ b \in A_\sigma.
\]

**Definition 6.6.** An \( R \)-linear \( \hat{L}(g,\sigma)_R \)-module \( M_R \) is called smooth if for any \( v \in M_R \), there exists an integer \( d \) (depending upon \( v \)) such that

\[
x[f] \cdot v = 0, \ \text{for all} \ f \in t^n O_R \text{ and} \ x[f] \in (g \otimes_\mathbb{C} O_R)^\sigma.
\]

Observe that this definition does not depend upon the choice of the parameter \( t \).

The generalized Verma module \( \hat{M}(V,c)_R \) (and hence the quotient module \( \mathcal{H}(V)_R \)) is clearly smooth.

We construct the following \( R \)-linear *Sugawara operators* on any smooth representation \( M_R \) of \( \hat{L}(g,\sigma)_R \) of level \( c \neq -h \) (which depends on the choice of \( t \)),

\[
(79) \quad L_0^t := \frac{1}{2(c + h)} \left( \sum_{a \in A_\sigma} u_a u^a + \sum_{n=0}^{m-1} u_a[t^n] u^a[t^n] + \frac{1}{2m^2} \sum_{n=0}^{m-1} n(m - n) \dim g_n \right).
\]

\[
(80) \quad L_k^t := \frac{1}{mk} [ -m^{k+1} \partial_t, L_0^t ], \quad \text{for} \ k \neq 0,
\]
where $\tilde{h}$ is the dual Coxeter number of $\mathfrak{g}$. Note that the smoothness ensures that $L^i_k$ is a well-defined operator on $M_R$ for each $k \in \mathbb{Z}$. Moreover, it is easy to see that $L^i_k$ does not depend upon the choice of the basis $\{u_a\}$ of $\mathfrak{g}$.

The following result can be found in [KW, §3.4], [W].

**Proposition 6.7.** For any $n, k \in \mathbb{Z}$ and $x \in \mathfrak{g}$, as operators on a smooth representation $M_R$ of $L(\mathfrak{g}, \sigma)_R$ of central charge $c \neq -\tilde{h}$,

(a) $[x[r^i], L^i_k] = \frac{n}{m} x[r^{i + mk}]$.

In particular, $L^i_0$ commutes with $\mathfrak{g}$. (b) $[L^i_n, L^j_k] = (n - k)L^i_{n + k} + \delta_{n, -k} \frac{n^2 - n}{12} \dim \mathfrak{g} - \frac{c}{c + \tilde{h}}$.

Let us recall the definition of the Virasoro algebra $\text{Vir}_R$ over $R$. It is the Lie algebra over $R$ with $R$-basis $\{d_n, \tilde{C}\}_{n \in \mathbb{Z}}$ and the commutation relation is given by

$$[d_n, d_m] = (n - k)d_{n + k} + \delta_{n, -k} \frac{n^2 - n}{12} \tilde{C}; \quad [d_n, \tilde{C}] = 0.$$  

An $R$-derivation of $\mathcal{K}_R$ is an $R$-linear map $\theta : \mathcal{K}_R \to \mathcal{K}_R$ such that $\theta(fg) = \theta(f)g + f\theta(g)$, for any $f, g \in \mathcal{K}_R$. Let $\Theta_{\mathcal{K}_R/R}$ denote the Lie algebra of all continuous $R$-derivations of $\mathcal{K}_R$, where we put the $m$-adic topology on $\mathcal{K}_R$, i.e., $\{f + m^n\}_{n \in \mathbb{Z}, f \in \mathcal{K}_R}$ is a basis of open subsets. (Here $m^n$ denotes $t^nO_R$, which does not depend upon the choice of $t$.) With the choice of the $R$-parameter $t$ in $O_R$, we have the equality $\Theta_{\mathcal{K}_R/R} = R((t))\partial_t$, where $\partial_t$ is the derivation on $\mathcal{K}_R$ such that $\partial_t(R) = 0$ and $\partial_t(1) = 1$. Let $\Theta_{\mathcal{K}_R/R}$ denote the $R$-Lie algebra of all continuous $\mathbb{C}$-linear derivations $\theta$ of $\mathcal{K}_R$ that are liftable from $R$, i.e., the restriction $\theta|_R$ is a $\mathbb{C}$-linear derivation of $R$. Let $\Theta_{\mathcal{K}_R/R}$ denote the Lie algebra of $\mathbb{C}$-linear derivations of $R$. There exists a short exact sequence:

$$0 \to \Theta_{\mathcal{K}_R/R} \to \Theta_{\mathcal{K}_R/R} \overset{\text{Res}}{\longrightarrow} \Theta_R \to 0,$$

where $\text{Res}$ denotes the restriction map of derivations from $\mathcal{K}_R$ to $R$. See more details in [L, Section 2]. It induces the following short exact sequence:

$$0 \to \Theta_{\mathcal{K}_R/R} \to \Theta_{\mathcal{K}_R/R} \overset{\text{Res}}{\longrightarrow} \Theta_R \to 0,$$

where $\Theta_{\mathcal{K}_R/R}$ (resp. $\Theta_{\mathcal{K}_R/R}^{\sigma}$) is the space of $\sigma$-equivariant derivations in $\Theta_{\mathcal{K}_R/R}$ (resp. $\Theta_{\mathcal{K}_R/R}$). Then, $\Theta_{\mathcal{K}_R/R}^{\sigma} = R((t^m))\partial_t$. We define a central extension

$$\Theta_{\mathcal{K}_R/R}^{\sigma} := \Theta_{\mathcal{K}_R/R}^{\sigma} \oplus R\tilde{C}$$

of the $R$-Lie algebra $\Theta_{\mathcal{K}_R/R}^{\sigma}$ by

$$[f\partial_t, g\partial_t] = (f\partial_t(g) - g\partial_t(f))\partial_t + \text{Res}_{t=0} \left( t^{3m}\partial_t^2 + t^{-1}\partial_t \right) \frac{\tilde{C}}{12m},$$

for $f\partial_t, g\partial_t \in R((t^m))\partial_t$, where $A$ is the operator $t^{-m}(m + t\partial_t)$. Observe that this bracket corresponds to the bracket of the Virasoro algebra defined by the identity (81) if we take
Let $d_k = -\frac{1}{m} t^{mk+1} \partial_t$ for any $k \in \mathbb{Z}$. Therefore, $\Theta_{K/\mathbb{R}}^{\sigma}$ defines a completed version of the Virasoro algebra over $R$.

For any $\theta \in \Theta_{K/\mathbb{R}}^{\sigma}$ with $\theta = \sum_{k \geq -N} a_{mk+1} t^{mk+1} \partial_t$, we define a Sugawara operator associated to $\theta$

$$L_\theta' := \sum_{k \geq -N} (-ma_{mk+1}) L_k'$$

on smooth modules of $\hat{L}(\theta, \sigma)_R$.

In the following lemma, the operator $L_\theta'$ is described more explicitly on any smooth module.

**Lemma 6.8.** For any $\theta \in R((t^n)) t \partial_t$, the operator $L_\theta'$ acts on any smooth module $M_R$ as follows:

$$L_\theta'(u_1[f_1] \cdots u_n[f_n] \cdot v) = u_1[f_1] \cdots u_n[f_n] \cdot L_\theta'(v) + \sum_{i=1}^n (u_1[f_1] \cdots u_i[\theta(f_i)] \cdots u_n[f_n] \cdot v)$$

where $u_1[f_1], \ldots, u_n[f_n] \in \hat{L}(\theta, \sigma)_R$ and $v \in M_R$.

**Proof.** It is enough to show that $[L_\theta', u_i[f_i]] = u_i[\theta(f_i)]$ for each $i = 1, \ldots, n$:

$$[L_\theta', u_i[f_i]] = \sum_{k \geq -N} (-ma_{mk+1}) [L_k', u_i[f_i]]$$

$$= \sum_{k \geq -N} (-a_{mk+1}) u_i[-t^{mk+1} \partial_t(f_i)]$$

$$= u_i[\theta(f_i)],$$

where the first equality follows from the definition (85), and the second equality follows from part (a) of Proposition 6.7. \hfill \square

Note that the choice of a $\sigma$-equivariant $R$-parameter $t$ gives the $R$-module splitting $\Theta_{K/\mathbb{R}}^{\sigma} = \Theta_{K/\mathbb{R}}^{\sigma} \oplus t_1(\Theta_R)$, where (for $\delta \in \Theta_R$) $t_1(\delta)(f) = \sum_k \delta(a_k) t^k$ if $f = \sum_k a_k t^k$. For any $f \delta \in \Theta_{K/\mathbb{R}}^{\sigma}$ and $\delta \in \Theta_R$.

$$[\tau_1(\delta), f \partial_t] = [\tau_1(\delta)(f) \partial_t, [\tau_1(\delta_1), \tau_1(\delta_2)] = \tau_1[\delta_1, \delta_2], [\tau_1(\delta), r \tilde{C}] = \delta(r) \tilde{C}$$

for $r \in R$.

The $\mathbb{C}$-linear brackets (84) and (88) define a completed extended Virasoro algebra $\hat{\Theta}_{K/\mathbb{R}}^{\sigma}$ over $R$, where

$$\hat{\Theta}_{K/\mathbb{R}}^{\sigma} = \Theta_{K/\mathbb{R}}^{\sigma} \oplus t_1(\Theta_R).$$

Take any smooth $\hat{L}(\theta, \sigma)_R$-module $M_R$ with $\mathbb{C}$-lattice $M_\mathbb{C}$ (i.e., $M_\mathbb{C} \otimes R = M_R$) stable under $\hat{L}(\theta, \sigma)_\mathbb{C}$. (Observe that $\hat{L}(\theta, \sigma)_\mathbb{C}$ depends upon the choice of the parameter $t$.) Let $\delta$ act $\mathbb{C}$-linearly on $M_R$ via its action only on the $R$-factor under the decomposition $M_R = M_\mathbb{C} \otimes R$. We denote this action on $M_R$ by $L_\theta'$. Observe that $L_\theta'$ depends upon the choice of the parameter $t$ as well as the choice of the $\mathbb{C}$-lattice $M_\mathbb{C}$ in $M_R$.

For any $\theta \in \Theta_{K/\mathbb{R}}^{\sigma}$, write $\theta = \theta' + t_1(\theta'')$ (uniquely), where $\theta' \in \Theta_{K/\mathbb{R}}^{\sigma}$ and $\theta'' \in \Theta_R$.

We define the extended Sugawara operator $L_\theta'$ associated to $\theta$ acting on any smooth
Let $M_R := M_C \otimes \mathbb{C} R$ (with $\mathbb{C}$-lattice $M_C$ as above) by
\begin{equation}
L_{q'}^t := L_t^q + L_t^{q'}. \tag{89}
\end{equation}
Then,
\begin{equation}
L_{q'}^t(u_1[f_1] \cdots u_n[f_n] \cdot v) = \sum_{i=1}^n u_1[f_1] \cdots u_i(\theta^t)(f_i) \cdots u_n[f_n] \cdot v, \tag{90}
\end{equation}
for $v \in M_C$ and $u_i[f_i] \in \hat{L}(\mathfrak{g}, \sigma)_R$.

The following proposition follows easily from Proposition 6.7 and the definition of the operator $L_{q'}^t$.

**Proposition 6.9.** (1) Let $M_R$ be a smooth module of $\hat{L}(\mathfrak{g}, \sigma)_R$ with $\mathbb{C}$-lattice $M_C$ as above with respect to a $\sigma$-equivariant $R$-parameter $t$ and central charge $c \neq -\hbar$. Then, we have a $\mathbb{C}$-Lie algebra homomorphism
\[ \Psi : \Theta_{r_{K,R}}^\sigma \to \text{End}_{\mathbb{C}}(M_R) \]
given by
\begin{equation}
r\tilde{C} \mapsto r \left( \frac{c \dim \mathfrak{g}}{c + \hbar} \right) I_{M_C}; \quad \theta \mapsto L_t^q, \quad \text{for any } \theta \in \Theta_{r_{K,R}}^\sigma, r \in R. \tag{91}
\end{equation}
Moreover, $\Psi$ is an $R$-module map under the $R$-module structure on $\text{End}_{\mathbb{C}}(M_R)$ given by
\[ (r \cdot f)(v) = r \cdot f(v), \quad \text{for } r \in R, v \in M_R, f \in \text{End}_{\mathbb{C}}(M_R). \]

(2) Further, for any $\theta \in \Theta_{r_{K,R}}^\sigma$, $v \in M_R$ and $a \in R$,
\begin{equation}
L_t^q(a \cdot v) = \theta(a) \cdot v + a \cdot L_t^q(v). \tag{92}
\end{equation}

The following lemma shows that the representation of $\Theta_{r_{K,R}}^\sigma$ on $\hat{M}(V, c)_R$ (and hence on $\mathcal{H}(V)_R$) is independent of the choice of the $\sigma$-equivariant $R$-parameters up to a multiple of the identity operator.

**Lemma 6.10.** Let $V = V(\lambda)$ be an irreducible $\mathfrak{g}$-module with highest weight $\lambda \in D_c$. Let $\theta'$ be another $\sigma$-equivariant $R$-parameter in $\mathcal{O}_R$. For any $\theta \in \Theta_{r_{K,R}}^\sigma$, there exists $b(\theta, \lambda, t, t') \in \mathbb{C}$ such that $L_t^q = L_t^q + b(\theta, \lambda, t, t') \text{Id}$ on $\hat{M}(V, c)_R$ and hence on $\mathcal{H}(V)_R$.

Here, with the choice of the parameter $t$, we have chosen the $\mathbb{C}$-lattice $\hat{M}(V, c)_\mathbb{C}$ of $\hat{M}(V, c)_R$ to be $U ((\mathfrak{g} \otimes \mathbb{C}((t)))^\sigma \otimes \mathbb{C}) \cdot V$ and the $\mathbb{C}$-lattice of $\mathcal{H}(V)_R$ to be the image of $\hat{M}(V, c)_\mathbb{C}$.

**Proof.** Assume first that $\theta \in \Theta_{r_{K,R}}^\sigma$. Let $L_t^q$ and $L_t^{q'}$ denote the Sugawara operators associated to $\theta$ with respect to the parameters $t$ and $t'$ respectively. For any $u[f] \in \hat{L}(\mathfrak{g}, \sigma)_R$, from the identity (87), we have the following formula:
\begin{equation}
[u[f], L_t^q] = -u[\theta(f)], \quad [u[f], L_t^{q'}] = -u[\theta(f)]. \tag{93}
\end{equation}
This gives
\begin{equation}
[u[f], L_t^q - L_t^{q'}] = 0 \quad \text{for any } u[f] \in \hat{L}(\mathfrak{g}, \sigma)_R. \tag{94}
\end{equation}
Applying the equation (95) to \( \theta \)

\[ \theta(t) \theta'(t) = \theta'(t) \theta(t) \]

where \( \theta(t) = \exp(\lambda(t) \partial_t + \mu(t \partial_t + u)) \).

For any \( \theta \in \Theta_{\mathcal{M} / R}^T \), we may write uniquely

\[ \theta = \theta_1 + \mu(t \partial_t + u) \theta_1 \]

where \( \theta_1 \in \mathcal{R}(\mathcal{M})t \partial_t, \theta_2 \in \mathcal{R}(\mathcal{M})t \partial_t, \) \( \theta_3, \theta_4 \in \mathcal{R}_R \).

Observe that

\[ \theta_3 = \theta_4 = \theta_R \]

Applying the equation (95) to \( \theta' \), we get

\[ \theta' = \theta_1 + \mu(t \partial_t + u) \theta_1 \]

where \( u = t' / t \in \mathcal{O}_R \).

As proved above, \( L_{\theta_1} - L_{\theta_2} \) is a scalar operator, as \( \theta_1 \in \Theta_{\mathcal{M} / R}^T \). So, to prove that

\[ L_{\theta_1} - L_{\theta_2} \]

is a scalar operator, it suffices to prove that \( L_{\theta_1} + L_{\theta_2} - L_{\theta_3} \) is a scalar operator, since \( L_{\theta_1} = L_{\theta_2} - L_{\theta_3} \), where \( \beta = \theta_1 - \theta_2 \).

Now, for \( u_1[f_1], \ldots, u_n[f_n] \in \hat{L}(\mathcal{M}, \sigma)_R \) and \( v \in \mathcal{R}_R \), by the identities (86) and (90), we get

\[ \left( L_{\theta_1} + L_{\theta_2} - L_{\theta_3} \right) (u_1[f_1] \ldots u_n[f_n] \cdot v) \]

\[ = u_1[f_1] \ldots u_n[f_n] \cdot (L_{\theta_1} v) + \sum_{i=1}^n u_1[f_1] \ldots u_i[f_i] \beta(f_i) + \mu(t \partial_t + u) \theta_1 \]

\[ = u_1[f_1] \ldots u_n[f_n] \cdot (L_{\theta_1} v), \quad \text{since } \beta \]}

\[ = m \theta_{\beta}^{u_1}(u_0) u_1[f_1] \ldots u_n[f_n] \cdot (L_{\theta_1} v), \quad \text{since } L_{\theta_1} v = 0 \text{ for all } k > 0 \text{ and } \beta \in \mathcal{R}[i \mathcal{M}] t \partial_t \]

\[ = \frac{m \theta_{\beta}^{u_1}(u_0)}{u_0} u_1[f_1] \ldots u_n[f_n] \cdot dv, \quad \text{for some constant } d \in \mathbb{C}, \]

by the definition of \( L_{\theta_1} \) since \( \sum_{u \in \mathcal{M}} u du^\theta \) is the Casimir operator of \( g^{\sigma} \). This proves the lemma.

7. Projectively flat connection on sheaf of twisted covacua

We define the sheaf of twisted covacua for a family \( \Sigma_T \) of \( s \)-pointed \( \Gamma \)-curves. We further show that this sheaf is locally free of finite rank for a smooth family \( \Sigma_T \) over a smooth base \( T \). In fact, we prove that it admits a projectively flat connection.
In this section, we take the parameter space $T$ to be a connected scheme over $\mathbb{C}$ and let $\Gamma$ be a finite group. We fix a group homomorphism $\phi: \Gamma \to \text{Aut}(g)$.

**Definition 7.1.** A family of curves over $T$ is a proper and flat morphism $\xi: \Sigma_T \to T$ such that every geometric fiber is a connected reduced curve (but not necessarily irreducible). For any $b \in T$ the fiber $\xi^{-1}(b)$ is denoted by $\Sigma_b$.

Let $\Gamma$ act faithfully on $\Sigma_T$ and that $\xi$ is $\Gamma$-invariant (where $\Gamma$ acts trivially on $T$). Let $\pi: \Sigma_T \to \Sigma_T/\Gamma = \tilde{\Sigma}_T$ be the quotient map, and let $\tilde{\xi}: \tilde{\Sigma}_T \to T$ be the induced family of curves over $T$. Observe that $\tilde{\xi}$ is also flat and proper. For any section $p$ of $\tilde{\xi}$, denote by $\pi^{-1}(p)$ the set of sections $q$ of $\xi$ such that $\pi \circ q = p$.

**Definition 7.2.** A family of $s$-pointed $\Gamma$-curves over $T$ is a family of curves $\xi: \Sigma_T \to T$ over $T$ with an action of a finite group $\Gamma$ as above, and a collection of sections $\tilde{q} := (q_1, \ldots, q_s)$ of $\xi$, such that

1. $p_1, \ldots, p_s$ are mutually distinct and non-intersecting to each other, and, for each $i$, $\pi^{-1}(p_i(T))$ is contained in the smooth locus of $\xi$ and $\pi^{-1}(p_i(T)) \to T$ is étale, where $p_i = \pi \circ q_i$ is the section of $\tilde{\xi}$;
2. for any geometric point $b \in T$, $\tilde{\xi}(b) = (\tilde{\Sigma}_b, p_1(b), \ldots, p_s(b))$ is an $s$-pointed curve in the sense of Definition 3.4. Moreover, $\pi_b : \Sigma_b \to \tilde{\Sigma}_b$ is a $\Gamma$-cover in the sense of Definition 3.1.

Let $\Sigma_T^p$ denote the open subset $\Sigma_T \setminus \bigcup_i \pi^{-1}(p_i(T))$ of $\Sigma_T$. Let $\xi^o : \Sigma_T^o \to T$ denote the restriction of $\xi$.

**Lemma 7.3.** The morphism $\xi^o : \Sigma_T^o \to T$ is affine.

**Proof.** See a proof by R. van Dobben de Bruyn on mathoverflow [vDdB].

Let $f: T' \to T$ be a morphism of schemes. Then, we can pull-back the triple $(\Sigma_T, \Gamma, \tilde{q})$ to $T'$ to get a family $(f^*(\Sigma_T), \Gamma, f^*(\tilde{q}))$ of pointed $\Gamma$-curves over $T'$, where $f^*(\Sigma_T) = T' \times_T \Sigma_T$, $f^*(\tilde{q}) = (f^*q_1, \ldots, f^*q_s)$.

**Lemma 7.4.** For any section $p$ of $\tilde{\xi}$ such that $\pi^{-1}(p(T))$ is contained in the smooth locus of $\xi$, if $\pi^{-1}(p)$ is nonempty (where $\pi^{-1}(p) = \{\text{sections } q \text{ of } \Sigma_T \to T \text{ such that } \pi \circ q = p\}$), then

1. For any $q \neq q' \in \pi^{-1}(p)$, $q(T)$ and $q'(T)$ are disjoint.
2. $\Gamma$ acts on $\pi^{-1}(p)$ transitively, and the stabilizer group $\Gamma_q$ is equal to the stabilizer group $\Gamma_{q(b)}$ at the point $q(b) \in \Sigma_b$ for any geometric point $b \in T$.

**Proof.** It is easy to see that $\pi^{-1}(p)$ is finite (it also follows from the equation (97)). Let $\pi^{-1}(p) = \{q_1, q_2, \ldots, q_k\}$. For each $b \in T$, $\{q_1(b), q_2(b), \ldots, q_k(b)\}$ is a $\Gamma$-stable set and it is contained in the fiber $\pi^{-1}(p(b))$. Since $\Gamma$ acts on $\pi^{-1}(p(b))$ transitively, it follows that $\pi^{-1}(p(b)) = \{q_1(b), q_2(b), \ldots, q_k(b)\}$.

Set $Z := \xi(\cup_{j \neq i} q_i(T) \cap q_j(T))$. Then, $Z$ is a proper closed subset of $T$. Let $U$ be the open subset $T \setminus Z$ of $T$. Then, $\{q_1(U), q_2(U), \ldots, q_k(U)\}$ are mutually disjoint to each other. In particular,
\( k = \frac{|\Gamma|}{|\Gamma_{q(b)}|}, \) for any \( b \in U. \)

By [BR, Lemma 4.2.1], for each \( 1 \leq i \leq k, \) the order of the stabilizer group \( \Gamma_{q(b)} \) is constant along \( T. \) For any \( b' \in \mathbb{Z}, \) there exists \( i \neq j \) such that \( q_i(b') = q_j(b'). \) It follows that \( \frac{|\Gamma|}{|\Gamma_{q(b')}|} = |\pi^{-1}(p(b'))| < k, \) which is a contradiction. Therefore, \( T = U, \) i.e. \( \{q_1(T), q_2(T), \ldots, q_i(T)\} \) are disjoint to each other. It finishes the proof of part (1).

Let \( \Gamma_q \) be the stabilizer group of \( q \in \pi^{-1}(p). \) It is clear that
\[ \Gamma_q \subset \Gamma_{q(b)}, \quad \text{for any geometric point} \ b \in T. \]

We have
\[ k = \frac{|\Gamma|}{|\Gamma_{q(b)}|} \leq \frac{|\Gamma|}{|\Gamma_q|} \leq k, \]
where the first equality follows from the equation (97) (since \( U = T \)) and the second inequality follows from the equation (98). The third inequality follows since \( k := |\pi^{-1}(p)|. \) Thus, we get \( \Gamma_q = \Gamma_{q(b)}, \) for any geometric point \( b \in T \) by the equation (98), and, moreover, \( \Gamma \) acts transitively on \( \pi^{-1}(p). \) It concludes part (2) of the lemma. \( \square \)

**Definition 7.5.** (1) A formal disc over \( T \) is a formal scheme \((T, O_T)\) over \( T \) (in the sense of [H, §II.9]), where \( O_T \) is an \( \mathcal{O}_T \)-algebra which has the following property: For any point \( b \in T \) there exists an affine open subset \( U \subset T \) containing \( b \) such that \( O_T(U) \) is a complete local \( \mathcal{O}_T(U) \)-algebra (see Definition 6.1 (a)).

Let \((T, \mathcal{K}_T)\) be the locally ringed space over \( T \) defined so that \( \mathcal{K}_T(U) \) is the \( \mathcal{O}_T(U) \)-algebra containing \( O_T(U) \) obtained by inverting a (and hence any) \( \mathcal{O}_T(U) \)-parameter \( t_U \) of \( O_T(U). \) Then, \((T, \mathcal{K}_T)\) is called the associated formal punctured disc over \( T. \)

(2) A rotation of a formal disc \((T, O_T)\) over \( T \) of order \( m \) is an \( \mathcal{O}_T(U) \)-module automorphism \( \sigma \) of \((T, O_T)\) of order \( m \) such that, for any \( b \in T, \sigma(t_U) = \epsilon^j t_U \) for some local parameter \( t_U \) around \( b, \) where \( \epsilon := \epsilon^\frac{2\pi}{m}. \)

**Lemma 7.6.** With the assumption and notation as in Definition 7.1, let \( q \) be a section of \( \xi : \Sigma_T \to T \) such that \( q(T) \) is contained in the smooth locus of \( \xi. \) Then,

(1) The formal scheme \((T, \xi, \hat{O}_{\Sigma_T,q(T)})\) is a formal disc over \( T, \) where \( \hat{O}_{\Sigma_T,q(T)} \) denotes the formal completion of \( \Sigma_T \) along \( q(T) \) (cf. [H, Chap. II, §9]).

(2) The stabilizer group \( \Gamma_q \) is a cyclic group of rotations acting faithfully on the formal disc \((T, \xi, \hat{O}_{\Sigma_T,q(T)})\). Moreover, the action of \( \Gamma_q \) on local \( \sigma \)-equivariant parameters is given by a primitive character \( \chi. \)

**Proof.** Part (1) follows from [EGA, 16.9.9, 17.12.1 (c’)]. For part (2), choose a local parameter \( t_U \in \left(\xi, \hat{O}_{\Sigma_T,q(T)}\right)(U). \) Let \( \sigma_q \in \Gamma_q \) be a generator. Set \( \tilde{\iota}_U := \frac{1}{|\Gamma_q|} \sum_{j=0}^{|\Gamma_q|-1} \epsilon_q^{-ij} \sigma_q^j(t_U), \)
where \( \epsilon_q := \epsilon^\frac{2\pi}{|\Gamma_q|} \) and \( \sigma_q(t_U) = \epsilon_q^j t_U + \text{higher terms}. \) Then, \( \tilde{\iota}_U \) is a local parameter in \( \left(\xi, \hat{O}_{\Sigma_T,q(T)}\right)(U) \) such that
\[ \sigma_q(\tilde{\iota}_U) = \epsilon_q^j \tilde{\iota}_U. \]
Since $\Gamma$ acts faithfully on $\Sigma_T$, the action of $\Gamma_q$ is faithful on the formal disc $(T, \xi, \hat{\Theta}_{\Sigma_T,q}(T))$. In particular, by equation (99), $\epsilon_q$ is a primitive $|\Gamma_q|$-th root of unity. Thus, we can find a generator $\tilde{\sigma}_q(U) \in \Gamma_q$ such that

$$\tilde{\sigma}_q(U)\hat{\iota}_U = \epsilon_q^{-1}\hat{\iota}_U.$$  
(100)

In fact, $\tilde{\sigma}_q(U)$ is the unique generator of $\Gamma_q$ satisfying the above equation (100) for any local parameter $\hat{\iota}_U$. From this it is easy to see that the generator $\tilde{\sigma}_q(U)$ does not depend upon $U$. We denote it by $\tilde{\sigma}_q$. It determines the primitive character $\chi$ of $\Gamma_q$, which satisfies $\chi(\tilde{\sigma}_q) = \epsilon_q$.

Denote by $O_q$ the sheaf of $\mathcal{O}_T$-algebra $\xi, \hat{\Theta}_{\Sigma_T,q}(T)$ over $T$, and let $(T, \mathcal{K}_q)$ be the associated formal punctured disc over $T$. For any section $q$ of $\xi$, define the sheaf of $\hat{\Theta}_{\Sigma_T,q}(T)$, the sheaf of twisted conformal blocks

$$\mathcal{H}(\lambda)_T$$
for any $\lambda \in D_{c,q} := D_{c,\sigma_q}$ we define a sheaf of integrable representation $\mathcal{H}(\lambda)_T$ over $T$ as follows: For any open affine subset $U \subset T$ such that $O_q(U)$ is a complete local $\mathcal{O}_T$-algebra,

$$U \mapsto \mathcal{H}(\lambda)|_{\mathcal{O}_T(U)}.$$  

By Lemma 6.5, this gives a well-defined sheaf over $T$. For each section $p$ of $\tilde{\xi}$ such that $\pi^{-1}(p)$ is non-empty and some (and hence any) $q \in \pi^{-1}(p)$ is contained in the smooth locus of $\xi$, we may define the following sheaf of Lie algebras over $\mathcal{O}_T$ (cf. Definition 3.1),

$$\hat{\mathfrak{g}}_p := (\oplus_{q \in \pi^{-1}(p)} \mathfrak{g} \otimes \mathcal{K}_q)^T \oplus \mathcal{O}_T \cdot C, \quad \text{and} \quad \mathfrak{g}_p := (\oplus_{q \in \pi^{-1}(p)} \mathfrak{g} \otimes \mathfrak{c} \otimes \mathcal{K}_q)^T.$$

The restriction gives an isomorphism $\hat{\mathfrak{g}}_p \cong \hat{\mathfrak{g}}(\mathfrak{g}, \Gamma_q)_T$, and $\mathfrak{g}_p \cong \mathfrak{g} \otimes \mathcal{O}_T$ as in Lemmas 3.2 and 3.3. For any $\lambda \in D_{c,q}$, we still denote by $\mathcal{H}(\lambda)_T$ the associated representation of $\hat{\mathfrak{g}}_p$ via the isomorphism $\hat{\mathfrak{g}}_p \cong \hat{\mathfrak{g}}(\mathfrak{g}, \Gamma_q)_T$.

**Definition 7.7** (Sheaf of twisted conformal blocks). Let $(\Sigma_T, \Gamma, \tilde{q})$ be a family of $\mathfrak{s}$-pointed $\Gamma$-curves over a connected scheme $T$. Set $\tilde{p} = \pi \circ \tilde{q}$. Let $\tilde{\lambda} = (\lambda_1, \ldots, \lambda_s)$ be a $s$-tuple of highest weights, where $\lambda_i \in D_{c,\tilde{q}_i}$ for each $i$.

Now, let us consider the sheaf of $\mathcal{O}_T$-module:

$$\mathcal{H}(\tilde{\lambda})_T := \mathcal{H}(\lambda_1)_T \otimes_{\mathcal{O}_T} \mathcal{H}(\lambda_2)_T \otimes_{\mathcal{O}_T} \cdots \otimes_{\mathcal{O}_T} \mathcal{H}(\lambda_s)_T,$$  
(101)

and

$$\hat{\mathfrak{g}}_{\tilde{p}} := \left( \oplus_{i=1}^s \left( \oplus_{q \in \pi^{-1}(p_i)} \mathfrak{g} \otimes \mathcal{K}_q^T \right) \right) \otimes \mathcal{O}_T \cdot C.$$  
(102)

We can define a $\mathcal{O}_T$-linear bracket in $\hat{\mathfrak{g}}_{\tilde{p}}$ as in (11); in particular, $C$ is a central element of $\hat{\mathfrak{g}}_{\tilde{p}}$. Then, $\hat{\mathfrak{g}}_{\tilde{p}}$ is a sheaf of $\mathcal{O}_T$-Lie algebra. There is a natural $\mathcal{O}_T$-linear Lie algebra homomorphism

$$\oplus_{i=1}^s \hat{\mathfrak{g}}_{p_i} \rightarrow \hat{\mathfrak{g}}_{\tilde{p}}, \quad \text{where} \quad C_i \mapsto C.$$  

The componentwise action of $\oplus_{i=1}^s \hat{\mathfrak{g}}_{p_i}$ on $\mathcal{H}(\tilde{\lambda})_T$ induces an action of $\hat{\mathfrak{g}}_{\tilde{p}}$ on $\mathcal{H}(\tilde{\lambda})_T$. We also introduce the following $\mathcal{O}_T$-Lie algebra under the pointwise bracket:

$$\mathfrak{g}(\Sigma_T^0)^T := \left( \mathfrak{g} \otimes \mathfrak{c} \otimes \mathcal{K}_T^T \right), \quad \text{where} \quad \Sigma_T^0 = \Sigma_T \setminus (\bigcup_{i=1}^s \pi^{-1}(p_i(T))).$$  
(103)
There is an embedding of sheaves of $\mathcal{O}_T$-Lie algebras:

$$\beta : \mathfrak{g}(\Sigma^0_T)^F \hookrightarrow \mathcal{H}(\hat{\lambda})_{\mathcal{O}_T}, \quad \sum_k x_k[f_k] \mapsto \sum_{\varphi \in \mathcal{O}_T} \sum_k x_k[(f_k)_{\varphi}],$$

for $x_k \in \mathfrak{g}$ and $f_k \in \xi, \mathcal{O}_{\Sigma^0_T}$ such that $\sum_k x_k[f_k] \in \mathfrak{g}[(\Sigma^0_T)^F$, where $(f_k)_\varphi$ denotes the image of $f_k$ in $\mathcal{K}_\varphi$ via the localization map $\xi, \mathcal{O}_{\Sigma^0_T} \to \mathcal{K}_\varphi$.

By the Residue Theorem, $\beta$ is indeed a Lie algebra embedding. (Observe that Lemma 7.3 has been used to show that $\beta$ is an embedding.)

Finally, define the sheaf of twisted covacua (also called the sheaf of twisted dual conformal blocks) $\mathcal{V}_{\Sigma, \Gamma, \phi}(\vec{q}, \vec{\lambda})$ over $T$ as the quotient sheaf of $\mathcal{O}_T$-module

$$\mathcal{V}_{\Sigma, \Gamma, \phi}(\vec{q}, \vec{\lambda}) := \mathcal{H}(\hat{\lambda})_{\mathcal{O}_T}/\mathfrak{g}(\Sigma^0_T)^F \cdot \mathcal{H}(\hat{\lambda})_{\mathcal{O}_T},$$

where $\mathfrak{g}(\Sigma^0_T)^F$ acts on $\mathcal{H}(\hat{\lambda})_{\mathcal{O}_T}$ via the embedding $\beta$ (given by (104)) and $\mathfrak{g}(\Sigma^0_T)^F \cdot \mathcal{H}(\hat{\lambda})_{\mathcal{O}_T} \subset \mathcal{H}(\hat{\lambda})_{\mathcal{O}_T}$ denotes the image sheaf under the sheaf homomorphism

$$\alpha_{\mathcal{O}_T} : \mathfrak{g}(\Sigma^0_T)^F \otimes \mathcal{O}_T \mathcal{H}(\hat{\lambda})_{\mathcal{O}_T} \to \mathcal{H}(\hat{\lambda})_{\mathcal{O}_T}$$

induced from the action of $\mathfrak{g}(\Sigma^0_T)^F$ on $\mathcal{H}(\hat{\lambda})_{\mathcal{O}_T}$.

Here we use the notation $\mathcal{V}_{\Sigma, \Gamma, \phi}(\vec{q}, \vec{\lambda})$ to denote the sheaf of twisted covacua (see Remark 3.6).

**Theorem 7.8.** (1) The sheaf $\mathcal{V}_{\Sigma, \Gamma, \phi}(\vec{q}, \vec{\lambda})$ is a coherent $\mathcal{O}_T$-module.

(2) For any morphism $f : T' \to T$ between schemes, there exists a natural isomorphism

$$\mathcal{O}_{T'} \otimes_{\mathcal{O}_T} (\mathcal{V}_{\Sigma, \Gamma, \phi}(\vec{q}, \vec{\lambda})) \cong (\mathcal{V}_{\Sigma, \Gamma, \phi}(f^*(\vec{q}), \varphi), \vec{\lambda}).$$

In particular, for any point $b \in T$ the restriction $\mathcal{V}_{\Sigma, \Gamma, \phi}(\vec{q}, \vec{\lambda})|_b$ is the space of twisted dual conformal blocks attached to $(\Sigma_b, \Gamma, \phi, p(b), \vec{\lambda})$.

**Proof.** We first prove part (1). Recall the embedding $\beta : \mathfrak{g}(\Sigma^0_T)^F \hookrightarrow \mathcal{H}(\hat{\lambda})$ of $\mathcal{O}_T$-Lie algebras from (104). Also, consider the $\mathcal{O}_T$-Lie subalgebra

$$\hat{\mathfrak{g}}_{\beta} := \left[ \cap_{\varphi \in \mathcal{O}_T} \mathfrak{g} \otimes \xi, \mathcal{O}_{\Sigma, \phi(T)} \right] \oplus \mathcal{O}_T C$$

of $\mathcal{H}(\hat{\lambda})$ and let $\hat{\mathfrak{g}}_{\beta} + \mathfrak{g}_{\beta}$ be the $\mathcal{O}_T$-subsheaf of $\mathcal{H}(\hat{\lambda})$ spanned by $\text{Im} \beta$ and $\mathfrak{g}_{\beta}$. Then, as can be seen, the quotient sheaf $\mathcal{H}(\mathfrak{g}(\Sigma^0_T)^F + \mathfrak{g}_{\beta})$ is a coherent $\mathcal{O}_T$-module (cf. [L, Lemma 5.1]). Thus, locally we can find a finite set of elements $\{x_j\}$ of $\hat{\mathfrak{g}}_{\beta}$ such that each $x_j$ acts locally finitely on $\mathcal{H}(\hat{\lambda})_{\mathcal{O}_T}$ and

$$\hat{\mathfrak{g}}_{\beta} = \mathfrak{g}(\Sigma^0_T)^F + \mathfrak{g}_{\beta} + \mathcal{O}_T x_j$$

(cf. [Ku, Proof of Lemma 10.2.2]). Now, following the proof of Lemma 3.7 and recalling that the Poincaré-Birkhoff-Witt theorem holds for any Lie algebra $\mathfrak{s}$ over a commutative ring $R$ such that $\mathfrak{s}$ is free as an $R$-module (cf. [CE, Theorem 3.1, Chapter XIII]), we get part (1) of the theorem.
We now prove part (2). By the definition of the sheaf of covacua, \( \mathcal{V}_{\Sigma_T, g, q}(\tilde{q}, \tilde{\lambda}) \) is the cokernel of the \( \mathcal{O}_T \)-morphism \( \sigma_T : g(\Sigma^o_T)^\Gamma \otimes_{\mathcal{O}_T} \mathcal{H}(\tilde{\lambda})_T \rightarrow \mathcal{H}(\tilde{\lambda})_T \), which gives rise to the exact sequence (on tensoring with \( \mathcal{O}_T \)):

\[
\begin{align*}
\mathcal{O}_T \otimes_{\mathcal{O}_T} g(\Sigma^o_T)^\Gamma & \longrightarrow \mathcal{O}_T \otimes_{\mathcal{O}_T} \mathcal{H}(\tilde{\lambda})_T \\
0 & \longrightarrow \mathcal{O}_T \otimes_{\mathcal{O}_T} \mathcal{V}_{\Sigma_T, g, q}(\tilde{q}, \tilde{\lambda}) \longrightarrow 0
\end{align*}
\]

where we have identified the bottom left term of the above under

\[
\left( \mathcal{O}_T \otimes_{\mathcal{O}_T} g(\Sigma^o_T)^\Gamma \right) \otimes_{\mathcal{O}_T} \mathcal{O}_T \otimes_{\mathcal{O}_T} \mathcal{H}(\tilde{\lambda})_T \approx g(\Sigma^o_T)^\Gamma \otimes_{\mathcal{O}_T} \mathcal{H}(\tilde{\lambda})_T,
\]

and the second vertical isomorphism is obtained by Lemma 6.5. The right most vertical isomorphism follows from the Five Lemma, proving the second part of the lemma.

\[
\square
\]

**In the rest of this section we assume that the family \( \xi : \Sigma_T \rightarrow T \) of s-pointed \( \Gamma \)-curves is such that \( T \) is a smooth and irreducible scheme over \( \mathbb{C} \) and \( \xi : \Sigma_T \rightarrow T \) is a smooth morphism.** Because of the existence of sections of \( \xi \), \( \xi \) is a smooth morphism if and only if \( \Sigma_T \) is a smooth scheme (cf. [H, Chap. III, proposition 10.4]).

Let \( \Theta_T \) be the sheaf of vector fields on \( T \). Let \( \Theta_{\Sigma^o_T/T} \) denote the \( \mathcal{O}_T \)-module of vertical vector fields on \( \Sigma^o_T \) with respect to \( \xi^o \), and let \( \Theta_{\Sigma^o_T, T} \) denote the \( \mathcal{O}_T \)-module of vector fields \( V \) on \( \Sigma^o_T \) that are locally liftable from vector fields on \( T \) (i.e., there exists an open cover \( U_i \) of \( T \) such that \( (d\xi^o)(V_{\xi^o-1(U_i)}) \) is a vector field on \( U_i \)). Since \( \xi^o : \Sigma^o_T \rightarrow T \) is an affine and smooth morphism, there exists a short exact sequence:

\[
(107) \quad 0 \rightarrow \Theta_{\Sigma^o_T/T} \rightarrow \Theta_{\Sigma^o_T, T} \xrightarrow{d\xi^o} \Theta_T \rightarrow 0.
\]

This short exact sequence induces the following short exact sequence:

\[
(108) \quad 0 \rightarrow \Theta_{\Sigma^o_T/T}^\Gamma \rightarrow \Theta_{\Sigma^o_T, T}^\Gamma \xrightarrow{d\xi^o} \Theta_T \rightarrow 0,
\]

where \( \Theta_{\Sigma^o_T/T}^\Gamma \) (resp. \( \Theta_{\Sigma^o_T, T}^\Gamma \)) denotes the \( \mathcal{O}_T \)-module of \( \Gamma \)-invariant vector fields in \( \Theta_{\Sigma^o_T/T} \) (resp. \( \Theta_{\Sigma^o_T, T} \)).

For any \( b \in T \), we can find an affine open subset \( b \in U \subset T \), and a \( s \)-tuple of formal parameters \( \tilde{t} := (t_1, t_2, \ldots, t_s) \) where \( t_i \) is a formal \( \Gamma^o_\eta \)-equivariant \( \mathcal{O}_T(U) \)-parameter around \( q \) (cf. Lemma 7.6). For any \( \theta \in \Theta_{\Sigma^o_T, T}^\Gamma(U) := \Theta_{\Sigma^o_T, T}(U)^\Gamma \), we denote by \( \theta_\tilde{t} \) the image in \( \Theta_{\Sigma^o_T, T}^\Gamma(U)^\tilde{t} \), where \( \Theta_{\Sigma^o_T, T}(U) \) is the space of continuous \( \mathbb{C} \)-linear derivations of \( \mathcal{K}_{q_0}^\Gamma \), under the \( m \)-adic topology (given below Proposition 6.7) that are liftable from the
vector fields on $U$. We define the operator $L^\theta_0$ on $\mathcal{H}(\lambda)_U$ by
\begin{equation}
(109) \quad L^\theta_0(h_1 \otimes \cdots \otimes h_s) := \sum_i h_1 \otimes \cdots \otimes L^\theta_0 \cdot h_i \otimes \cdots \otimes h_s,
\end{equation}
where, for $1 \leq i \leq s$, $L^\theta_0$ is the extended Sugawara operator associated to $\theta_i$ with respect to the Kac-Moody algebra $\hat{L}(\xi, \Gamma_q)_U$ defined by (89), where we choose the $\mathbb{C}$-lattice in $\mathcal{H}(\lambda)_U$ as in Lemma 6.10.

**Lemma 7.9.** For any $\theta \in \Theta_{\Sigma_T}(U)^T$, the operator $L^\theta_0$ preserves $g(\Sigma_U^0)^T \cdot \mathcal{H}(\lambda)_U$, where $\Sigma_U^0 := \xi_{\sigma}^{-1}(U)$.

**Proof.** For any $x[f] \in g(\Sigma_U^0)$, let $A(x[f])$ denote the average $\sum_{\sigma \in T} \sigma(x)[\sigma(f)] \in g(\Sigma_U^0)^T$. For any $\tilde{h} := h_1 \otimes \cdots \otimes h_s \in \mathcal{H}(\lambda)_U$, and $\theta \in \Theta_{\Sigma_T}(U)^T$, by the formulae (86) and (90), one can easily check that
\[ L^\theta_0(A(x[f]) \cdot \tilde{h}) = A(x[\theta(f)])(\tilde{h}) + A(x[f])(L^\theta_0 \cdot \tilde{h}). \]
It follows thus that $L^\theta_0$ preserves $g(\Sigma_U^0)^T \cdot \mathcal{H}(\lambda)_U$.

From the above lemma, the operator $L^\theta_0$ induces an operator denoted $\nabla^\theta_0$ on $\mathcal{V}_{\Sigma_T, \Gamma, \phi}(\tilde{q}, \tilde{\lambda})|_U$.

**Theorem 7.10.** With the same notation and assumptions as in Theorem 7.8, assume, in addition, that $\xi : \Sigma_T \to T$ is a smooth morphism and $T$ is smooth. Then, $\mathcal{V}_{\Sigma_T, \Gamma, \phi}(\tilde{q}, \tilde{\lambda})$ is a locally free $\mathcal{O}_T$-module of finite rank.

**Proof.** It is enough to show that the space of twisted covacua $\mathcal{V}_{\Sigma_T, \Gamma, \phi}(\tilde{q}, \tilde{\lambda})|_U$ is locally free for any affine open subset $U \subset T$ with a $s$-tuple of formal parameters $\tilde{t} := (t_1, \cdots, t_s)$ around $\tilde{t} := (q_1, \cdots, q_s)$, where $t_i$ is a $\Gamma_q$-equivariant $\mathcal{O}_T(U)$-parameter. From the short exact sequence (108), we may assume (by shrinking $U$ if necessary) that there exists a $\mathcal{O}_T(U)$-linear section $a : \Theta_T(U) \to \Theta_{\Sigma_T, \Gamma}(U)^T$ of $d\xi_U : \Theta_{\Sigma_T, \Gamma}(U)^T \to \Theta_T(U)$. By part (2) of Proposition 6.9, the following map
\[ \theta \mapsto \nabla^\theta_0 : \mathcal{V}_{\Sigma_T, \Gamma, \phi}(\tilde{q}, \tilde{\lambda})|_U \to \mathcal{V}_{\Sigma_T, \Gamma, \phi}(\tilde{q}, \tilde{\lambda})|_U \]
defines a connection on $\mathcal{V}_{\Sigma_T, \Gamma, \phi}(\tilde{q}, \tilde{\lambda})|_U$. Thus, by the same proof as in [HTT, Theorem 1.4.10], $\mathcal{V}_{\Sigma_T, \Gamma, \phi}(\tilde{q}, \tilde{\lambda})|_U$ is locally free. By Theorem 7.8, $\mathcal{V}_{\Sigma_T, \Gamma, \phi}(\tilde{q}, \tilde{\lambda})$ is a coherent $\mathcal{O}_T$-module and hence it is of finite rank.

Let $\mathcal{V}$ be a locally free $\mathcal{O}_T$-module of finite rank. Let $D_1(\mathcal{V})$ denote the $\mathcal{O}_T$-module of operators $P : \mathcal{V} \to \mathcal{V}$ such that for any $P \in D_1(\mathcal{V})$ and $f \in \mathcal{O}_T$, the Lie bracket $[P, f]$ is an $\mathcal{O}_T$-module morphism from $\mathcal{V}$ to $\mathcal{V}$.

**Definition 7.11 ([L]).** A projectively flat connection over $\mathcal{V}$ is a sheaf of $\mathcal{O}_T$-Lie algebra $\mathcal{L} \subset D_1(\mathcal{V})$ containing $\mathcal{O}_T$ (where $\mathcal{O}_T$ acts on $\mathcal{V}$ by multiplication) such that
\begin{equation}
(110) \quad 0 \longrightarrow \mathcal{O}_T \longrightarrow i \mathcal{L} \overset{\text{Symb}}{\longrightarrow} \mathcal{O}_T \longrightarrow 0
\end{equation}
is a short exact sequence, where Symb denotes the symbol map defined by $(\text{Symb } P)(f) = [P, f]$ for $P \in \mathcal{L}$ and $f \in \mathcal{O}_T$. 

\[ \quad 0 \longrightarrow \mathcal{O}_T \longrightarrow i \mathcal{L} \overset{\text{Symb}}{\longrightarrow} \mathcal{O}_T \longrightarrow 0 \]
Following this definition, choose a local section $\nabla : \Theta_U \to \mathcal{L}|_U$ of Symb on some open subset $U \subset T$. Then, $\nabla$ defines a connection on $\mathcal{Y}$ over $U$, since for any $X \in \Theta_U$, $f \in \mathcal{O}_U$ and $v \in \mathcal{V}$,

$$\nabla_X(f \cdot v) = f\nabla_X v + X(f) \cdot v.$$ 

For any $X, Y \in \Theta_U$, the curvature $\mathcal{K}(X, Y) := [\nabla_X, \nabla_Y] - \nabla_{[X, Y]} \in \mathcal{O}_U$.

We now construct a projectively flat connection $\mathcal{L}_T$ on the sheaf of covacua $\mathcal{Y}_{\Sigma, \Gamma, \phi}(\vec{q}, \vec{\lambda})$. Let $U$ be any affine open subset of $T$ with a $s$-tuple of parameters $\vec{r}$ around $\vec{q}$ as above. Define $\mathcal{L}_T(U)$ to be the $\mathcal{O}_T(U)$-module spanned by $\{\nabla^r_{\vec{r}} | \theta \in \Theta_{\Sigma, \Gamma, \phi}(U)\}$ and $\mathcal{O}_T(U)$. By Lemma 6.10, $\mathcal{L}_T(U)$ does not depend on the choice of parameters $\vec{r}$. Therefore, the assignment $U \mapsto \mathcal{L}_T(U)$ glues to be a sheaf $\mathcal{L}_T$ over $T$.

**Theorem 7.12.** With the same notation and assumptions as in Theorem 7.10, the sheaf $\mathcal{L}_T$ of operators on $\mathcal{Y}_{\Sigma, \Gamma, \phi}(\vec{q}, \vec{\lambda})$ is a projectively flat connection on $\mathcal{Y}_{\Sigma, \Gamma, \phi}(\vec{q}, \vec{\lambda})$.

**Proof.** For any $b \in T$, choose an affine open subset $U \subset T$ with an $s$-tuple of parameters $\vec{r}$ around $\vec{q}$. Given $\theta_1, \theta_2 \in \Theta_{\Sigma, \Gamma, \phi}(U)$, by Proposition 6.9 and the formula (109), the difference $\nabla^r_{\vec{r}[\theta_1, \theta_2]} - [\nabla^r_{\vec{r}}, \nabla^r_{\vec{r}]})$ is a $\mathcal{O}_T(U)$-scalar operator and so is $[\nabla^r_{\vec{r}}, f]$ for $f \in \mathcal{O}_T(U)$. It follows that $\mathcal{L}_T$ is a sheaf of $\mathcal{O}_T$-Lie algebra acting on $\mathcal{Y}_{\Sigma, \Gamma, \phi}(\vec{q}, \vec{\lambda})$.

Note that $\Theta_{\Sigma, \Gamma, \phi}^T|_b \simeq \Theta(\Sigma^0 b)^T$, where $\Sigma^0_b$ is the affine curve $\Sigma_b \pi^{-1}(\vec{p}(b))$ where $\vec{p} = \pi \circ \vec{q}$ and $\Theta(\Sigma^0 b)^T$ is the Lie algebra of $\Gamma$-invariant vector fields on $\Sigma^0_b$. In view of the part (2) of Theorem 7.8, $\mathcal{Y}_{\Sigma, \Gamma, \phi}(\vec{q}, \vec{\lambda})|_b \simeq \mathcal{Y}_{\Sigma, \Gamma, \phi}(\vec{q}(b), \vec{\lambda})$. Therefore, the $\mathcal{O}_T(U)$-linear map $\nabla^r_{\vec{r}} : \Theta_{\Sigma, \Gamma, \phi}(U)^T \to \text{End}_{\mathcal{O}_T(U)}(\mathcal{Y}_{\Sigma, \Gamma, \phi}(\vec{q}, \vec{\lambda})|_U)$ induces a projective representation of $\Theta(\Sigma^0 b)^T$ on the space of covacua $\mathcal{Y}_{\Sigma, \Gamma, \phi}(\vec{q}(b), \vec{\lambda})$ attached to the $s$-pointed curve $(\Sigma_b, \vec{q}(b))$. By Lemma 6.10, the map $\nabla^r_{\vec{r}}$ is independent of the choice of $\vec{r}$ if we consider it projectively as a map $\nabla^r_{\vec{r}} : \Theta_{\Sigma, \Gamma, \phi}(U)^T \to \text{End}_{\mathcal{O}_T(U)}(\mathcal{Y}_{\Sigma, \Gamma, \phi}(\vec{q}, \vec{\lambda})|_U)/\mathcal{O}_T(U)$.

Note that $\Theta(\Sigma^0 b)^T$ is isomorphic to the Lie algebra $\Theta(\Sigma^0 b)$ of vector fields on the affine curve $\Sigma_b \pi^{-1}(\vec{p}(b))$. By [BFM, Lemma 2.5.1], $\Theta(\Sigma^0 b)$ and hence $\Theta(\Sigma^0 b)^T$ is an infinite dimensional simple Lie algebra. Since $\mathcal{Y}_{\Sigma, \Gamma, \phi}(\vec{q}(b), \vec{\lambda})$ is a finite-dimensional vector space, $\Theta(\Sigma^0 b)^T$ can only act by scalars on $\mathcal{Y}_{\Sigma, \Gamma, \phi}(\vec{q}(b), \vec{\lambda})$. Therefore, for any $\theta \in \Theta_{\Sigma, \Gamma, \phi}(U)^T$, the operator $\nabla^r_{\vec{r}}$ acts via multiplication by an element of $\mathcal{O}_T(U)$ on $\mathcal{Y}_{\Sigma, \Gamma, \phi}(\vec{q}(b), \vec{\lambda})|_U$. It follows that $\mathcal{L}_T$ is indeed a projectively flat connection on $\mathcal{Y}_{\Sigma, \Gamma, \phi}(\vec{q}, \vec{\lambda})$. 

---

**8. Local freeness of the sheaf of twisted conformal blocks on stable compactification of Hurwitz stacks**

We consider families of $\Gamma$-stable $s$-pointed $\Gamma$-curves and we show that the sheaf of twisted covacua over the stable compactification of Hurwitz stack is locally free.

In this section, we fix a group homomorphism $\phi : \Gamma \to \text{Aut}(\mathfrak{g})$ such that $\Gamma$ stabilizes a Borel subalgebra $b$ of $\mathfrak{g}$.

**Definition 8.1.** We say that a family of $s$-pointed $\Gamma$-curves $(\Sigma_T, \vec{q})$ over a scheme $T$ (see Definition 7.2) is $\Gamma$-stable if
(1) the family of $s$-pointed curves $(\tilde{\Sigma}_T, \tilde{\rho})$ is stable where $\tilde{\rho} := \pi \circ \tilde{q}$, i.e., for any geometric point $b \in T$ the fiber $\tilde{\Sigma}_b$ is a connected reduced curve with at most nodal singularity and the automorphism group of the pointed curve $(\tilde{\Sigma}_b, \tilde{\rho}(b))$ is finite;

(2) the action of $\Gamma$ on any node of each geometric fiber $\Sigma_b$ is stable in the sense of Definition 5.1.

A $s$-pointed $\Gamma$-curve is called $\Gamma$-stable if it is $\Gamma$-stable as a family over a point.

**Remark 8.2.** For a family of $s$-pointed $\Gamma$-curves $(\Sigma_T, \tilde{q})$ over $T$ satisfying the property (2) as above, the stability of $(\tilde{\Sigma}_T, \tilde{\rho})$ is equivalent to the stability of $(\Sigma_T, \Gamma \cdot \tilde{q})$ (cf. [BR, Proposition 5.1.3]).

Let $(C_o, \tilde{q}_o)$ be a $\Gamma$-stable $s$-pointed $\Gamma$-curve. Let $\tilde{C}_o$ be the normalization of $C_o$ at the points $\Gamma \cdot r$, where $r$ is a $\Gamma$-stable nodal point of $C_o$ (cf. Definition 5.1). The nodal point $r$ splits into two smooth points $r', r''$ in $\tilde{C}_o$. The following lemma shows that there exists a canonical smoothing deformation of $(C_o, \tilde{q}_o)$ over a formal disc $\mathbb{D}_r := \text{Spec} \mathbb{C}[[\tau]]$. We denote by $\mathbb{D}_r^\times$ the associated punctured formal disc $\text{Spec} \mathbb{C}[[\tau]]$.

**Lemma 8.3.** With the same notation as above, we assume that the stabilizer group $\Gamma$, at $r$ is cyclic and does not exchange the branches. Then, there exist a $\Gamma$-stable formal deformation of the $\Gamma$-stable $s$-pointed $\Gamma$-curve $(C_o, \tilde{q}_o)$ over a formal disc $\mathbb{D}_r$, with the formal parameter $\tau$, and a

$\Gamma$-stable families of $s$-pointed $\Gamma$-curves $(C, \tilde{q})$, $(\tilde{C}, \tilde{q})$ over $\mathbb{D}_r$, and a morphism $\xi : \tilde{C} \to C$ of families of $s$-pointed $\Gamma$-curves over $\mathbb{D}_r$, such that the following properties hold:

1. over the closed point $o \in \mathbb{D}_r$, $\xi_o : \tilde{C}_o \to C_o$ is the normalization of $C_o$ at the points $\Gamma \cdot r$, and over the formal punctured disc $\mathbb{D}_r^\times$, we have $\tilde{C}_o \cong C_o \times \mathbb{D}_r^\times$;

2. for each $i = 1, \ldots, s$, $\xi \circ \tilde{q}_i = q_i$ and $\tilde{q}(o) = \tilde{q}_o$ (we also use $\tilde{q}$ to denote the sections $\tilde{q}_i$ in $\tilde{C}$ if there is no confusion);

3. the completed local ring $\hat{\mathcal{O}}_{C, r}$ of $\mathcal{O}_C$ at $r$ is isomorphic to $\mathbb{C}[[z', z'', \tau]]/(\tau - z'z'') \cong \mathbb{C}[[z', z'']],$ where $\Gamma_r$ acts on $z'$ (resp. $z''$) via a primitive character $\chi$ (resp. $\chi^{-1}$). Moreover, $(z', \tau/z'')$ (resp. $(z'', \tau/z')$) gives a formal coordinate around $r'$ (resp. $r''$) in $\tilde{C}$, where we still denote by $z'$ (resp. $z''$) the function around $r'$ (resp. $r''$) by pulling back $z'$ (resp. $z''$) via $\xi$;

4. there exists a $\Gamma$-equivariant isomorphism of algebras

\[
\kappa : \hat{\mathcal{O}}_{C, \Gamma \cdot \{r', r''\}, C_o, \Gamma \cdot \{r', r''\}} \cong \hat{\mathcal{O}}_{C_o, \Gamma \cdot \{r', r''\}}[[\tau]],
\]

where $\hat{\mathcal{O}}_{C, \Gamma \cdot \{r', r''\}, C_o, \Gamma \cdot \{r', r''\}}$ is the completion of $\mathcal{O}_{C, \Gamma \cdot \{r', r''\}}$ along $C \setminus \{r', r''\}$. Moreover, $\tilde{q}$ in $\tilde{C}$ and $\tilde{q}_o$ in $\tilde{C}_o$ are compatible under this isomorphism, i.e., the points $q_{o,i} \in C_o \setminus \Gamma \cdot \{r', r''\}$ (for $i = 1, \ldots, s$) identified with the algebra homomorphisms $\beta_i : \hat{\mathcal{O}}_{C_o, \Gamma \cdot \{r', r''\}} \to \mathbb{C}$ extended to

\[
\beta_i : \hat{\mathcal{O}}_{C, \Gamma \cdot \{r', r''\}, C_o, \Gamma \cdot \{r', r''\}} \to \mathbb{C}[[\tau]]
\]

under the identification $\tilde{q}_i$ of $\tilde{C} \to \mathbb{D}_r$.

**Proof.** In the non-equivariant case, this smoothing construction as formal deformation is sketched by Looijenga in [L, Section 6], and detailed argument from formal deformation
to algebraic deformation can be found in [D, §6.1]. These constructions/arguments can be easily generalized to the equivariant setting when \( \Gamma \) acts on the node stably and does not exchange the branches.

Let \( \hat{L}(g, \Gamma_r) \) (resp. \( \hat{L}(g, \Gamma_{r''}) \)) be the Kac-Moody algebra attached to the point \( r' \) (resp. \( r'' \)) in \( \hat{C}_r \). Recall that (Lemma 5.3) \( \mu' \in D_{e,r'} \) if and only if \( \mu \in D_{e,r''} \) where \( V(\mu') = V(\mu)^* \). (By Lemma 5.2, \( \Gamma_r = \Gamma_{r''} \) and hence \( g_{r'} = g_{r''} \).) Let \( \mathcal{H}(\mu') \) (resp. \( \mathcal{H}(\mu) \)) be the highest weight integrable representation of \( \hat{L}(g, \Gamma_r) \) (resp. \( \hat{L}(g, \Gamma_{r''}) \)) as usual.

**Lemma 8.4.** There exists a non-degenerate pairing \( b_\mu : \mathcal{H}(\mu') \times \mathcal{H}(\mu) \to \mathbb{C} \) such that for any \( h_1 \in \mathcal{H}(\mu'), h_2 \in \mathcal{H}(\mu), \) and \( x[z'^n] \in \hat{L}(g, \Gamma_r) \),

\[
b_\mu(x[z'^n] \cdot h_1, h_2) + b_\mu(h_1, x[z'^n] \cdot h_2) = 0.
\]

Note that \( x[z'^n] \in \hat{L}(g, \Gamma_r) \) if and only if \( x[z'^{n''}] \in \hat{L}(g, \Gamma_{r''}) \).

**Proof.** From Lemma 5.3 (especially see ‘another proof of Lemma 5.3 Part (2)’), there exists a non-degenerate pairing \( b_\mu : \mathcal{H}(\mu') \times \mathcal{H}(\mu) \to \mathbb{C} \) such that \( b_\mu(x[z'^n] \cdot h_1, h_2) + b_\mu(h_1, x[z'^n] \cdot h_2) = 0 \), for any \( x[f] \in \hat{L}(g, \Gamma_r), h_1, h_2 \in \mathcal{H}(\mu') \),

where \( \sigma \) is the Cartan involution of \( \hat{L}(g, \Gamma_r) \) mapping \( x[i][z'^{n''}] \) (resp. \( y[i][z'^{n''}] \)) to \( -y[i][z'^{n''}] \) (resp. \( -x[i][z'^{n''}] \)) for any \( i \in \hat{L}(g, \Gamma_r) \), see these notation in the second proof of Lemma 5.3 part (2). Observe that the composition \( \hat{\omega} \circ \sigma : \hat{L}(g, \Gamma_r) \to \hat{L}(g, \Gamma_{r''}) \) is an isomorphism of Lie algebras mapping \( x[z'^n] \) to \( x[z'^{n''}] \). Hence, the lemma follows after we identify the second copy of \( \mathcal{H}(\mu') \) in \( \hat{b}_\mu \) with \( \mathcal{H}(\mu) \) via \( \hat{\omega}^{-1} \) mentioned above.

There exist direct sum decompositions by \( t \)-degree (putting the \( t \)-degree of the highest weight vectors at 0):

\[
\mathcal{H}(\mu') = \bigoplus_{d=0}^{\infty} \mathcal{H}(\mu')_{-d}, \quad \mathcal{H}(\mu) = \bigoplus_{d=0}^{\infty} \mathcal{H}(\mu)_{-d}.
\]

The non-degenerate pairing \( b_\mu \) in Lemma 8.4 induces a non-degenerate pairing \( b_{\mu,d} : \mathcal{H}(\mu')_{-d} \times \mathcal{H}(\mu)_{-d} \to \mathbb{C} \) for each \( d \geq 0 \). Let \( b_{\mu,d} \in (\mathcal{H}(\mu')_{-d})^* \otimes (\mathcal{H}(\mu)_{-d})^* \) be the dual of \( b_{\mu,d} \). The contravariant form \( \hat{b}_\mu \) on \( \mathcal{H}(\mu')^* \) with respect to \( \hat{L}(g, \Gamma_r) \) induces an isomorphism \( c'_\mu : (\mathcal{H}(\mu')_{-d})^* \simeq \mathcal{H}(\mu')_{-d} \). Similarly, the contravariant form on \( \mathcal{H}(\mu) \) with respect to \( \hat{L}(g, \Gamma_{r''}) \) induces an isomorphism \( c''\mu : (\mathcal{H}(\mu)_{-d})^* \simeq \mathcal{H}(\mu)_{-d} \) (the Cartan involution on \( \hat{L}(g, \Gamma_r) \) taken here is obtained from \( \sigma \) on \( \hat{L}(g, \Gamma_{r''}) \) via the isomorphism \( \hat{\omega} \)). Set \( \Delta_{\mu,d} := (c'_\mu \otimes c''\mu)(b_{\mu,d}) \in \mathcal{H}(\mu')_{-d} \otimes \mathcal{H}(\mu)_{-d} \) if \( d \geq 0 \) and \( 0 \) if \( d < 0 \). Note that \( \Delta_{\mu,0} \) is exactly the element \( I_\mu \) induced from the identity map on \( V(\mu) \) (see the formula (43)). In view of Lemma 8.4, \( \Delta_{\mu,d} \) satisfies the following property (for any \( d, n \in \mathbb{Z} \))

\[
(x[z'^n] \otimes 1) \cdot \Delta_{\mu,d+n} + (1 \otimes x[z'^{n''}]) \cdot \Delta_{\mu,d} = 0, \quad \text{for any } x[z'^n] \in \hat{L}(g, \Gamma_r).
\]
We now construct the following ‘gluing’ tensor element (following [L, Lemma 6.5] in the non-equivariant setting),
\[
\Delta_\mu := \sum_{d \geq 0} \Delta_{\mu, d} r^d \in (\mathcal{H}(\mu^* \otimes \mathcal{H}(\mu))[\tau]).
\]

Let \(\theta', \theta''\) be the maps of pulling-back functions via the map \(\zeta : \tilde{C} \to C\)
\[
\theta' : \hat{\mathcal{O}}_{C, r} \to \hat{\mathcal{O}}_{C, r'} \subset \mathbb{C}((z'))[[\tau]], \text{ and } \theta'' : \hat{\mathcal{O}}_{C, r} \to \hat{\mathcal{O}}_{C, r'} \subset \mathbb{C}((z''))[[\tau]],
\]
where \(\hat{\mathcal{O}}_{C, r}\) is the completion of \(\mathcal{O}_C\) along \(r\), and \(\hat{\mathcal{O}}_{C, r'}\) and \(\hat{\mathcal{O}}_{C, r''}\) are defined similarly.

For any \(f(z', z'') = \sum_{i,j \geq 0} a_{ij} z'^i z''^j \in \hat{\mathcal{O}}_{C, r}\), we have
\[
\theta'(f) = f(z', \tau/z') = \sum_{i,j \geq 0} (\sum a_{ij} z''^j) z'^i,
\]
and
\[
\theta''(f) = f(\tau/z'', z'') = \sum_{i,j \geq 0} (\sum a_{ij} z'^i) z''^j.
\]

The morphisms \(\theta', \theta''\) induce a \(\mathbb{C}[[\tau]]\)-module morphism \(\theta : (\mathfrak{g} \otimes \hat{\mathcal{O}}_{C, r})^F_r \to (\mathfrak{g} \otimes \mathbb{C}((z')))^F_r[[\tau]] \oplus (\mathfrak{g} \otimes \mathbb{C}((z'')))^F_r[[\tau]],\) where \(\tau\) acts on \(\hat{\mathcal{O}}_{C, r}\) via
\[
\tau \cdot f(z', z'') = z' z'' f(z', z'').
\]

Thus, we get an injective map from \((\mathfrak{g} \otimes \hat{\mathcal{O}}_{C, r})^F_r\) into \(\hat{\mathcal{L}}(\mathfrak{g}, \Gamma_r)[[\tau]] \oplus \hat{\mathcal{L}}(\mathfrak{g}, \Gamma_r)[[\tau]]\) (but not a Lie algebra homomorphism), which acts on \((\mathcal{H}(\mu^* \otimes \mathcal{H}(\mu))[\tau])\).

**Lemma 8.5.** The element \(\Delta_\mu \in (\mathcal{H}(\mu^* \otimes \mathcal{H}(\mu))[\tau])\) is annihilated by \((\mathfrak{g} \otimes \hat{\mathcal{O}}_{C, r})^F_r\) via the morphism \(\theta\) defined as above.

**Proof.** For any \(x[z'^i z''^j] \in \mathfrak{g} \otimes \hat{\mathcal{O}}_{C, r}\),
\[
x[z'^i z''^j] \cdot \Delta_\mu = \sum_{d \in \mathbb{Z}} (x[z''^j] \otimes 1) \Delta_{\mu, d} r^{d+j} + \sum_{d \in \mathbb{Z}} (1 \otimes x[z'^i]) \Delta_{\mu, d} r^{d+i}
\]
\[
= -\sum_{d \in \mathbb{Z}} (1 \otimes x[z''^j]) \Delta_{\mu, d+j-i} r^{d+j} + \sum_{d \in \mathbb{Z}} (1 \otimes x[z'^i]) \Delta_{\mu, d} r^{d+i}, \quad \text{by (112)}
\]
\[
= 0.
\]
From this it is easy to see that \(x[f] \cdot \Delta_\mu = 0\) for any \(x[f] \in (\mathfrak{g} \otimes \hat{\mathcal{O}}_{C, r})^F_r\). This proves the lemma. \(\square\)

For each \(i = 1, \ldots, s\), let \(\mathcal{H}(\lambda_i)_{\mathbb{D}_r}\) (resp. \(\mathcal{H}(\lambda_i)\)) denote the integrable representation of \(\hat{\mathcal{L}}(\mathfrak{g}, \Gamma_{q_i})_{\mathbb{D}_r}\) (resp. \(\hat{\mathcal{L}}(\mathfrak{g}, \Gamma_{q_i})\)) attached to \(\hat{\mathcal{O}}_{C, q_i}\) (resp. \(\hat{\mathcal{O}}_{C, q_i}^r\)) as in Section 6, and let \(\mathcal{H}(\lambda)_{\mathbb{D}_r}\) (resp. \(\mathcal{H}(\lambda)\)) denote their tensor product over \(\mathbb{C}[[\tau]]\) (resp. \(\mathbb{C}\)). For each \(i = 1, \ldots, s\), we choose a \((\Gamma_{q_i}, \chi_i)\)-equivariant formal parameter \(z_i\) around \(q_i\), i.e. \(\hat{\mathcal{O}}_{C, q_i} = \mathbb{C}[[\tau]][[z_i]]\), where \(\chi_i\) is a primitive character of \(\Gamma_{q_i}\). It gives rise to a trivialization (cf. Formula (75))
\[
t_i : \mathcal{H}(\lambda)_{\mathbb{D}_r} \simeq \mathcal{H}(\lambda) \otimes_{\mathbb{C}} \mathbb{C}[[\tau]].
\]
We now construct a morphism of \( \mathbb{C}[[\tau]] \)-modules:

\[
F_{\tilde{\lambda}} : \mathcal{H}(\tilde{\lambda}) \otimes_{\mathbb{C}} \mathbb{C}[[\tau]] \to \bigoplus_{\mu \in \mathbb{D}_{r^t}} (\mathcal{H}(\tilde{\lambda}) \otimes \mathcal{H}(\mu') \otimes \mathcal{H}(\mu))[[\tau]]
\]
given by

\[
\sum_{i=0}^{\infty} h_i \tau^i \mapsto \sum_{i,d=0}^{\infty} (h_i \otimes \Delta_{\mu_d}) \tau^{i+d},
\]
where, for each \( i, h_i \in \mathcal{H}(\tilde{\lambda}) \). Finally, we set

\[
F_{\tilde{\lambda}} := \tilde{F}_{\tilde{\lambda}} \circ t_{\tilde{\lambda}} : \mathcal{H}(\tilde{\lambda})_{\mathbb{D}_r} \to \bigoplus_{\mu \in \mathbb{D}_{r^t}} (\mathcal{H}(\tilde{\lambda}) \otimes \mathcal{H}(\mu') \otimes \mathcal{H}(\mu))[[\tau]].
\]

Consider the following canonical homomorphisms (obtained by pull-back and restrictions):

\[
\mathcal{O}_{C[G,q]} \to \mathcal{O}_{C[G,q]} \to \mathcal{O}_{\mathcal{C}_{q}(-q^{(q',r^s)})} \to \mathcal{O}_{C[G,q]} \cup \mathcal{O}_{\mathcal{C}_{q}(-q^{(q',r^s)})} \approx \mathcal{O}_{C[G,q]} \cup \mathcal{O}_{\mathcal{C}_{q}(-q^{(q',r^s)})}[[\tau]],
\]
where the last isomorphism is obtained from the isomorphism \( \kappa \) of Lemma 8.3 (see the isomorphism (111)). This gives rise to a Lie algebra homomorphism (depending upon the isomorphism \( \kappa \)):

\[
\kappa_q : g[C \setminus G \cdot q]^\Gamma \to \left(g \otimes \mathcal{O}_{\mathcal{C}_{q}(-q^{(q',r^s)})}\right)^\Gamma[[\tau]].
\]

Hence, the Lie algebra \( g[C \setminus G \cdot q]^\Gamma \) acts on \( (\mathcal{H}(\tilde{\lambda}) \otimes \mathcal{H}(\mu') \otimes \mathcal{H}(\mu))[[\tau]] \) via the action of \( g[C \setminus G \cdot q]^\Gamma \) on \( \mathcal{H}(\tilde{\lambda}) \otimes \mathcal{H}(\mu') \otimes \mathcal{H}(\mu) \) at the points \( \{\tilde{q}^0, r^t, r^s\} \) as given just before Theorem 5.4 and extending it \( \mathbb{C}[[\tau]] \)-linearly.

Recall from Definition 7.7 the action of \( g[C \setminus G \cdot q]^\Gamma \) on \( \mathcal{H}(\tilde{\lambda})_{\mathbb{D}_r} \). Further, \( g[C \setminus G \cdot q]^\Gamma \) acts on \( (\mathcal{H}(\mu') \otimes \mathcal{H}(\mu))[[\tau]] \) via the Lie algebra homomorphism (obtained by the restriction):

\[
g[C \setminus G \cdot q]^\Gamma \to \left(g \otimes \mathcal{O}_{\mathcal{C}_{q}}\right)^\Gamma,
\]
and the action of \( \left(g \otimes \mathcal{O}_{\mathcal{C}_{q}}\right)^\Gamma \) on \( (\mathcal{H}(\mu') \otimes \mathcal{H}(\mu))[[\tau]] \) (which is a Lie algebra action only projectively) is given just before Lemma 8.5.

**Theorem 8.6.** *We have the following:

1. the morphism \( F_{\tilde{\lambda}} \) is \( g[C \setminus G \cdot q]^\Gamma \)-equivariant;
2. the morphism \( F_{\tilde{\lambda}} \) induces an isomorphism of sheaf of covacua over \( \mathbb{D}_r \):

\[
F_{\tilde{\lambda}} : \mathcal{V}_{C,G,q}(\tilde{\lambda})_{\mathbb{D}_r} \to \bigoplus_{\mu \in \mathbb{D}_{r^t}} \mathcal{V}_{C,G,q}(\tilde{\lambda}, \mu', \mu)[[\tau]].
\]

Note that here we take slightly different notation for the spaces/sheaves of covacua, see Remark 3.6.

**Proof.** By Lemma 8.5, the morphism

\[
F_{\tilde{\lambda}}' : \mathcal{H}(\tilde{\lambda})_{\mathbb{D}_r} \to \mathcal{H}(\tilde{\lambda})_{\mathbb{D}_r} \otimes_{\mathbb{C}[[\tau]]} \bigoplus_{\mu \in \mathbb{D}_{r^t}} (\mathcal{H}(\mu') \otimes \mathcal{H}(\mu))[[\tau]]
\]
given by \( h \mapsto \sum_{\mu \in D_q, r'} h \otimes \Delta_{\mu} \), is a morphism of \( g[C \setminus \Gamma \cdot \bar{q}]^! \)-modules. Moreover, there exists an embedding obtained from the isomorphism \( t_\gamma \):
\[
i : \mathcal{H}(\tilde{\Lambda})_{D_q} \otimes \mathbb{C}[[\tau]] (\mathcal{H}(\mu^s) \otimes \mathcal{H}(\mu))[[\tau]] \hookrightarrow (\mathcal{H}(\tilde{\Lambda}) \otimes \mathcal{H}(\mu^s) \otimes \mathcal{H}(\mu))[[\tau]].
\]
Observe that \( F_\gamma = i \circ F'_\gamma \). It concludes part (1) of the theorem.

We now proceed to prove part (2) of the theorem. Using part (1) of the theorem and the morphism \( \kappa_{\bar{q}} \), we get the \( \mathbb{C}[[\tau]] \)-morphism (113). Taking quotient by \( \tau \), by the Factorization Theorem (Theorem 5.4) the morphism \( F_\gamma \) gives rise to an isomorphism
\[
\mathcal{V}_{\mathcal{C}, \Gamma, \phi}(\bar{q}, \bar{\lambda}) \rightarrow \bigoplus_{\mu \in D_q, r''} \mathcal{V}_{\mathcal{C}, \Gamma, \phi}(\bar{q}, r', r'', (\bar{\lambda}, \mu^s, \mu)).
\]
As a consequence of the Nakayama Lemma (cf. [AM, Exercise 10, Chap. 2]), \( F_\gamma \) is surjective. (Observe that by Theorem 7.8, both the domain and the range of \( F_\gamma \) are finitely generated \( \mathbb{C}[[\tau]] \)-modules.) Now, since the range of \( F_\gamma \) is a free \( \mathbb{C}[[\tau]] \)-module, we get that \( F_\gamma \) splits over \( \mathbb{C}[[\tau]] \). Thus, applying the Nakayama lemma (cf. [AM, Proposition 2.6]) again to the kernel \( K \) of \( F_\gamma \), we get that \( K = 0 \). Thus, \( F_\gamma \) is an isomorphism, proving (2).

\begin{definition}
We say that a \( \Gamma \)-stable \( s \)-pointed \( \Gamma \)-curve \( (\Sigma, q_1, \ldots, q_s) \) has marking data \( \eta = ((\Gamma_1, \chi_1), (\Gamma_2, \chi_2), \ldots, (\Gamma_s, \chi_s)) \) if for each \( i \), the stabilizer group at \( q_i \) is a (cyclic) subgroup \( \Gamma_i \subset \Gamma \) and \( \chi_i \) is the induced (automatically primitive) character of \( \Gamma_i \) on the tangent space \( T_{q_i} \Sigma \).
\end{definition}

We now introduce the moduli stack \( \overline{\mathcal{M}}_{g, \Gamma, \eta} \), which associates to each \( \mathbb{C} \)-scheme \( T \) the groupoid of \( \Gamma \)-stable family \( \xi : \Sigma_T \rightarrow T \) of \( s \)-pointed \( \Gamma \)-curves over \( T \), such that each geometric fiber is of genus \( g \) and is of marking data \( \eta \). We further require that \( \bigcup_{i=1}^s \Gamma \cdot q_i(T) \) contains the ramification divisor of \( \pi : \Sigma_T \rightarrow \Sigma_T/\Gamma \) (cf. [BR, Definition 4.1.6]).

Note that for any \( \Gamma \)-stable family of \( s \)-pointed \( \Gamma \)-curves in \( \overline{\mathcal{M}}_{g, \Gamma, \eta} \), its geometric fibers contain at worst only nodal singularity such that their stabilizer groups are cyclic which do not exchange the branches (cf. [BR, Corollary 4.3.3], and the comment after [BR, Definition 6.2.3]).

\begin{theorem}[Bertin-Romagny]
\( \overline{\mathcal{M}}_{g, \Gamma, \eta} \) is a proper and smooth Deligne-Mumford stack of finite type.
\end{theorem}

\begin{proof}
(Sketch) We can associate to \( (\Sigma_T, \bar{q}) \) (a \( \Gamma \)-stable family \( \xi : \Sigma_T \rightarrow T \) of \( s \)-pointed \( \Gamma \)-curves) the \( \Gamma \)-stable relative Cartier divisor \( \bigcup_{i} \Gamma \cdot (q_{i}(T)) \) in \( \Sigma_T \) which is étale over \( T \). The \( \Gamma \)-conjugacy classes \([\eta]\) of \( \eta \) is the marking type of \( \bigcup_{\gamma \in \Gamma} \gamma(q(T)) \). Let \([\xi]\) be the subclass of those conjugacy classes \([\Gamma_i, \chi_i]\) such that \( \Gamma_i \) is nontrivial. Then, \([\xi]\) is the associated ramification datum of \( \Gamma \)-stable \( s \)-pointed \( \Gamma \)-curves in \( \overline{\mathcal{M}}_{g, \Gamma, \eta} \). Let \( \overline{\mathcal{M}}_{g, \Gamma, [\xi], [\eta]} \) be the stable compactification of Hurwitz stack defined in [BR, Definition 6.2.3]. The natural morphism \( \overline{\mathcal{M}}_{g, \Gamma, \eta} \rightarrow \overline{\mathcal{M}}_{g, \Gamma, [\xi], [\eta]} \) described above is clearly representable, étale and essentially surjective. By [BR, Theorem 6.3.1], \( \overline{\mathcal{M}}_{g, \Gamma, [\xi], [\eta]} \) is a smooth proper Deligne-Mumford stack, and hence so is \( \overline{\mathcal{M}}_{g, \Gamma, \eta} \).
\end{proof}
Let $D_{c,i}$ be the set of dominant weights of $\mathfrak{g}^{\Gamma}$ associated to the irreducible integrable representations of $\hat{L}(\mathfrak{g}, \Gamma, \chi_i)$. Choose a collection $\alpha = (\lambda_1, \cdots, \lambda_s)$ of dominant weights, where $\lambda_i \in D_{c,i}$ for each $i$. Recall that being a Deligne-Mumford stack, $\mathcal{M}_{g,\Gamma,\eta}$ has an atlas $\phi: X \to \mathcal{M}_{g,\Gamma,\eta}$, i.e., $\phi$ is étale and surjective. By Theorem 8.8, $X$ is a smooth (but not necessarily connected) scheme of finite type over $\mathbb{C}$.

We can attach to $\phi: X \to \mathcal{M}_{g,\Gamma,\eta}$ the coherent sheaf $\mathcal{V}_{\Sigma, x, \Gamma, \phi}(\bar{q}, \check{\lambda})$ of conformal blocks, where $(\Sigma, \bar{q})$ is the associated $\Gamma$-stable family of $s$-pointed $\Gamma$-curves over $X$. This attachment can be done componentwise on $X$ via Definition 7.7.

For any two atlases $X, Y$ of $\mathcal{M}_{g,\Gamma,\eta}$ and a morphism $f: Y \to X$ compatible with the atlas structures, by Theorem 7.8, there exists a canonical isomorphism

$$\alpha_f: f^* \mathcal{V}_{\Sigma, x, \Gamma, \phi}(\bar{q}, \check{\lambda}) \simeq \mathcal{V}_{\Sigma, y, \Gamma, \phi}(\bar{q}, \check{\lambda}),$$

where $(\Sigma_x, \bar{q}_x)$ and $(\Sigma_y, \bar{q}_y)$ are the families of $\Gamma$-stable $s$-pointed $\Gamma$-curves associated to these two atlases $X, Y$. Given three atlases $X, Y, Z$ and morphisms $g: Z \to Y$ and $f: Y \to X$, Theorem 7.8 ensures the obvious cocycle condition. Therefore, we get a coherent sheaf $\mathcal{V}_{g, \Gamma, \phi}(\eta, \check{\lambda})$ on $\mathcal{M}_{g,\Gamma,\eta}$ such that $\phi^* \mathcal{V}_{g, \Gamma, \phi}(\eta, \check{\lambda}) \simeq \mathcal{V}_{\Sigma, x, \Gamma, \phi}(\bar{q}, \check{\lambda})$ for any atlas $\phi: X \to \mathcal{M}_{g,\Gamma,\eta}$. Some basics of coherent sheaves on Deligne-Mumford stacks can be found in [Va].

**Theorem 8.9.** For any genus $g \geq 0$, any marking data $\eta = ((\Gamma_1, \chi_1), \cdots, (\Gamma_s, \chi_s))$ and any set of dominant weights $\check{\lambda} = (\lambda_1, \cdots, \lambda_s)$ with $\lambda_i \in D_{c,i}$, the sheaf of conformal blocks $\mathcal{V}_{\Sigma, x, \Gamma, \phi}(\bar{q}, \check{\lambda})$ is locally free over $\mathcal{M}_{g,\Gamma,\eta}$.

**Proof.** It suffices to show that the coherent sheaf $\mathcal{V}_{\Sigma, x, \Gamma, \phi}(\bar{q}, \check{\lambda})$ is locally free, where $X$ is an atlas of $\mathcal{M}_{g,\Gamma,\eta}$. Since $X$ is a disjoint union of smooth irreducible schemes, we can work with a fixed component $X_\eta$ of $X$, and show that the associated sheaf of conformal blocks restricted to $X_\eta$ is locally free.

We introduce a filtration on $\mathcal{M}_{g,\Gamma,\eta}$:

$$\mathcal{M}^0_{g,\Gamma,\eta} \subset \mathcal{M}^1_{g,\Gamma,\eta} \subset \cdots \subset \mathcal{M}^k_{g,\Gamma,\eta} = \mathcal{M}_{g,\Gamma,\eta},$$

where $\mathcal{M}^i_{g,\Gamma,\eta}$ is the open substack of $\mathcal{M}_{g,\Gamma,\eta}$ with each geometric fiber consisting of at most $i$ many $\Gamma$-orbits of nodal points. Note that $\mathcal{M}^0_{g,\Gamma,\eta}$ consists of smooth $s$-pointed $\Gamma$-curves. With the restriction on the genus to be fixed $g$, there exists $k \geq 0$ such that the number of orbits of nodal points is bounded by $k$. This filtration induces an open filtration on $X_\eta$ via $\phi$, $\mathcal{X}^0_\eta \subset \mathcal{X}^1_\eta \subset \cdots \subset \mathcal{X}^k_\eta = X_\eta$.

We now prove inductively that the coherent sheaf $\mathcal{V}_{\Sigma, x, \Gamma, \phi}(\bar{q}_X, \check{\lambda})$ is locally free, where $\bar{q}_X$ is the restriction of $\bar{q}$ to $X^i_\eta$. When $i = 0$, in view of Theorem 7.10, $\mathcal{V}_{\Sigma, x, \Gamma, \phi}(\bar{q}_X, \check{\lambda})$ is locally free. Assume that $\mathcal{V}_{\Sigma, x, \Gamma, \phi}(\bar{q}_X, \check{\lambda})$ is locally free where $i \geq 1$. By the smoothing construction in Lemma 8.3, for any $x \in \mathcal{X}^i_\eta \setminus \mathcal{X}^{i-1}_\eta$, there exists a morphism $\beta_x: \mathcal{D}_x \to \mathcal{M}^i_{g,\Gamma,\eta}$ such that $\beta_x(\eta) = \phi(x)$ and $\beta_x(g_\tau) \in \mathcal{M}^{i-1}_{g,\Gamma,\eta} \setminus \mathcal{M}^{i-2}_{g,\Gamma,\eta}$, where $g_\tau$ is the generic
point of \( \mathbb{D}_r \). Recall that \( \phi : X \to \mathcal{H}M_{g,\Gamma,\eta} \) is étale and surjective, hence \( \beta_i \) can be lifted to \( \beta'_i : \mathbb{D}_r \to X_\alpha \) such that \( \phi \circ \beta'_i = \beta_i \) and \( \beta'_i(o) = x \). It follows that \( \beta'_i(\bar{q}_x) \in X_{\alpha i}^{\prime -1} \setminus X_{\alpha i}^{\prime -2} \). By Theorems 7.8 and 8.6, the rank of \( \mathcal{V}_{\Sigma_{X_{\alpha i}^{\prime -1}},\Gamma,\delta}(\bar{q}_x,\bar{\lambda}) \) is equal to the dimension of \( \mathcal{V}_{\Sigma_{X_{\alpha i}^{\prime -1}},\Gamma,\delta}(\bar{q},\bar{\lambda}) \). It follows that \( \mathcal{V}_{\Sigma_{X_{\alpha i}^{\prime -1}},\Gamma,\delta}(\bar{q}_x,\bar{\lambda}) \) is also locally free (cf. [E, Exercise 20.13]).

It concludes the proof of the theorem. \( \square \)

The dimension of \( \mathcal{H}M_{g,\Gamma,\eta} \) is equal to \( 3\bar{g} - 3 + s \) (cf. [BR, Theorem 5.1.5]) if it is not empty, where by Riemann-Hurwitz formula the genus \( \bar{g} \) of \( \Sigma/\Gamma \) for any \( \Sigma \in \mathcal{H}M_{g,\Gamma,\eta} \), satisfies the following equation (cf. [H, Corollary 2.4, Chap. IV]):

\[
2g - 2 = |\Gamma|(2\bar{g} - 2) + \sum_{i=1}^{s} |\Gamma_i|(|\Gamma_i| - 1).
\]

If \( \dim \mathcal{H}M_{g,\Gamma,\eta} = 0 \), then we must have \( \bar{g} = 0 \) and \( s = 3 \).

**Lemma 8.10.** If \( \dim \mathcal{H}M_{g,\Gamma,\eta} > 0 \), then any stable \( s \)-pointed curve \( (\Sigma, \bar{q}) \) of genus \( \bar{g} \) and consisting of only one node, admits a \( \Gamma \)-stable \( s \)-pointed \( \Gamma \)-cover \( (\bar{\Sigma}, \bar{\bar{q}}) \in \mathcal{H}M_{g,\Gamma,\eta} \).

**Proof.** By assumption, \( 3\bar{g} - 3 + s > 0 \). It follows that either \( \bar{g} \geq 1 \) or \( s \geq 4 \). Hence, we can always find a stable \( s \)-pointed curve \( \bar{\bar{C}} \) of genus \( \bar{g} \) over \( \mathbb{C}[\![\tau]\!] \) such that the special fiber \( \bar{\bar{C}}_o \) has only one node, and the generic fiber \( \bar{\bar{C}}_K \) is smooth, where \( K = \mathbb{C}[\![\tau]\!] \). Let \( \mathcal{M}_{\bar{g}, s} \) be the moduli stack of stable \( s \)-pointed curves of genus \( \bar{g} \). By [BR, Proposition 6.5.2 iii), the morphism \( \mathcal{H}M_{g,\Gamma,\eta} \to \mathcal{M}_{\bar{g}, s} \) given by \( (\Sigma, \bar{q}) \mapsto (\bar{\bar{\Sigma}}, \bar{\bar{\bar{q}}}) \), where \( \bar{\bar{\Sigma}} = \Sigma/\Gamma \) and \( \bar{\bar{\bar{q}}} \) is the image of \( \bar{\bar{q}} \) in \( \bar{\bar{\Sigma}} \), is surjective. It follows that after a finite base change of \( K \), \( \bar{\bar{C}}_K \) has a Galois cover \( C_K \) with Galois group \( \Gamma \) and the prescribed marking data. By semi-stable reduction theorem (cf. [BR, Proposition 5.2.2]), after another finite base change of \( K \), \( C_K \) can be uniquely extended to a \( \Gamma \)-stable \( s \)-pointed \( \Gamma \)-curve \( \bar{\bar{C}}_\eta \) and consisting of only one node, admits a \( \Gamma \)-stable \( s \)-pointed \( \Gamma \)-curve \( \bar{\bar{C}}_\eta \) whose quotient \( \bar{\bar{C}}_\eta/\Gamma \) is exactly the given stable \( s \)-pointed curve \( \bar{\bar{C}}_o \). It concludes the proof of the lemma. \( \square \)

**Remark 8.11.** (1) In the case \( \Gamma \) is cyclic, \( \mathcal{H}M_{g,\Gamma,\eta} \) is irreducible (which can be deduced from the irreducibility of \( \mathcal{H}M_{g,\Gamma,\eta,\xi,\eta,\eta} \) proved in [BR, Corollary 6.4.3]). Then, by Lemma 8.10, Theorem 8.9 and the Factorization Theorem (Theorem 5.4), one can see that to compute the dimension of the space of conformal blocks on smooth \( \Gamma \)-stable \( s \)-pointed \( \Gamma \)-curve, we are reduced to considering the case: cyclic covers over \( \mathbb{P}^1 \) with \( s = 3 \).

(2) For any \( s \)-pointed smooth \( \Gamma \)-curve \( (\Sigma, \bar{q}) \) and weights \( \bar{\lambda} = (\lambda_1, \ldots, \lambda_s) \) with \( \lambda_i \in D_{c, q_i} \), as in Definition 3.5, we can attach the space of conformal blocks. Here we do not need to assume that \( \cup_i \Gamma \cdot q_i \) contains all the ramified points of \( \Sigma \to \bar{\Sigma} \). Thanks to the Propagation Theorem (Theorem 4.3), the dimension of the space of conformal blocks in this case can be reduced to the case that all the ramified points are contained in \( \cup_i \Gamma \cdot q_i \) when \( 0 \in D_{c, q_i} \) for any ramified point \( q_i \) in \( \Sigma \).

(3) The morphism \( f : \overline{\mathcal{H}M}_{g,\Gamma,\eta} \to \overline{\mathcal{M}}_{g,s'} \) given by mapping the \( \Gamma \)-stable \( s \)-pointed \( \Gamma \)-curve \( (\Sigma, \bar{q}) \) to the stable \( s' \)-pointed curve \( (\Sigma, \cup_i \gamma \cdot q_i) \) (cf. Remark 8.2) (where
s' = \sum_{s=1}^{s} \prod_{i=1}^{s} p_i is representable and finite (cf. [BR, Proposition 6.5.2]). By taking the pushforward, the locally free sheaf of conformal blocks on \( \overline{\mathcal{M}}_{g, \Sigma, \eta} \) gives rise to many characteristic classes in the cohomology of \( \overline{\mathcal{M}}_{g, s'} \). It could give rise to interesting applications.

9. Connectedness of \( \text{Mor}_\Gamma(\Sigma^*, G) \)

We prove the connectedness of the ind-group \( \text{Mor}_\Gamma(\Sigma^*, G) \). In particular, we show that the twisted Grassmannian \( X^g = G(\mathbb{D}_g)^{\Gamma} / G(\mathbb{D}_g)^{\Gamma} \) is irreducible.

In this section, we fix a group homomorphism \( \phi : \Gamma \to \text{Aut}(g) \) such that \( \Gamma \) stabilizes a Borel subalgebra \( \mathfrak{b} \) of \( g \). Moreover, \( \Sigma \) denotes a smooth irreducible projective curve with a faithful action of \( \Gamma \) with the projection \( \pi : \Sigma \to \Sigma := \Sigma / \Gamma \) and \( G \) the simply-connected simple algebraic group over \( \mathbb{C} \) with Lie algebra \( g \).

Let \( \mathcal{A}\text{-lg} \) be the category of commutative algebras with identity over \( \mathbb{C} \) (which are not necessarily finitely generated) and all \( \mathbb{C} \)-algebra homomorphisms between them.

**Definition 9.1.** A \( \mathbb{C} \)-space functor (resp. \( \mathbb{C} \)-group functor) is a covariant functor

\[ \mathcal{F} : \mathcal{A}\text{-lg} \to \mathcal{E}\text{t} \quad (\text{resp. } \mathcal{G}\text{roup}) \]

which is a sheaf for the fppf (faithfully flat of finite presentation - Fidélement Plat de Présentation Finie) topology, i.e., for any \( R \in \mathcal{A}\text{-lg} \) and any faithfully flat finitely presented \( R \)-algebra \( R' \), the diagram

\[ \mathcal{F}(R) \to \mathcal{F}(R') \Rightarrow \mathcal{F}(R \otimes_R R') \]

in exact, where \( \mathcal{E}\text{t} \) (resp. \( \mathcal{G}\text{roup} \)) is the category of sets (resp. groups).

From now on we shall abbreviate faithfully flat finitely presented \( R \)-algebra by fppf \( R \)-algebra.

By a \( \mathbb{C} \)-functor morphism \( \varphi : \mathcal{F} \to \mathcal{F}' \) between two \( \mathbb{C} \)-space functors, we mean a set map \( \varphi_R : \mathcal{F}(R) \to \mathcal{F}'(R) \) for any \( R \in \mathcal{A}\text{-lg} \) such that the following diagram is commutative for any algebra homomorphism \( R \to S \):

\[ \mathcal{F}(R) \xrightarrow{\varphi_R} \mathcal{F}'(R) \]

\[ \mathcal{F}(S) \xrightarrow{\varphi_S} \mathcal{F}'(S) \]

Direct limits exist in the category of \( \mathbb{C} \)-space (\( \mathbb{C} \)-group) functors. For any ind-scheme \( X = (X_n)_{n\geq 0} \) over \( \mathbb{C} \), the functor \( \mathcal{E}_X \) is a \( \mathbb{C} \)-space functor by virtue of the Faithfully Flat Descent (cf. [G, VIII 5.1, 1.1 and 1.2]), where \( \mathcal{E}_X(R) \) is the set of all the morphisms \( \text{Mor}(\text{Spec} \, R, X) \). This allows us to realize the category of ind-schemes over \( \mathbb{C} \) as a full subcategory of the category of \( \mathbb{C} \)-space functors.

We recall the following well-known lemma.

**Lemma 9.2.** Let \( \mathcal{F}^o : \mathcal{A}\text{-lg} \to \mathcal{E}\text{t} \) be a covariant functor. Assume that

\[ \mathcal{F}^o(R) \to \mathcal{F}^o(R') \text{ is one to one} \]

for any \( R \in \mathcal{A}\text{-lg} \) and any fppf \( R \)-algebra \( R' \).
Then, there exists a $\mathbb{C}$-space functor $\mathcal{F}$ containing $\mathcal{F}^o$ (i.e., $\mathcal{F}^o(R) \subseteq \mathcal{F}(R)$ for any $R$) satisfying (115) such that for any $\mathbb{C}$-space functor $\mathcal{G}$ satisfying (115) and a natural transformation $\theta^o : \mathcal{F}^o \to \mathcal{G}$, there exists a unique natural transformation $\theta : \mathcal{F} \to \mathcal{G}$ extending $\theta^o$.

Moreover, such a $\mathcal{F}$ is unique up to a unique isomorphism extending the identity map of $\mathcal{F}^o$.

We call such a $\mathcal{F}$ the fppf-sheafification of $\mathcal{F}^o$.

If $\mathcal{F}^o$ is a $\mathbb{C}$-group functor, then its fppf-sheafification $\mathcal{F}$ is a $\mathbb{C}$-group functor.

We recall the following result communicated by Faltings. A detailed proof (due to B. Conrad) can be found in [Ku2, Theorem 1.2.22].

**Theorem 9.3.** Let $\mathcal{G} = (\mathcal{G}_n)_{n \geq 0}$ be an ind-affine group scheme filtered by (affine) finite type schemes over $\mathbb{C}$ and let $\mathcal{G}^\text{red} = (\mathcal{G}^\text{red}_n)_{n \geq 0}$ be the associated reduced ind-affine group scheme. Assume that the canonical ind-group morphism $i : \mathcal{G}^\text{red} \to \mathcal{G}$ induces an isomorphism $(di)_* : \text{Lie}(\mathcal{G}^\text{red}) \to \text{Lie} \mathcal{G}$ of the associated Lie algebras (cf. [Ku, §4.2]). Then, $i$ is an isomorphism of ind-groups, i.e., $\mathcal{G}$ is a reduced ind-scheme.

**Definition 9.4.** (Twisted affine Grassmannian) Recall that for any affine scheme $Y = \text{Spec} S$ with the action of a group $H$, the closed subset $Y^H$ acquires a closed subscheme structure by taking

$$Y^H := \text{Spec} \left( S / \langle g \cdot f - f \rangle_{g \in H, f \in S} \right),$$

where $\langle g \cdot f - f \rangle$ denotes the ideal generated by the collection $g \cdot f - f$. With this scheme structure, $Y^H$ represents the functor $R \rightsquigarrow \text{Mor}_H(\text{Spec} R, Y)$.

For any point $q \in \Sigma$, let $\sigma_q$ be the generator of the stabilizer $\Gamma_q$ such that $\chi_q(\sigma_q) = e_q$, where $e_q := e^{2\pi i}$ (cf. Definition 3.1). Choose a formal parameter $z_q$ at $q$ in $\Sigma$ such that

$$\sigma_q \cdot z_q^{-1} = e_q z_q^{-1}, \quad \text{cf. the identity (27)}.$$

Consider the functor

$$R \rightsquigarrow G \left( R(\langle z_q \rangle) \right)^{\Gamma_q} / G \left( R[\langle z_q \rangle] \right)^{\Gamma_q}.$$

Its fppf-sheafification is denoted by the functor $\mathcal{X}'^q = \mathcal{X}'(G, q, \Gamma)$.

Recall that there exists an open subset $\forall \subset G((\langle z_q \rangle))$ (where $G((\langle z_q \rangle)) := G \left( \mathbb{C}((\langle z_q \rangle)) \right)$) such that the product map

$$G \left( \mathbb{R}[z_q^{-1}] \right)^\times \times G \left( R[\langle z_q \rangle] \right) \cong \forall(R)$$

is a bijection for any $R$,

where $G \left( \mathbb{R}[z_q^{-1}] \right)^\times$ is the kernel of $G \left( \mathbb{R}[z_q^{-1}] \right) \to G(\mathbb{R}), z_q^{-1} \mapsto 0$. Moreover, the functor $G \left( \mathbb{R}[z_q^{-1}] \right)^\times$ is represented by an ind-group variety (in particular, reduced) structure on $G[\langle z_q^{-1} \rangle]$ (cf. [Fa, Corollary 3] and [Ku2, Corollary 1.2.3 and Theorem 1.2.23]). This gives rise to a bijection

$$\left( G \left( \mathbb{R}[z_q^{-1}] \right)^\times \times G \left( R[\langle z_q \rangle] \right) \right)^{\Gamma_q} \cong \forall(R)^{\Gamma_q}.$$

Declare $\{ g^y \mathcal{X}'^q / G[\langle z_q \rangle]^{\Gamma_q} \}_{g \in G((\langle z_q \rangle))^{\Gamma_q}}$ as an open cover of

$$X^q = X(G, q, \Gamma) := G((\langle z_q \rangle))^{\Gamma_q} / G[\langle z_q \rangle]^{\Gamma_q}.$$
and put the ind-scheme structure on $g^\gamma T_q / G[[z_q]]^\Gamma_q$ via its bijection  
$$g^\gamma T_q / G[[z_q]]^\Gamma_q \simeq \left( G[z_q^{-1}]^\Gamma_q \right)^{\text{red}}$$  
induced from the identification (117)  
with the closed ind-subgroup scheme structure on $\left( G[z_q^{-1}]^\Gamma_q \right)^{\text{red}}$ coming from $G[z_q^{-1}]$. In particular, $\left( G[z_q^{-1}]^\Gamma_q \right)^{\text{red}}$ represents the functor $\left( G(R[z_q^{-1}])^\Gamma_q \right)^{\text{red}}$. Thus, we get an ind-scheme structure on $X^q$ such that the projection $G((z_q))^\Gamma_q \to X^q$ admits local sections in the Zariski topology. Moreover, the injection $X^q \hookrightarrow G((z_q))/G[[z_q]]$ is a closed embedding. Further, $X^q$ represents the functor $\mathcal{P}$ (cf. (118)) since the ind-projective variety $G((z_q))/G[[z_q]]$ represents the fppf-sheafification of the functor $G(R((z_q)))/G(R[[z_q]])$. In particular,  
$$\mathcal{P}((\mathbb{C})) = X^q.$$  
Let $U$ (resp. $U^-$) be the unipotent radical of $B$ (resp. of the opposite Borel subgroup $B^\circ$). By considering the ind-subgroup schemes $\left( U[z_q^{-1}] \right)^\Gamma_q$ and $\left( U^- [z_q^{-1}] \right)^\Gamma_q$ of $\left( G[z_q^{-1}] \right)^\Gamma_q$, it is easy to see that the Lie algebras  
$$\text{Lie}\left( \left( G[z_q^{-1}]^\Gamma_q \right)^{\text{red}} \right) = \text{Lie}\left( \left( G[z_q^{-1}]^\Gamma_q \right)^{\text{red}} \right),$$  
since the Lie subalgebras $\left( u \otimes [z_q^{-1}] \right)^\Gamma_q$ and $\left( u^- \otimes [z_q^{-1}] \right)^\Gamma_q$ generate the Lie algebra $\left( g \otimes [z_q^{-1}] \right)^\Gamma_q$ (cf. Section 2), where $Y_{\text{red}}$ denotes the corresponding reduced ind-subscheme. Thus, $\left( G[z_q^{-1}]^\Gamma_q \right)^{\text{red}}$ is a (reduced) ind-group variety (cf. Theorem 9.3). Hence, $X^q$ also is a (reduced) ind-projective variety.  
Observe that $X^q$ being reduced, it is the (twisted) affine Grassmannian considered in [Ku, §7.1] based at $q$ corresponding to the twisted affine Lie algebra $\hat{G}(\Gamma_q) \simeq \hat{A}(q, \Gamma_q)$ (cf. Section 2 and Lemma 3.3) and its parabolic subalgebra $\hat{L}(q, \Gamma_q)_{\geq 0}$. To prove this, follow the same argument as in [LS, Proof of Proposition 4.7] and the construction of the projective representation of $G((z_q))^\Gamma_q$ given by (subsequent) Theorem 10.3.  
Let $\tilde{q} = \{q_1, \ldots, q_s\}$ be a set of points of $\Sigma$ (for $s \geq 1$) with distinct $\Gamma$-orbits and let $\Sigma^\ast := \Sigma \setminus \Gamma \cdot \tilde{q}$. Recall that $\Xi = \Xi_q := \text{Mor}_\Gamma(\Sigma^\ast, G)$ is an ind-affine group scheme, which is a closed ind-subgroup scheme of $\text{Mor}(\Sigma^\ast, G)$ as $\Gamma$-fixed points. We abbreviate  
$$\text{Mor}_\Gamma(\Sigma^\ast, G) = G(\mathbb{C}(\Sigma^\ast))^\Gamma_q$$  
by $G(\Sigma^\ast)^\Gamma_q$. Then, $\Xi$ represents the functor:  
$$R \in \text{Alg} \rightsquigarrow \Xi(R) := \text{Mor}_\Gamma(\Sigma^\ast, G),$$  
where $\Sigma^\ast := \Sigma \times \text{Spec} R$, with the trivial action of $\Gamma$ on $R$. This follows from the corresponding result for the functor $R \rightsquigarrow \text{Mor}(\Sigma^\Gamma, G)$ (without the $\Gamma$-action) which is represented by $G(\Sigma^\ast)$ (cf. [Ku2, Lemma 5.2.10]).  
Let $\Xi^{an}$ denote the group $\Xi$ with the analytic topology. The following result in the non-equivariant case (i.e. $\Gamma = (1)$) is due to Drinfeld. We adapt his arguments (cf. [LS, §5]) along with the construction of the projective representation of $G((z_q))^\Gamma_q$ as in (subsequent) Theorem 10.3.  

**Theorem 9.5.** The group $\Xi^{an}$ is path-connected and hence $\Xi$ is irreducible.
Proof. Take any points $q'_1, \ldots, q'_n, q'_{n+1} \in \Sigma \setminus \Gamma \cdot \bar{q}$ with distinct $\Gamma$-orbits and set (for any $0 \leq i \leq n+1$)

$$
\Xi_i = \Xi_{q'_i, \ldots, q'_{i+1}} = G(\Sigma_i)^\Gamma, \text{ where } \Sigma_i := \Sigma^i \setminus \{q'_i, \ldots, q'_i\}.
$$

Consider the functor

$$
\mathcal{F}^\circ : R \mapsto \Xi_{n+1}(R)/\Xi_n(R).
$$

It is easy to see that

$$(120) \quad \mathcal{F}^\circ(R) \hookrightarrow \mathcal{F}^\circ(R'), \quad \text{for any } \mathbb{C}\text{-algebras } R \subset R'.
$$

Let $\Xi_{n+1}/\Xi_n$ be the fppf-sheafification of $\mathcal{F}^\circ$ (cf. Lemma 9.2).

We claim that as the $\mathbb{C}$-space functors

$$(120) \quad \Xi_{n+1}/\Xi_n \simeq \mathcal{F}^q,
$$

where $q = q'_{n+1}$. Define the morphism

$$
\Xi_{n+1}(R) \to \mathcal{F}^q(R), \quad \gamma \mapsto q\gamma,
$$

where $q\gamma$ is the power series expansion of $\gamma$ at $q$ in the parameter $z_q$. The above morphism clearly factors through

$$
\Xi_{n+1}(R)/\Xi_n(R) \to \mathcal{F}^q(R)
$$

and hence we get a morphism of $\mathbb{C}$-space functors

$$
\hat{\theta} : \Xi_{n+1}/\Xi_n \to \mathcal{F}^q.
$$

Conversely, we define a map $\hat{\psi} : \mathcal{F}^q \to \Xi_{n+1}/\Xi_n$ as follows. Fix $R \in \mathcal{A}\lg$. Take $\gamma_R \in G\left(R((z_q))^{\Gamma}\right)$. Take a $\Gamma$-equivariant $G$-bundle $E_{\gamma R}$ over $\Sigma_R := \Sigma \times \text{Spec } R$ (which respects the actions of $\Gamma$ on $G$ and $\Sigma$), together with a $\Gamma$-equivariant section $\sigma_R$ over $(\Sigma \setminus \Gamma \cdot q)_R$ and a $\Gamma$-equivariant section $\mu_R$ over $(\pi^1 M(q))^\Gamma_R$ such that

$$(121) \quad \mu_R = \sigma_R \cdot \gamma_R, \quad \text{over } (\mathcal{D}^q_R).
$$

This is possible since $\gamma_R$ extends uniquely to an element of $\left(G(\pi^1 M(q))^\Gamma\right)_R$ (cf. Definition 3.1). There exists an $R$-algebra $R'$ with $\text{Spec } R' \to \text{Spec } R$ an étale cover (in particular, $R'$ is a fppf $R$-algebra) such that the pull-back bundle $E_{\gamma' R}$ over $\Sigma_{R'}$ admits a $\Gamma$-invariant section $\theta_{R'}$ over $(\Sigma')_{R'}$ (cf. [He, Theorem 4] and (subsequent) Definition 11.1). Define

$$
\theta_{R'} = \sigma_{R'} \cdot \psi_{\theta_{R'}}(\gamma_{R'}), \quad \text{over } (\Sigma'_{n+1})_{R'}.
$$

where $\sigma_{R'}$ is the pull-back of the section $\sigma_R$ to $(\Sigma \setminus \Gamma \cdot q)_{R'}$ and $\gamma_{R'}$ denotes the image of $\gamma_R$ in $G\left(R((z_q))^{\Gamma}\right)^{\Gamma}_{R'}$. Now, set $\hat{\psi}(\gamma_{R'}) = \psi_{\theta_{R'}}(\gamma_{R'})$ mod $\Xi_n(R')$. It is easy to see that $\hat{\psi}(\gamma_{R'})$ does not depend upon the choices of $\sigma_R, \mu_R$ and $\theta_{R'}$ satisfying the equation (121). Moreover, $\hat{\psi}$ factors through $G\left(R[[z_q]]^{\Gamma}\right)^{\Gamma}_{R'}$. Thus, we get a $\mathbb{C}$-functor morphism (still denoted by) $\hat{\psi} : \mathcal{F}^q \to \Xi_{n+1}/\Xi_n$. Further, it is easy to see that $\hat{\theta}$ and $\hat{\psi}$ are inverses of each other. This proves the assertion (120). In particular, the functor $\Xi_{n+1}/\Xi_n$ is also representable represented by its $\mathbb{C}$-points $\Xi_{n+1}/\Xi_n(\mathbb{C})$. We abbreviate $\Xi_n(\mathbb{C})$ by $\Xi_i$. From the equation (120), we see that

$$
\Xi_{n+1}/\Xi_n \hookrightarrow \Xi_{n+1}/\Xi_n(\mathbb{C}) \simeq \mathcal{F}^q(\mathbb{C}).
$$
Moreover, from the above definition and the identity (118), \( \psi : \mathcal{X}^q(\mathbb{C}) \to \Xi_{n+1}/\Xi_n(\mathbb{C}) \) lands inside \( \Xi_{n+1}/\Xi_n \). Thus, we get

\[
\Xi_{n+1}/\Xi_n = \Xi_{n+1}/\Xi_n(\mathbb{C}) \cong \mathcal{X}^q(\mathbb{C}).
\]

This identification gives rise to an ind-variety structure on \( \Xi_{n+1}/\Xi_n \) transported from that of \( X^q = \mathcal{X}^q(\mathbb{C}) \). Moreover, with this ind-variety structure, \( \Xi_{n+1}/\Xi_n \) represents the functor \( \Xi_{n+1}/\Xi_n \). It is easy to see (by considering the corresponding map at \( R \)-points) that with this ind-variety structure on \( \Xi_{n+1}/\Xi_n \), the action map:

\[
\Xi_{n+1} \times (\Xi_{n+1}/\Xi_n) \to \Xi_{n+1}/\Xi_n
\]

is a morphism of ind-schemes.

For any morphism \( f : \text{Spec } R \to \Xi_{n+1}/\Xi_n \), there exists an étale cover \( \text{Spec } S \to \text{Spec } R \) such that the projection \( \Xi_{n+1}/\Xi_n \to \Xi_{n+1}/\Xi_n \) splits over \( \text{Spec } S \). (This follows since \( \Xi_{n+1}/\Xi_n \) represents the functor \( \Xi_{n+1}/\Xi_n \).) From this it is easy to see that \( (\Xi_{n+1}/\Xi_n)^{an} \) has the quotient topology induced from \( \Xi_{n+1}^{an} \). Moreover, for any ind-variety \( Y = (Y_n)_{n \geq 0} \), any compact subset of \( Y^{an} \) lies in some \( Y_N \) (which is easy to verify). Thus, \( \Xi_{n+1}/\Xi_n \) is a Serre fibration. This gives rise to an exact sequence (cf. [Sp, Chap. 7, Theorem 10])

\[
\pi_1((X^q)^{an}) \to \pi_0(\Xi_n^{an}) \to \pi_0(\Xi_{n+1}^{an}) \to \pi_0((X^q)^{an}).
\]

But,

\[
\pi_1((X^q)^{an}) = \pi_0((X^q)^{an}) = 0,
\]

from the Bruhat decomposition (cf. [Ku, Proposition 7.4.16]). Thus, we get

\[
\pi_0(\Xi_n^{an}) \simeq \pi_0(\Xi_{n+1}^{an}).
\]

Now, we are ready to prove the theorem. Take

\[
\sigma \in \Xi_{\hat{q}} := \text{Mor}_K(\Sigma^*, G) = G(\mathbb{C}[\Sigma^*])^F \subset G(K)^F,
\]

where \( K \) is the quotient field of \( \mathbb{C}[\Sigma^*] \). Since \( G \) is simply-connected, by the following lemma, \( G(K)^F \) is generated by subgroups \( U(K)^F \) and \( U^-(K)^F \), where (as before) \( U \) (resp. \( U^- \)) is the unipotent radical of \( B \) (resp. of the opposite Borel subgroup \( B^- \)). Moreover, \( U \) and \( U^- \) being unipotent groups and \( K \subset \mathbb{C} \), \( U(K)^F \simeq u(K)^F \) under the exponential map (and similarly for \( U^- \)). Thus, we can write

\[
\sigma = \text{Exp}(x_1) \ldots \text{Exp}(x_d), \quad \text{for some } x_i \in u(K)^F \cup u^-(K)^F.
\]

Thus, there exists a finite set \( \hat{q}' = \{ q'_1, \ldots, q'_{n+1} \} \subset \Sigma^* \) with disjoint \( \Gamma \)-orbits such that all the poles of any \( x_i \) (which means the poles of \( f_i^j \) writing \( x_i = \sum_j e^j \otimes f_i^j \) for a basis \( e^j \) of \( u \) or \( u^- \)) are contained in \( \Gamma \cdot \hat{q}' \). Thus, \( \sigma \in \Xi_{n+1} \). Consider the curve

\[
\hat{\sigma} : [0, 1] \to \Xi_{n+1}^{an}, \quad t \mapsto \text{Exp}(tx_1) \ldots \text{Exp}(tx_d) \) joining \( e \) to \( \sigma \).
\]

Since

\[
\pi_0(\Xi_n^{an}) \simeq \pi_0(\Xi_{n+1}^{an}), \quad \text{by (125)},
\]

we get that \( e \) and \( \sigma \) lie in the same path component of \( \Xi^{an} \), thus \( \Xi^{an} \) is path-connected. Using [Ku, Lemma 4.2.5] we get that \( \Xi \) is irreducible. \( \square \)
Our original proof of the following lemma was more direct (and involved). We thank Philippe Gille for pointing out the following argument relying on results of Borel-Tits and Steinberg.

**Lemma 9.6.** Let $G, \Sigma, \Gamma$ be as in the beginning of this section and let $K$ be the function field of $\Sigma$. Let $U$ (resp. $U^-$) be the unipotent radical of $B$ (resp. of the opposite Borel subgroup $B^-$). Then, $(G(K))^\Gamma$ is generated (as an abstract group) by $U(K)^\Gamma$ and $U^-(K)^\Gamma$.

**Proof.** Denote $K_o = K^\Gamma$. Then, $(G(K))^\Gamma$ can be considered as a group scheme over $K_o$. Moreover, since $\Gamma$ stabilizes the Borel subgroup $B$ of $G(\mathbb{C})$, $(G(K))^\Gamma$ is a quasisplit group scheme (over $K_o$). Moreover, $G(K \otimes_{K_o} \bar{K}_o)^\Gamma$ with the trivial action of $\Gamma$ on $\bar{K}_o$ can be identified with $G(\bar{K}_o)$ since $\Gamma$ acts faithfully on $K$, where $\bar{K}_o$ is the algebraic closure of $K$. Now, the lemma follows from combining the results [BT, Proposition 6.2 and Remark 6.6] and [St2, Lemma 64].

**Remark 9.7.** The above lemma is also true (by the same proof) for $K$ replaced by $\mathbb{C}((q))$ and $\Gamma$ replaced by $\Gamma_q$. In particular, this gives another proof of $\pi_0((X^q)^{an}) = 0$.

As a special case of Theorem 9.5, we get the following. The connectedness of $X^q$ in a more general setting is obtained by Pappas-Rapoport [PR1, Theorem 0.1].

**Corollary 9.8.** With the notation as in Definition 9.4, the (twisted) infinite Grassmannian $X^q$ is an irreducible ind-projective (reduced) variety.

**Proof.** Let $\Sigma = \mathbb{P}^1, \bar{q} = \{\infty, 0\}$, and the action of $\Gamma = \Gamma_q$ given as follows: Let $\sigma_q$ be any generator of $\Gamma_q$ (of order $e_q := |\Gamma_q|$). Define the action of $\Gamma_q$ on $\mathbb{P}^1$ by setting

$$\sigma_q \cdot z = e^{2\pi i / e_q} z,$$

for any $z \in \mathbb{P}^1$.

Consider the natural transformation between the functors

$$G \left( R[z, z^{-1}] \right)^{\Gamma_q} \rightarrow G \left( R((z)) \right)^{\Gamma_q} / G \left( R[[z]] \right)^{\Gamma_q}.$$

This gives rise to the morphism between the corresponding ind-schemes:

$$\theta : G \left( \mathbb{C}[z, z^{-1}] \right)^{\Gamma_q} \rightarrow X^q = G((z))^{\Gamma_q} / G[[z]]^{\Gamma_q}.$$

From the isomorphism (cf. equation (122) of Theorem 9.5):

$$\Xi_{n+1}/\Xi_n \cong X^q$$

applied to the above example of $\Sigma = \mathbb{P}^1, \bar{q} = \{\infty, 0\}$ and the action of $\Gamma$ as above, we get that $\theta$ is surjective. Since $G \left( \mathbb{C}[z, z^{-1}] \right)^{\Gamma_q}$ is irreducible (by Theorem 9.5) and hence so is $X^q$. Observe that in the proof of Theorem 9.5 we used the connectedness and simply-connectedness of $(X^q)^{an}$; in particular, this corollary builds upon the connectedness of $(X^q)^{an}$ to prove the stronger result.

**10. Central extension of twisted loop group and its splitting over $\Xi$**

We construct the central extensions of the twisted loop group $G(D^*_q)^{\Gamma_q}$. We introduce the notion of ‘canonical’ splitting and prove the existence of its canonical splitting over $\Xi := \text{Mor}_c(\Sigma \setminus \Gamma \cdot q, G)$ when $c$ is divisible by $|\Gamma|$.
We continue to have the same assumptions on $G, \Gamma$ and $\Sigma$ as in the beginning of Section 9. Fix any base point $q \in \Sigma$ and let $\Sigma^* := \Sigma \setminus \gamma \cdot q$ and $\Xi := \Xi_q := \text{Mor}_i(\Sigma^*, G)$. Then, $\Xi$ is an irreducible ind-affine group scheme (cf. Theorem 9.5). Let $z_q$ be a local parameter on $\Sigma$ around $q$ satisfying the condition (116). This gives rise to a morphism

$$\Xi \hookrightarrow \mathcal{L}^q_G,$$

obtained by taking the Laurent series expansion at $q$ (with respect to the parameter $z_q$ at $q$), where $\mathcal{L}^q_G := G((z_q))^{\Gamma_q}$.

**Definition 10.1 (Adjoint action of $\mathcal{L}^q_G$).** Define the $R$-linear *Adjoint action* of the group functor $\mathcal{L}^q_G(R) := G \left( R((z_q)) \right)^{\Gamma_q}$ on the Lie-algebra functor $\hat{L}(g, \Gamma_q)(R) := \left( g \otimes R((z_q)) \right)^{\Gamma_q} \otimes R.C$ (extending $R$-linearly the bracket in $\hat{L}(g, \Gamma_q)(R)$) by:

$$(\mathcal{A} \gamma)(x \otimes sC) = \gamma x \gamma^{-1} + \left( s + \frac{1}{[\gamma_q]} \text{Res}(\gamma^{-1}d\gamma, x) \right)C,$$

for $\gamma \in \mathcal{L}^q_G(R), x \in \left( g \otimes R((z_q)) \right)^{\Gamma_q}$ and $s \in R$, where $\langle, \rangle$ is the $R((t))$ - bilinear extension of the normalized invariant form on $g$ (normalized as in Section 2) and taking an embedding $i : G \hookrightarrow \text{SL}_N$ we view $G(R((z_q)))$ as a subgroup of $N \times N$ invertible matrices over the ring $R((z_q))$. From the functoriality of the conjugation, $\gamma x \gamma^{-1} \in \left( g \otimes R((z_q)) \right)^{\Gamma_q}$ and it does not depend upon the choice of the embedding $i$. A similar remark applies to $\gamma^{-1}d\gamma$.

Here $d\gamma$ for $\gamma = (\gamma_i) \in M_N(R((z_q)))$ denotes $d\gamma := \left\{ \frac{d\gamma_i}{dz_q} \right\}$.

It is easy to check that for any $\gamma \in \mathcal{L}^q_G(R), \mathcal{A} \gamma : \hat{L}(g, \Gamma_q)(R) \to \hat{L}(g, \Gamma_q)(R)$ is a $R$-linear Lie algebra homomorphism. Moreover, for $\gamma_1, \gamma_2 \in \mathcal{L}^q_G(R),$

$$\mathcal{A} \gamma_1 \gamma_2 = \mathcal{A}(\gamma_1) \mathcal{A}(\gamma_2).$$

One easily sees that for any finite dimensional $C$-algebra $R$ and $x \in \left( g \otimes R((z_q)) \right)^{\Gamma_q}$, the derivative

$$\mathcal{A} \gamma(x)(y) = [x, y], \quad \text{for any } y \in \hat{L}(g, \Gamma_q)(R).$$

Let $\mathcal{H}(\lambda)$ be an integrable highest weight (irreducible) representation of $\hat{L}(g, \Gamma_q)$ (with central charge $c$). It clearly extends to a $R$-linear representation $\tilde{\mathcal{H}}_R$ of $\hat{L}(g, \Gamma_q)(R)$ in $\mathcal{H}(\lambda)_R := \mathcal{H}(\lambda) \otimes_C R$. A proof of the following result is parallel to the proof due to Faltings in the untwisted case (cf. [BL, Lemma A.3]).

**Proposition 10.2.** For any $R \in \mathcal{A} \text{lg} g$ and $\gamma \in G \left( R((z_q)) \right)^{\Gamma_q}$, locally over Spec $R$, there exists an $R$-linear automorphism $\tilde{\mathcal{H}}_R(\gamma)$ of $\mathcal{H}(\lambda)_R$ uniquely determined up to an invertible element of $R$ satisfying

$$\tilde{\mathcal{H}}_R(\gamma) \tilde{\mathcal{H}}_R(x) \tilde{\mathcal{H}}_R(\gamma)^{-1} = \tilde{\mathcal{H}}_R(\mathcal{A} \gamma \cdot x), \quad \text{for any } x \in \hat{L}(g, \Gamma_q)(R).$$

As a corollary of the above Proposition, we get the following.
Theorem 10.3. There exists a homomorphism $\rho_R : G\left(R((z_q))\right)^\Gamma_q \to \mathcal{P}GL_{\mathcal{H}(\lambda)}(R)$ of group functors such that

$$\hat{\rho} = \hat{\rho}(\mathbb{C}) : T_1(\mathcal{L}_G^q(\mathbb{C})) = \left(\mathfrak{g} \otimes \mathbb{C}((z_q))\right)^\Gamma_q \to \text{End}_C(\mathcal{H}(\lambda))/\mathbb{C} \cdot \text{Id}_{\mathcal{H}(\lambda)}$$

coincides with the projective representation $\mathcal{H}(\lambda)$ of $\left(\mathfrak{g} \otimes \mathbb{C}((z_q))\right)^\Gamma_q$.

Definition 10.4 (Central extension). Let $0 \in D_{c,q}$, where $D_{c,q}$ denotes $D_c$ for the twisted affine Lie algebra $\hat{L}(\mathfrak{g}, \Gamma_q)$ (cf. Lemma 2.1). By the above theorem, we have a homomorphism of group functors:

$$\rho_R : \mathcal{L}_G^q(R) \to \mathcal{P}GL_{\mathcal{H}^c}(R),$$

where $\mathcal{H}^c := \mathcal{H}(0)$ with central charge $c$ for the twisted affine Lie algebra $\hat{L}(\mathfrak{g}, \Gamma_q)$. Also, there is a canonical homomorphism of group functors

$$\pi_R : \mathcal{L}_{\mathcal{H}^c}(R) \to \mathcal{P}GL_{\mathcal{H}^c}(R).$$

From this we get the fiber product group functor $\hat{\mathcal{L}}_c^q$:

$$\hat{\mathcal{L}}_c^q(R) := \mathcal{L}_G^q(R) \times_{\mathcal{P}GL_{\mathcal{H}^c}(R)} \mathcal{L}_{\mathcal{H}^c}(R).$$

By definition, we get homomorphisms of group functors

$$p_R : \hat{\mathcal{L}}_c^q(R) \to \mathcal{L}_G^q(R) \quad \text{and} \quad \hat{\rho}_R : \hat{\mathcal{L}}_c^q(R) \to \mathcal{L}_{\mathcal{H}^c}(R)$$

making the following diagram commutative:

$$\begin{array}{ccc}
\hat{\mathcal{L}}_c^q(R) & \xrightarrow{p_R} & \mathcal{L}_{\mathcal{H}^c}(R) \\
p_R \downarrow & & \downarrow \pi_R \\
\mathcal{L}_G^q(R) & \xrightarrow{\hat{\rho}_R} & \mathcal{P}GL_{\mathcal{H}^c}(R).
\end{array}$$

The following is the central extension we are seeking:

$$1 \to \mathbb{C}^* \to \hat{\mathcal{L}}_c^q \xrightarrow{p} \mathcal{L}_G^q \to 1, \quad \text{where} \quad \hat{\mathcal{L}}_c^q := \hat{\mathcal{L}}_c^q(\mathbb{C}).$$

It is easy to see that the Lie algebra $\hat{\mathcal{L}}_c^q(R) := T_1(\hat{\mathcal{L}}_c^q)_R$ is identified with the fiber product Lie algebra:

$$\hat{\mathcal{L}}_c^q(R) = \left(\mathfrak{g} \otimes R((z_q))\right)^\Gamma_q \times_{\text{End}_R((\mathcal{H}^c)_R)} \text{End}_R((\mathcal{H}^c)_R),$$

for any finite dimensional $\mathbb{C}$-algebra $R$.

Lemma 10.5. The Lie algebra $\hat{\mathcal{L}}_c^q := \text{Lie}(\hat{\mathcal{L}}_c^q(\mathbb{C}))$ can canonically be identified with the twisted affine Lie algebra $\hat{L}(\mathfrak{g}, \Gamma_q)$.

Proof. Define

$$\psi : \hat{L}(\mathfrak{g}, \Gamma_q) \to \hat{\mathcal{L}}_c^q, \quad x + z\mathbb{C} \mapsto (x, \hat{\rho}(x) + z\text{Id}), \quad \text{for} \ x \in \mathfrak{g}((z_q))^\Gamma_q \text{and} \ z \in \mathbb{C}.\,$$

From the definition of the bracket in $\hat{L}(\mathfrak{g}, \Gamma_q)$ and Theorem 10.3, $\psi$ is an isomorphism of Lie algebras.

Combining Theorem 10.3, Definition 10.4 and Lemma 10.5, we get the following.
Corollary 10.6. We have a homomorphism of group functors:
\[ \hat{\rho} : \hat{G_c} \to GL_{\mathcal{H}_c} \]
such that its derivative at \( R = \mathbb{C} \):
\[ \dot{\hat{\rho}} : \hat{g} \to \text{End}_{\mathbb{C}}(\mathcal{H}_c) \]
under the identification of Lemma 10.5 coincides with the Lie algebra representation
\[ \dot{\rho} : \hat{L}(g, \Gamma_q) \to \text{End}_{\mathbb{C}}(\mathcal{H}_c). \]
Moreover, for any \( \hat{\gamma} \in \hat{G}_c(R) \) and \( x \in \hat{g}(R) \),
\[ (131) \quad \dot{\hat{\rho}}(\hat{\gamma})\dot{\rho}_R(x)\dot{\rho}_R(\hat{\gamma})^{-1} = \dot{\rho}_R(\mathcal{A}(p_R(\hat{\gamma}))x), \]
as operators on \((\mathcal{H}_c)_R\).

Theorem 10.7. (1) The central extension \( p : \hat{\mathcal{G}}_c \to \mathcal{L}_G^q \) (as in Definition 10.4) splits over \( G[[z_q]]^{F_q} \) for any \( c \geq 1 \) such that \( 0 \in D_c = D_{c,q} \). Moreover, we can choose the splitting so that the corresponding tangent map is the identity via Lemma 10.5.

(2) The above central extension splits over \( \Xi \) if \( c \) is a multiple of \( |\Gamma| \). Moreover, we can choose the splitting so that the corresponding tangent map is the identity via Lemma 10.5.

(By Corollary 2.2, if \( |\Gamma| \) divides \( c \) then \( 0 \in D_c \).)

We call the unique splitting satisfying the above property canonical.

Proof. We first prove part (1) of the theorem. By Proposition 10.2 (using the fact that the annihilator of \( g[[z_q]]^{F_q} \) in \( \mathcal{H}_c \) is exactly \( C_{v_+} \)), the map
\[ \rho : G((z_q))^{F_q} \to PGL_{\mathcal{H}_c} \]
restricted to \( G[[z_q]]^{F_q} \) lands inside \( PGL_{\mathcal{H}_c}^+ \) consisting of those (projective) automorphisms which take the highest weight vector \( v_+ \) of \( \mathcal{H}_c \) to \( C_{v_+} \). Take the subgroup \( GL_{\mathcal{H}_c}^+ \) consisting of those automorphisms which take \( v_+ \mapsto v_+ \). Then, the map \( GL_{\mathcal{H}_c}^+ \to PGL_{\mathcal{H}_c}^+ \) is an isomorphism providing the splitting of \( GL_{\mathcal{H}_c} \to PGL_{\mathcal{H}_c} \) over \( PGL_+^{\mathcal{H}_c} \). Thus, the central extension \( p : \hat{\mathcal{G}}_c \to \mathcal{L}_G^q \) splits over \( G[[z_q]]^{F_q} \). Denote this splitting by \( \sigma \).

We next prove that \( \sigma \) (via Lemma 10.5) is the identity map: Let
\[ \sigma(x) = x + \lambda(x)C, \quad \text{for } x \in g[[z_q]]^{F_q}, \]
where \( \lambda : g[[z_q]]^{F_q} \to \mathbb{C} \) is a \( \mathbb{C} \)-linear map. Thus, for any \( x \in g[[z_q]]^{F_q} \),
\[ (132) \quad \dot{\rho} \circ \sigma(x)(v_+) = x \cdot v_+ + \lambda(x)cv_+ = \lambda(x)cv_+. \]
But, since \( \dot{\rho}(\text{Im}(\sigma)) \subset GL_{\mathcal{H}_c}^+ \),
\[ (133) \quad \dot{\rho} \circ \sigma(x)(v_+) = 0, \quad \text{for all } x \in g[[z_q]]^{F_q}. \]
Combining (132) and (133), we get \( \lambda \equiv 0 \). This proves that \( \sigma \) is the identity map.

We now prove part (2) of the theorem. Consider the embedding obtained via the restriction:
\[ i_q : \Xi = G(\Sigma \\Gamma \cdot q)^F \hookrightarrow G(D_q)^{F_q}. \]
Also, consider the embedding

\[ j_q = \prod_{y \in \Gamma/q} j_q' \colon G(\mathbb{D}_q^*) \leftrightarrow \prod_{y \in \Gamma/q} G(\mathbb{D}_{yq}^*), \]

where \( j_q' : G(\mathbb{D}_q^*) \rightarrow G(\mathbb{D}_{yq}^*) \) is defined by

\[ j_q'(f)(yz) = \gamma \cdot f(z), \quad \text{for } \gamma \in \Gamma/q, z \in \mathbb{D}_q^*, \text{ and } f \in G(\mathbb{D}_q^*). \]

Here \( \Gamma/q \) denotes a (fixed) set of coset representatives of the cosets \( \Gamma/q \).

Let \( \mathcal{H}_1 \) denote the integrable highest weight module of highest weight 0 and central charge 1 of the untwisted affine Lie algebra \( \hat{\mathfrak{L}}(\mathfrak{g}) \) based at \( q, \) i.e., the central extension of \( \mathfrak{g}(\mathfrak{z}_q) \), where \( \mathfrak{z}_q \) is a local parameter for \( \Sigma \) at \( q \). Identifying \( G(\mathbb{D}_{yq}^*) \) with \( G(\mathbb{D}_{q}^*) \) via \( j_q' \), we get a projective representation \( \rho \) and the following commutative diagram:

\[
\begin{array}{c}
\mathcal{H}_1 \xrightarrow{j_q} \mathcal{H}_1 \oplus \cdots \oplus \mathcal{H}_1 \\
\mathcal{H}_1 \xrightarrow{j_q} \prod_{y \in \Gamma/q} G(\mathbb{D}_{yq}^*) \xrightarrow{\rho} \text{PGL}(\mathcal{H}_1 \oplus \cdots \oplus \mathcal{H}_1),
\end{array}
\]

where we take \( |\Gamma/q| \) copies of \( \mathcal{H}_1 \) and \( \hat{\rho} \) (resp. \( \hat{j}_q \)) is the pull-back of \( \text{GL}(\mathcal{H}_1 \oplus \cdots \oplus \mathcal{H}_1) \) (resp. \( \mathcal{H}_1 \)) induced from \( \rho \) (resp. \( j_q \)). Since \( \mathcal{H}_1 \) has central charge 1, it is easy to see that \( j_q \) is the central extension of \( G(\mathbb{D}_q^*) \) corresponding to the central charge \( |\Gamma/q| \).

Now, any splitting \( \sigma \) of \( p_{\Gamma/q} \) over \( G(\Sigma \backslash \Gamma \cdot q) \) clearly induces a splitting \( \hat{\sigma} \) of the central extension \( p_q : G(\mathbb{D}_q^*) \rightarrow G(\mathbb{D}_q^*) \) over \( G(\Sigma \backslash \Gamma \cdot q)^F \). \( \hat{\sigma} = \text{Id} \). This follows trivially from the fact that

\[ g(\Sigma \backslash \Gamma \cdot q), g(\Sigma \backslash \Gamma \cdot q)] = g(\Sigma \backslash \Gamma \cdot q). \]

Now, the induced splitting \( \hat{\sigma} \) of the central extension \( p_q : G(\mathbb{D}_q^*) \rightarrow G(\mathbb{D}_q^*) \) over \( G(\Sigma \backslash \Gamma \cdot q)^F \) clearly satisfies \( \hat{\sigma} = \text{Id} \). This proves the theorem.

**Remark 10.8.** Because of the possible existence of nontrivial characters of \( G^{F_q} \) (resp. \( G^{F_q} \) for \( q' \in \Sigma \backslash \Gamma \cdot q) \), the splittings of \( \hat{\mathfrak{g}}^\circ \rightarrow \mathcal{L}^\circ_G \) over \( G[[\mathfrak{z}_q]]^{F_q} \) (resp. \( G(\Sigma \backslash \Gamma \cdot q)^F \)) may not be unique.

By a theorem of Steinberg (cf. [St1]), the fixed subgroup \( G^\sigma \) is connected for any finite order automorphism \( \sigma \) of \( G \).
Proposition 10.9. Let $\sigma$ be a finite order automorphism of $\mathfrak{g}$ of order $m$ and let $m$ divide $\bar{s}c$, where $\bar{s}$ is defined above Corollary 2.2. For any $\lambda \in D_{\ast}$, the irreducible $\mathfrak{g}^n$-module $V(\lambda)$ integrates to a representation of $G^\tau$.

Proof. Decompose $\sigma = \tau e^{ad}$ as in (3). To prove that $V(\lambda)$ integrates to a $G^\tau$-module, it suffices to show that the torus $H^\tau = H^\tau$ acts on $V(\lambda)$, where $H$ is the maximal torus of $G$ with Lie algebra $\mathfrak{h}$ ($\mathfrak{h}$ being a compatible Cartan subalgebra). By the definition of $D_{\ast}$, since $m$ divides $\bar{s}c$ (by assumption), $\lambda(\alpha_i) \in \mathbb{Z}$ for all the simple coroots $\alpha_i$ of $\mathfrak{g}^\tau$, i.e., $\lambda$ belongs to the weight lattice of $\mathfrak{g}^\tau$. So, if $G^\tau$ is simply-connected, $\lambda$ gives rise to a character of $H^\tau$. Thus, in this case $H^\tau$ acts on $V(\lambda)$. Recall that for a diagram automorphism $\tau$, $G^\tau$ is simply-connected unless $(\mathfrak{g}, r) = (A_{2n}, 2)$, where $r$ is the order of $\tau$. In this case $G^\tau = SO(2n + 1)$ and following the notation of the identity (6),

$$[x_{\alpha}, y_{\alpha}] = -(\alpha_1^\vee + \cdots + \alpha_{m-1}^\vee + \alpha_n^\vee/2)$$

with the Bourbaki convention [Bo, Peanche II]. Since $n_{ij}$ is required to lie in $\mathbb{Z}_{\geq 0}$, for all $i \in \mathcal{I}(\mathfrak{g}, \sigma)$, and $m$ divides $\bar{s}c$, we get

$$\lambda(\alpha_i^\vee)/2 \in \mathbb{Z}.$$

Thus, $\lambda$ belongs to the root lattice of $\mathfrak{g}^\tau$ and hence $\lambda$ gives rise to a character of $H^\tau$. This proves that $H^\tau$ acts on $V(\lambda)$, proving the proposition. \qed

11. Uniformization theorem (A review)

We continue to have the same assumptions on $G, \Gamma, \Sigma$ as in the beginning of Section 9. We recall some results due to Heinloth [He] (conjectured by Pappas-Rapoport [PR1, PR2]) only in the generality we need and in the form suitable for our purposes. In particular, we recall the uniformization theorem due to Heinloth for the parahoric Bruhat-Tits group schemes $\mathcal{G}$ in our setting. We introduce the moduli stack $\text{Parbun}_G$ of quasi-parabolic $\mathcal{G}$-torsors over $\bar{\Sigma}$ and construct the line bundles over $\text{Parbun}_G$.

Definition 11.1 (Parahoric Bruhat-Tits group scheme). Consider the $\Gamma$-invariant Weil restriction $\mathcal{G} = \mathcal{G}(\Sigma, \Gamma, \phi)$ via $\pi : \Sigma \to \bar{\Sigma} := \Sigma/\Gamma$ of the constant group scheme $\Sigma \times G \to \Sigma$ over $\Sigma$. More precisely, $\mathcal{G}$ is given by the following group functor over $\bar{\Sigma}$:

$$U \mapsto G(U \times_{\bar{\Sigma}} \Sigma)^\Gamma,$$

for any scheme $U$ over $\bar{\Sigma}$, where $U \times_{\bar{\Sigma}} \Sigma$ is the fiber product of $U$ and $\Sigma$ over $\bar{\Sigma}$. Then, $\mathcal{G} \to \bar{\Sigma}$ is a smooth affine group scheme over $\bar{\Sigma}$.

This provides a class of examples of parahoric Bruhat-Tits group schemes.

For any point $p \in \bar{\Sigma}$, the fiber $\mathcal{G}_p \simeq G$ if $p$ is an unramified point. However, if $p$ is a ramified point, the group $\mathcal{G}_p$ has unipotent radical $U_p$ and

$$\mathcal{G}_p/U_p \simeq G^{F_q}, \text{ for any } q \in \pi^{-1}(p).$$

Take any point $q \in \pi^{-1}(p)$ and let $\mathcal{D}_p \subset \bar{\Sigma}$ (resp. $\mathcal{D}_q \subset \Sigma$) be the formal disc around $p$ in $\bar{\Sigma}$ (resp. around $q$ in $\Sigma$). Then,

$$\mathcal{G}(\mathcal{D}_p) \simeq G(\mathcal{D}_q)^{F_q}. $$
Similarly, for the punctured discs $D_p^\times$ and $D_q^\times$,

\begin{equation}
\mathcal{G}(D_p^\times) \cong G(D_q^\times)^{\Gamma_q}.
\end{equation}

Thus,

\[ \mathcal{G}(D_p^\times)/\mathcal{G}(D_q^\times) \cong X^q, \quad \text{(cf. Definition 9.4).} \]

In particular, it is also an irreducible (reduced) ind-projective variety (cf. Corollary 9.8).

**Definition 11.2** (Moduli stack of $\mathcal{G}$-torsors). Consider the stack $\mathcal{Bun}_q$ assigning to a commutative ring $R$ the category of $\mathcal{G}_R$-torsors over $\Sigma_R := \tilde{\Sigma} \times \text{Spec}R$, where $\mathcal{G}_R$ is the pull-back of $\mathcal{G}$ via the projection from $\tilde{\Sigma}_R$ to $\tilde{\Sigma}$. Then, as proved by Heinloth [He, Proposition 1], $\mathcal{Bun}_q$ is a smooth algebraic stack, which is locally of finite type.

We need the following parabolic generalization of $\mathcal{Bun}_q$. Let $\vec{\mathcal{P}} = (p_1, \cdots, p_s)$ be a set of distinct points in $\tilde{\Sigma}$. Label the points $\vec{\mathcal{P}}$ by parabolic subgroups $\vec{\mathcal{P}} = (P_1, \cdots, P_s)$, where $P_i$ is a parabolic subgroup of $G_{p_i}$. Via the isomorphism (135), we can think of $P_i$ as a parabolic subgroup $P_i^{\vec{\mathcal{P}}}$ of $G^{\vec{\mathcal{P}}}$ for any $q_i \in \pi^{-1}(p_i)$.

A quasi-parabolic $\mathcal{G}$-torsor of type $\vec{\mathcal{P}}$ over $(\tilde{\Sigma}, \vec{\mathcal{P}})$ is, by definition, a $\mathcal{G}$-torsor $\mathcal{E}$ over $\tilde{\Sigma}$ together with points $\sigma_i$ in $\mathcal{E}_{p_i}/P_i$. This gives rise to the stack: $\tilde{\Sigma} \to \mathcal{Bun}_q$.

We recall the following uniformization theorem. It was proved by Heinloth [He, Theorem 4, Proposition 4, and Theorem 7] (and conjectured by Pappas-Rapoport [PR2]) for $\mathcal{Bun}_q$ (in fact, he proved a more general result). Its extension to $\mathcal{Parbun}_q$ follows by the same proof. (Observe that $\mathcal{G}$ is simply-connected by the proof of Lemma 9.6.)

**Theorem 11.3.** Take any $q_i \in \pi^{-1}(p_i)$, $q \in \Sigma \setminus \pi^{-1}\{p_1, \cdots, p_s\}$ and any parabolic type $\vec{\mathcal{P}}$ at the points $\vec{\mathcal{P}}$. Then, as stacks,

\begin{equation}
\mathcal{Parbun}_q(\vec{\mathcal{P}}) \cong \left[ G(\Sigma \setminus \Gamma \cdot q)^\Gamma \left\{ X^q \times \prod_{i=1}^s (G^{\Gamma_{q_i}/P_i^{\vec{\mathcal{P}}}}) \right\} \right],
\end{equation}

where $G(\Sigma \setminus \Gamma \cdot q)^\Gamma$ acts on $X^q$ via its restriction to $D_q^\times$ and it acts on $G^{\Gamma_{q_i}/P_i^{\vec{\mathcal{P}}}}$ via its evaluation at $q_i$. Here $[G(\Sigma \setminus \Gamma \cdot q)^\Gamma \left\{ X^q \times \prod_{i=1}^s (G^{\Gamma_{q_i}/P_i^{\vec{\mathcal{P}}}}) \right\}]$ denotes the quotient stack obtained by taking the quotient of the projective ind-variety $X^q \times \prod_{i=1}^s (G^{\Gamma_{q_i}/P_i^{\vec{\mathcal{P}}}})$ by the ind-group $G(\Sigma \setminus \Gamma \cdot q)^\Gamma$.

Moreover, the projection $X^q \times \prod_{i=1}^s (G^{\Gamma_{q_i}/P_i^{\vec{\mathcal{P}}}}) \to \mathcal{Parbun}_q(\vec{\mathcal{P}})$ is locally trivial in the smooth topology.

**Remark 11.4.** Even though we will not use, there is also an isomorphism of stacks:

\[ \mathcal{Parbun}_q(\vec{\mathcal{P}}) \cong \left[ G(\Sigma \setminus \vec{\mathcal{q}})^\Gamma \left\{ \prod_{i=1}^s (X^{\vec{\mathcal{q}}}(P_i^{\vec{\mathcal{P}}})) \right\} \right], \]

where $\Gamma \cdot \vec{\mathcal{q}} := \bigcup_{i=1}^s \Gamma \cdot q_i$ and $X^{\vec{\mathcal{q}}}(P_i^{\vec{\mathcal{P}}})$ is the partial twisted affine flag variety which is by definition $G(D_{\vec{\mathcal{q}}})^{\Gamma_{\vec{\mathcal{q}}}} / \mathcal{P}_i$ and $\mathcal{P}_i$ is the inverse image of $P_i^{\vec{\mathcal{P}}}$ under the surjective evaluation map $G(D_{\vec{\mathcal{q}}})^{\Gamma_{\vec{\mathcal{q}}}} \to G^{\Gamma_{\vec{\mathcal{q}}}}$. 
Theorem 10.7 (2), take the canonical splitting. This provides a line bundle $L$ on the line bundle $L_c$ corresponding to the central charge. Observe that the canonical splitting of $G a \times G v$ rem 10.7 (1): where $v$ Consider the morphism Let us consider the following canonical homomorphism: Moreover, from the see-saw principle, since $X^q$ is ind-projective and Pic of each factor is discrete, Let us consider the following canonical homomorphism: Consider the morphism \[\hat{G}^q_c \rightarrow \mathcal{H} \setminus \{0\}, \quad g \mapsto gv^+,\] where $v^+$ is a highest weight vector of $\mathcal{H}$. This factors through a morphism (via Theorem 10.7 (1)): \[X^q = \hat{G}^q_c / (G(\mathbb{D}_q)^{\Gamma z} \times \mathbb{C}^\times) \rightarrow \mathbb{F} (\mathcal{H}).\]

Pulling back the dual of the tautological line bundle on $\mathbb{F} (\mathcal{H})$, we get a $\hat{G}^q_c$-equivariant line bundle $L^q_c$ on $X^q$ given by the character \[G(\mathbb{D}_q)^{\Gamma z} \times \mathbb{C}^\times \rightarrow \mathbb{C}^\times, \quad (g, z) \mapsto z.\] Observe that the canonical splitting of $G(\mathbb{D}_q)^{\Gamma z}$ is taken for the central extension $\hat{G}^q_c$ corresponding to the central charge $c$.

Now, if $c$ is a multiple of $|\Gamma|$, the central extension $\hat{G}^q_c \rightarrow G(\mathbb{D}_q)^{\Gamma z}$ splits over $\Xi$. As in Theorem 10.7 (2), take the canonical splitting. This provides a $\Xi$-equivariant structure on the line bundle $L^q_c$ over $X^q$.

Similarly, for any $\lambda_i \in D_{c,q}$, the $G^{\Gamma q}$-module $V(\lambda_i)$ with highest weight $\lambda_i$ integrates to a $G^{\Gamma q}$-module $V(\lambda_i)$ if $|\Gamma|$ divides $c$ (cf. Proposition 10.9). Take the highest weight vector $v^+ \in \mathcal{H} (\lambda_i)$ which is an (irreducible) integrable highest weight $\hat{L}(\mathbb{C}, \Gamma q) = \hat{g}_q$-module with highest weight $\lambda_i$ and central charge $c$. Then, $V(\lambda_i)$ is the $G^{\Gamma q}$-submodule of $\mathcal{H} (\lambda_i)$.
generally by \( \nu_+ \). Let \( P_i^\nu \) be the parabolic subgroup of \( G^\Gamma \), which stabilizes the line \( \mathbb{C} \nu_+ \).

Define the \( G^\Gamma \)-equivariant ample line bundle

\[
\mathcal{L}^\mu(\lambda_i) := G^\Gamma \times_{P_i^\nu(\mathbb{C} \nu_+)} (\mathcal{O} \nu_+)^\vee \to G^\Gamma / P_i^\nu.
\]

Then, \( \mathcal{L}^\mu(\lambda_i) \) is \( \Xi \)-equivariant line bundle by virtue of the following evaluation map at \( q_i \):

\[
e_i : \Xi := G(\Sigma \cdot q)^\Gamma \to G^\Gamma.
\]

Thus, we obtain the \( \Xi \)-equivariant line bundle

\[
\mathcal{L}_\mathcal{F}^\mu \otimes \mathcal{L}^\mu(\lambda_1) \otimes \cdots \otimes \mathcal{L}^\mu(\lambda_s)
\]

over \( X^\mu \times \prod_{i=1}^s (G^\Gamma / P_i^\nu), \) for any \( c \) divisible by \( |\Gamma| \) and \( \lambda_i \in D_{c,q_i} \).

Thus, under the isomorphism (139), we get the corresponding line bundle \( \mathcal{L}(c; \mathcal{F}) \) over the stack \( \mathcal{P}arbun_{q}(\mathcal{F}) \), where \( \mathcal{F} = (P_1^\nu, \ldots, P_s^\nu) \) and \( P_i^\nu \) is the stabilizer in \( G^\Gamma \) of the line \( \mathbb{C} \cdot \nu_+ \subset \mathcal{H}(\lambda_i) \).

12. Identification of Twisted Conformal Blocks with the Space of Global Sections of Line Bundles on Moduli Stack

In this final section, we establish the identification of twisted conformal blocks and generalized theta functions on the moduli stack \( \mathcal{P}arbun_{q} \).

We continue to have the same assumptions on \( G, \Gamma, \Sigma \) and \( \phi : \Gamma \to \text{Aut}(q) \) as in the beginning of Section 9. Let \( \mathcal{F} = (q_1, \ldots, q_s) \) (\( s \geq 1 \)) be marked points on \( \Sigma \) with distinct \( \Gamma \)-orbits and let \( \mathcal{F} = (\lambda_1, \ldots, \lambda_s) \) be weights with \( \lambda_i \in D_{c,q_i} \) attached to the points \( q_i \). Let \( P_i^\nu \) be the stabilizer of the line \( \mathbb{C} \cdot \nu_+ \subset \mathcal{H}(\lambda_i) \) in \( G^\Gamma \).

Recall the definition of the moduli stack \( \mathcal{P}arbun_{q}(\mathcal{F}) \) of quasi-parabolic \( \mathcal{G} \)-torsors over \( (\Sigma, \mathcal{F}) \) of type \( \mathcal{F} = (P_1^\nu, \ldots, P_s^\nu) \) from Definition 11.6, where \( \mathcal{F} = (\pi(q_1), \ldots, \pi(q_s)) \).

Also, recall from Definition 11.6 the definition of the line bundle \( \mathcal{L}^\mu(\mathcal{F}) \) over \( \mathcal{F} \) for any \( c \) such that \( 0 \in D_{c,q_i} \), and any \( q \in \Sigma \setminus \bigcup_{i=1}^s \Gamma \cdot q_i \) and the definition of the ample homogeneous line bundle \( \mathcal{L}_\mathcal{F}^\mu(\lambda_i) \) over the flag variety \( G^\Gamma / P_i^\nu \). When \( |\Gamma| \) divides \( c \), these line bundles give rise to a line bundle \( \mathcal{L}(c; \mathcal{F}) \) over the stack \( \mathcal{P}arbun_{q}(\mathcal{F}) \) (cf. Definition 11.6).

The following result confirms a conjecture by Pappas-Rapoport [PR2, Conjecture 3.7] in the case of the parahoric Bruhat-Tits group schemes considered in our paper.

**Theorem 12.1.** Assume that \( |\Gamma| \) divides \( c \) and \( \Gamma \) stabilizes a Borel subgroup of \( G \). Then, there is a canonical isomorphism:

\[
H^0(\mathcal{P}arbun_{q}(\mathcal{F}), \mathcal{L}(c, \mathcal{F})) \simeq \mathcal{V}_{\Sigma, \Gamma, q}(\mathcal{F}, \mathcal{F}),
\]

where \( \mathcal{V}_{\Sigma, \Gamma, q}(\mathcal{F}, \mathcal{F}) \) is the space of (twisted) vacua (cf. Identity (17)).

**Proof.** From the uniformization theorem (Theorem 11.3), there is an isomorphism of stacks:

\[
\mathcal{P}arbun_{q}(\mathcal{F}) \simeq [G(\Sigma \cdot q)^\Gamma / \bigg\{ X^q \times \prod_{i=1}^s (G^\Gamma / P_i^\nu) \bigg\}],
\]
Moreover, by definition, the line bundle \( \mathcal{L}(c, \tilde{\lambda}) \) over \( \mathcal{P}arbun_{\mathbf{g}}(\tilde{P}) \) is the descent of the line bundle

\[
\mathcal{O}^q \boxtimes \mathcal{L}^{q_i}((\lambda_1)) \boxtimes \cdots \boxtimes \mathcal{L}^{q_i}((\lambda_s))
\]

over \( X^q \times \prod_{i=1}^{s}(G^T_{\mathbf{g}}/P_i^q) \) (Definition 11.6). Thus, we have the following isomorphisms:

\[
H^0(\mathcal{P}arbun_{\mathbf{g}}(\tilde{P}), \mathcal{L}(c, \tilde{\lambda})) \cong H^0(X^q \times \prod_{i=1}^{s}(G^T_{\mathbf{g}}/P_i^q), \mathcal{O}^q \boxtimes \mathcal{L}^{q_i}((\lambda_1)) \boxtimes \cdots \boxtimes \mathcal{L}^{q_i}((\lambda_s))^{G(\Sigma \backslash G)^F})
\]

\[
\cong (\mathcal{O}^c \otimes V(\lambda_1)^* \otimes \cdots \otimes V(\lambda_s)^*)^{G(\Sigma \backslash G)^F}
\]

\[
\cong (\mathcal{O}^c \otimes V(\lambda_1)^* \otimes \cdots \otimes V(\lambda_s)^*)^{\eta(\Sigma \backslash G)^F}
\]

\[
\cong \mathcal{Y}_{\Sigma, \Gamma, \mathbf{g}}(\tilde{P}, \tilde{\lambda})^\dagger
\]

where the first isomorphism follows from [BL, Lemma 7.2]; the second isomorphism follows from the standard Borel-Weil theorem and its generalization for the Kac-Moody case due to Kumar as well as Mathieu [Ku, Corollary 8.3.12]; the third isomorphism follows from [BL, Proposition 7.4] since \( G(\Sigma \backslash G)^F \) is reduced and irreducible (Corollary 11.5) and \( X^q \) is reduced and irreducible by Corollary 9.8; and the last isomorphism follows from Propagation of Vacua (Corollary 4.5 (b)). It finishes the proof of the theorem. \( \square \)

**Remark 12.2.** (a) If we drop the assumption that

\[ \Gamma \text{ stabilizes a Borel subgroup of } G, \]

we still have the isomorphism:

\[ H^0(\mathcal{P}arbun_{\mathbf{g}}(\tilde{P}), \mathcal{L}(c, \tilde{\lambda})) \cong (\mathcal{O}^c \otimes V(\lambda_1)^* \otimes \cdots \otimes V(\lambda_s)^*)^{\eta(\Sigma \backslash G)^F} \]

since Theorem 11.3 remains valid without the assumption (*). Since our Propagation Theorem (Corollary 4.5 (a)) requires the assumption (*), the space on the right side of the equation (140) is not known to be isomorphic with \( \mathcal{Y}_{\Sigma, \Gamma, \mathbf{g}}(\tilde{P}, \tilde{\lambda})^\dagger \) in general.

(b) The condition \(|\Gamma| \text{ divides } c^*\) can not, in general, be dropped since for \( \lambda_i \in D_{\mathbf{g}^q} \), to be a dominant integral weight of \( g_\mathbf{g}^q \), imposes some divisibility condition on \( c \) with respect to \( \Gamma_q \) (cf. Lemma 2.1 and Proposition 10.9).

Also, Heinloth’s example [He, Remark 19 (4)] shows that the line bundle

\[
\mathcal{O}^q \boxtimes \mathcal{L}^{q_i}((\lambda_1)) \boxtimes \cdots \boxtimes \mathcal{L}^{q_i}((\lambda_s))
\]

does not, in general, descend to the moduli stack \( \mathcal{P}arbun_{\mathbf{g}}(\tilde{P}) \) for an arbitrary \( c \).

**References**


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