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Representation ring of Levi subgroups versus cohomology ring of flag varieties

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Abstract Recall the classical result that the cup product structure constants for the singular cohomology with integral coefficients $H^*(\mathrm{Gr}(r,n))$ of the Grassmannian of r-planes coincide with the Littlewood-Richardson tensor product structure constants for GL_r. Specifically, the result asserts that there is an explicit surjective ring homomorphism $\xi : \mathrm{Rep}_{\mathrm{poly}}(\mathrm{GL}_r) \to H^*(\mathrm{Gr}(r,n))$, where $\mathrm{Gr}(r,n)$ denotes the Grassmannian of r-planes in $\mathbb{C}^n$ and $\mathrm{Rep}_{\mathrm{poly}}(\mathrm{GL}_r)$ denotes the polynomial representation ring of GL_r. This work seeks to achieve one possible generalization of this classical result for GL_r and the Grassmannian $\mathrm{Gr}(r,n)$ to the Levi subgroups of any reductive group G and the corresponding flag varieties.

1 Introduction

Let us begin by recalling the classical result that the cup product structure constants for the singular cohomology with integral coefficients H^* of the Grassmannian of r-planes coincide with the Littlewood–Richardson tensor product structure constants for GL_r. Specifically, the result asserts that there is a $\mathbb{Z}$-algebra homomorphism ϕ : $\mathrm{Rep}_{\mathrm{poly}}(\mathrm{GL}_r) \to H^*(\mathrm{Gr}(r,n))$, where $\mathrm{Gr}(r,n)$ denotes the Grassmannian of r-planes in $\mathbb{C}^n$, $\mathrm{Rep}_{\mathrm{poly}}(\mathrm{GL}_r)$ denotes the polynomial representation ring of GL_r and ϕ takes the irreducible polynomial representation $V(\lambda)$ of GL_r corresponding to the partition $\lambda : \lambda_1 \geq \cdots \geq \lambda_r \geq 0$ to the Schubert class $\epsilon_{v_A(\lambda)}$ corresponding to the same partition λ if $\lambda_1 \leq n-r$, where $v_A(\lambda)$ is defined in Sect. 5. If $\lambda_1 > n-r$, then $\phi(V(\lambda)) = 0$.

This work seeks to achieve one possible generalization of this classical result for GL_r *and the Grassmannian* $\mathrm{Gr}(r,n)$ *to the Levi subgroups of any reductive group* G

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and the corresponding flag varieties. We also refer to the work of Tamvakis [9] for another generalization.

Let G be a connected reductive group over $\mathbb{C}$ with a Borel subgroup B and maximal torus $T \subset B$. Let P be a standard parabolic subgroup with the Levi subgroup L containing T. Let W (resp. W_L) be the Weyl group of G (resp. L). Let $V(\lambda)$ be an irreducible almost faithful representation of G with highest weight λ [i.e., the corresponding map $\rho_\lambda : G \to \mathrm{Aut}(V(\lambda))$ has finite kernel]. Then, Springer defined an adjoint-equivariant regular map with Zariski dense image $\theta_\lambda : G \to \mathfrak{g}$ (depending upon λ) (cf. Sect. 3). By Lemma 2, θ_λ takes the maximal torus T to its Lie algebra $\mathfrak{t}$. This induces a $\mathbb{C}$-algebra homomorphism $(\theta_{\lambda|T})^* : \mathbb{C}[\mathfrak{t}] \to \mathbb{C}[T]$ between the corresponding affine coordinate rings. Since θ_λ is equivariant under the adjoint actions, $(\theta_{\lambda|T})^*$ takes $\mathbb{C}[\mathfrak{t}]^{W_L} = S(\mathfrak{t}^*)^{W_L}$ to $\mathbb{C}[T]^{W_L}$. Moreover, $(\theta_{\lambda|T})^*$ is injective. Let $\mathrm{Rep}^{\mathbb{C}}(L)$ be the complexified representation ring of the Levi subgroup L. As it is well known,

$$\mathrm{Rep}^{\mathbb{C}}(L) \simeq \mathbb{C}[T]^{W_L}$$

induced from the restriction of the character to T. We call the image of $\mathbb{C}[\mathfrak{t}]^{W_L}$, under $(\theta_{\lambda|T})^*$, the *λ-polynomial subring* $\mathrm{Rep}^{\mathbb{C}}_{\lambda\text{-poly}}(L)$ of $\mathrm{Rep}^{\mathbb{C}}(L)$ (cf. Definition 4).

For $G = \mathrm{GL}_n$ and $V(\lambda)$ the defining representation $\mathbb{C}^n$, the ring $\mathrm{Rep}_{\lambda\text{-poly}}(G) := \mathrm{Rep}^{\mathbb{C}}_{\lambda\text{-poly}}(G) \cap \mathrm{Rep}(G)$ coincides with the standard notion of polynomial representation ring of GL_n (cf. Sect. 5).

The Borel homomorphism $\beta : S(\mathfrak{t}^*) \to H^*(G/B, \mathbb{C})$ (which is surjective) from the symmetric algebra of $\mathfrak{t}^*$ restricticted to the W_L-invariants gives a surjective $\mathbb{C}$-algebra homomorphism $\beta^P : S(\mathfrak{t}^*)^{W_L} \to H^*(G/P, \mathbb{C})$. Thus, we get a surjective $\mathbb{C}$-algebra homomorphism $\xi^P_\lambda : \mathrm{Rep}^{\mathbb{C}}_{\lambda\text{-poly}}(L) \to H^*(G/P, \mathbb{C})$ (cf. Theorem 5), which is our main result.

Specializing the above result to the case when $G = \mathrm{GL}_n$, λ is the first fundamental weight [so that $V(\lambda)$ is the standard defining representation $\mathbb{C}^n$] and $P = P_r$ (for any $1 \le r \le n-1$) is the maximal parabolic subgroup so that the flag variety G/P_r is the Grassmannian $\mathrm{Gr}(r, n)$, we recover the above classical result as shown in Sect. 5 (cf. Theorem 8).

We determine the λ-polynomial representation ring $\mathrm{Rep}^{\mathbb{C}}_{\lambda\text{-poly}}(G)$, for λ the first fundamental weight ω_1 [i.e., $V(\lambda)$ is the defining representation] of the classical groups: SO_n, Sp_{2n} in Sect. 6 (cf. Proposition 12). In this case, the Springer morphism coincides with the classical Cayley transform. Recall that the defining representations of the classical groups have minimum Dynkin index (cf. [8, §4]). We believe that for the exceptional groups as well, the irreducible representation $V(\lambda)$ with minimum Dynkin index might be most 'appropriate' to consider the Springer morphism. Recall (loc. cit.) that for the exceptional groups: G_2, F_4, E_6, E_7, E_8, the representation $V(\lambda)$ has minimum Dynkin index for $\lambda = \omega_1, \omega_4, \omega_1$ (and ω_6), ω_7, ω_8 respectively.

We partially determine the homomorphism $\xi^P_\lambda : \mathrm{Rep}^{\mathbb{C}}_{\lambda\text{-poly}}(L) \to H^*(G/P, \mathbb{C})$ (with respect to the defining representation: $\lambda = \omega_1$) for all the maximal parabolic subgroups P in the classical groups Sp_{2n}, SO_{2n+1} and SO_{2n} in Sects. 7, 8 and 9 (cf. Propositions 19, 20 and 21 respectively).

We determine the homomorphism $\xi^B_{\omega_1} : \mathrm{Rep}^{\mathbb{C}}_{\omega_1\text{-poly}}(T) \to H^*(G/B, \mathbb{C})$ for the Borel subgroups B in the classical groups Sp_{2n}, SO_{2n+1} and SO_{2n} in Sect. 10 (cf. Proposition 24). (In this case, T is of course the Levi subgroup of B.)

2 Notation

Let G be a connected reductive group over $\mathbb{C}$ with a Borel subgroup B and maximal torus $T \subset B$. Let P be a standard parabolic subgroup with the Levi subgroup L containing T. We denote their Lie algebras by the corresponding Gothic characters: $\mathfrak{g}, \mathfrak{b}, \mathfrak{t}, \mathfrak{p}, \mathfrak{l}$ respectively. We denote by $\Delta = \{\alpha_1, \ldots, \alpha_\ell\} \subset \mathfrak{t}^*$ the set of simple roots. The fundamental weights of $\mathfrak{g}$ are denoted by $\{\omega_1, \ldots, \omega_\ell\} \subset \mathfrak{t}^*$. Let W (resp. W_L) be the Weyl group of G (resp. L). Then, W is generated by the simple reflections $\{s_i\}_{1 \le i \le \ell}$. Let W^P denote the set of smallest coset representatives in the cosets in W/W_L. *Throughout the paper we follow the indexing convention as in* [4, Planche I–IX]. Let $X(T)$ be the group of characters of T and let $D \subset X(T)$ be the set of dominant characters (with respect to the given choice of B and hence positive roots, which are the roots of $\mathfrak{b}$). Then, the isomorphism classes of finite dimensional irreducible representations of G are bijectively parameterized by D under the correspondence $\lambda \in D \rightsquigarrow V(\lambda)$, where $V(\lambda)$ is the irreducible representation of G with highest weight λ. We call $V(\lambda)$ *almost faithful* if the corresponding map $\rho_\lambda : G \to \mathrm{Aut}(V(\lambda))$ has finite kernel.

Recall the Bruhat decomposition for the flag variety:

$$G/P = \sqcup_{w \in W^P} \Lambda^P_w, \quad \text{where } \Lambda^P_w := BwP/P.$$

Let $\bar{\Lambda}^P_w$ denote the closure of Λ^P_w in G/P. We denote by $[\bar{\Lambda}^P_w] \in H_{2\ell(w)}(G/P, \mathbb{Z})$ its fundamental class. Let $\{\epsilon^P_w\}_{w \in W^P}$ denote the Kronecker dual basis of the cohomology, i.e.,

$$\epsilon^P_w([\bar{\Lambda}^P_v]) = \delta_{w,v}, \quad \text{for any } v, w \in W^P.$$

Thus, ϵ^P_w belongs to the singular cohomology:

$$\epsilon^P_w \in H^{2\ell(w)}(G/P, \mathbb{Z}).$$

3 Springer morphism

Definition 1 Let $V(\lambda)$ be any almost faithful irreducible representation of G. Following Springer (cf. [2, §9]), define the map

$$\theta_\lambda : G \to \mathfrak{g} \quad \text{(depending upon } \lambda)$$

as follows:

$$\begin{array}{ccc} G & \xrightarrow{\rho_\lambda} & \mathrm{Aut}(V(\lambda)) \subset \mathrm{End}(V(\lambda)) = \mathfrak{g} \oplus \mathfrak{g}^\perp \\ & \searrow^{\theta_\lambda} & \downarrow \pi \\ & & \mathfrak{g} \end{array}$$

where $\mathfrak{g}$ sits canonically inside $\mathrm{End}(V(\lambda))$ via the derivative $d\rho_\lambda$, the orthogonal complement $\mathfrak{g}^\perp$ is taken with respect to the standard conjugate $\mathrm{Aut}(V(\lambda))$-invariant form on $\mathrm{End}(V(\lambda))$: $\langle A, B\rangle := \mathrm{tr}(AB)$, and π is the projection to the $\mathfrak{g}$-factor. (By considering a compact form K of G, it is easy to see that $\mathfrak{g} \cap \mathfrak{g}^\perp = \{0\}$.)

Since $\pi \circ d\rho_\lambda$ is the identity map, θ_λ is a local diffeomorphism at 1 (and hence with Zariski dense image). Of course, by construction, θ_λ is an algebraic morphism. Moreover, since the decomposition $\mathrm{End}(V(\lambda)) = \mathfrak{g} \oplus \mathfrak{g}^\perp$ is G-stable, it is easy to see that θ_λ is G-equivariant under conjugation.

Lemma 2 *The above morphism restricts to* $\theta_{\lambda|T} : T \to \mathfrak{t}$.

Proof Take any $t \in T$ and write

$$\theta_\lambda(t) = h + \sum_{\alpha \in R} x_\alpha, \quad \text{for } h \in \mathfrak{t}, \quad \text{and } x_\alpha \in \mathfrak{g}_\alpha,$$

where R is the set of all the roots and $\mathfrak{g}_\alpha$ is the root space of $\mathfrak{g}$ corresponding to the root α. Now, since θ_λ is conjugation invariant, we get that

$$\theta_\lambda(t) = h + \sum_{\alpha \in R} (\mathrm{Ad}\, s) \cdot x_\alpha, \quad \text{for any } \ s \in T.$$

From this we see that each $x_\alpha = 0$, i.e., $\theta_\lambda(t) \in \mathfrak{t}$, proving the lemma. □

Example 3 The Springer morphism $\theta_\lambda : G \to \mathfrak{g}$, in general, indeed depends upon the choice of λ. For example, the Springer morphism $\theta_{\omega_1} : \mathrm{SL}_2 \to \mathfrak{s}l_2$ restricted to the diagonal torus can easily seen to be

$$\theta_{\omega_1} \begin{pmatrix} z & 0 \\ 0 & z^{-1} \end{pmatrix} = \begin{pmatrix} \frac{z - z^{-1}}{2} & 0 \\ 0 & -\frac{z - z^{-1}}{2} \end{pmatrix}.$$

On the other hand, the Springer morphism $\theta_{2\omega_1} : \mathrm{SL}_2 \to \mathfrak{s}l_2$ restricted to the diagonal torus is given by

$$\theta_{2\omega_1} \begin{pmatrix} z & 0 \\ 0 & z^{-1} \end{pmatrix} = \begin{pmatrix} \frac{z^2 - z^{-2}}{2} & 0 \\ 0 & -\frac{z^2 - z^{-2}}{2} \end{pmatrix}.$$

4 Main result

We follow the notation and assumptions as in Sect. 2. In particular, G is a connected reductive group and P a standard parabolic subgroup with Levi subgroup L containing the chosen maximal torus T. Take an almost faithful irreducible G-module $V(\lambda)$. As in Sect. 3, this gives rise to the Springer morphism $\theta_\lambda : G \to \mathfrak{g}$, which restricts to $\theta_{\lambda|T} : T \to \mathfrak{t}$ (cf. Lemma 2).

For any $\mu \in X(T)$, we have a G-equivariant line bundle $\mathcal{L}(\mu)$ on G/B associated to the principal B-bundle $G \to G/B$ via the one dimensional B-module μ^{-1}. (Any $\mu \in X(T)$ extends uniquely to a character of B.) The one dimensional B-module μ is also denoted by $\mathbb{C}_\mu$. Recall the surjective Borel homomorphism

$$\beta : S(\mathfrak{t}^*) \to H^*(G/B, \mathbb{C}),$$

which takes a character $\mu \in X(T)$ to the first Chern class of the line bundle $\mathcal{L}(\mu)$. [We realize $X(T)$ as a lattice in $\mathfrak{t}^*$ via taking derivative.] We then extend this map linearly over $\mathbb{C}$ to $\mathfrak{t}^*$ and extend further as a graded algebra homomorphism from $S(\mathfrak{t}^*)$ (doubling the degree). Under the Borel homomorphism, as, e.g., in [7, Exercise 11.3.E.1],

$$\beta(\omega_i) = \epsilon^B_{s_i}, \quad \text{for any fundamental weight } \omega_i. \tag{1}$$

Fix a compact form K of G. In particular, $T_o := K \cap T$ is a (compact) maximal torus of K. Then, $W \simeq N(T_o)/T_o$, where $N(T_o)$ is the normalizer of T_o in K. Recall that β is W-equivariant under the standard action of W on $S(\mathfrak{t}^*)$ and the W-action on $H^*(G/B, \mathbb{C})$ induced from the W-action on $G/B \simeq K/T_o$ via

$$(nT_o) \cdot (kT_o) := kn^{-1}T_o, \quad \text{for } n \in N(T_o) \quad \text{and } k \in K.$$

Thus, for any standard parabolic subgroup P with the Levi subgroup L containing T, restricting β, we get a surjective graded algebra homorphism:

$$\beta^P : S(\mathfrak{t}^*)^{W_L} \to H^*(G/B, \mathbb{C})^{W_L} \simeq H^*(G/P, \mathbb{C}),$$

where the last isomorphism, which is induced from the projection $G/B \to G/P$, can be found, e.g., in [7, Corollary 11.3.14].

Now, the Springer morphism $\theta_{\lambda|T} : T \to \mathfrak{t}$ (restricted to T) gives rise to the corresponding W-equivariant injective algebra homomorphism on the affine coordinate rings:

$$(\theta_{\lambda|T})^* : \mathbb{C}[\mathfrak{t}] = S(\mathfrak{t}^*) \to \mathbb{C}[T].$$

Thus, on restriction to W_L-invariants, we get an injective algebra homomorphism

$$\theta_\lambda(P)^* : \mathbb{C}[\mathfrak{t}]^{W_L} = S(\mathfrak{t}^*)^{W_L} \to \mathbb{C}[T]^{W_L}.$$

[Since W_L-invariants depend upon the choice of the parabolic subgroup P, we have included P in the notation of $\theta_\lambda(P)^*$.] Now, let $\mathrm{Rep}(L)$ be the representation ring of L

and let $\mathrm{Rep}^{\mathbb{C}}(L) := \mathrm{Rep}(L) \otimes_{\mathbb{Z}} \mathbb{C}$ be its complexification. Then, as it is well known,

$$\mathrm{Rep}^{\mathbb{C}}(L) \simeq \mathbb{C}[T]^{W_L} \tag{2}$$

obtained from taking the character of an L-module restricted to T.

A representation V of L, thought of as an element of $\mathrm{Rep}(L)$, is denoted by $[V]$. *We will often identify a virtual representation of L with its character restricted to T (which is automatically W_L-invariant).*

Definition 4 We will call a virtual character $\chi \in \mathrm{Rep}^{\mathbb{C}}(L)$ of L a *λ-polynomial character* if the corresponding function in $\mathbb{C}[T]^{W_L}$ is in the image of $\theta_\lambda(P)^*$. The set of all λ-polynomial characters of L, which is, by definition, a subalgebra of $\mathrm{Rep}^{\mathbb{C}}(L)$ isomorphic to the algebra $S(\mathfrak{t}^*)^{W_L}$, is denoted by $\mathrm{Rep}^{\mathbb{C}}_{\lambda\text{-}\mathrm{poly}}(L)$. Of course, the map $\theta_\lambda(P)^*$ induces an algebra isomorphism (still denoted by)

$$\theta_\lambda(P)^* : S\left(\mathfrak{t}^*\right)^{W_L} \simeq \mathrm{Rep}^{\mathbb{C}}_{\lambda\text{-}\mathrm{poly}}(L),$$

under the identification (2).

Putting together the above identifications, we get the following main result of this note.

Theorem 5 *Let $V(\lambda)$ be an almost faithful irreducible G-module and let P be any standard parabolic subgroup. Then, the above maps (specifically $\beta^P \circ (\theta_\lambda(P)^*)^{-1}$) give rise to a surjective $\mathbb{C}$-algebra homomorphism*

$$\xi^P_\lambda : \mathrm{Rep}^{\mathbb{C}}_{\lambda\text{-}\mathrm{poly}}(L) \to H^*(G/P, \mathbb{C}).$$

Moreover, let Q be another standard parabolic subgroup with Levi subgroup R containing T such that $P \subset Q$ (and hence $L \subset R$). Then, we have the following commutative diagram:

$$\begin{array}{ccc} \mathrm{Rep}^{\mathbb{C}}_{\lambda\text{-}\mathrm{poly}}(R) & \xrightarrow{\xi^Q_\lambda} & H^*(G/Q, \mathbb{C}) \\ \downarrow{\scriptstyle \gamma} & & \downarrow{\scriptstyle \pi^*} \\ \mathrm{Rep}^{\mathbb{C}}_{\lambda\text{-}\mathrm{poly}}(L) & \xrightarrow{\xi^P_\lambda} & H^*(G/P, \mathbb{C}), \end{array}$$

where π^* is induced from the standard projection $\pi : G/P \to G/Q$ and γ is induced from the restriction of representations.

Example 6 The subalgebra $\mathrm{Rep}^{\mathbb{C}}_{\lambda\text{-}\mathrm{poly}}(G) \subset \mathrm{Rep}^{\mathbb{C}}(G)$, in general, indeed depends upon the choice of λ. For example, for $G = \mathrm{SL}_2$, following Example 3,

$$\mathrm{Rep}^{\mathbb{C}}_{\omega_1\text{-}\mathrm{poly}}(\mathrm{SL}_2) = \mathbb{C}[(z - z^{-1})^2],$$

whereas

$$\mathrm{Rep}^{\mathbb{C}}_{2\omega_1\text{-poly}}(\mathrm{SL}_2) = \mathbb{C}[(z^2 - z^{-2})^2],$$

for the maximal torus in SL_2 given by

$$T = \left\{ \begin{pmatrix} z & 0 \\ 0 & z^{-1} \end{pmatrix} : z \in \mathbb{C}^* \right\}.$$

5 Specialization of Theorem 5 to the case of $G = \mathrm{GL}_n$

Theorem 5 specialized to $G = \mathrm{GL}_n$ and P the standard maximal parabolic subgroups gives the following classical result. As mentioned in the Introduction, our motivation behind this work was to seek a generalization of this classical result for an arbitrary reductive group G and *any* parabolic subgroup P.

For $G = \mathrm{GL}_n$, we take B to be the (standard) Borel subgroup consisting of invertible upper triangular matrices and T to be the subgroup consisting of invertible diagonal matrices. Then,

$$\mathfrak{t} = \{\bar{\mathbf{t}} = \mathrm{diag}(t_1, \ldots, t_n) : t_i \in \mathbb{C}\}.$$

The simple roots and simple coroots are given respectively by

$$\alpha_i(\bar{\mathbf{t}}) = t_i - t_{i+1} \quad \text{and} \quad \alpha_i^\vee = \mathrm{diag}(0, \ldots, 0, 1, -1, 0, \ldots, 0), \quad \text{for any } 1 \le i \le n-1,$$

where 1 is placed in the i-th place. We have the fundamental weights:

$$\omega_i(\bar{\mathbf{t}}) = t_1 + \cdots + t_i, \quad \text{for any } 1 \le i \le n.$$

The Weyl group W can be identified with the symmetric group S_n, which acts via the permutation of the coordinates of $\bar{\mathbf{t}}$. Let $\{s_1, \ldots, s_{n-1}\} \subset S_n$ be the (simple) reflections corresponding to the simple roots $\{\alpha_1, \ldots, \alpha_{n-1}\}$ respectively. Then,

$$s_i = (i, i+1).$$

For any $1 \le r \le n-1$, let $P_r \supset B$ be the (standard) maximal parabolic subgroup of GL_n such that its unique Levi subgroup L_r containing T has for its simple roots $\{\alpha_1, \ldots, \hat{\alpha}_r, \ldots, \alpha_{n-1}\}$. Then, GL_n/P_r can be identified with the Grassmannian $\mathrm{Gr}(r, n)$ of r-dimensional subspaces of $\mathbb{C}^n$. Moreover, the set of minimal coset representatives W^{P_r} of W/W_{L_r} can be identified with the set of r-tuples

$$S(r, n) = \{A := 1 \le a_1 < \cdots < a_r \le n\}.$$

Any such r-tuple A represents the permutation

$$v_A = (a_1, \ldots, a_r, a_{r+1}, \ldots, a_n), \;\; i \mapsto a_i,$$

where $\{a_{r+1} < \cdots < a_n\} = [n]\backslash\{a_1, \ldots, a_r\}$ and $[n] := \{1, \ldots, n\}$.

Recall that an irreducible representation of GL_r is called a *polynomial representation* if its character $\mathrm{GL}_r \to \mathbb{C}$ extends as a regular map $\mathfrak{gl}_r \to \mathbb{C}$, where $\mathfrak{gl}_r$ denotes the space of all $r \times r$-matrices over $\mathbb{C}$. Let $\mathrm{Rep}_{\mathrm{poly}}(\mathrm{GL}_r)$ denote the subring of the representation ring $\mathrm{Rep}(\mathrm{GL}_r)$ spanned by the (irreducible) polynomial representations of GL_r. [$\mathrm{Rep}_{\mathrm{poly}}(\mathrm{GL}_r)$ is indeed a subring [5, §8.3].] As in [5, Theorem 2, §8.2], the (irreducible) polynomial representations of GL_r are parameterized by the partitions $\lambda = (\lambda_1 \geq \cdots \geq \lambda_r \geq 0)$, where the corresponding irreducible representation $V(\lambda)$ has highest weight $\bar{\lambda}$,

$$\bar{\lambda}(\mathbf{t}) := t_1^{\lambda_1} \ldots t_r^{\lambda_r}, \quad \text{for any invertible diagonal matrix } \mathbf{t} = (t_1, \ldots, t_r).$$

Define the $\mathbb{Z}$-linear map

$$\xi : \mathrm{Rep}_{\mathrm{poly}}(\mathrm{GL}_r) \to H^*(\mathrm{Gr}(r,n), \mathbb{Z}), \ \ [V(\lambda)] \mapsto \epsilon^{P_r}_{v_{A(\lambda)}}, \ \text{ if } \lambda_1 \leq n-r,$$

and $\xi([V(\lambda)]) := 0$, if $\lambda_1 > n-r$, where $A(\lambda) := (1+\lambda_r < \cdots < r+\lambda_1)$. Now, we recall the following classical result (cf. [5, §9.4; Proposition 9 of §10.6; and the identity (1) of §10.2]).

Theorem 7 *The above map ξ is a surjective algebra homomorphism.*

Take $G = \mathrm{GL}_n$ and $\lambda = \omega_1$ [so that $V(\lambda)$ is the standard representation $\mathbb{C}^n$]. Then, clearly,

$$\theta_{\omega_1} : \mathrm{GL}_n \to \mathfrak{gl}_n \quad \text{is the canonical inclusion.} \tag{3}$$

Thus, in this case, it is easy to see that

$$\mathrm{Rep}_{\omega_1\text{-}\mathrm{poly}}(\mathrm{GL}_n) = \mathrm{Rep}_{\mathrm{poly}}(\mathrm{GL}_n), \tag{4}$$

where $\mathrm{Rep}_{\omega_1\text{-}\mathrm{poly}}(\mathrm{GL}_n) := \mathrm{Rep}^{\mathbb{C}}_{\omega_1\text{-}\mathrm{poly}}(\mathrm{GL}_n) \cap \mathrm{Rep}(\mathrm{GL}_n)$. For $1 \leq r \leq n-1$, the Levi subgroup L_r of P_r containing T is the subgroup $\mathrm{GL}_r \times \mathrm{GL}_{n-r}$ of GL_n. From the identity (4), it is easy to see that

$$\mathrm{Rep}^{\mathbb{C}}_{\omega_1\text{-}\mathrm{poly}}(L_r) \simeq \Big(\mathrm{Rep}_{\mathrm{poly}}(\mathrm{GL}_r) \otimes \mathrm{Rep}_{\mathrm{poly}}(\mathrm{GL}_{n-r})\Big) \otimes_{\mathbb{Z}} \mathbb{C}. \tag{5}$$

Following Theorem 5, we have the $\mathbb{C}$-algebra homomorphism:

$$\xi^{P_r}_{\omega_1} : \mathrm{Rep}^{\mathbb{C}}_{\omega_1\text{-}\mathrm{poly}}(L_r) \to H^*(\mathrm{Gr}(r,n), \mathbb{C}).$$

Of course, we have a ring homomorphism

$$i : \mathrm{Rep}_{\mathrm{poly}}(\mathrm{GL}_r) \to \mathrm{Rep}_{\mathrm{poly}}(\mathrm{GL}_r) \otimes \mathrm{Rep}_{\mathrm{poly}}(\mathrm{GL}_{n-r})$$

obtained from tensoring a GL_r-module with the trivial one dimensional GL_{n-r}-module. Then, we have the following result:

Theorem 8 *The* $\mathbb{C}$*-algebra homomorphism* $\xi_{\omega_1}^{P_r} : \mathrm{Rep}^{\mathbb{C}}_{\omega_1\text{-poly}}(L_r) \to H^*(\mathrm{Gr}(r,n),\mathbb{C})$ *restricted to* $\mathrm{Rep}_{\mathrm{poly}}(\mathrm{GL}_r)$ *via i (under the isomorphism (5)) coincides with the homomorphism* ξ *of Theorem 7.*

Proof Since $\xi_{\omega_1}^{P_r}$ and ξ are both algebra homomorphisms and the ring $\mathrm{Rep}_{\mathrm{poly}}(\mathrm{GL}_r)$ is generated by the fundamental representations $[V(\omega_1)],\ldots,[V(\omega_r)]$, it suffices to prove that

$$\xi_{\omega_1}^{P_r}([V(\omega_i)]) = \xi([V(\omega_i)]), \quad \text{for all } 1 \le i \le r. \tag{6}$$

Now, by definition,

$$\xi([V(\omega_i)]) = \epsilon^{P_r}_{s_{r-i+1}\ldots s_r}. \tag{7}$$

Moreover, from the definition of $\xi_{\omega_1}^{P_r}$, by (3),

$$\xi_{\omega_1}^{P_r}([V(\omega_i)]) = \beta\left(e_i(x_1,\ldots,x_r)\right), \tag{8}$$

where $x_k \in \mathfrak{t}^*$ is the linear form which takes $\bar{\mathbf{t}} = (t_1,\ldots,t_r,t_{r+1},\ldots,t_n) \in \mathfrak{t}$ to t_k, e_i is the i-th elementary symmetric function and β is the Borel homomorphism defined in Sect. 4. Now, by [5, Proposition 8, §10.6],

$$\beta\left(e_i(x_1,\ldots,x_r)\right) = \epsilon^{P_r}_{s_{r-i+1}\ldots s_r}. \tag{9}$$

Combining the Eqs. (7)–(9), we get the Eq. (6). This proves the theorem. □

Remark 9 Observe that our Theorem 5 in the case of $G = \mathrm{GL}_n$ and $\lambda = \omega_1$ extends the classical Theorem 7 for $H^*(\mathrm{Gr}(r,n))$ to the cohomology $H^*(\mathrm{GL}_n/B)$ of the full flag variety.

6 Determination of $\mathrm{Rep}_{\omega_1\text{-poly}}(L)$ for other classical groups

From now on we only consider the Springer morphism for classical groups with respect to the first fundamental weight $\lambda = \omega_1$*. So, we will abbreviate* θ_{ω_1} *by* θ *and* $\mathrm{Rep}^{\mathbb{C}}_{\omega_1\text{-poly}}(G)$ *by* $\mathrm{Rep}^{\mathbb{C}}_{\mathrm{poly}}(G)$*. In this case,* θ_{ω_1} *is the classical Cayley transform.*

We choose quadratic forms on $\mathbb{C}^{2n}$, $\mathbb{C}^{2n+1}$ (resp. alternating form on $\mathbb{C}^{2n}$) so that SO_{2n}, SO_{2n+1} (resp. Sp_{2n}) are given respectively by

$$\begin{aligned} \mathrm{SO}_{2n} &= \{g \in \mathrm{SL}_{2n} : (g^t)^{-1} = E_D g E_D^{-1}\} \\ \mathrm{SO}_{2n+1} &= \{g \in \mathrm{SL}_{2n+1} : (g^t)^{-1} = E_B g E_B^{-1}\} \\ \mathrm{Sp}_{2n} &= \{g \in \mathrm{SL}_{2n} : (g^t)^{-1} = E_C g E_C^{-1}\}, \end{aligned}$$

where E_D is the antidiagonal matrix with all its antidiagonal entries 1; E_B is the antidiagonal matrix with all its antidiagonal entries 1 except the $(n+1,n+1)$-th entry which is 2; E_C is the block matrix

$$E_C = \begin{pmatrix} 0 & -J_n \\ J_n & 0 \end{pmatrix},$$

where J_n is the antidiagonal $n \times n$ matrix with all its antidiagonal entries 1. (The suffix D, B, C refers to the types of the corresponding groups.)

Depending upon the case, denote E_D, E_B or E_C by the common symbol E. We recall the expression of the Springer morphism for these groups with respect to their standard representations $V(\omega_1)$, where ω_1 is the first fundamental weight.

Lemma 10 *The Springer morphism* $\theta : G \to \mathfrak{g}$ *for* $G = \mathrm{SO}_{2n}$, SO_{2n+1} *or* Sp_{2n} *is given by*

$$g \mapsto \frac{g - E^{-1} g^t E}{2}, \quad \textit{for } g \in G.$$

(Observe that this is the Cayley transform.)

Proof The lemma follows immediately since under the decomposition

$$\mathrm{End}(V(\omega_1)) = \mathfrak{g} \oplus \mathfrak{g}^{\perp},$$

any $A \in \mathrm{End}(V(\omega_1))$ decomposes as

$$A = \frac{(A - E^{-1} A^t E)}{2} + \frac{(A + E^{-1} A^t E)}{2}.$$

□

Take the maximal tori in Sp_{2n}, SO_{2n} and SO_{2n+1} respectively as follows:

$$T_C = T_D = \left\{ \mathbf{t} = \mathrm{diag}\big(t_1, \ldots, t_n, t_n^{-1}, \ldots, t_1^{-1}\big) : t_i \in \mathbb{C}^* \right\} \tag{10}$$

$$T_B = \left\{ \mathbf{t} = \mathrm{diag}\big(t_1, \ldots, t_n, 1, t_n^{-1}, \ldots, t_1^{-1}\big) : \ t_i \in \mathbb{C}^* \right\}. \tag{11}$$

Their Lie algebras are given respectively by

$$\mathfrak{t}_C = \mathfrak{t}_D = \big\{ \bar{\mathbf{t}} = \mathrm{diag}\big(x_1, \ldots, x_n, -x_n, \ldots, -x_1\big) : x_i \in \mathbb{C} \big\} \tag{12}$$

$$\mathfrak{t}_B = \big\{ \bar{\mathbf{t}} = \mathrm{diag}\big(x_1, \ldots, x_n, 0, -x_n, \ldots, -x_1\big) : \ x_i \in \mathbb{C} \big\}. \tag{13}$$

From the description of the Springer morphism given above, we immediately get the following:

Corollary 11 *Restricted to the maximal torus as above, we get the following description of the Springer map* θ*:*

(a) $G = \mathrm{SO}_{2n} : \theta(\mathbf{t}) = \mathrm{diag}\big(\frac{t_1 - t_1^{-1}}{2}, \ldots, \frac{t_n - t_n^{-1}}{2}, -(\frac{t_n - t_n^{-1}}{2}), \ldots, -(\frac{t_1 - t_1^{-1}}{2})\big)$

(b) $G = \mathrm{Sp}_{2n}$ *: Same as in the above case of* $G = \mathrm{SO}_{2n}$.

(c) $G = \mathrm{SO}_{2n+1} : \theta(\mathbf{t}) = \mathrm{diag}\big(\frac{t_1 - t_1^{-1}}{2}, \ldots, \frac{t_n - t_n^{-1}}{2}, 0, -(\frac{t_n - t_n^{-1}}{2}), \ldots, -(\frac{t_1 - t_1^{-1}}{2})\big)$.

The following result follows easily from Corollary 11 together with the description of the Weyl group.

Proposition 12 *Let $G = \mathrm{SO}_{2n}$, SO_{2n+1} or Sp_{2n} and let $f : T \to \mathbb{C}$ be a regular map. Then, $f \in \mathrm{Rep}^{\mathbb{C}}_{\mathrm{poly}}(G)$ if and only if the following is satisfied:*

(a) $G = \mathrm{Sp}_{2n}$ or SO_{2n+1}: there exists a symmetric polynomial $P_f(x_1, \ldots, x_n)$ such that

$$f(\mathbf{t}) = P_f\left(\left(\frac{t_1 - t_1^{-1}}{2}\right)^2, \ldots, \left(\frac{t_n - t_n^{-1}}{2}\right)^2\right),$$
$$\textit{for } \mathbf{t} \in T_C \textit{ given by } (10) \textit{ or } \mathbf{t} \in T_B \textit{ given by } (11).$$

(b) $G = \mathrm{SO}_{2n}$: There exist symmetric polynomials $P_f(x_1, \ldots, x_n)$ and $Q_f(x_1, \ldots, x_n)$ satisfying

$$f(\mathbf{t}) = P_f\left(\left(\frac{t_1 - t_1^{-1}}{2}\right)^2, \ldots, \left(\frac{t_n - t_n^{-1}}{2}\right)^2\right) +$$
$$\left(\left(\frac{t_1 - t_1^{-1}}{2}\right) \cdots \left(\frac{t_n - t_n^{-1}}{2}\right)\right) Q_f\left(\left(\frac{t_1 - t_1^{-1}}{2}\right)^2, \ldots, \left(\frac{t_n - t_n^{-1}}{2}\right)^2\right),$$
$$\textit{for } \mathbf{t} \in T_D \textit{ given by } (10).$$

We recall the definition of λ-rings due to Grothendieck from [1, §1].

Definition 13 A *special λ-ring* is, by definition, a commutative ring R with identity with a map

$$\lambda : R \to R[[q]], \quad x \mapsto \sum_{i \geq 0} \lambda^i(x) q^i,$$

which satisfies the following:

(1) $\lambda^0(x) = 1$
(2) $\lambda^1(x) = x$, for all $x \in R$
(3) $\lambda(x + y) = \lambda(x)\lambda(y)$, for all $x, y \in R$
(4) $\lambda(1) = 1 + q$, and
(5) There are universal (independent of R) polynomials P_k and $P_{k,l}$ over $\mathbb{Z}$ such that

$$\lambda^k(xy) = P_k\left(\lambda^1(x), \ldots, \lambda^k(x), \lambda^1(y), \ldots, \lambda^k(y)\right)$$
$$\text{and} \quad \lambda^k(\lambda^l x) = P_{k,l}\left(\lambda^1(x), \ldots, \lambda^{kl}(x)\right), \quad \text{for all } k, l \geq 1.$$

The following example can be found in [6, Exercise 23.39].

Example 14 For any reductive group G, the representation ring $\mathrm{Rep}(G)$ is a special λ-ring, where for any representation V,

$$\lambda([V]) := \sum_{i \geq 0} [\Lambda^i(V)] q^i$$

and extend it to all of Rep(G) by demanding the properly (3) of the definition of special λ-rings.

Lemma 15 *Let* $G = \mathrm{SO}_{2n+1}$ *or* Sp_{2n}. *Then,*

(a) The subring $\mathrm{Rep}_{\mathrm{poly}}(G) \subset \mathrm{Rep}(G)$ *of* ω_1*-polynomial characters is a special* λ*-subring, where*

$$\mathrm{Rep}_{\mathrm{poly}}(G) := \mathrm{Rep}^{\mathbb{C}}_{\mathrm{poly}}(G) \cap \mathrm{Rep}(G).$$

(b) Moreover, the character

$$\chi(\mathbf{t}) = \sum_{i=1}^{n} \left(t_i^2 + t_i^{-2}\right) \in \mathrm{Rep}_{\mathrm{poly}}(G), \quad \textit{for } \mathbf{t} \in T_C \ \ \textit{given by } (10)$$
$$\textit{or } \mathbf{t} \in T_B \ \ \textit{given by } (11)$$

generates $\mathrm{Rep}_{\mathrm{poly}}(G)$ *as a* λ*-ring, i.e.,* $\chi(\mathbf{t}), \lambda^2(\chi(\mathbf{t})), \ldots, \lambda^n(\chi(\mathbf{t}))$ *generate the ring* $\mathrm{Rep}_{\mathrm{poly}}(G)$. *(By the following identity* (15)*,* $\chi \in \mathrm{Rep}_{\mathrm{poly}}(G)$.*)*

Proof (a): Since

$$\mathrm{Rep}(G) = \mathrm{Rep}(T)^W, \quad \text{for any connected reductive group } G, \tag{14}$$

(cf. [3, Proposition 2.1, Chapter VI]), by Proposition 12, it is easy to see that

$$\mathrm{Rep}_{\mathrm{poly}}(G) = \mathbb{Z}_{\mathrm{sym}}[t_1^2+t_1^{-2}, \ldots, t_n^2+t_n^{-2}], \quad \text{for } G = \mathrm{SO}_{2n+1} \ \text{ or } \ \mathrm{Sp}_{2n}, \tag{15}$$

where $\mathbb{Z}_{\mathrm{sym}}[t_1^2 + t_1^{-2}, \ldots, t_n^2 + t_n^{-2}]$ denotes the subring of the polynomial ring $\mathbb{Z}[t_1^2 + t_1^{-2}, \ldots, t_n^2 + t_n^{-2}]$ consisting of symmetric polynomials. Further, the ring $\mathbb{Z}_{\mathrm{sym}}[t_1^2+t_1^{-2}, \ldots, t_n^2+t_n^{-2}]$ is generated by the elementary symmetric polynomials $e_1, \ldots, e_n$ in the variables $x_1 = t_1^2 + t_1^{-2}, \ldots, x_n = t_n^2 + t_n^{-2}$ (cf. [5, §6.2]). In particular, to prove that $\mathrm{Rep}_{\mathrm{poly}}(G)$ is a λ-subring of Rep(G), by the axioms (3) and (5) of the definition of special λ-rings as in Definition 13, it suffices to prove that $\lambda^d(e_k) \in \mathrm{Rep}_{\mathrm{poly}}(G)$, for any $d \geq 1$ and $1 \leq k \leq n$. Now,

$$\lambda(t_i^2 + t_i^{-2}) = 1 + (t_i^2 + t_i^{-2})q + q^2, \quad \text{for any } 1 \leq i \leq n. \tag{16}$$

Thus, for any $1 \leq k \leq n$, using the axioms (3) and (5) of Definition 13,

$$\begin{aligned}\lambda(e_k) &= \prod_{1 \leq i_1 < \cdots < i_k \leq n} \lambda(x_{i_1} \ldots x_{i_k}) \\ &= \prod_{1 \leq i_1 < \cdots < i_k \leq n} \left(\sum_{d \geq 0} P_d(x_{i_1}, \ldots, x_{i_k}) q^d \right), \quad \text{by the Eq. (16),}\end{aligned}$$

for certain symmetric polynomials P_d in k variables with integral coefficients. From this it is easy to see that the coefficient of any q^d in $\lambda(e_k)$ belongs to $\mathbb{Z}_{\text{sym}}[t_1^2+t_1^{-2},\ldots,t_n^2+t_n^{-2}]$. This proves the (a)-part from the identity (15).

(b): By the identity (16),

$$\lambda\left(\sum_{i=1}^{n}(t_i^2+t_i^{-2})\right)=\prod_{i=1}^{n}\left(1+(t_i^2+t_i^{-2})q+q^2\right)$$
$$=\sum_{k=0}^{2n}\left(e_k(t_1^2+t_1^{-2},\ldots,t_n^2+t_n^{-2})+f_k(t_1^2+t_1^{-2},\ldots,t_n^2+t_n^{-2})\right)q^k,$$

where f_k is a symmetric polynomial with integral coefficients of degree $< k$. From this the (b) part of the lemma follows by using the identity (15).

□

Lemma 16 *For* SO_{2n} $(n \geq 3)$, $\prod_{i=1}^{n}(t_i-t_i^{-1})$ *is the character of the virtual representation*

$$[V(2\omega_n)]-[V(2\omega_{n-1})],$$

where ω_i *is the* i*-th fundamental representation of* Spin_{2n}.

Proof We first recall (cf. [3, §5.5, Chapter VI]) that

$$\Lambda^n(\mathbb{C}^{2n})=V'\oplus V'',\quad \text{where } V'=V(2\omega_n),\ V''=V(2\omega_{n-1})$$

as representations of SO_{2n}. Recall from [4, Planche IV] that

$$2\omega_n=t_1\ldots t_n \text{ and } 2\omega_{n-1}=t_1\ldots t_{n-1}t_n^{-1}.$$

Moreover, the diagram automorphism of SO_{2n} gives rise to a linear isomorphism between V' and V''. Under this isomorphism any weight space of V' of weight $t_1^{d_1}\ldots t_{n-1}^{d_{n-1}}t_n^{d_n}$ corresponds to the weight space of V'' of weight $t_1^{d_1}\ldots t_{n-1}^{d_{n-1}}t_n^{-d_n}$. Any weight vector of $\Lambda^n(\mathbb{C}^{2n})$ clearly has weight $\mathbf{t^d}:=t_1^{d_1}\ldots t_n^{d_n}$, for $d_i\in\{0,\pm1\}$. By using the invariance of the dimension of weight spaces under W-action, we see that

$$\dim\left([V']_{\mathbf{t^d}}\right)=\dim\left([V'']_{\mathbf{t^d}}\right),\quad \text{if at least one of } d_i=0,$$

where $[V']_{\mathbf{t^d}}$ denotes the weight space of V' of weight $\mathbf{t^d}$. So, such weights $\mathbf{t^d}$, with at least one $d_i=0$, do not contribute to the character of $[V(2\omega_n)]-[V(2\omega_{n-1})]$. Let us now consider the weight spaces of V' and V'' of weight $\mathbf{t^d}$ where none of d_i is zero. In this case, define

$$\mathbf{d}_-:=\sharp\{1\leq i\leq n: d_i=-1\}.$$

Again, using the invariance of the dimension of weight spaces under W-action, we see that

$$\dim[V']_{\mathbf{t}^{\mathbf{d}}} = 1 \text{ if } \mathbf{d}_- \text{ is even,} \tag{17}$$

and zero otherwise. Similarly,

$$\dim[V'']_{\mathbf{t}^{\mathbf{d}}} = 1 \text{ if } \mathbf{d}_- \text{ is odd,} \tag{18}$$

and zero otherwise. Combining the Eqs. (17) and (18), we get the lemma. □

Remark 17 (a) It is easy to see that for the case of $G = \mathrm{SO}_{2n}$, $\mathrm{Rep}_{\mathrm{poly}}(G) := \mathrm{Rep}^{\mathbb{C}}_{\mathrm{poly}}(G) \cap \mathrm{Rep}(G)$ is *not* a λ-subring of $\mathrm{Rep}(G)$, by considering the function $\prod_{i=1}^{n}(t_i - t_i^{-1}) \in \mathrm{Rep}_{\mathrm{poly}}(G)$.

(b) Observe that $\chi(\mathrm{t}) = \sum_{i=1}^{n}(t_i^2 + t_i^{-2})$ is the character of the following virtual representation:

(1) $G = \mathrm{Sp}_{2n}$: $[S^2V] - [\Lambda^2 V]$, where V is the standard representation of Sp_{2n} in $V = \mathbb{C}^{2n}$.

(2) $G = \mathrm{SO}_{2n+1}$: $[S^2V] - [\Lambda^2 V] - [\epsilon]$, where V is the standard representation of SO_{2n+1} in $V = \mathbb{C}^{2n+1}$ and ϵ is the one dimensional trivial representation.

Lemma 18 *Let* $G = \mathrm{Sp}_{2n}(n \geq 2)$, $\mathrm{SO}_{2n+1}(n \geq 2)$ *or* $\mathrm{SO}_{2n}(n \geq 4)$. *Then, no non-trivial irreducible representation* $V(\lambda)$ *(i.e.,* $\lambda \neq 0$*) belongs to* $\mathrm{Rep}^{\mathbb{C}}_{\mathrm{poly}}(G)$.

Proof We first prove the lemma for $G = \mathrm{Sp}_{2n}$. Write $\lambda = \sum_{i=1}^{n} d_i\omega_i$, where ω_i are the fundamental weights. The highest weight of $V(\lambda)$ is given by $t_1^{d_1+\cdots+d_n} t_2^{d_2+\cdots+d_n} \dots t_n^{d_n}$. By Proposition 12, for $[V(\lambda)]$ to lie in $\mathrm{Rep}^{\mathbb{C}}_{\mathrm{poly}}(G)$, $[V(\lambda)]$ must, in particular, lie in $\mathbb{C}[t_1^{\pm 2}, \dots, t_n^{\pm 2}]$. In particular, each d_i should be even. Choose i_o such that $d_{i_o} \neq 0$. If $i_o < n$, then the exponent of t_{i_o} in the weight of $f_{i_o} \cdot v_+$ (which is a nonzero vector) is odd, where v_+ is a highest weight vector of $V(\lambda)$ and f_{i_o} is a root vector of negative weight $-\alpha_{i_o}$. So, assume that $\lambda = 2d\omega_n$. In this case, $f_n^{2d} \cdot v_+ \neq 0$ and $f_n^{2d+1} \cdot v_+ = 0$. Thus, there exists an $i < n$ such that $f_i f_n^{2d} \cdot v_+ \neq 0$. Again, the exponent of t_i in the weight of $f_i f_n^{2d} \cdot v_+$ is odd. This proves the lemma for $G = \mathrm{Sp}_{2n}$.

We next consider the case of $G = \mathrm{SO}_{2n+1}$. In this case any irreducible module has highest weight $\lambda = \sum_{i=1}^{n} d_i\omega_i$, where d_n is even. The highest weight of $V(\lambda)$ is given by $t_1^{d_1+\cdots+d_{n-1}+\frac{d_n}{2}} t_2^{d_2+\cdots+d_{n-1}+\frac{d_n}{2}} \dots t_n^{\frac{d_n}{2}}$. Again, by Proposition 12, for $[V(\lambda)]$ to lie in $\mathrm{Rep}^{\mathbb{C}}_{\mathrm{poly}}(G)$, $[V(\lambda)]$ must, in particular, lie in $\mathbb{C}[t_1^{\pm 2}, \dots, t_n^{\pm 2}]$. For the exponent of each t_i in the highest weight of $V(\lambda)$ to be even, we must have each d_i to be even and d_n must be divisible by 4. Assuming this restriction on λ, choose i_o such that $d_{i_o} \neq 0$. Then, the exponent of t_{i_o} in the weight of the nonzero vector $f_{i_o} \cdot v_+$ is odd. This proves the lemma for $G = \mathrm{SO}_{2n+1}$.

We finally consider the case of $G = \mathrm{SO}_{2n}$. In this case any irreducible module has highest weight $\lambda = \sum_{i=1}^{n} d_i\omega_i$, where $d_{n-1} + d_n$ is even, say equal to $2m$. The highest weight of $V(\lambda)$ is given by $t_1^{d_1+\cdots+d_{n-2}+m} t_2^{d_2+\cdots+d_{n-2}+m} \dots t_{n-2}^{d_{n-2}+m} t_{n-1}^{m} t_n^{\frac{d_n - d_{n-1}}{2}}$. By

Proposition 12, for $[V(\lambda)]$ to lie in $\mathrm{Rep}^{\mathbb{C}}_{\mathrm{poly}}(G)$, $[V(\lambda)]$ must, in particular, lie in

$$\mathbb{C}[t_1^{\pm 2}, \dots, t_n^{\pm 2}] \oplus (t_1 \dots t_n)\mathbb{C}[t_1^{\pm 2}, \dots, t_n^{\pm 2}].$$

In particular, each of $d_1, \dots, d_n$ is even. Assuming this restriction on λ, choose i_o such that $d_{i_o} \neq 0$. Then, the weight of the nonzero vector $f_{i_o} \cdot v_+$ has exactly two exponents either odd or exactly two exponents even. This proves the lemma for $G = \mathrm{SO}_{2n}$ (for $n \geq 4$). □

7 Specialization of Theorem 5 to $G = \mathrm{Sp}_{2n}$ and P any maximal parabolic

We follow the notation from Sect. 6 and take $n \geq 2$.

Let $V = \mathbb{C}^{2n}$ be equipped with the nondegenerate symplectic form $\langle\, , \,\rangle$ so that its matrix $\big(\langle e_i, e_j\rangle\big)_{1\leq i,j\leq 2n}$ in the standard basis $\{e_1, \dots, e_{2n}\}$ is given by the matrix E_C of Sect. 6. For $1 \leq r \leq n$, we let $\mathrm{IG}(r, 2n)$ to be the set of r-dimensional isotropic subspaces of V with respect to the form $\langle\, , \,\rangle$, i.e.,

$$\mathrm{IG}(r, 2n) := \{M \in \mathrm{Gr}(r, 2n) : \langle v, v'\rangle = 0, \ \forall\, v, v' \in M\}.$$

We take $B_C := B \cap \mathrm{Sp}_{2n}$ as the Borel subgroup of Sp_{2n}, where B is the standard Borel subgroup of SL_{2n} consisting of upper triangular matrices of determinant 1. Then, $\mathrm{IG}(r, 2n)$ is the quotient Sp_{2n} / P_r^C of Sp_{2n} by the standard maximal parabolic subgroup P_r^C with $\Delta\backslash\{\alpha_r\}$ as the set of simple roots of its Levi component L_r^C. (Again we take L_r^C to be the unique Levi subgroup of P_r^C containing T_C.) Then,

$$L_r^C \simeq \mathrm{GL}_r \times \mathrm{Sp}_{2(n-r)}\,.$$

In this case, by the identity (4), Corollary 11 and Proposition 12 (a),

$$\mathrm{Rep}^{\mathbb{C}}_{\mathrm{poly}}(L_r^C) \simeq \mathbb{C}_{\mathrm{sym}}\left[\left(\frac{t_1 - t_1^{-1}}{2}\right), \dots, \left(\frac{t_r - t_r^{-1}}{2}\right)\right]$$
$$\otimes_{\mathbb{C}} \mathbb{C}_{\mathrm{sym}}\left[\left(\frac{t_{r+1} - t_{r+1}^{-1}}{2}\right)^2, \dots, \left(\frac{t_n - t_n^{-1}}{2}\right)^2\right],$$

where $\mathbb{C}_{\mathrm{sym}}$ denotes the subalgebra of the polynomial ring consisting of symmetric polynomials. Further, by Lemma 15, the identity (15) and Remark 17 (b),

$$\mathbb{C}_{\mathrm{sym}}\left[\left(\frac{t_{r+1} - t_{r+1}^{-1}}{2}\right)^2, \dots, \left(\frac{t_n - t_n^{-1}}{2}\right)^2\right]$$

is generated (as a $\mathbb{C}$-algebra) by the virtual representations:

$$\left\{\lambda^d\left([S^2(V_{2(n-r)})] - [\Lambda^2(V_{2(n-r)})]\right)\right\}_{1\leq d\leq n-r},$$

where $V_{2(n-r)} = \mathbb{C}^{2(n-r)}$ is the standard representation of $\mathrm{Sp}_{2(n-r)}$ and λ is the λ-ring structure on $\mathrm{Rep}(G)$ as in Example 14.

Proposition 19 *The map* $\xi^P : \mathrm{Rep}^{\mathbb{C}}_{\mathrm{poly}}(L^C_r) \to H^*(\mathrm{IG}(r, 2n), \mathbb{C})$ *of Theorem 5, where* $P = P^C_r$*, takes*

$$(t_1 - t_1^{-1}) + \cdots + (t_r - t_r^{-1}) \mapsto 2\epsilon^P_{s_r},$$

and

$$(t_{r+1} - t_{r+1}^{-1})^2 + \cdots + (t_n - t_n^{-1})^2 = [S^2(V_{2(n-r)})] - [\Lambda^2(V_{2(n-r)})] - 2(n-r)[\epsilon]$$
$$\mapsto 4\left(\left(\epsilon^P_{s_r}\right)^2 + 2\left(\sum_{j=r+1}^{n-1}\left(\epsilon^P_{s_j}\right)^2\right) + \left(\epsilon^P_{s_n}\right)^2 - 2\sum_{j=r}^{n-1}\left(\epsilon^P_{s_j}\epsilon^P_{s_{j+1}}\right)\right),$$

where $\{\epsilon^P_w\}_{w \in W^P}$ *is the Schubert basis of* $H^*(G/P)$ *as in Sect. 2.*

Proof For $1 \le i \le n$, let $x_i : \mathfrak{t} \to \mathbb{C}$ be the linear map which takes

$$\mathrm{diag}(x_1, \ldots, x_n, -x_n, \ldots, -x_1) \in \mathfrak{t} \ \text{ to } x_i$$

[cf. the identity (12)]. By Corollary 11, the homomorphism $(\theta_{|T})^* : \mathbb{C}[\mathfrak{t}] \to \mathbb{C}[T]$, induced from the Springer morphism θ takes

$$2\,(x_1 + \cdots + x_r) \mapsto (t_1 - t_1^{-1}) + \cdots + (t_r - t_r^{-1}).$$

Now, the weight $x_1 + \cdots + x_r$ corresponds to the fundamental weight ω_r (cf. [4, Planche III]). Hence, the first part of the proposition follows from the definition of ξ^P and the identity (1).

Similarly, under $(\theta_{|T})^*$,

$$4\left(x^2_{r+1} + \cdots + x^2_n\right) \mapsto \left(t_{r+1} - t^{-1}_{r+1}\right)^2 + \cdots + (t_n - t_n^{-1})^2.$$

Thus, the second part follows again from the identity (1) and Remark 17 (b). □

8 Specialization of Theorem 5 to $G = \mathrm{SO}_{2n+1}$ and P any maximal parabolic

We follow the notation from Sect. 6 and take $n \ge 2$.

Let $V' = \mathbb{C}^{2n+1}$ be equipped with the nondegenerate symmetric form $\langle\,,\,\rangle$ so that its matrix $E_B = \left(\langle e_i, e_j\rangle\right)_{1 \le i,j \le 2n+1}$ (in the standard basis $\{e_1, \ldots, e_{2n+1}\}$) is the one given in Sect. 6. Note that the associated quadratic form on V' is given by

$$Q\left(\sum t_i e_i\right) = t^2_{n+1} + \sum_{i=1}^{n} t_i t_{2n+2-i}.$$

For $1 \leq r \leq n$, we let $\mathrm{OG}(r, 2n+1)$ to be the set of r-dimensional isotropic subspaces of V' with respect to the quadratic form Q, i.e.,

$$\mathrm{OG}(r, 2n+1) := \{M \in \mathrm{Gr}(r, V') : Q(v) = 0, \ \forall\, v \in M\}.$$

We take $B_B := B \cap \mathrm{SO}_{2n+1}$ as the Borel subgroup of SO_{2n+1}, where B is the standard Borel subgroup of SL_{2n+1} consisting of upper triangular matrices of determinant 1. Then, $\mathrm{OG}(r, 2n+1)$ is the quotient $\mathrm{SO}(2n+1)/P_r^B$ of $\mathrm{SO}(2n+1)$ by the standard maximal parabolic subgroup P_r^B with $\Delta \backslash \{\alpha_r\}$ as the set of simple roots of its Levi component L_r^B. (Again we take L_r^B to be the unique Levi subgroup of P_r^B containing T_B.) Then,

$$L_r^B \simeq \mathrm{GL}_r \times \mathrm{SO}_{2(n-r)+1}\,.$$

In this case, by the identity (4), Corollary 11 and Proposition 12 (a),

$$\mathrm{Rep}_{\mathrm{poly}}^{\mathbb{C}}(L_r^B) \simeq \mathbb{C}_{\mathrm{sym}}\left[\left(\frac{t_1 - t_1^{-1}}{2}\right), \ldots, \left(\frac{t_r - t_r^{-1}}{2}\right)\right] \otimes_{\mathbb{C}} \mathbb{C}_{\mathrm{sym}}\left[\left(\frac{t_{r+1} - t_{r+1}^{-1}}{2}\right)^2, \ldots, \left(\frac{t_n - t_n^{-1}}{2}\right)^2\right].$$

Further, by Lemma 15, the identity (15) and Remark 17 (b), $\mathbb{C}_{\mathrm{sym}}[(\frac{t_{r+1}-t_{r+1}^{-1}}{2})^2, \ldots, (\frac{t_n-t_n^{-1}}{2})^2]$ is generated (as a $\mathbb{C}$-algebra) by the virtual representations:

$$\{\lambda^d \left([S^2(V'_{2(n-r)+1})] - [\Lambda^2(V'_{2(n-r)+1})] - [\epsilon]\right)\}_{1 \leq d \leq n-r},$$

where $V'_{2(n-r)+1} = \mathbb{C}^{2(n-r)+1}$ is the standard representation of $\mathrm{SO}_{2(n-r)+1}$ and λ is the special λ-ring structure on $\mathrm{Rep}(G)$ as in Example 14.

Proposition 20 *The map* $\xi^P : \mathrm{Rep}_{\mathrm{poly}}^{\mathbb{C}}(L_r^B) \to H^*(\mathrm{OG}(r, 2n+1), \mathbb{C})$ *of Theorem 5, where* $P = P_r^B$, *takes*

$$(t_1 - t_1^{-1}) + \cdots + (t_r - t_r^{-1}) \mapsto 2\epsilon_{s_r}^P, \quad \textit{if } r < n,$$

$$(t_1 - t_1^{-1}) + \cdots + (t_r - t_r^{-1}) \mapsto 4\epsilon_{s_r}^P, \quad \textit{if } r = n,$$

and

$$(t_{r+1} - t_{r+1}^{-1})^2 + \cdots + (t_n - t_n^{-1})^2 = [S^2(V'_{2(n-r)+1})] - [\Lambda^2(V'_{2(n-r)+1})] - (2(n-r)+1)[\epsilon]$$

$$\mapsto 4\left(\left(\epsilon^P_{s_r}\right)^2 + 2\left(\sum_{j=r+1}^{n-1}\left(\epsilon^P_{s_j}\right)^2\right) + 4\left(\epsilon^P_{s_n}\right)^2 \right.$$
$$\left. -2\left(\sum_{j=r}^{n-2}\left(\epsilon^P_{s_j}\epsilon^P_{s_{j+1}}\right)\right) - 4\left(\epsilon^P_{s_{n-1}}\epsilon^P_{s_n}\right)\right).$$

Proof The proof is the same as that of Proposition 19. □

9 Specialization of Theorem 5 to $G = \mathrm{SO}_{2n}$ and P any maximal parabolic

We follow the notation from Sect. 6 and take $n \geq 4$.

(a) For $1 \leq r \leq n-2$, we let $\mathrm{OG}(r, 2n)$ to be the set of r-dimensional isotropic subspaces of $V = \mathbb{C}^{2n}$ with respect to the quadratic form $Q = \sum_{i=1}^n t_i t_{2n+1-i}$, i.e.,

$$\mathrm{OG}(r, 2n) := \{M \in \mathrm{Gr}(r, V) : Q(v) = 0,\ \forall\, v \in M\}.$$

We take $B_D := B \cap \mathrm{SO}_{2n}$ as the Borel subgroup of SO_{2n}, where B is the standard Borel subgroup of SL_{2n} consisting of upper triangular matrices of determinant 1. Then, $\mathrm{OG}(r, 2n)$ is the quotient $\mathrm{SO}(2n)/P_r^D$ of $\mathrm{SO}(2n)$ by the standard maximal parabolic subgroup P_r^D with $\Delta\backslash\{\alpha_r\}$ as the set of simple roots of its Levi component L_r^D. (Again we take L_r^D to be the unique Levi subgroup of P_r^D containing T_D.) Then,

$$L_r^D \simeq \mathrm{GL}_r \times \mathrm{SO}_{2(n-r)}\,.$$

In this case, by the identity (4), Corollary 11 and Proposition 12 (b),

$$\mathrm{Rep}^{\mathbb{C}}_{\mathrm{poly}}(L_r^D) \simeq \mathbb{C}_{\mathrm{sym}}\left[\left(\frac{t_1 - t_1^{-1}}{2}\right), \ldots, \left(\frac{t_r - t_r^{-1}}{2}\right)\right]$$
$$\otimes_{\mathbb{C}}\left(R \oplus \left(\left(\frac{t_{r+1} - t_{r+1}^{-1}}{2}\right)\cdots\left(\frac{t_n - t_n^{-1}}{2}\right)\right)R\right),$$

where $R := \mathbb{C}_{\mathrm{sym}}[(\frac{t_{r+1}-t_{r+1}^{-1}}{2})^2, \ldots, (\frac{t_n - t_n^{-1}}{2})^2]$. Further, as for the case of $G = \mathrm{Sp}_{2n}$ in Sect. 6, R is generated (as a $\mathbb{C}$-algebra) by the virtual representations:

$$\{\lambda^d\left([S^2(V_{2(n-r)})] - [\Lambda^2(V_{2(n-r)})]\right)\}_{1 \leq d \leq n-r},$$

where $V_{2(n-r)} = \mathbb{C}^{2(n-r)}$ is the standard representation of $\mathrm{SO}_{2(n-r)}$. Moreover, by Lemma 16, $\left((t_{r+1} - t_{r+1}^{-1})\ldots(t_n - t_n^{-1})\right)$ corresponds to the virtual representation $[V(2\omega_{n-r})] - [V(2\omega_{n-r-1})]$ of $\mathrm{SO}_{2(n-r)}$ (for $r \leq n-3$).

Proposition 21 *For* $1 \leq r \leq n-2$*, the map* $\xi^P : \mathrm{Rep}^{\mathbb{C}}_{\mathrm{poly}}(L^D_r) \to H^*(\mathrm{OG}(r, 2n), \mathbb{C})$ *of Theorem 5, where* $P = P^D_r$*, takes*

$$(t_1 - t_1^{-1}) + \cdots + (t_r - t_r^{-1}) \mapsto 2\epsilon^P_{s_r},$$

$$(t_{r+1} - t_{r+1}^{-1})^2 + \cdots + (t_n - t_n^{-1})^2 = [S^2(V_{2(n-r)})] - [\Lambda^2(V_{2(n-r)})] - 2(n-r)[\epsilon]$$
$$\mapsto 4\left(\left(\epsilon^P_{s_r}\right)^2 + 2\left(\sum_{j=r+1}^{n}\left(\epsilon^P_{s_j}\right)^2\right) - 2\left(\sum_{j=r}^{n-2}\left(\epsilon^P_{s_j}\epsilon^P_{s_{j+1}}\right)\right) - 2\epsilon^P_{s_{n-2}}\epsilon^P_{s_n}\right),$$

and

$$\left((t_{r+1} - t_{r+1}^{-1})\dots(t_n - t_n^{-1})\right) \mapsto 2^{n-r}(\epsilon^P_{s_{r+1}} - \epsilon^P_{s_r})\dots$$
$$(\epsilon^P_{s_{n-2}} - \epsilon^P_{s_{n-3}})(\epsilon^P_{s_n} + \epsilon^P_{s_{n-1}} - \epsilon^P_{s_{n-2}})(\epsilon^P_{s_n} - \epsilon^P_{s_{n-1}}).$$

Proof The proof is similar to that of Proposition 19 and hence omitted. □

(b) For $r = n$, let P^D_n be the maximal standard parabolic subgroup with $\Delta\backslash\{\alpha_n\}$ as the set of simple roots of its Levi component L^D_n. Then, the partial flag variety SO_{2n}/P^D_n can be identified with a connected component $\mathrm{OG}(n, 2n)_+$ of the set of n-dimensional isotropic subspaces of V. Moreover, the Levi subgroup

$$L^D_n \simeq \mathrm{GL}_n\,.$$

In this case, by the identity (4) and Corollary 11,

$$\mathrm{Rep}^{\mathbb{C}}_{\mathrm{poly}}(L^D_n) \simeq \mathbb{C}_{\mathrm{sym}}\left[\left(\frac{t_1 - t_1^{-1}}{2}\right), \dots, \left(\frac{t_n - t_n^{-1}}{2}\right)\right].$$

Proposition 22 *The map* $\xi^{P^D_n} : \mathrm{Rep}^{\mathbb{C}}_{\mathrm{poly}}(L^D_n) \to H^*(\mathrm{OG}(n, 2n)_+, \mathbb{C})$ *of Theorem 5 takes*

$$(t_1 - t_1^{-1}) + \cdots + (t_n - t_n^{-1}) \mapsto 4\epsilon^{P^D_n}_{s_n}\,.$$

Proof The proof is the same as that of Proposition 19. □

Of course, the case of $r = n-1$ is parallel to the above case of $r = n$ due to the diagram automorphism taking the n-th node of D_n to the $(n-1)$-th node.

10 Specialization of Theorem 5 to the classical groups and $P = B$

In this section G is any of Sp_{2n}, SO_{2n+1} or SO_{2n}. We get the following lemma immediately from Corollary 11.

Lemma 23 *Under the coordinates* (10) *and* (11) *on the maximal torus* T *of* Sp_{2n}, SO_{2n+1} *or* SO_{2n},

$$\mathrm{Rep}^{\mathbb{C}}_{\mathrm{poly}}(T) = \mathbb{C}\left[\left(\frac{t_1 - t_1^{-1}}{2}\right), \ldots, \left(\frac{t_n - t_n^{-1}}{2}\right)\right],$$

where T *is thought of as the Levi subgroup of* B.

Proposition 24 *Under the homomorphism* $\xi^B : \mathrm{Rep}^{\mathbb{C}}_{\mathrm{poly}}(T) \to H^*(G/B, \mathbb{C})$ *of Theorem* 5,

(a) $G = \mathrm{Sp}_{2n}$ $(n \geq 2)$:

$$t_i - t_i^{-1} \mapsto 2(\epsilon^B_{s_i} - \epsilon^B_{s_{i-1}}), \quad \textit{for any } 1 \leq i \leq n.$$

(b) $G = \mathrm{SO}_{2n+1}$ $(n \geq 2)$:

$$t_i - t_i^{-1} \mapsto 2(\epsilon^B_{s_i} - \epsilon^B_{s_{i-1}}), \quad \textit{for any } 1 \leq i < n,$$

and

$$t_n - t_n^{-1} \mapsto 2\left(2\epsilon^B_{s_n} - \epsilon^B_{s_{n-1}}\right).$$

(c) $G = \mathrm{SO}_{2n}$ $(n \geq 4)$:

$$t_i - t_i^{-1} \mapsto 2\left(\epsilon^B_{s_i} - \epsilon^B_{s_{i-1}}\right), \quad \textit{for any } 1 \leq i \leq n-2,$$

$$t_{n-1} - t_{n-1}^{-1} \mapsto 2\left(\epsilon^B_{s_{n-1}} + \epsilon^B_{s_n} - \epsilon^B_{s_{n-2}}\right),$$

and

$$t_n - t_n^{-1} \mapsto 2\left(\epsilon^B_{s_n} - \epsilon^B_{s_{n-1}}\right).$$

Proof The proof is similar to the proof of Proposition 19 and hence omitted. □

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