Additive Eigenvalue Problem

Shrawan Kumar (University of North Carolina, Chapel Hill, NC 27599-3250, USA)

1 Introduction

The classical Hermitian eigenvalue problem addresses the following question: What are the possible eigenvalues of the sum $A + B$ of two Hermitian matrices A and B, provided we fix the eigenvalues of A and B. A systematic study of this problem was initiated by H. Weyl (1912). By virtue of contributions from a long list of mathematicians, notably Weyl (1912), Horn (1962), Klyachko (1998) and Knutson-Tao (1999), the problem is finally settled. The solution asserts that the eigenvalues of $A + B$ are given in terms of certain system of linear inequalities in the eigenvalues of A and B. These inequalities are given explicitly in terms of certain triples of Schubert classes in the singular cohomology of Grassmannians and the standard cup product. Belkale (2001) gave an optimal set of inequalities for the problem in this case. The Hermitian eigenvalue problem has been extended by Berenstein-Sjamaar (2000) and Kapovich-Leeb-Millson (2005) for any semisimple complex algebraic group G. Their solution is again in terms of a system of linear inequalities obtained from certain triples of Schubert classes in the singular cohomology of the partial flag varieties G/P (P being a maximal parabolic subgroup) and the standard cup product. However, their solution is far from being optimal. In a joint work with P. Belkale, we defined a deformation of the cup product in the cohomology of G/P and used this new product to generate a system of inequalities which solves the problem for any G optimally (as shown by Ressayre). This article is a brief survey of this additive eigenvalue problem. The eigenvalue problem is equivalent to the *saturated tensor product problem*.

This note was written during my visit to the University of Sydney, hospitality of which is gratefully acknolwledged. Also, partial support from NSF grant number DMS- 1501094 is gratefully acknowledged.

2 Main Results

We now explain the classical Hermitian eigenvalue problem and its generalization to an arbitrary connected reductive group more precisely.

For any $n \times n$ Hermitian matrix A, let $\lambda_A = (\lambda_1 \geq \cdots \geq \lambda_n)$ be its set of eigenvalues written in descending order. Recall the following classical problem, known as the *Hermitian eigenvalue problem*: Given two n-tuples of nonincreasing real numbers: $\lambda = (\lambda_1 \geq \cdots \geq \lambda_n)$ and $\mu = (\mu_1 \geq \cdots \geq \mu_n)$, determine all possible $\nu = (\nu_1 \geq \cdots \geq \nu_n)$ such that there exist Hermitian matrices A, B, C with $\lambda_A = \lambda, \lambda_B = \mu, \lambda_C = \nu$ and $C = A + B$. This problem has a long history starting with the work of Weyl (1912) followed by works of Fan (1949), Lidskii (1950), Wielandt (1955), and culminating into the following conjecture given by Horn (1962). (Also see Thompson-Freede (1971).)

For any positive integer $r < n$, inductively define the set S^n_r as the set of triples (I, J, K) of subsets of $[n] := \{1, \ldots, n\}$ of cardinality r such that

$$\sum_{i\in I} i + \sum_{j\in J} j = r(r+1)/2 + \sum_{k\in K} k \qquad (1)$$

and for all $0 < p < r$ and $(F, G, H) \in S^r_p$ the following inequality holds:

$$\sum_{f\in F} i_f + \sum_{g\in G} j_g \leq p(p+1)/2 + \sum_{h\in H} k_h. \qquad (2)$$

Conjecture 1. *A triple λ, μ, ν occurs as eigenvalues of Hermitian $n \times n$ matrices A, B, C respectively such that $C = A + B$ if and only if*

$$\sum_{i=1}^{n} \nu_i = \sum_{i=1}^{n} \lambda_i + \sum_{i=1}^{n} \mu_i,$$

and for all $1 \leq r < n$ and all triples $(I, J, K) \in S^n_r$, we have

$$\sum_{k\in K} \nu_k \leq \sum_{i\in I} \lambda_i + \sum_{j\in J} \mu_j.$$

Horn's above conjecture was settled in the affirmative (cf. Corollary 11) by combining the work of Klyachko [Kly] (1998) with the work of Knutson-Tao [KT] (1999) on the 'saturation' problem.

The above system of inequalities is overdetermined. Belkale (2001) proved that a certain subset of the above set of inequalities suffices. Subsequently, Knutson-Tao-Woodward (2004) proved that the subsystem of inequalities given by Belkale forms an irredundant system of inequalities.

Now, we discuss a generalization of the above Hermitian eigenvalue problem (which can be rephrased in terms of the special unitary group $\mathrm{SU}(n)$ and its complexified Lie algebra $\mathfrak{sl}(n)$) to an arbitrary complex semisimple group. Let G be a connected, semisimple complex algebraic group. We fix a Borel subgroup B, a maximal torus H, and a maximal compact subgroup K. We denote their Lie algebras by the corresponding Gothic characters: $\mathfrak{g}, \mathfrak{b}, \mathfrak{h}, \mathfrak{k}$ respectively. Let R^+ be the set of positive roots (i.e., the set of roots of $\mathfrak{b}$) and let $\Delta = \{\alpha_1, \dots, \alpha_\ell\} \subset R^+$ be the set of simple roots. There is a natural homeomorphism $\delta : \mathfrak{k}/K \to \mathfrak{h}_+$, where K acts on $\mathfrak{k}$ by the adjoint representation and $\mathfrak{h}_+ := \{h \in \mathfrak{h} : \alpha_i(h) \geq 0 \,\forall\, i\}$ is the positive Weyl chamber in $\mathfrak{h}$. The inverse map δ^{-1} takes any $h \in \mathfrak{h}_+$ to the K-conjugacy class of $\sqrt{-1}h$.

For any positive integer s, define the *eigencone*

$$\bar{\Gamma}_s(\mathfrak{g}) := \{(h_1, \dots, h_s) \in \mathfrak{h}_+^s \mid \exists (k_1, \dots, k_s) \in \mathfrak{k}^s : \sum_{j=1}^s k_j = 0 \text{ and } \delta(k_j) = h_j \forall j\}.$$

By virtue of the general convexity result in symplectic geometry, the subset $\bar{\Gamma}_s(\mathfrak{g}) \subset \mathfrak{h}_+^s$ is a convex rational polyhedral cone (defined by certain inequalities with rational coefficients). The aim of the *general additive eigenvalue problem* is to find the inequalities describing $\bar{\Gamma}_s(\mathfrak{g})$ explicitly. (The case $\mathfrak{g} = \mathfrak{sl}(n)$ and $s = 3$ specializes to the Hermitian eigenvalue problem if we replace C by $-C$.)

Let $\Lambda = \Lambda(H)$ denote the character group of H and let $\Lambda_+ := \{\lambda \in \Lambda : \lambda(\alpha_i^\vee) \geq 0 \,\forall$ simple coroots $\alpha_i^\vee\}$ denote the set of all the dominant characters. Then, the set of isomorphism classes of irreducible (finite dimensional) representations of G is parameterized by Λ_+ via the highest weights of irreducible representations. For $\lambda \in \Lambda_+$, we denote by $[\lambda]$ the corresponding irreducible representation (of highest weight λ).

Similar to the eigencone $\bar{\Gamma}_s(\mathfrak{g})$, one defines the *saturated tensor semigroup*

$$\Gamma_s(G) = \{(\lambda_1, \dots, \lambda_s) \in \Lambda_+^s : ([N\lambda_1] \otimes \cdots \otimes [N\lambda_s])^G \neq 0, \text{ for some } N \geq 1\}.$$

Then, under the identification $\varphi : \mathfrak{h} \xrightarrow{\sim} \mathfrak{h}^*$ (via the Killing form)

$$\varphi(\bar{\Gamma}_s(\mathfrak{g})) \cap \Lambda_+^s = \Gamma_s(G). \tag{3}$$

(cf. Theorem 5).

For any $1 \leq j \leq \ell$, define the element $x_j \in \mathfrak{h}$ by

$$\alpha_i(x_j) = \delta_{i,j}, \ \forall\, 1 \leq i \leq \ell. \tag{4}$$

Let $P \supset B$ be a standard parabolic subgroup with Lie algebra $\mathfrak{p}$ and let $\mathfrak{l}$ be its unique Levi component containing the Cartan subalgebra $\mathfrak{h}$. Let $\Delta(P) \subset \Delta$ be the set of simple roots contained in the set of roots of $\mathfrak{l}$. Let W_P be the Weyl group of P (which is, by definition, the Weyl Group of the Levi component L) and let W^P be the set of the minimal length representatives in the cosets of W/W_P. For any $w \in W^P$, define the Schubert variety:

$$X_w^P := \overline{BwP/P} \subset G/P.$$

It is an irreducible (projective) subvariety of G/P of dimension $\ell(w)$. Let $\mu(X_w^P)$ denote the fundamental class of X_w^P considered as an element of the singular homology with integral coefficients $H_{2\ell(w)}(G/P, \mathbb{Z})$ of G/P. Then, from the Bruhat decomposition, the elements $\{\mu(X_w^P)\}_{w \in W^P}$ form a $\mathbb{Z}$-basis of $H_*(G/P, \mathbb{Z})$. Let $\{[X_w^P]\}_{w \in W^P}$ be the Poincaré dual basis of the singular cohomology $H^*(G/P, \mathbb{Z})$. Thus,

$$[X_w^P] \in H^{2(\dim G/P - \ell(w))}(G/P, \mathbb{Z}).$$

Write the standard cup product in $H^*(G/P, \mathbb{Z})$ in the $\{[X_w^P]\}$ basis as follows:

$$[X_u^P] \cdot [X_v^P] = \sum_{w \in W^P} c_{u,v}^w [X_w^P]. \tag{5}$$

Introduce the indeterminates τ_i for each $\alpha_i \in \Delta \setminus \Delta(P)$ and define a deformed cup product $\odot$ as follows:

$$[X_u^P] \odot [X_v^P] = \sum_{w \in W^P} \Big(\prod_{\alpha_i \in \Delta \setminus \Delta(P)} \tau_i^{(w^{-1}\rho - u^{-1}\rho - v^{-1}\rho - \rho)(x_i)} \Big) c_{u,v}^w [X_w^P], \tag{6}$$

where ρ is the (usual) half sum of positive roots of $\mathfrak{g}$. By Corollary 16 and the identity (13), whenever $c_{u,v}^w$ is nonzero, the exponent of τ_i in the above is a nonnegative integer. Moreover, the product $\odot$ is associative (and clearly commutative). The cohomology algebra of G/P obtained by setting each $\tau_i = 0$ in $(H^*(G/P, \mathbb{Z}) \otimes \mathbb{Z}[\tau_i], \odot)$ is denoted by $(H^*(G/P, \mathbb{Z}), \odot_0)$. Thus, as a $\mathbb{Z}$-module, this is the same as the singular cohomology $H^*(G/P, \mathbb{Z})$ and under the product $\odot_0$ it is associative (and commutative). Moreover, it continues to satisfy the Poincaré duality (cf. [BK$_1$, Lemma 16(d)]). The definition of the deformed product $\odot_0$ (now known as the *Belkale-Kumar product*) was arrived at from the crucial concept of Levi-movability as in Definition 14. For a cominuscule maximal parabolic P, the product $\odot_0$ coincides with the standard cup product (cf. Lemma 17).

Now we are ready to state the main result on solution of the eigenvalue problem for any connected semisimple G. For a maximal parabolic P, let α_{i_P} be the unique simple root not in the Levi of P and let $\omega_P := \omega_{i_P}$ be the corresponding fundamental weight.

Theorem 2. *Let $(h_1, \dots, h_s) \in \mathfrak{h}_+^s$. Then, the following are equivalent:*

(a) $(h_1, \dots, h_s) \in \bar{\Gamma}_s(\mathfrak{g})$.

(b) For every standard maximal parabolic subgroup P in G and every choice of s-tuples $(w_1, \dots, w_s) \in (W^P)^s$ such that $[X_{w_1}^P] \cdots [X_{w_s}^P] = d[X_e^P]$ for some $d \neq 0$, the following inequality holds:

$$I_{(w_1, \dots, w_s)}^P : \quad \omega_P\Big(\sum_{j=1}^s w_j^{-1} h_j\Big) \leq 0.$$

(c) For every standard maximal parabolic subgroup P in G and every choice of s-tuples $(w_1, \dots, w_s) \in (W^P)^s$ such that $[X_{w_1}^P] \cdots [X_{w_s}^P] = [X_e^P]$,

the above inequality $I^P_{(w_1,\ldots,w_s)}$ holds.

(d) For every standard maximal parabolic subgroup P in G and every choice of s-tuples $(w_1, \ldots, w_s) \in$

$$(W^P)^s \text{ such that } [X^P_{w_1}] \odot_0 \cdots \odot_0 [X^P_{w_s}] = [X^P_e],$$

the above inequality $I^P_{(w_1,\ldots,w_s)}$ holds.

The equivalence of (a) and (b) in the above theorem for general G is due to Berenstein-Sjamaar (2000). Kapovich-Leeb-Millson (2009) showed the equivalence of (a) and (c). The equivalence of (a) and (d) is due to Belkale-Kumar (2006). If we specialize the above Theorem for $G = \mathrm{SL}(n)$, then, in the view of Theorem 10, the equivalence of (a) and (b) is nothing but the Horn's conjecture (Corollary 11) solved by combining the work of Klyachko (1998) with the work of Knutson-Tao (1999). In this case, the equivalence of (a) and (c) is due to Belkale (2001). In this case, every maximal parabolic subgroup P is cominuscule and hence the deformed product $\odot_0$ in $H^*(G/P)$ coincides with the standard cup product. Hence the parts (c) and (d) are the same in this case.

Because of the identification (3), the above theorem allows us to determine the saturated tensor semigroup $\Gamma_s(G)$ (see Theorem 18 for a precise statement).

The following result was proved by Ressayre [R] (2010). As mentioned above, for $\mathfrak{g} = \mathfrak{sl}(n)$ it was proved by Knutson-Tao-Wodward. Ressayre's proof relies on the notion of well-covering pairs, which is equivalent to the notion of Levi-movability with cup product 1.

Theorem 3. *The inequalities given by the (d) part of the above theorem form an irredundant system of inequalities determining the cone $\bar{\Gamma}_s(\mathfrak{g})$ (cf. Theorem 23 for a more precise statement).*

As shown by Kumar-Leeb-Millson (2003), the (c) part of the above theorem gives rise to 126 inequalities for $\mathfrak{g}$ of type B_3 or C_3, whereas by the (d) part one gets only 93 inequalities.

We refer the reader to the survey article of Fulton [F] on the Hermitian eigenvalue problem; and for general G the survey articles by Brion [Br] and by Kumar [K_3].

3 Determination of the eigencone (A Weaker Result)

We give below an indication of the proof of the equivalence of (a) and (b) in Theorem 2 .

Definition 4. Let S be any (not necessarily reductive) algebraic group acting on a (not necessarily projective) variety $\mathbb{X}$ and let $\mathbb{L}$ be an S-equivariant line bundle on $\mathbb{X}$. Any algebraic group morphism $\mathbb{G}_m \to S$ is called a *one parameter subgroup* (for short OPS) in S. Let $O(S)$ be the set of all the OPS in S. Take any $x \in \mathbb{X}$ and $\delta \in O(S)$ such that the limit $\lim_{t\to 0} \delta(t)x$ exists in $\mathbb{X}$ (i.e., the morphism $\delta_x : \mathbb{G}_m \to X$ given by $t \mapsto \delta(t)x$ extends to a morphism $\widetilde{\delta_x} : \mathbb{A}^1 \to X$). Then, following Mumford, define a number $\mu^{\mathbb{L}}(x, \delta)$ as follows: Let $x_o \in X$ be the point $\widetilde{\delta_x}(0)$. Since x_o is $\mathbb{G}_m$-invariant via δ, the fiber of $\mathbb{L}$ over x_o is a $\mathbb{G}_m$-module; in particular, is given by a character of $\mathbb{G}_m$. This integer is defined as $\mu^{\mathbb{L}}(x, \delta)$.

Under the identification $\varphi : \mathfrak{h} \xrightarrow{\sim} \mathfrak{h}^*$ (via the Killing form) $\Gamma_s(G)$ corresponds to the set of integral points of $\overline{\Gamma}_s(\mathfrak{g})$. Specifically, we have the following result essentially following from Mumford [N, Appendix] (also see [Sj, Theorem 7.6] and [Br, Théorème 1.3]).

Theorem 5.

$$\varphi(\overline{\Gamma}_s(\mathfrak{g})) \cap \Lambda^s_+ = \Gamma_s(G).$$

Let P be any standard parabolic subgroup of G acting on P/B_L via the left multiplication, where L is the Levi subgroup of P containing H and $B_L := B \cap L$ is a Borel subgroup of L. We call $\delta \in O(P)$ *P-admissible* if, for all $x \in P/B_L$, $\lim_{t\to 0} \delta(t) \cdot x$ exists in P/B_L. If $P = G$, then $P/B_L = G/B$ and any $\delta \in O(G)$ is G-admissible since G/B is a projective variety.

Observe that, B_L being the semidirect product of its commutator $[B_L, B_L]$ and H, any $\lambda \in \Lambda$ extends uniquely to a character of B_L. Thus, for any $\lambda \in \Lambda$, we have a P-equivariant line bundle $\mathcal{L}_P(\lambda)$ on P/B_L associated to the principal B_L-bundle $P \to P/B_L$ via the one dimensional B_L-module λ^{-1}. We abbreviate $\mathcal{L}_G(\lambda)$ by $\mathcal{L}(\lambda)$. We have taken the following lemma from [BK_1, Lemma 14]. It is a generalization of the corresponding result in [BS, Section 4.2].

Lemma 6. *Let $\delta \in O(H)$ be such that $\dot{\delta} \in \mathfrak{h}_+$. Then, δ is P-admissible and, moreover, for any $\lambda \in \Lambda$ and $x = ulB_L \in P/B_L$ (for $u \in U, l \in L$), we have the following formula:*

$$\mu^{\mathcal{L}_P(\lambda)}(x, \delta) = -\lambda(w\dot{\delta}),$$

where U is the unipotent radical of P, $P_L(\delta) := P(\delta) \cap L$ and $w \in W_P/W_{P_L(\delta)}$ is any coset representative such that $l^{-1} \in B_L w P_L(\delta)$.

Let $\lambda = (\lambda_1, \ldots, \lambda_s) \in \Lambda^s_+$ and let $\mathbb{L}(\lambda)$ denote the G-linearized line bundle $\mathcal{L}(\lambda_1) \boxtimes \cdots \boxtimes \mathcal{L}(\lambda_s)$ on $(G/B)^s$ (under the diagonal action of G). Then, there exist unique standard parabolic subgroups $P_1, \ldots, P_s$ such that the line bundle $\mathbb{L}(\lambda)$ descends as an ample line bundle $\bar{\mathbb{L}}(\lambda)$ on $\mathbb{X}(\lambda) := G/P_1 \times \cdots \times G/P_s$. We call a point $x \in (G/B)^s$ *G-semistable* (with respect to, not necessarily ample, $\mathbb{L}(\lambda)$) if its image in $\mathbb{X}(\lambda)$ under the canonical map $\pi : (G/B)^s \to \mathbb{X}(\lambda)$ is semistable with respect to the ample line bundle $\bar{\mathbb{L}}(\lambda)$. Now, one has the following fundamental theorem due to Klyachko [Kly] for $G = \mathrm{SL}(n)$, extended to general G by Berenstein-Sjamaar [BS].

Theorem 7. *Let $\lambda_1, \dots, \lambda_s \in \Lambda_+$. Then, the following are equivalent:*

(a) $(\lambda_1, \dots, \lambda_s) \in \Gamma_s(G)$

(b) *For every standard maximal parabolic subgroup P and every Weyl group elements $w_1, \dots, w_s \in W^P \simeq W/W_P$ such that*

$$[X^P_{w_1}] \dots [X^P_{w_s}] = d[X^P_e], \text{ for some } d \neq 0, \quad (7)$$

the following inequality is satisfied:

$$I^P_{(w_1,\dots,w_s)} : \qquad \sum_{j=1}^{s} \lambda_j(w_j x_P) \leq 0,$$

where α_{i_P} is the unique simple root not in the Levi of P and $x_P := x_{i_P}$.

The equivalence of (a) and (b) in Theorem 2 follows easily by combining Theorems 7 and 5.

Remark 8. As proved by Belkale [B_1] for $G = \mathrm{SL}(n)$ and extended for an arbitrary G by Kapovich-Leeb-Millson [KLM], Theorem 7 remains true if we replace d by 1 in the identity (7). A much sharper (and optimal) result for an arbitrary G is obtained in Theorem 18.

4 Specialization of Results to $G = \mathrm{SL}(n)$: Horn Inequalities

We first need to recall the Knutson-Tao saturation theorem [KT], conjectured by Zelevinsky [Z]. Other proofs of their result are given by Derksen-Weyman [DK], Belkale [B_3] and Kapovich-Millson [KM_2].

Theorem 9. *Let $G = \mathrm{SL}(n)$ and let $(\lambda_1, \dots, \lambda_s) \in \Gamma_s(G)$ be such that $\lambda_1 + \dots + \lambda_s$ belongs to the root lattice. Then,*

$$([\lambda_1] \otimes \dots \otimes [\lambda_s])^G \neq 0.$$

Specializing Theorem 7 to $G = \mathrm{SL}(n)$, as seen below, we obtain the classical Horn inequalities.

In this case, the partial flag varieties corresponding to the maximal parabolics P_r are precisely the Grassmannians of r-planes in n-space $G/P_r = \mathrm{Gr}(r, n)$, for $0 < r < n$. The Schubert cells in $\mathrm{Gr}(r, n)$ are parameterized by the subsets of cardinality r:

$$I = \{i_1 < \dots < i_r\} \subset \{1, \dots, n\}.$$

The corresponding Weyl group element $w_I \in W^{P_r}$ is nothing but the permutation

$$1 \mapsto i_1, \quad 2 \mapsto i_2, \cdots, r \mapsto i_r$$

and $w_I(r+1), \dots, w_I(n)$ are the elements in $\{1, \dots, n\}\backslash I$ arranged in ascending order.

Let I' be the 'dual' set

$$I' = \{n + 1 - i, \ \ i \in I\},$$

arranged in ascending order.

Then, the Schubert class $[X_I := X^{P_r}_{w_I}]$ is Poincaré dual to the Schubert class $[X_{I'}] \in H^*(\mathrm{Gr}(r, n), \mathbb{Z})$. Moreover,

$$\dim X_I = \operatorname{codim} X_{I'} = (\sum_{i \in I} i) - \frac{r(r+1)}{2}. \quad (8)$$

For $0 < r < n$, recall the definition of the set S^r_n of triples (I, J, K) of subsets of $\{1, \dots, n\}$ of cardinality r from Section 2. The following theorem follows from Theorem 7 for $G = \mathrm{SL}(n)$ (proved by Klyachko) and Theorem 9 (proved by Knutson-Tao). Belkale [B_3] gave another geometric proof of the theorem.

Theorem 10. *For subsets (I, J, K) of $\{1, \dots, n\}$ of cardinality r, the product*

$$[X_{I'}] \cdot [X_{J'}] \cdot [X_K] = d[X^{P_r}_e], \text{ for some } d \neq 0$$
$$\Leftrightarrow (I, J, K) \in S^r_n.$$

For a Hermitian $n \times n$ matrix A, let $\lambda_A = (\lambda_1 \geq \dots \geq \lambda_n)$ be its set of eigenvalues (which are all real). Let $\mathfrak{a}$ be the standard Cartan subalgebra of $sl(n)$ consisting of traceless diagonal matrices and let $\mathfrak{b} \subset sl(n)$ be the standard Borel subalgebra consisting of traceless upper triangular matrices (where $sl(n)$ is the Lie algebra of $\mathrm{SL}(n)$ consisting of traceless $n \times n$-matrices). Then, the Weyl chamber

$$\mathfrak{a}_+ = \{\mathrm{diag}\,(e_1 \geq \dots \geq e_n) : \sum e_i = 0\}.$$

Define the *Hermitian eigencone*

$$\bar{\Gamma}(n) = \{(a_1, a_2, a_3) \in \mathfrak{a}^3_+ : \text{there exist } n \times n \text{ Hermitian matrices } A, B, C \text{ with} \lambda_A = a_1, \lambda_B = a_2, \lambda_C = a_3 \text{ and } A + B = C\}.$$

It is easy to see that $\bar{\Gamma}(n)$ essentially coincides with the eigencone $\bar{\Gamma}_3(sl(n))$. Specifically,

$$(a_1, a_2, a_3) \in \bar{\Gamma}(n) \Leftrightarrow (a_1, a_2, a^*_3) \in \bar{\Gamma}_3(sl(n)),$$

where $(e_1 \geq \dots \geq e_n)^* := (-e_n \geq \dots \geq -e_1)$.

Combining Theorems 7 and 5 for $sl(n)$ with Theorem 10, we get the following Horn's conjecture [Ho] established by the works of Klyachko (equivalence of (a) and (b) in Theorem 2 for $\mathfrak{g} = sl(n)$) and Knutson-Tao (Theorem 9).

Corollary 11. *For $(a_1, a_2, a_3) \in \mathfrak{a}^3_+$, the following are equivalent.*

(a) $(a_1, a_2, a_3) \in \bar{\Gamma}(n)$

(b) *For all $0 < r < n$ and all $(I, J, K) \in S^r_n$,*

$$|a_3(K)| \leq |a_1(I)| + |a_2(J)|,$$

where for a subset $I = (i_1 < \dots < i_r) \subset \{1, \dots, n\}$ and $a = (e_1 \geq \dots \geq e_n) \in \mathfrak{a}_+$, $a(I) := (e_{i_1} \geq \dots \geq e_{i_r})$, and $|a(I)| := e_{i_1} + \dots + e_{i_r}$.

We have the following representation theoretic analogue of the above corollary, obtained by combining Theorems 7, 9 and 10.

Corollary 12. *Let* $\lambda = (\lambda_1 \geq \cdots \geq \lambda_n \geq 0), \mu = (\mu_1 \geq \cdots \geq \mu_n \geq 0)$ *and* $\nu = (\nu_1 \geq \cdots \geq \nu_n \geq 0)$ *be three partitions such that* $|\lambda| + |\mu| - |\nu| \in n\mathbb{Z}$*, where* $|\lambda| := \lambda_1 + \cdots + \lambda_n$*. Then, the following are equivalent:*

(a) $[\nu]$ *appears as a* $\mathrm{SL}(n)$*-submodule of* $[\lambda] \otimes [\mu]$*.*

(b) For all $0 < r < n$ *and all* $(I, J, K) \in S_n^r$*,*

$$|\nu(K)| \leq |\lambda(I)| + |\mu(J)| - \frac{r}{n}(|\lambda| + |\mu| - |\nu|),$$

where for a subset $I = (i_1 < \cdots < i_r) \subset \{1, \ldots, n\}$*,* $\lambda(I)$ *denotes* $(\lambda_{i_1} \geq \cdots \geq \lambda_{i_r})$ *and* $|\lambda(I)| := \lambda_{i_1} + \cdots + \lambda_{i_r}$*.*

5 Deformed product

This section is based on the work [BK_1] due to Belkale-Kumar. Consider the shifted Bruhat cell:

$$\Phi_w^P := w^{-1}BwP \subset G/P.$$

Let $T^P = T(G/P)_e$ be the tangent space of G/P at $e \in G/P$. It carries a canonical action of P. For $w \in W^P$, define T_w^P to be the tangent space of Φ_w^P at e. We shall abbreviate T^P and T_w^P by T and T_w respectively when the reference to P is clear. It is easy to see that B_L stabilizes Φ_w^P keeping e fixed. Thus,

$$B_L T_w \subset T_w. \tag{9}$$

The following result follows from the Kleiman transversality theorem by observing that $g\Phi_w^P$ passes through $e \Leftrightarrow g\Phi_w^P = p\Phi_w^P$ for some $p \in P$.

Proposition 13. Take any $(w_1, \ldots, w_s) \in (W^P)^s$ such that

$$\sum_{j=1}^{s} \operatorname{codim} \Phi_{w_j}^P \leq \dim G/P. \tag{10}$$

Then, the following three conditions are equivalent:

(a) $[X_{w_1}^P] \ldots [X_{w_s}^P] \neq 0 \in H^*(G/P)$.

(b) For general $(p_1, \ldots, p_s) \in P^s$, the intersection

$$p_1\Phi_{w_1}^P \cap \cdots \cap p_s\Phi_{w_s}^P \text{ is transverse at } e.$$

(c) For general $(p_1, \ldots, p_s) \in P^s$,

$$\dim(p_1 T_{w_1} \cap \cdots \cap p_s T_{w_s}) = \dim G/P - \sum_{j=1}^{s} \operatorname{codim} \Phi_{w_j}^P.$$

The set of s-tuples in (b) as well as (c) is an open subset of P^s.

Definition 14. Let $w_1, \ldots, w_s \in W^P$ be such that

$$\sum_{j=1}^{s} \operatorname{codim} \Phi_{w_j}^P = \dim G/P. \tag{11}$$

We then call the s-tuple $(w_1, \ldots, w_s)$ *Levi-movable* for short *L-movable* if, for general $(l_1, \ldots, l_s) \in L^s$, the intersection $l_1\Phi_{w_1}^P \cap \cdots \cap l_s\Phi_{w_s}^P$ is transverse at e.

By Proposition 13, if $(w_1, \ldots, w_s)$ is L-movable, then $[X_{w_1}^P] \ldots [X_{w_s}^P] = d[X_e^P]$ in $H^*(G/P)$, for some nonzero d.

For $w \in W^P$, define the character $\chi_w \in \Lambda$ by

$$\chi_w = \sum_{\beta \in (R^+ \setminus R_1^+) \cap w^{-1}R^+} \beta. \tag{12}$$

Then, from [K_1, 1.3.22.3],

$$\chi_w = \rho - 2\rho^L + w^{-1}\rho, \tag{13}$$

where ρ (resp. ρ^L) is half the sum of roots in R^+ (resp. in R_1^+).

Proposition 15. Assume that $(w_1, \ldots, w_s) \in (W^P)^s$ satisfies equation (11). Then, the following are equivalent.

(a) $(w_1, \ldots, w_s)$ is L-movable.

(b) $[X_{w_1}^P] \ldots [X_{w_s}^P] = d[X_e^P]$ in $H^*(G/P)$, for some nonzero d, and for each $\alpha_i \in \Delta \setminus \Delta(P)$, we have

$$((\sum_{j=1}^{s} \chi_{w_j}) - \chi_1)(x_i) = 0.$$

Corollary 16. *For any* $u, v, w \in W^P$ *such that* $c_{u,v}^w \neq 0$ *(cf. equation (5)), we have*

$$(\chi_w - \chi_u - \chi_v)(x_i) \geq 0, \quad \textit{for each } \alpha_i \in \Delta \setminus \Delta(P). \tag{14}$$

The above corollary together with the identity (13) justifies the definition of the deformed product $\odot_0$ given in Section 2. This deformed product is used in determining the facets (codimension 1 faces) of $\bar{\Gamma}_s(\mathfrak{g})$.

Lemma 17. *Let* P *be a cominuscule maximal standard parabolic subgroup of* G *(i.e., the unique simple root* $\alpha_P \in \Delta \setminus \Delta(P)$ *appears with coefficient* 1 *in the highest root of* R^+*). Then, the product* $\odot$ *coincides with the cup product in* $H^*(G/P)$*.*

6 Efficient determination of the eigencone

This section is again based on the work [BK_1] due to Belkale-Kumar. The following theorem [BK_1, Theorem 22] determines the saturated tensor semigroup $\Gamma_s(G)$ efficiently. Specifically, as proved by Ressayre (see Theorem 23), the set of inequalities given by (b) of the following theorem is an irredundant set of inequalities determining $\Gamma_s(G)$.

For $G = \mathrm{SL}(n)$, each maximal parabolic P is cominuscule, and hence, by Lemma 17, $\odot_0$ coincides with the standard cup product in $H^*(G/P)$. Thus, the following theorem in this case reduces to Theorem 7 with $d = 1$ in the identity (7).

It may be mentioned that replacing the product $\odot_0$ in the (b)-part of the following theorem by the standard cup product (i.e., Theorem 7 with $d = 1$ in the identity

(7); cf. Remark 8), we get, in general, 'far more' inequalities for simple groups other than SL(n). For example, for G of type B_3 (or C_3), Theorem 7 with $d = 1$ gives rise to 126 inequalities, whereas the following theorem gives only 93 inequalities (cf. [KuLM]).

Theorem 18. *Let G be a connected semisimple group and let $(\lambda_1, \dots, \lambda_s) \in \Lambda_+^s$. Then, the following are equivalent:*

(a) $\lambda = (\lambda_1, \dots, \lambda_s) \in \Gamma_s(G)$.

(b) For every standard maximal parabolic subgroup P in G and every choice of s-tuples $(w_1, \dots, w_s) \in (W^P)^s$ such that

$$[X_{w_1}^P] \odot_0 \cdots \odot_0 [X_{w_s}^P] = [X_e^P] \in (H^*(G/P, \mathbb{Z}), \odot_0),$$

the following inequality holds:

$$I^P_{(w_1,\dots,w_s)} : \qquad \sum_{j=1}^{s} \lambda_j(w_j x_P) \leq 0,$$

where α_{i_P} is the (unique) simple root in $\Delta \setminus \Delta(P)$ and $x_P := x_{i_P}$.

We briefly recall some of the main ingredients which go into the proof of the above theorem and which are of independent interest.

Definition 19. (Maximally destabilizing one parameter subgroups) Let $\mathbb{X}$ be a projective variety with the action of a connected reductive group S and let $\mathbb{L}$ be a S-linearized ample line bundle on $\mathbb{X}$. Introduce the set $M(S)$ of fractional OPS in S. This is the set consisting of the ordered pairs (δ, a), where $\delta \in O(S)$ and $a \in \mathbb{Z}_{>0}$, modulo the equivalence relation $(\delta, a) \simeq (\gamma, b)$ if $\delta^b = \gamma^a$. The equivalence class of (δ, a) is denoted by $[\delta, a]$. An OPS δ of S can be thought of as the element $[\delta, 1] \in M(S)$. The group S acts on $M(S)$ via conjugation: $g \cdot [\delta, a] = [g\delta g^{-1}, a]$. Choose a S-invariant norm $q : M(S) \to \mathbb{R}_+$, where norm means that $q_{|_{M(H)}}$ is the square root of a positive definite quadratic form on the $\mathbb{Q}$-vector space $M(H)$ for a maximal torus H of S. We can extend the definition of $\mu^{\mathbb{L}}(x, \delta)$ to any element $\hat{\delta} = [\delta, a] \in M(S)$ and $x \in \mathbb{X}$ by setting $\mu^{\mathbb{L}}(x, \hat{\delta}) = \frac{\mu^{\mathbb{L}}(x,\delta)}{a}$.

For any unstable (i.e., nonsemistable) point $x \in \mathbb{X}$, define

$$q^*(x) = \inf_{\hat{\delta} \in M(S)} \{q(\hat{\delta}) \mid \mu^{\mathbb{L}}(x, \hat{\delta}) \leq -1\},$$

and the *optimal class*

$$\Lambda(x) = \{\hat{\delta} \in M(S) \mid \mu^{\mathbb{L}}(x, \hat{\delta}) \leq -1, q(\hat{\delta}) = q^*(x)\}.$$

Any $\hat{\delta} \in \Lambda(x)$ is called *Kempf's OPS associated to x.*

By a theorem of Kempf (cf. [Ki, Lemma 12.13]), $\Lambda(x)$ is nonempty and the parabolic $P(\hat{\delta}) := P(\delta)$ (for $\hat{\delta} = [\delta, a]$) does not depend upon the choice of $\hat{\delta} \in \Lambda(x)$. The parabolic $P(\hat{\delta})$ for $\hat{\delta} \in \Lambda(x)$ will be denoted by $P(x)$ and called the *Kempf's parabolic associated to the unstable point x.* Moreover, $\Lambda(x)$ is a single conjugacy class under $P(x)$.

We recall the following theorem due to Ramanan-Ramanathan [RR, Proposition 1.9].

Theorem 20. *For any unstable point $x \in \mathbb{X}$ and $\hat{\delta} = [\delta, a] \in \Lambda(x)$, let*

$$x_o = \lim_{t \to 0} \delta(t) \cdot x \in \mathbb{X}.$$

Then, x_o is unstable and $\hat{\delta} \in \Lambda(x_o)$.

Indication of the Proof of Theorem 18: The implication $(a) \Rightarrow (b)$ of Theorem 18 is of course a special case of Theorem 7.

To prove the implication $(b) \Rightarrow (a)$ in Theorem 18, we need to recall the following result due to Kapovich-Leeb-Millson [KLM]. Suppose that $x = (\bar{g}_1, \dots, \bar{g}_s) \in (G/B)^s$ is an unstable point and $P(x)$ the Kempf's parabolic associated to $\pi(x)$, where $\pi : \mathbb{L} \to \mathbb{L}(\lambda)$ is the map defined above Theorem 7. Let $\hat{\delta} = [\delta, a]$ be a Kempf's OPS associated to $\pi(x)$. Express $\delta(t) = f\gamma(t)f^{-1}$, where $\dot{\gamma} \in \mathfrak{h}_+$. Then, the Kempf's parabolic $P(\gamma)$ is a standard parabolic. Define $w_j \in W/W_{P(\gamma)}$ by $fP(\gamma) \in g_j B w_j P(\gamma)$ for $j = 1, \dots, s$. Let P be a maximal parabolic containing $P(\gamma)$.

Theorem 21. *(i) The intersection $\bigcap_{j=1}^{s} g_j B w_j P \subset G/P$ is the singleton $\{fP\}$.*

(ii) For the simple root $\alpha_{i_P} \in \Delta \backslash \Delta(P)$, $\sum_{j=1}^{s} \lambda_j(w_j x_{i_P}) > 0$.

The equivalence of (a) and (d) in Theorem 2 follows easily by combining Theorems 18 and 5.

Remark 22. The cone $\bar{\Gamma}_3(\mathfrak{g}) \subset \mathfrak{h}_+^3$ is quite explicitly determined for any simple $\mathfrak{g}$ of rank 2 in [KLM, §7]; any simple $\mathfrak{g}$ of rank 3 in [KuLM]; and for $\mathfrak{g} = so(8)$ in [KKM]. It has $12(6+6); 18(9+9); 30(15+15); 41(10+21+10); 93(18+48+27); 93(18+48+27); 294(36+186+36+36); 1290(126+519+519+126); 26661(348+1662+4857+14589+4857+348)$ facets inside $\mathfrak{h}_+^3$ (intersecting the interior of $\mathfrak{h}_+^3$) for $\mathfrak{g}$ of type $A_2; B_2; G_2; A_3; B_3; C_3; D_4; F_4; E_6$ respectively. The notation $30(15+15)$ means that there are 15 (irredundant) inequalities coming from G/P_1 and there are 15 inequalities coming from G/P_2. (The indexing convention is as in [Bo, Planche I - IX].)

The following result is due to Ressayre [R]. In the case $G = \mathrm{SL}(n)$, the result was earlier proved by Knutson-Tao-Woodward [KTW].

Theorem 23. *Let $s \geq 3$. The set of inequalities provided by the (b)-part of Theorem 18 is an irredundant system of inequalities describing the cone $\Gamma_s(G)_{\mathbb{R}}$ generated by $\Gamma_s(G)$ inside $\Lambda_+(\mathbb{R})^s$, i.e., the hyperplanes* *given by the equality in $I^P_{(w_1,\dots,w_s)}$ are precisely those facets of the cone $\Gamma_s(G)_{\mathbb{R}}$ which intersect the interior of $\Lambda_+(\mathbb{R})^s$.*

By Theorem 5, the same result is true for the cone $\bar{\Gamma}_s(\mathfrak{g})$.

Let $\mathfrak{g}$ be a simple simply-laced Lie algebra and let $\sigma : \mathfrak{g} \to \mathfrak{g}$ be a diagram automorphism with fixed subalgebra $\mathfrak{k}$ (which is necessarily a simple Lie algebra again). Let $\mathfrak{b}$ (resp. $\mathfrak{h}$) be a Borel (resp. Cartan) subalgebra of $\mathfrak{g}$ such that they are stable under σ. Then, $\mathfrak{b}^{\mathfrak{k}} := \mathfrak{b}^{\sigma}$ (resp. $\mathfrak{h}^{\mathfrak{k}} := \mathfrak{h}^{\sigma}$) is a Borel (resp. Cartan) subalgebra of $\mathfrak{k}$. Let $\mathfrak{h}_+$ and $\mathfrak{h}^{\mathfrak{k}}_+$ be the dominant chambers in $\mathfrak{h}$ and $\mathfrak{h}^{\mathfrak{k}}$ respectively. Then,

$$\mathfrak{h}^{\mathfrak{k}}_+ = \mathfrak{h}_+ \cap \mathfrak{k}.$$

We have the following result originally conjectured by Belkale-Kumar [BK$_2$] and proved by Belkale-Kumar [BK$_2$], Braley [Bra] and Lee [Le] (case by case).

Theorem 24. *For any $s \geq 1$,*

$$\bar{\Gamma}_s(\mathfrak{k}) = \bar{\Gamma}_s(\mathfrak{g}) \cap (\mathfrak{h}^{\mathfrak{k}}_+)^s.$$

7 Saturation Problem

In Section 2, we defined the saturated tensor semigroup $\Gamma_s(G)$ (for any integer $s \geq 1$) and determined it by describing its facets (cf. Theorems 18 and 23).

Define the *tensor semigroup* for G:

$$\hat{\Gamma}_s(G) = \left\{(\lambda_1, \dots, \lambda_s) \in \Lambda^s_+ : ([\lambda_1] \otimes \cdots \otimes [\lambda_s])^G \neq 0\right\}.$$

It is indeed a semigroup by [K$_2$, Lemma 3.9]. The *saturation problem* aims at comparing these two semigroups. We recall the following result (cf. [K$_2$, Lemma 3.9]).

Lemma 25. *There exists a uniform integer $d > 0$ (depending only upon s and G) such that for any $\lambda = (\lambda_1, \dots, \lambda_s) \in \Gamma_s(G)$, $d\lambda = (d\lambda_1, \dots, d\lambda_s) \in \hat{\Gamma}_s(G)$.*

We now begin with the following definition. We take $s = 3$ as this is the most relevant case to the tensor product decomposition.

Definition 26. An integer $d \geq 1$ is called a *saturation factor* for G, if for any $(\lambda, \mu, \nu) \in \Gamma_3(G)$ such that $\lambda + \mu + \nu \in Q$, we have $(d\lambda, d\mu, d\nu) \in \hat{\Gamma}_3(G)$, where Q is the root lattice of G. Of course, if d is a saturation factor then so is its any multiple. If $d = 1$ is a saturation factor for G, we say that the *saturation property holds for G*.

The *saturation theorem* of Knutson-Tao (cf. Theorem 9) asserts that the saturation property holds for $G = \mathrm{SL}(n)$.

The following general result (though not optimal) on saturation factor is obtained by Kapovich-Millson [KM$_2$] by using the geometry of geodesics in Euclidean buildings and Littelmann's path model. A weaker form of the following theorem was conjectured by Kumar in a private communication to J. Millson (also see [KT, Conjecture]).

Theorem 27. *For any connected simple G, $d = k^2_{\mathfrak{g}}$ is a saturation factor, where $k_{\mathfrak{g}}$ is the least common multiple of the coefficients of the highest root θ of the Lie algebra $\mathfrak{g}$ of G written in terms of the simple roots $\{\alpha_1, \dots, \alpha_\ell\}$.*

Observe that the value of $k_{\mathfrak{g}}$ is 1 for $\mathfrak{g}$ of type $A_\ell (\ell \geq 1)$; it is 2 for $\mathfrak{g}$ of type $B_\ell(\ell \geq 2), C_\ell(\ell \geq 3), D_\ell(\ell \geq 4)$; and it is 6, 12, 60, 12, 6 for $\mathfrak{g}$ of type E_6, E_7, E_8, F_4, G_2 respectively.

Kapovich-Millson determined $\hat{\Gamma}_3(G)$ explicitly for $G = \mathrm{Sp}(4)$ and G_2 (cf. [KM$_1$, Theorems 5.3, 6.1]). In particular, from their description, the following theorem follows easily.

Theorem 28. *The saturation property does not hold for either $G = \mathrm{Sp}(4)$ or G_2. Moreover, 2 is a saturation factor (and no odd integer d is a saturation factor) for $\mathrm{Sp}(4)$, whereas both of 2, 3 are saturation factors for G_2 (and hence any integer $d > 1$ is a saturation factor for G_2).*

It was known earlier that the saturation property fails for G of type B_ℓ (cf. [E]).

Kapovich-Millson [KM$_1$] made the following very interesting conjecture:

Conjecture 29. *If G is simply-laced, then the saturation property holds for G.*

Apart from $G = \mathrm{SL}(n)$, the only other simply-connected, simple, simply-laced group G for which the above conjecture is known so far is $G = \mathrm{Spin}(8)$, proved by Kapovich-Kumar-Millson [KKM, Theorem 5.3] by explicit calculation using Theorem 18.

Finally, we have the following improvement of Theorem 27 for the classical groups $\mathrm{SO}(n)$ and $\mathrm{Sp}(2\ell)$. It was proved by Belkale-Kumar [BK$_2$, Theorems 25 and 26] for the groups $\mathrm{SO}(2\ell + 1)$ and $\mathrm{Sp}(2\ell)$ by using geometric techniques. Sam [S] proved it for $\mathrm{SO}(2\ell)$ (and also for $\mathrm{SO}(2\ell + 1)$ and $\mathrm{Sp}(2\ell)$) via the quiver approach following the proof by Derksen-Weyman [DW] for $G = \mathrm{SL}(n)$. Hong-Shen [HS] show that the spin group $\mathrm{Spin}(2\ell + 1)$ has saturation factor 2.

Theorem 30. *For the groups $\mathrm{SO}(n)$ $(n \geq 7)$, Spin$(2\ell + 1)$ and $\mathrm{Sp}(2\ell)$ $(\ell \geq 2)$, 2 is a saturation factor.*

References

[B$_1$] P. Belkale, *Local systems on $\mathbb{P}^1 - S$ for S a finite set*, Compositio Math. **129** (2001), 67–86.

[B_2] P. Belkale, *Invariant theory of $GL(n)$ and intersection theory of Grassmannians*, IMRN **2004**, no. 69, 3709–3721.

[B_3] P. Belkale, *Geometric proofs of Horn and saturation conjectures*, J. Alg. Geom. **15** (2006), 133–173.

[BK_1] P. Belkale and S. Kumar, *Eigenvalue problem and a new product in cohomology of flag varieties*, Inventiones Math. **166** (2006), 185–228.

[BK_2] P. Belkale and S. Kumar, *Eigencone, saturation and Horn problems for symplectic and odd orthogonal groups*, J. of Algebraic Geom. **19** (2010), 199–242.

[BS] A. Berenstein and R. Sjamaar, *Coadjoint orbits, moment polytopes, and the Hilbert-Mumford criterion*, Journ. Amer. Math. Soc. **13** (2000), 433–466.

[Bra] E. Braley, *Eigencone Problems for Odd and Even Orthogonal Groups*, Ph.D. Thesis (under the supervision of P. Belkale), University of North Carolina, 2012.

[Br] M. Brion, *Restriction de represéntations et projections d'orbites coadjointes (d'apres Belkale, Kumar et Ressayre)*, Séminaire Bourbaki, 64ème année **n^{o}1043**, 2011-2012.

[DW] H. Derksen and J. Weyman, *Semi-invariants of quivers and saturation for Littlewood-Richardson coefficients*, J. Amer. Math. Soc. **13** (2000), 467–479.

[E] A. G. Elashvili, *Invariant algebras*, In: "Lie Groups, their Discrete Subgroups, and Invariant Theory" (ed. E. B. Vinberg), Advances in Soviet Math. **8**, Amer. Math. Soc., Providence, 1992, 57–64.

[Fa] K. Fan, *On a theorem of Weyl concerning eigenvalues of linear transformations, invariant factors, highest weights, and Schubert calculus*, Proc. Natl. Acad. Sci. USA, **35** (1949), 652–655.

[F] W. Fulton, *Eigenvalues, invariant factors, highest weights, and Schubert calculus*, Bull. Amer. Math. Soc. (N.S.) **37** (2000), 209–249.

[HS] J. Hong and L. Shen, *Tensor invariants, Saturation problems, and Dynkin automorphisms*, Preprint (2014).

[Ho] A. Horn, *Eigenvalues of sums of Hermitian matrices*, Pacific J. Math. **12** (1962), 225–241.

[KKM] M. Kapovich, S. Kumar and J. J. Millson, *The eigencone and saturation for Spin(8)*, Pure and Applied Math. Quarterly **5** (2009), 755–780.

[KLM] M. Kapovich, B. Leeb and J. J. Millson, *Convex functions on symmetric spaces, side lengths of polygons and the stability inequalities for weighted configurations at infinity*, Journal of Differential Geometry **81** (2009), 297–354.

[KM_1] M. Kapovich and J. J. Millson, *Structure of the tensor product semigroup*, Asian J. of Math. **10** (2006), 492–540.

[KM_2] M. Kapovich and J. J. Millson, *A path model for geodesics in Euclidean buildings and its applications to representation theory*, Groups, Geometry and Dynamics **2** (2008), 405–480.

[Ki] F. Kirwan, *Cohomology of Quotients in Symplectic and Algebraic Geometry*, Princeton University Press, 1984.

[Kly] A. Klyachko, *Stable bundles, representation theory and Hermitian operators*, Selecta Mathematica **4** (1998), 419–445.

[KT] A. Knutson and T. Tao, *The honeycomb model of $GL_n(\mathbb{C})$ tensor products I: Proof of the saturation conjecture*, J. Amer. Math. Soc. **12** (1999), 1055–1090.

[KTW] A. Knutson, T. Tao and C. Woodward, *The honeycomb model of $GL_n(\mathbb{C})$ tensor products II: Puzzles determine facets of the Littlewood-Richardson cone*, J. Amer. Math. Soc. **17** (2004), 19–48.

[K_1] S. Kumar, *Kac-Moody Groups, their Flag Varieties and Representation Theory*, Progress in Mathematics, vol. **204**, Birkhäuser, 2002.

[K_2] S. Kumar, *Tensor product decomposition*, Proc. of the International Congress of Mathematicians, Hyderabad (India), (2010), 1226–1261.

[K_3] S. Kumar, *A Survey of the Additive Eigenvalue Problem (with Appendix by M. Kapovich)*, Transformation Groups **19** (2014), 1051–1148.

[KuLM] S. Kumar, B. Leeb and J. J. Millson, *The generalized triangle inequalities for rank 3 symmetric spaces of noncompact type*, Contemp. Math. **332** (2003), 171–195.

[Le] B. Lee, *A Comparison of Eigencones Under Certain Diagram Automorphisms*, Ph.D. Thesis (under the supervision of S. Kumar), University of North Carolina, 2012.

[Li] B.V. Lidskii, *The proper values of the sum and product of symmetric matrices*, Dokl. Acad. Nauk SSSR **74** (1950), 769–772.

[N] L. Ness, *A stratification of the null cone via the moment map (with an appendix by D. Mumford)*, Amer. J. Math. **106** (1984), 1281–1329.

[RR] S. Ramanan and A. Ramanathan, *Some remarks on the instability flag*, Tohoku Math. J. **36** (1984), 269–291.

[R] N. Ressayre, *Geometric invariant theory and the generalized eigenvalue problem*, Inventiones Math. **180** (2010), 389–441.

[Sj] R. Sjamaar, *Convexity properties of the moment mapping reexamined*, Advances Math. **138** (1998), 46–91.

[S] S. Sam, *Symmetric quivers, invariant theory, and saturation theorems for the classical groups*, Adv. Math. **229** (2012), 1104–1135.

[TF] R.C. Thompson and L. Freede, *On the eigenvalues of a sum of Hermitian matrices*, *Linear Algebra Appl.* **4** (1971), 369–371.

[W] H. Weyl, *Das asymptotische verteilungsgesetz eigenwerte lineare partieller differential-gleichungen*, Math. Annalen **71** (1912), 441–479.

[Wi] H. Wielandt, *An extremum property of sums of eigenvalues*, Proc. Amer. Math. Soc. **6** (1955), 106–110.

[Z] A. Zelevinsky, *Littlewood-Richardson semigroups*, In: "New Perspectives in Algebraic Combinatorics", MSRI Publ. **38** (1999), Cambridge Univ. Press, Cambridge, 337–345.

Shrawan Kumar [shrawan@email.unc.edu] is John R. and Louise S. Parker Distinguished Professor at the University of North Carolina, Chapel Hill, USA. He was an invited speaker at the ICM, 2010. He is a Fellow of the American Math Society.