

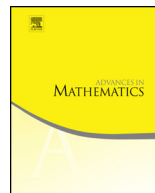


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The multiplicative eigenvalue problem and deformed quantum cohomology



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ABSTRACT

We construct deformations of the small quantum cohomology rings of homogeneous spaces G/P , and obtain an irredundant set of inequalities determining the multiplicative eigen polytope for the compact form K of G .

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1. Introduction

Let G be a simple, connected, simply-connected complex algebraic group. We choose a Borel subgroup B and a maximal torus $H \subset B$. We denote their Lie algebras by \mathfrak{g} , \mathfrak{b} , \mathfrak{h} respectively. Let $R = R_{\mathfrak{g}} \subset \mathfrak{h}^*$ be the set of roots of \mathfrak{g} and let R^+ be the set of positive roots (i.e., the set of roots of \mathfrak{b}). Let $\Delta = \{\alpha_1, \dots, \alpha_\ell\} \subset R^+$ be the set of simple roots.

Consider the *fundamental alcove* $\mathcal{A} \subset \mathfrak{h}$ defined by

$$\mathcal{A} = \{\mu \in \mathfrak{h} : \alpha_i(\mu) \geq 0 \text{ for all simple roots } \alpha_i \text{ and } \theta_o(\mu) \leq 1\},$$

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where θ_o is the highest root of \mathfrak{g} . Then, \mathcal{A} parameterizes the K -conjugacy classes of K under the map $C : \mathcal{A} \rightarrow K/\text{Ad } K$,

$$\mu \mapsto c(\text{Exp}(2\pi i\mu)),$$

where K is a maximal compact subgroup of G and $c(\text{Exp}(2\pi i\mu))$ denotes the K -conjugacy class of $\text{Exp}(2\pi i\mu)$. Fix a positive integer $n \geq 3$ and define the *multiplicative eigen polytope*

$$\mathcal{C}_n := \{(\mu_1, \dots, \mu_n) \in \mathcal{A}^n : 1 \in C(\mu_1) \dots C(\mu_n)\}.$$

Then, \mathcal{C}_n is a rational convex polytope with nonempty interior in \mathfrak{h}^n . Our aim is to describe the facets (i.e., the codimension one faces) of \mathcal{C}_n which meet the interior of \mathcal{A}^n .

We need to introduce some more notation before we can state our results. Let P be a standard parabolic subgroup (i.e., $P \supset B$) and let $L \subset P$ be its Levi subgroup containing H . Then, $B_L := B \cap L$ is a Borel subgroup of L . We denote the Lie algebras of P, L, B_L by the corresponding Gothic characters: $\mathfrak{p}, \mathfrak{l}, \mathfrak{b}_L$ respectively. Let $R_{\mathfrak{l}}$ be the set of roots of \mathfrak{l} and $R_{\mathfrak{l}}^+$ be the set of roots of \mathfrak{b}_L . We denote by Δ_P the set of simple roots contained in $R_{\mathfrak{l}}$ and we set

$$S_P := \Delta \setminus \Delta_P.$$

For any $1 \leq j \leq \ell$, define the element $x_j \in \mathfrak{h}$ by

$$\alpha_i(x_j) = \delta_{i,j}, \quad \forall 1 \leq i \leq \ell.$$

Let W be the Weyl group of G and let W^P be the set of the minimal length representatives in the cosets of W/W_P , where W_P is the Weyl group of P . For any $w \in W^P$, let $X_w^P := \overline{BwP/P} \subset G/P$ be the corresponding Schubert variety and let $\{\sigma_w^P\}_{w \in W^P}$ be the Poincaré dual (dual to the fundamental class of X_w^P) basis of $H^*(G/P, \mathbb{Z})$.

We begin with the following theorem. It was proved by Biswas [14] in the case $G = \text{SL}_2$; by Belkale [5] for $G = \text{SL}_m$ (and in this case a slightly weaker result by Agnihotri–Woodward [1] where the inequalities were parameterized by $\langle \sigma_{u_1}^P, \dots, \sigma_{u_n}^P \rangle_d \neq 0$); and by Teleman–Woodward [39] for general G . It may be recalled that the precursor to these results was the result due to Klyachko [23] determining the additive eigencone for SL_m .

Theorem 1.1. *Let $(\mu_1, \dots, \mu_n) \in \mathcal{A}^n$. Then, the following are equivalent:*

- (a) $(\mu_1, \dots, \mu_n) \in \mathcal{C}_n$.
- (b) For any standard maximal parabolic subgroup P of G , any $u_1, \dots, u_n \in W^P$, and any $d \geq 0$ such that the Gromov–Witten invariant (cf. Definition 2.1)

$$\langle \sigma_{u_1}^P, \dots, \sigma_{u_n}^P \rangle_d = 1,$$

the following inequality is satisfied:

$$\mathcal{J}_{(u_1, \dots, u_n; d)}^P : \sum_{k=1}^n \omega_P(u_k^{-1} \mu_k) \leq d,$$

where ω_P is the fundamental weight ω_{i_P} such that α_{i_P} is the unique simple root in S_P .

Even though this result describes the inequalities determining the polytope \mathcal{C}_n , however for groups other than G of type A_ℓ , the above system of inequalities has redundancies. The aim of our work is to give an irredundant subsystem of inequalities determining the polytope \mathcal{C}_n .

To achieve this, similar to the notion of Levi-movability of Schubert varieties in $X = G/P$ (for any parabolic P) introduced in [12], which gives rise to a deformed product in the cohomology $H^*(X)$, we have introduced here the notion of *quantum Levi-movability* resulting into a deformed product in the quantum cohomology $QH^*(X)$ parameterized by $\{\tau_i\}_{\alpha_i \in S_P}$ as follows. As a $\mathbb{Z}[q, \tau]$ -module, it is the same as $H^*(X, \mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{Z}[q, \tau]$, where q (resp. τ) stands for multi variables $\{q_i\}_{\alpha_i \in S_P}$ (resp. $\{\tau_i\}_{\alpha_i \in S_P}$). For $u, v \in W^P$, define the $\mathbb{Z}[q, \tau]$ -linear *quantum deformed product* by

$$\sigma_u^P \circledast \sigma_v^P = \sum_{d \geq 0 \in H_2(X, \mathbb{Z}); w \in W^P} \left(\prod_{\alpha_i \in S_P} \tau_i^{A_i(u, v, w, d)} \right) q^d \langle \sigma_u^P, \sigma_v^P, \sigma_w^P \rangle_d \sigma_{w_o w w_o^P}^P, \tag{1}$$

where w_o (resp. w_o^P) is the longest element of W (resp. W^P),

$$A_i(u, v, w, d) = (\chi_e - \chi_u - \chi_v - \chi_w)(x_i) + \frac{2a_i g^*}{\langle \alpha_i, \alpha_i \rangle},$$

$\chi_w = \sum_{\beta \in (R^+ \setminus R_1^+) \cap w^{-1}R^+} \beta$, g^* is the dual Coxeter number of \mathfrak{g} , $\langle \cdot, \cdot \rangle$ is the normalized Killing form so that $\langle \theta_o, \theta_o \rangle = 2$ and a_i is defined by the identity (7). It is shown that, for a cominuscule maximal parabolic subgroup P , the deformed product coincides with the original product in the quantum cohomology of X (cf. Lemma 3.6).

We have the following result obtained by crucially using deformation theory (cf. Theorem 3.3 and Lemma 3.4).

Theorem 1.2. *Let $u_1, \dots, u_n \in W^P$ and $d = (a_i)_{\alpha_i \in S_P} \in H_2(X, \mathbb{Z})$ be such that $\langle \sigma_{u_1}^P, \sigma_{u_2}^P, \dots, \sigma_{u_n}^P \rangle_d \neq 0$. Then, for any $\alpha_i \in S_P$,*

$$(\chi_e - \sum_{k=1}^n \chi_{u_k})(x_i) + g^* \langle x_i, \tilde{d} \rangle \geq 0,$$

where $\tilde{d} = \sum_{\alpha_j \in S_P} a_j \alpha_j^\vee$ and α_j^\vee are the simple coroots.

Evaluating each $\tau_i = 0$ in the identity (1) (which is well defined because of the above theorem and the identity (11)), we get

$$\sigma_u^P \otimes_0 \sigma_v^P = \sum_{d,w} q^d \langle \sigma_u^P, \sigma_v^P, \sigma_w^P \rangle_d \sigma_{w_0 w w_0^P}^P,$$

where the sum is restricted over those $d \geq 0 \in H_2(X, \mathbb{Z})$ and $w \in W^P$ so that $A_i(u, v, w, d) = 0$ for all $\alpha_i \in S_P$. We shall denote the coefficient of $q^d \sigma_{w_0 w w_0^P}^P$ in $\sigma_u^P \otimes_0 \sigma_v^P$ by $\langle \sigma_u^P, \sigma_v^P, \sigma_w^P \rangle_d^{\otimes 0}$. Similarly, we shall denote the coefficient of $q^d \sigma_{w_0 u_n w_0^P}^P$ in $\sigma_{u_1}^P \otimes_0 \dots \otimes_0 \sigma_{u_{n-1}}^P$ by $\langle \sigma_{u_1}^P, \dots, \sigma_{u_n}^P \rangle_d^{\otimes 0}$. We give an equivalent characterization of when $\langle \sigma_{u_1}^P, \dots, \sigma_{u_n}^P \rangle_d^{\otimes 0} \neq 0$ in Theorem 3.16. Now our first main theorem on the multiplicative eigen problem is the following (cf. Theorem 4.1):

Theorem 1.3. *Let $(\mu_1, \dots, \mu_n) \in \mathcal{A}^n$. Then, the following are equivalent:*

(a) $(\mu_1, \dots, \mu_n) \in \mathcal{C}_n$.

(b) For any standard maximal parabolic subgroup P of G , any $u_1, \dots, u_n \in W^P$, and any $d \geq 0$ such that

$$\langle \sigma_{u_1}^P, \dots, \sigma_{u_n}^P \rangle_d^{\otimes 0} = 1,$$

the following inequality is satisfied:

$$\mathcal{I}_{(u_1, \dots, u_n; d)}^P : \sum_{k=1}^n \omega_P(u_k^{-1} \mu_k) \leq d.$$

The role of the flag varieties $(G/B)^n$ in [12] is replaced here by the quasi-parabolic moduli stack Parbun_G of principal G -bundles on \mathbb{P}^1 with parabolic structure at the marked points $b_1, \dots, b_n \in \mathbb{P}^1$. The proof makes crucial use of the canonical reduction of parabolic G -bundles and a certain *Levification process* of principal P -bundles (cf. Subsection 3.8), which allows degeneration of a principal P -bundle to a L -bundle (a process familiar in the theory of vector bundles as reducing the structure group to a Levi subgroup of P).

Our second main theorem on the multiplicative eigen problem (cf. Theorem 8.1) asserts that the inequalities given by the (b)-part of the above theorem provide an irredundant system of inequalities defining the polytope \mathcal{C}_n . Specifically, we have the following result. This result for $G = \text{SL}_m$ was proved by Belkale combining the works [8, 9] (see Remark 8.6). It is the multiplicative analogue of Ressayre’s result [37]. Our proof is a certain adaptation of Ressayre’s proof. There are additional technical subtleties involving essential use of the moduli stack of G -bundles and its smoothness, conformal field theory, affine flag varieties and the classification of line bundles on them.

Theorem 1.4. *Let $n \geq 3$. The inequalities*

$$\mathcal{I}_{(u_1, \dots, u_n; d)}^P : \sum_{k=1}^n \omega_P(u_k^{-1} \mu_k) \leq d,$$

given by part (b) of the above theorem (as we run through the standard maximal parabolic subgroups P , n -tuples $(u_1, \dots, u_n) \in (W^P)^n$ and non-negative integers d such that $\langle \sigma_{u_1}^P, \dots, \sigma_{u_n}^P \rangle_d^{\otimes 0} = 1$) are pairwise distinct (even up to scalar multiples) and form an irredundant system of inequalities defining the eigen polytope \mathcal{C}_n inside \mathcal{A}^n , i.e., the hyperplanes given by the equality in $\mathcal{S}_{(u_1, \dots, u_n; d)}^P$ are precisely the (codimension one) facets of the polytope \mathcal{C}_n which intersect the interior of \mathcal{A}^n .

To show that the inequality $\mathcal{S}_{(u_1, \dots, u_n; d)}^P$ can not be dropped, we produce (following Ressayre’s general strategy [37]) a collection of points of \mathcal{C}_n for which the above inequality is an equality, and such that their convex span has the dimension of a facet (i.e., $-1 + n \dim \mathfrak{h}$). This is achieved by the parabolic analogue of Narasimhan–Seshadri theorem for the Levi subgroup L resulting in a description of \mathcal{C}_n for L in terms of the non-vanishing of the space of global sections of certain line bundles on the moduli stack $\text{Parbun}_L(d)$ of quasi-parabolic L -bundles of degree d (cf. Theorem 5.2 applied to the semisimple part of L and Corollary 7.6). To be able to use the parabolic analogue of Narasimhan–Seshadri theorem, we need a certain *Levi twisting*, which produces an isomorphism of $\text{Parbun}_L(d)$ with $\text{Parbun}_L(d \pm 1)$ (cf. Lemma 7.4).

Section 8 of the paper is greatly influenced by Ressayre [37] as will be clear to any informed reader.

It may be remarked that our work completes the multiplicative eigenvalue problem for compact simply-connected groups in the sense that we determine the multiplicative eigen polytope \mathcal{C}_n by giving an irredundant system of inequalities defining it. The problem of a recursive description of \mathcal{C}_n in terms of eigen polytopes of “smaller groups” remains open for general G (for $G = \text{SL}_m$ this has been carried out in [10]).

2. Notation and preliminaries

2.1. Notation

Let G be a semisimple, connected, simply-connected complex algebraic group. We choose a Borel subgroup B and a maximal torus $H \subset B$ and let $W = W_G := N_G(H)/H$ be the associated Weyl group, where $N_G(H)$ is the normalizer of H in G . Let $P \supseteq B$ be a (standard) parabolic subgroup of G and let $U = U_P$ be its unipotent radical. Consider the Levi subgroup $L = L_P$ of P containing H , so that P is the semi-direct product of U and L . Then, $B_L := B \cap L$ is a Borel subgroup of L . Let $\Lambda = \Lambda(H)$ denote the character group of H , i.e., the group of all the algebraic group morphisms $H \rightarrow \mathbb{G}_m$. Clearly, W acts on Λ . We denote the Lie algebras of G, B, H, P, U, L, B_L by the corresponding Gothic characters: $\mathfrak{g}, \mathfrak{b}, \mathfrak{h}, \mathfrak{p}, \mathfrak{u}, \mathfrak{l}, \mathfrak{b}_L$ respectively. We will often identify an element λ of Λ (via its derivative $\dot{\lambda}$) by an element of \mathfrak{h}^* . Let $R = R_{\mathfrak{g}} \subset \mathfrak{h}^*$ be the set of roots of \mathfrak{g} with respect to the Cartan subalgebra \mathfrak{h} and let R^+ be the set of positive roots (i.e., the set of roots of \mathfrak{b}). Similarly, let $R_{\mathfrak{l}}$ be the set of roots of \mathfrak{l} with respect to \mathfrak{h} and $R_{\mathfrak{l}}^+$ be the set of roots of \mathfrak{b}_L . Let $\Delta = \{\alpha_1, \dots, \alpha_\ell\} \subset R^+$ be the set of simple roots, $\{\alpha_1^\vee, \dots, \alpha_\ell^\vee\} \subset \mathfrak{h}$ the

corresponding simple coroots and $\{s_1, \dots, s_\ell\} \subset W$ the corresponding simple reflections, where ℓ is the rank of G . We denote by Δ_P the set of simple roots contained in R_1 and we set

$$S_P := \Delta \setminus \Delta_P.$$

For any $1 \leq j \leq \ell$, define the element $x_j \in \mathfrak{h}$ by

$$\alpha_i(x_j) = \delta_{i,j}, \quad \forall 1 \leq i \leq \ell. \tag{2}$$

Further, define the element \bar{x}_j by

$$\bar{x}_j = N_j x_j, \tag{3}$$

where N_j is the smallest positive integer such that $N_j x_j$ is in the coroot lattice $Q^\vee \subset \mathfrak{h}$ of G .

Recall that if W_P is the Weyl group of P (which is, by definition, the Weyl Group W_L of L), then each coset of W/W_P contains a unique member w of minimal length. This satisfies:

$$wB_L w^{-1} \subseteq B. \tag{4}$$

Let W^P be the set of the minimal length representatives in the cosets of W/W_P .

For any $w \in W^P$, define the Schubert cell:

$$C_w^P := BwP/P \subset X^P := G/P.$$

Then, it is a locally closed subvariety of the flag variety X^P , isomorphic with the affine space $\mathbb{A}^{\ell(w)}$, $\ell(w)$ being the length of w . Its closure is denoted by X_w^P , which is an irreducible (projective) subvariety of X^P of dimension $\ell(w)$. We denote the point $wP \in C_w^P$ by \dot{w} . We abbreviate X_w^B by X_w . We define the *shifted* Schubert cell $\Lambda_w^P := w^{-1}BwP/P$, and its closure is denoted by $\bar{\Lambda}_w^P$. Then, B_L keeps Λ_w^P (and hence $\bar{\Lambda}_w^P$) stable by (4).

Let $\mu(X_w^P)$ denote the fundamental class of X_w^P considered as an element of the singular homology with integral coefficients $H_{2\ell(w)}(X^P, \mathbb{Z})$ of X^P . Then, from the Bruhat decomposition, the elements $\{\mu(X_w^P)\}_{w \in W^P}$ form a \mathbb{Z} -basis of $H_*(X^P, \mathbb{Z})$. Let $\{\sigma_w^P\}_{w \in W^P}$ be the Poincaré dual basis of the singular cohomology with integral coefficients $H^*(X^P, \mathbb{Z})$. Thus, $\sigma_w^P \in H^{2(\dim X^P - \ell(w))}(X^P, \mathbb{Z})$.

An element $\lambda \in \Lambda$ is called dominant (resp. dominant regular) if $\dot{\lambda}(\alpha_i^\vee) \geq 0$ (resp. $\dot{\lambda}(\alpha_i^\vee) > 0$) for all the simple coroots α_i^\vee . Let Λ_+ (resp. Λ_{++}) denote the set of all the dominant (resp. dominant regular) characters. We denote the fundamental weights by $\{\omega_i\}_{1 \leq i \leq \ell}$, i.e.,

$$\omega_i(\alpha_j^\vee) = \delta_{i,j}.$$

For any $\lambda \in \Lambda$, we have a G -equivariant line bundle $\mathcal{L}(\lambda)$ on G/B associated to the principal B -bundle $G \rightarrow G/B$ via the one dimensional B -module λ^{-1} . (Any $\lambda \in \Lambda$ extends uniquely to a character of B .) The one dimensional B -module λ is also denoted by \mathbb{C}_λ . If λ vanishes on $\{\alpha_i^\vee\}_{\alpha_i \in \Delta_P}$, it defines a character of P and hence a line bundle $\mathcal{L}_P(\lambda)$ on X^P associated to the character λ^{-1} of P . It is easy to see that

$$\int_{X_{s_i}^P} c_1(\mathcal{L}_P(\lambda)) = \lambda(\alpha_i^\vee), \text{ for any } \alpha_i \in S_P. \tag{5}$$

For $w \in W^P$, define $\chi_w = \chi_w^P \in \mathfrak{h}^*$ by

$$\chi_w = \sum_{\beta \in (R^+ \setminus R_1^+) \cap w^{-1}R^+} \beta = \rho - 2\rho^L + w^{-1}\rho, \tag{6}$$

where ρ (resp. ρ^L) is half the sum of roots in R^+ (resp. in R_1^+).

All the schemes are considered over the base field of complex numbers \mathbb{C} . The varieties are reduced (but not necessarily irreducible) schemes.

2.2. Quantum cohomology of X^P

We refer the reader to [26,18] for the foundations of small quantum cohomology (also see [19]). Let $X = X^P$ be the flag variety as above, where P is any standard parabolic subgroup. Then,

$$\{\mu(X_{s_i}^P)\}_{\alpha_i \in S_P}$$

is a \mathbb{Z} -basis of $H_2(X, \mathbb{Z})$.

Introduce the variables q_i associated to each $\alpha_i \in S_P$. For

$$d = \sum_{\alpha_i \in S_P} a_i \mu(X_{s_i}^P) \in H_2(X, \mathbb{Z}), \tag{7}$$

let $q^d := \prod_{\alpha_i \in S_P} q_i^{a_i}$. We say $d \geq 0$ if each $a_i \geq 0$. We denote the class d by $(a_i)_{\alpha_i \in S_P}$.

Definition 2.1. Let $u_1, \dots, u_n \in W^P$ and $d \geq 0 \in H_2(X)$. Fix distinct points $b_1, \dots, b_n \in \mathbb{P}^1$, and a general point $(g_1, \dots, g_n) \in G^m$. Let

$$\langle \sigma_{u_1}^P, \sigma_{u_2}^P, \dots, \sigma_{u_n}^P \rangle_d \tag{8}$$

be the number of maps (count as 0 if infinite) $f : \mathbb{P}^1 \rightarrow X$ of degree d (i.e., $f_*[\mathbb{P}^1] = d \in H_2(X)$) such that $f(b_k) \in g_k C_{u_k}^P$, $k = 1, \dots, n$. These are called *Gromov–Witten numbers*.

Definition 2.2. Call a tuple $(P; u_1, \dots, u_n; d)$ as above *quantum non-null* if there are maps f (possibly infinitely many) in the setting of Definition 2.1. This notion will play a role in Section 4.

The space of maps $\mathbb{P}^1 \rightarrow X$ of degree d is a smooth variety of dimension $\dim X + \int_d c_1(T_X)$, where T_X is the tangent bundle of X . Therefore, (8) is zero unless

$$\sum_{k=1}^n \text{codim}[\bar{\Lambda}_{u_k}^P] = \dim X + \int_d c_1(T_X). \tag{9}$$

Let w_o (resp. w_o^P) be the longest element of the Weyl group W (resp. W_P). Now, the quantum product in $H^*(X, \mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{Z}[q_i]_{\alpha_i \in S_P}$ is defined by

$$\sigma_u^P \star \sigma_v^P = \sum_{d \geq 0} q^d \langle \sigma_u^P, \sigma_v^P, \sigma_w^P \rangle_d \sigma_{w_o w v u^P}^P, \tag{10}$$

giving rise to a graded associative and commutative ring, where we assign the degree of q_i to be $\int_{X_{S_i}^P} c_1(T_X)$, which is clearly equal to $2 - 2\rho^L(\alpha_i^\vee)$ by the equation (5).

We note that there exist maps $\mathbb{P}^1 \rightarrow X$ of any degree $d \geq 0$.

3. Quantum Levi-movability and a deformed product in the quantum cohomology of X^P

Consider a commutative and associative ring R over \mathbb{Z} freely (additively) generated by $\{e_u\}_{u \in I}$. Write

$$e_u \cdot e_v = \sum c_{u,v}^w e_w, \text{ for } c_{u,v}^w \in \mathbb{Z}.$$

Consider a multigrading $\gamma : I \rightarrow \mathbb{Z}^S$, where S is a set with m elements, such that whenever $c_{u,v}^w \neq 0$, we have

$$\gamma(w) - \gamma(u) - \gamma(v) \geq 0,$$

where an element $\vec{a} = (a_i)_{i \in S} \in \mathbb{Z}^S$ is ≥ 0 if each $a_i \geq 0$. Introduce $m = |S|$ variables $\tau_i, i \in S$. For $\vec{a} \in \mathbb{Z}^S$, define $\tau^{\vec{a}} = \prod_{i \in S} \tau_i^{a_i}$. Define a new product \odot_τ on $R \otimes_{\mathbb{Z}} \mathbb{Z}[\tau_i]_{i \in S}$ by

$$e_u \odot_\tau e_v = \sum \tau^{\gamma(w) - \gamma(u) - \gamma(v)} c_{u,v}^w e_w.$$

Lemma 3.1.

- (1) \odot_τ is a commutative and associative ring.
- (2) Putting all $\tau_i = 0$ gives a commutative and associative multigraded ring (i.e., the product respects the multigrading). More specifically, the ring structure \odot_0 is given by the following:

$$e_u \circ_0 e_v = \sum c_{u,v}^w e_w,$$

where the sum is restricted over those w such that $\gamma(w) = \gamma(u) + \gamma(v)$.

Example 3.2. The deformed product in $H^*(X^P)$ as introduced by Belkale–Kumar in [12] comes from such a situation with

$$\gamma(u) = (\chi_u(x_i))_{\alpha_i \in S_P},$$

and the Schubert basis $\{\sigma_w^P\}_{w \in W^P}$, where χ_u is defined by the identity (6).

Let $X = X^P$ be any flag variety. Recall the definition of the small quantum cohomology of X from Section 2. We give the definition of a certain deformation of the quantum product in X as below.

We begin with the following result, which will be proved towards the end of this section.

Theorem 3.3. *Let $u_1, \dots, u_n \in W^P$ and $d = (a_i)_{\alpha_i \in S_P} \in H_2(X, \mathbb{Z})$ be such that $\langle \sigma_{u_1}^P, \sigma_{u_2}^P, \dots, \sigma_{u_n}^P \rangle_d \neq 0$. Then, for any $\alpha_i \in S_P$,*

$$(\chi_e - \sum_{k=1}^n \chi_{u_k})(x_i) + \sum_{\alpha \in R^+ \setminus R_1^+} \alpha(x_i) \alpha(\tilde{d}) \geq 0,$$

where $\tilde{d} = \sum_{\alpha_j \in S_P} a_j \alpha_j^\vee$.

Consider the normalized Killing form $\langle \cdot, \cdot \rangle$ on \mathfrak{h}^* normalized so that $\langle \theta_o, \theta_o \rangle = 2$, where θ_o is the highest root of \mathfrak{g} . This gives rise to an identification

$$\kappa : \mathfrak{h}^* \rightarrow \mathfrak{h}.$$

It is easy to see that

$$\kappa(\omega_i) = \frac{\langle \alpha_i, \alpha_i \rangle}{2} x_i. \tag{11}$$

The following form is studied in [16, Chap. VI, §1.12]. Even though well-known, we did not find the following lemma in its present form in the literature (and hence we have included a proof).

Lemma 3.4. *For any $h, h' \in \mathfrak{h}$,*

$$\sum_{\alpha \in R} \alpha(h) \alpha(h') = 2g^* \langle h, h' \rangle,$$

where $g^* := 1 + \langle \rho, \theta_o^\vee \rangle$ is the dual Coxeter number of \mathfrak{g} .

Proof. Consider the bilinear form on \mathfrak{h} given by $\langle h, h' \rangle' := \sum_{\alpha \in R} \alpha(h)\alpha(h')$. It is W -invariant and hence it is a multiple of the original Killing form, i.e., $\langle h, h' \rangle' = z\langle h, h' \rangle$, for some constant z . To calculate z ,

$$\begin{aligned} 2z &= \langle \theta_o^\vee, \theta_o^\vee \rangle' \\ &= \sum_{\alpha \in R} \alpha(\theta_o^\vee)^2 \\ &= 4 + 2 \sum_{\alpha \in R^+} \alpha(\theta_o^\vee), \text{ since } \alpha(\theta_o^\vee) \in \{0, 1\} \forall \alpha \in R^+ \setminus \{\theta_o\} \\ &= 4 + 4\rho(\theta_o^\vee) \\ &= 4g^*. \quad \square \end{aligned}$$

Theorem 3.3 and the general deformation principle spelled out in **Lemma 3.1** allows us to give the following deformed product in the quantum cohomology of X .

Definition 3.5. Introduce the τ -deformation of the quantum cohomology of X as follows:

As a $\mathbb{Z}[q, \tau]$ -module, it is the same as $H^*(X, \mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{Z}[q, \tau]$, where q (resp. τ) stands for multi variables $\{q_i\}_{\alpha_i \in S_P}$ (resp. $\{\tau_i\}_{\alpha_i \in S_P}$). For $u, v \in W^P$, define the $\mathbb{Z}[q, \tau]$ -linear deformed product by

$$\sigma_u^P \circledast \sigma_v^P = \sum_{d \geq 0 \in H_2(X, \mathbb{Z}); w \in W^P} \left(\prod_{\alpha_i \in S_P} \tau_i^{A_i(u, v, w, d)} \right) q^d \langle \sigma_u^P, \sigma_v^P, \sigma_w^P \rangle_d \sigma_{w_o w w_o^P}^P,$$

where (for $d = (a_i)_{\alpha_i \in S_P}$)

$$A_i(u, v, w, d) = (\chi_e - \chi_u - \chi_v - \chi_w)(x_i) + \frac{2a_i g^*}{\langle \alpha_i, \alpha_i \rangle}. \tag{12}$$

Using **Lemma 3.4** and the equation (11) (and observing that for $\alpha \in R_1^+$, $\alpha(x_i) = 0$ for any $\alpha_i \in S_P$), we get another expression:

$$A_i(u, v, w, d) = (\chi_e - \chi_u - \chi_v - \chi_w)(x_i) + \sum_{\alpha \in R^+ \setminus R_1^+} \alpha(x_i) \alpha(\tilde{d}).$$

Evaluating each $\tau_i = 0$ in the above (which is well defined because of **Theorem 3.3**), we get

$$\sigma_u^P \circledast_0 \sigma_v^P = \sum_{d, w} q^d \langle \sigma_u^P, \sigma_v^P, \sigma_w^P \rangle_d \sigma_{w_o w w_o^P}^P,$$

where the sum is restricted over those $d \geq 0 \in H_2(X, \mathbb{Z})$ and $w \in W^P$ so that $A_i(u, v, w, d) = 0$ for all $\alpha_i \in S_P$. We shall denote the coefficient of $q^d \sigma_{w_o w w_o^P}^P$ in $\sigma_u^P \circledast_0 \sigma_v^P$ by $\langle \sigma_u^P, \sigma_v^P, \sigma_w^P \rangle_d^{\circledast_0}$.

From the general deformation principle given in [Lemma 3.1](#), taking the multigraded function $\gamma = (\gamma_i)_{\alpha_i \in S_P}$ defined by

$$\gamma_i(q^d \sigma_w^P) = \chi_w(x_i) + \frac{2a_i g^*}{\langle \alpha_i, \alpha_i \rangle} \text{ for } d = (a_j)_{\alpha_j \in S_P} \in H_2(X, \mathbb{Z}),$$

it follows that \otimes and \otimes_0 give associative (and commutative) products.

Lemma 3.6. *Let P be a cominuscule maximal standard parabolic subgroup of G (i.e., the unique simple root $\alpha_{i_P} \in S_P$ appears with coefficient 1 in the highest root of R^+). Then, the deformed product \otimes coincides with the quantum product \star in $H^*(X^P) \otimes_{\mathbb{Z}} \mathbb{Z}[q_{i_P}]$.*

Proof. By the definition of \otimes , it suffices to show that for any $u, v, w \in W^P$ and $d = a_{i_P} \in H_2(X^P, \mathbb{Z})$, such that $\langle \sigma_u^P, \sigma_v^P, \sigma_w^P \rangle_d \neq 0$,

$$A_{i_P}(u, v, w, d) = 0. \tag{13}$$

Since P is cominuscule, by [\[12\]](#), Proof of Lemma 19,

$$\chi_w(x_{i_P}) = \text{codim}(\Lambda_w^P : X^P). \tag{14}$$

Moreover, the quantum cohomological degree of q_{i_P} equals

$$\int_{X_{s_{i_P}}^P} c_1(T_{X^P}) = \sum_{\alpha \in R^+ \setminus R_1^+} \alpha(\alpha_{i_P}^\vee), \text{ by equation (5)}. \tag{15}$$

Thus, since $\langle \sigma_u^P, \sigma_v^P, \sigma_w^P \rangle_d \neq 0$, we get (by equating the cohomological degrees on the two sides of [\(10\)](#))

$$\text{codim}(\Lambda_u^P : X^P) + \text{codim}(\Lambda_v^P : X^P) = \dim \Lambda_w^P + a_{i_P} \cdot \text{degree } q_{i_P}. \tag{16}$$

Combining equations [\(14\)](#), [\(15\)](#) and [\(16\)](#) with [Lemma 3.4](#), we get equation [\(13\)](#) (since $\alpha(x_{i_P}) = 1$ for all $\alpha \in R^+ \setminus R_1^+$, P being cominuscule). \square

3.1. The enumerative problem of small quantum cohomology in terms of principal bundles

Let \mathcal{E} be a principal right G -bundle on \mathbb{P}^1 . It is well-known that sections $f : \mathbb{P}^1 \rightarrow \mathcal{E}/P$ are in one to one correspondence with reductions of the structure group of \mathcal{E} to P . (This correspondence is true for \mathbb{P}^1 replaced by any scheme Y .) This correspondence works as follows: Given f , let \mathcal{P} be the right P -bundle with fiber $f(x)P \subseteq \mathcal{E}_x$ over $x \in \mathbb{P}^1$. It is then easy to see that there is a canonical isomorphism of principal G -bundles $\mathcal{P} \times^P G \rightarrow \mathcal{E}$. For $\mathcal{E} = \epsilon_G$, the trivial bundle $\mathbb{P}^1 \times G \rightarrow \mathbb{P}^1$,

- (1) sections f correspond to maps $\bar{f} : \mathbb{P}^1 \rightarrow X^P$.
- (2) For $\alpha_i \in S_P$, let $\mathcal{E}(\omega_i) := \epsilon \boxtimes \mathcal{L}_P(\omega_i)$ be the corresponding line bundle on $\mathcal{E}/P = \mathbb{P}^1 \times X^P$, where ϵ is the trivial line bundle on \mathbb{P}^1 . Then, \bar{f} has degree $d = (a_i)_{\alpha_i \in S_P}$ if $c_1(f^*(\mathcal{E}(\omega_i))) = a_i$ (using the identity (5)).

Let \mathcal{E} be a principal G -bundle on \mathbb{P}^1 . We want to state an enumerative problem for \mathcal{E} that corresponds to that of Definition 2.1 for $\mathcal{E} = \epsilon_G$. Fix distinct points $b_1, \dots, b_n \in \mathbb{P}^1$, $u_1, \dots, u_n \in W^P$ and $d = (a_i)_{\alpha_i \in S_P} \geq 0 \in H_2(X^P)$. Fix general choices of $\bar{g}_k \in \mathcal{E}_{b_k}/B$, where \mathcal{E}_{b_k} is the fiber of \mathcal{E} over b_k . The enumerative problem, which gives the Gromov–Witten numbers $\langle \sigma_{u_1}^P, \dots, \sigma_{u_n}^P \rangle_d$ in the case $\mathcal{E} = \epsilon_G$, is the following: Count the number of sections $f : \mathbb{P}^1 \rightarrow \mathcal{E}/P$ (count as 0 if infinite), such that

- (1) $c_1(f^*(\mathcal{E}(\omega_i))) = a_i$ for each $\alpha_i \in S_P$, where $\mathcal{E}(\omega_i)$ is the line bundle $\mathcal{E} \times^P \mathbb{C}_{-\omega_i}$ on \mathcal{E}/P .
- (2) $f(b_k) \in \mathcal{E}_{b_k}/P$ and $\bar{g}_k \in \mathcal{E}_{b_k}/B$ are in relative position $u_k \in W^P$, $k = 1, \dots, n$, defined as follows. Pick a trivialization $e \in \mathcal{E}_{b_k}$ and write $f(b_k) = eh_kP$ and $\bar{g}_k = eg_kB$. Then, we want $h_k \in g_kBu_kP \subseteq X^P$. A different choice of e acts on h_k and g_k by a left multiplication and therefore does not affect the relative position.

The above enumerative problem may be degenerate for some \mathcal{E} in the sense that the number of such sections f may be infinite.

3.2. Tangent spaces

For the homogeneous space $X = X^P$, the tangent bundle T_X is globally generated (since $\mathfrak{g} \otimes \mathcal{O}_X$ surjects onto T_X).

Fix any $\alpha_i \in S_P$. We can filter $T_{\dot{e}} := T(X^P)_{\dot{e}} \simeq \mathfrak{g}/\mathfrak{p}$ by counting the multiplicity of α_i in the root spaces in $\mathfrak{g}/\mathfrak{p}$, where \dot{e} is the base point of X . Specifically, for any $r \geq 1$, let $T_{i,r} \subset T_{\dot{e}}$ be the P -submodule spanned by the root spaces $\mathfrak{g}_{-\alpha}$ of $T_{\dot{e}}$ such that $\alpha(x_i) < r$. Define the P -module $Q_{i,r}$ by the following:

$$0 \rightarrow T_{i,r} \rightarrow T_{\dot{e}} \rightarrow Q_{i,r} \rightarrow 0.$$

Let $\mathcal{Q}_{i,r}$ (resp. $\mathcal{T}_{i,r}$) be the vector bundle on X arising from the P -module $Q_{i,r}$ (resp. $T_{i,r}$). Since $\mathcal{Q}_{i,r}$ are quotients of T_X , they are globally generated. Let $\beta_i := \sum_{r \geq 1} c_1(\mathcal{Q}_{i,r})$, and define the integers $s_{i,j} = \int_{X^{s_j}} \beta_i$, for any $\alpha_i, \alpha_j \in S_P$. Then, it is easy to see (using the equation (5)) that

$$s_{i,j} = \sum_{\alpha \in R^+ \setminus R_1^+} \alpha(x_i)\alpha(\alpha_j^\vee). \tag{17}$$

This is a non-negative integer since $\mathcal{Q}_{i,r}$'s are globally generated.

3.3. Some deformation theory

Let \mathcal{E} be a principal G -bundle on a smooth projective curve C and let $P \subseteq G$ be a parabolic subgroup. Let $f : C \rightarrow \mathcal{E}/P$ be a section, and \mathcal{P} the corresponding P -bundle.

Let Z be the space of sections $f : C \rightarrow \mathcal{E}/P$. This is a subscheme of the scheme \mathcal{Z} of maps $\beta : C \rightarrow \mathcal{E}/P$. Let M be the scheme of maps $C \rightarrow C$. Then, we have the morphism $\phi : \mathcal{Z} \rightarrow M$, $\beta \mapsto \gamma \circ \beta$, where $\gamma : \mathcal{E}/P \rightarrow C$ is the canonical projection. By definition, Z is the fiber of ϕ over the identity map I_C .

Lemma 3.7. *For any $f \in Z$, the Zariski tangent space TZ_f is identified with $H^0(C, f^*T_v(\mathcal{E}/P))$, where T_v is the vertical tangent bundle.*

Proof. By deformation theory, the tangent space of \mathcal{Z} at the point f is $H^0(C, f^*T(\mathcal{E}/P))$. There is a natural exact sequence:

$$0 \rightarrow f^*T_v(\mathcal{E}/P) \rightarrow f^*T(\mathcal{E}/P) \rightarrow TC \rightarrow 0,$$

which allows us to conclude the proof. \square

Let $f \in Z$ (for $C = \mathbb{P}^1$) and let \mathcal{P} be the corresponding principal P -bundle. Then, by [25, Theorem I.2.16], Z is smooth at f of the expected dimension

$$\dim X + \int_{\mathbb{P}^1} f^*(c_1(T_v(\mathcal{E}/P)))$$

if $H^1(\mathbb{P}^1, f^*T_v(\mathcal{E}/P)) = 0$ (which happens if, for example, \mathcal{E} is trivial). If Z is smooth of the expected dimension at \mathcal{P} then f deforms with every deformation of \mathcal{E} (over a complete local ring).

We have the following simple result.

Lemma 3.8. $f^*T_v(\mathcal{E}/P) = \mathcal{P} \times^P T_{\dot{e}}$.

Also, note that for any character β of P ,

$$f^*\mathcal{E}(\beta) = \mathcal{P} \times^P \mathbb{C}_{-\beta} \text{ as a bundle over } C. \tag{18}$$

3.4. Tangent spaces of Schubert varieties

Let \mathcal{P} be a principal P -bundle on \mathbb{P}^1 and $x \in \mathbb{P}^1$. Given $\bar{p} \in \mathcal{P}_x/B_L$ and $u \in W^P$, we can construct a subspace $T(\bar{p}, u, x) \subseteq \mathcal{P}_x \times^P T_{\dot{e}}$ as follows. Fix a trivialization e of \mathcal{P}_x and write $\bar{p} = epB_L$. Then, the subspace $T(\bar{p}, u, x)$ is defined to be $e \times T(p\Lambda_u^P)_{\dot{e}} \subseteq \mathcal{P}_x \times^P T_{\dot{e}}$. In particular,

$$\dim T(\bar{p}, u, x) = \dim \Lambda_u^P. \tag{19}$$

A different choice of the coset representative of \bar{p} or the choice of e gives the same subspace.

Consider the evaluation map $e_{b_k} : Z \rightarrow \mathcal{E}_{b_k}/P$ at b_k . Fix $f \in Z$ such that the corresponding principal bundle is \mathcal{P} . Then, the differential map de_{b_k} on tangent spaces (under the identifications of [Lemmas 3.7 and 3.8](#))

$$H^0(\mathbb{P}^1, \mathcal{P} \times^P T_{\bar{e}}) \rightarrow T_{f(b_k)}(\mathcal{E}_{b_k}/P) = \mathcal{P}_{b_k} \times^P T_{\bar{e}}$$

is the evaluation map at b_k .

Fix an element $e_k \in \mathcal{P}_{b_k}$. Since we require that $f(b_k)$ and \bar{g}_k are in relative position $u_k \in W^P$ (cf. [§3.1](#)), we have the following:

$$\bar{g}_k = e_k p_k u_k^{-1} B, \quad \text{for some } p_k \in P. \tag{20}$$

To prove [\(20\)](#), observe that since $f(b_k) = e_k P$ and $\bar{g}_k = e_k g_k B \in \mathcal{E}_{b_k}/B$ are in relative position u_k , we get $1 \in g_k B u_k P$, i.e., $g_k c_k u_k p_k^{-1} = 1$ for some $c_k \in B$ and $p_k \in P$. From this we see that $\bar{g}_k = e_k p_k u_k^{-1} B$, proving [\(20\)](#).

3.5. Determinant of cohomology

The determinant of cohomology of a coherent sheaf \mathcal{F} on a projective curve C is the line

$$D(\mathcal{F}) = \det H^0(C, \mathcal{F})^* \otimes \det H^1(C, \mathcal{F}).$$

Automorphisms of \mathcal{F} act on $D(\mathcal{F})$. For example, multiplication by $t \neq 0$ on \mathcal{F} acts on $D(\mathcal{F})$ by $t^{-\chi(C, \mathcal{F})}$, where $\chi(C, \mathcal{F})$ is the Euler characteristic of \mathcal{F} . In the cases we consider here, \mathcal{F} is locally free.

Suppose $\chi(C, \mathcal{F}) = 0$, then, as, e.g., in [\[17\]](#), $D(\mathcal{F})$ carries a canonical element $\theta(\mathcal{F})$ which is non-vanishing if and only if

$$H^0(C, \mathcal{F}) = H^1(C, \mathcal{F}) = 0. \tag{21}$$

Automorphisms of \mathcal{F} act on $D(\mathcal{F})$ preserving $\theta(\mathcal{F})$. In particular, if an automorphism of \mathcal{F} acts non-trivially on $D(\mathcal{F})$, then $\theta(\mathcal{F}) = 0$.

As earlier, we have fixed distinct points $b_1, \dots, b_n \in \mathbb{P}^1$. Let $\text{Parbun}_P = \text{Parbun}_P(d)$ be the moduli-stack of quasi-parabolic principal P -bundles on \mathbb{P}^1 of degree $d = (a_i)_{\alpha_i \in S_P}$, i.e., data $\tilde{\mathcal{P}} = (\mathcal{P}; \bar{p}_1, \dots, \bar{p}_n)$ such that

- \mathcal{P} is a principal P -bundle on \mathbb{P}^1 such that $\mathcal{P} \times^P \mathbb{C}_{-\omega_i}$ has degree a_i , for each $\alpha_i \in S_P$.
- For $k = 1, \dots, n$, $\bar{p}_k \in \mathcal{P}_{b_k}/B_L$.

3.6. Transversality

Fix $u_1, \dots, u_n \in W^P$ satisfying the equation (9). For any $\tilde{\mathcal{P}} = (\mathcal{P}; \bar{p}_1, \dots, \bar{p}_n) \in \text{Parbun}_P(d)$, define a locally free sheaf $\mathcal{K} = \mathcal{K}(\tilde{\mathcal{P}}; u_1, \dots, u_n)$ on \mathbb{P}^1 by the exact sequence obtained from the evaluation maps at $b_k \in \mathbb{P}^1$:

$$0 \rightarrow \mathcal{K} \rightarrow \mathcal{P} \times^P T_{\dot{e}} \xrightarrow{\alpha} \bigoplus_{k=1}^n i_{b_k*} \frac{\mathcal{P}_{b_k} \times^P T_{\dot{e}}}{T(\bar{p}_k, u_k, b_k)} \rightarrow 0, \tag{22}$$

where i_{b_k} is the embedding $b_k \hookrightarrow \mathbb{P}^1$. (The sheaf morphism α is clearly surjective since locally, the map of a coherent sheaf to its fiber at any point (as a sky-scraper sheaf) is surjective. Moreover, \mathcal{K} is a locally free sheaf on \mathbb{P}^1 since it is a coherent subsheaf of a locally free sheaf on a smooth curve C .) By the condition (9) and the equation (19), the Euler characteristic of $\bigoplus_{k=1}^n i_{b_k*} \frac{\mathcal{P}_{b_k} \times^P T_{\dot{e}}}{T(\bar{p}_k, u_k, b_k)}$ equals $\dim X + \int_d c_1(T_X)$. Further, the Euler characteristic

$$\chi(\mathcal{P} \times^P T_{\dot{e}}) = \dim X + \int_d c_1(T_X) \tag{23}$$

by the following calculation:

Let $\det T_{\dot{e}}$ be the character $\mathbb{C}_{-\lambda}$ as a P -module. Write $\lambda = \sum_{\alpha_i \in S_P} \delta_i \omega_i$, where $\delta_i := \lambda(\alpha_i^\vee)$. Then,

$$\deg(\mathcal{P} \times^P T_{\dot{e}}) = \sum_{\alpha_i \in S_P} \delta_i a_i.$$

Further, by (5),

$$\int_d c_1(T_X) = \sum_{\alpha_i \in S_P} a_i \int_{X_{s_i}^P} c_1(\mathcal{L}_P(\lambda)) = \sum_{\alpha_i \in S_P} a_i \lambda(\alpha_i^\vee) = \sum_{\alpha_i \in S_P} a_i \delta_i.$$

This proves (23). Hence, \mathcal{K} has zero Euler characteristic.

Definition 3.9. The quasi-parabolic bundle $\tilde{\mathcal{P}}$ is said to satisfy the *transversality condition* if \mathcal{K} has non-vanishing θ -section $\theta(\mathcal{K}) \in D(\mathcal{K})$.

As will be seen below (see Lemma 3.11, and Equation (24)), the transversality condition relates to smoothness properties of certain spaces which appear in our study of the enumerative problem from Section 3.1. Note that this sort of reformulation of transversality appears in [7,10,12], and (in a related situation) in [11].

Define a line bundle \mathcal{R} on Parbun_P such that its fiber over a quasi-parabolic $\tilde{\mathcal{P}}$ is $D(\mathcal{K}(\tilde{\mathcal{P}}; u_1, \dots, u_n))$. The line bundle \mathcal{R} admits a canonical section θ given by

$$\theta(\tilde{\mathcal{P}}) = \theta(\mathcal{K}(\tilde{\mathcal{P}}; u_1, \dots, u_n)).$$

Remark 3.10. Note that if θ does not vanish at a quasi-parabolic $\tilde{\mathcal{P}}$ then, since

$$H^0(\mathbb{P}^1, \mathcal{K}) = H^1(\mathbb{P}^1, \mathcal{K}) = 0 \text{ by the identity (21),}$$

we get $H^1(\mathbb{P}^1, \mathcal{P} \times^P T_{\tilde{\mathcal{E}}}) = 0$ from the long exact cohomology sequence associated to the sheaf exact sequence (22).

3.7. The space of P -subbundles of a G -bundle

Let Parbun_G be the moduli-stack of quasi-parabolic principal G -bundles on \mathbb{P}^1 , i.e., data $\tilde{\mathcal{E}} = (\mathcal{E}; \bar{g}_1, \dots, \bar{g}_n)$, where \mathcal{E} is a principal G -bundle on \mathbb{P}^1 and $\bar{g}_k \in \mathcal{E}_{b_k}/B$.

For any $\tilde{\mathcal{E}} \in \text{Parbun}_G$, a standard parabolic P , elements $u_1, \dots, u_n \in W^P$ and degree $d = (a_i)_{\alpha_i \in S_P}$, define the schemes $Z_d(\mathcal{E})$ and $Z'_d(\tilde{\mathcal{E}})$ as follows:

$Z_d(\mathcal{E})$ is the space of sections f of \mathcal{E}/P of degree d , i.e., $c_1(f^*(\mathcal{E}(\omega_i))) = a_i \forall \alpha_i \in S_P$.

$$\begin{aligned} Z'_d(\tilde{\mathcal{E}}) &= Z'_d(\tilde{\mathcal{E}}; u_1, \dots, u_n) \\ &:= \{f \in Z_d(\mathcal{E}) : f(b_k) \text{ and } \bar{g}_k \text{ are in relative position } u_k \forall 1 \leq k \leq n\}. \end{aligned}$$

Then, for any $f \in Z_d(\mathcal{E})$, the Zariski tangent space

$$T(Z_d(\mathcal{E}))_f = H^0(\mathbb{P}^1, \mathcal{P}(f) \times^P T_{\tilde{\mathcal{E}}}),$$

where $\mathcal{P}(f)$ is the P -subbundle of \mathcal{E} associated to f . Thus, $f \in Z_d(\mathcal{E})$ is a smooth point if $H^1(\mathbb{P}^1, \mathcal{P}(f) \times^P T_{\tilde{\mathcal{E}}}) = 0$.

For any $f \in Z'_d(\tilde{\mathcal{E}})$, we have the canonical morphism

$$\phi_f : H^0(\mathbb{P}^1, \mathcal{P}(f) \times^P T_{\tilde{\mathcal{E}}}) \rightarrow \bigoplus_{k=1}^n \frac{\mathcal{P}(f)_{b_k} \times^P T_{\tilde{\mathcal{E}}}}{T(\bar{p}_k, u_k, b_k)},$$

induced from the evaluation maps $e_{b_k} : Z_d(\mathcal{E}) \rightarrow \mathcal{E}_{b_k}/P$ at b_k , where p_k satisfies the equation (20) and $\bar{p}_k := e_k p_k B_L \in \mathcal{P}(f)_{b_k}/B_L$. (Observe that p_k is unique modulo $u_k^{-1} B u_k \cap P$.) Then, for any $f \in Z'_d(\tilde{\mathcal{E}})$, the Zariski tangent space

$$T(Z'_d(\tilde{\mathcal{E}}))_f = \ker \phi_f. \tag{24}$$

Lemma 3.11. Let $u_1, \dots, u_n \in W^P$ and non-negative $d = (a_i)_{\alpha_i \in S_P} \in H_2(X, \mathbb{Z})$. Then, the following are equivalent under the condition (9).

- (a) $\langle \sigma_{u_1}^P, \sigma_{u_2}^P, \dots, \sigma_{u_n}^P \rangle_d \neq 0$.
- (b) There is a quasi-parabolic P -bundle $\tilde{\mathcal{P}} = (\mathcal{P}; \bar{p}_1, \dots, \bar{p}_n) \in \text{Parbun}_P(d)$ on \mathbb{P}^1 so that the canonical evaluation map

$$H^0(\mathbb{P}^1, \mathcal{P} \times^P T_{\tilde{e}}) \rightarrow \bigoplus_{k=1}^n \frac{\mathcal{P}_{b_k} \times^P T_{\tilde{e}}}{T(\bar{p}_k, u_k, b_k)}$$

is an isomorphism.

(c) The section $\theta \in H^0(\text{Parbun}_P(d), \mathcal{K})$ is nonzero.

Remark 3.12. Observe that the section θ on a quasi-parabolic $\tilde{\mathcal{P}}$ does not vanish if and only if the evaluation map as in (b) of the above lemma is an isomorphism. In particular, in this case, $H^1(\mathbb{P}^1, \mathcal{P} \times^P T_{\tilde{e}}) = 0$. (To prove this, use the identity (21) and the fact that $\chi(\mathbb{P}^1, \mathcal{K}) = 0$ together with the exact sequence (22).)

Proof of Lemma 3.11. We first prove (a) \implies (b): Let Z_d be the space $Z_d(\epsilon_G)$, where ϵ_G is the trivial G -bundle on \mathbb{P}^1 , i.e., Z_d is the space of all maps $f : \mathbb{P}^1 \rightarrow X$ of degree d . Then, Z_d is a smooth variety of dimension $= \dim X + \int_d c_1(T_X)$ (cf. §2). Let $\{\bar{g}_k\}_{1 \leq k \leq n}$ be general points of G/B . Then, by the assumption (a), there exists $f \in Z'_d = Z'_d(\tilde{\epsilon}_G; u_1, \dots, u_n)$, where $\tilde{\epsilon}_G = (\epsilon_G; \bar{g}_1, \dots, \bar{g}_n)$ is the quasi-parabolic G -bundle on \mathbb{P}^1 . Moreover, the subscheme Z'_d is finite and reduced. Fix $f \in Z'_d$. Then, Z'_d being finite and reduced, we get that ϕ_f is injective by the equation (24). Now, by the condition (9), the dimension of the domain is at least as much as the dimension of the range of ϕ_f (since $\chi(\mathbb{P}^1, \mathcal{P}(f) \times^P T_{\tilde{e}}) = \dim X + \int_{\mathbb{P}^1} c_1(T_X)$). Hence, being injective, ϕ_f is an isomorphism, proving (b).

(b) \implies (c): If the condition (b) holds for $\tilde{\mathcal{P}}$, then $H^0(\mathbb{P}^1, \mathcal{K}(\tilde{\mathcal{P}})) = 0$. But, since $\chi(\mathbb{P}^1, \mathcal{K}(\tilde{\mathcal{P}})) = 0$, we get that $H^1(\mathbb{P}^1, \mathcal{K}(\tilde{\mathcal{P}})) = 0$. Hence, as in §3.5, $\theta(\mathcal{K}(\tilde{\mathcal{P}})) \neq 0$, proving (c).

(c) \implies (a): Suppose $\tilde{\mathcal{P}} = (\mathcal{P}; \bar{p}_1, \dots, \bar{p}_n)$ is in $\text{Parbun}_P(d)$ such that $\theta(\mathcal{K}(\tilde{\mathcal{P}})) \neq 0$. Let $\mathcal{E} = \mathcal{P} \times^P G$ be the corresponding principal G -bundle. The reduction of \mathcal{E} to \mathcal{P} gives rise to the section $f : \mathbb{P}^1 \rightarrow \mathcal{E}/P$ of degree d . Consider the elements:

$$\bar{g}_k = \bar{p}_k u_k^{-1} B \in \mathcal{E}_{b_k}/B.$$

(Observe that \bar{g}_k does not depend upon the choice of \bar{p}_k in its B_L -orbit by the identity (4).) Then, $f(b_k)$ and $\bar{g}_k \in \mathcal{E}_{b_k}/B$ are in relative position u_k . To see this, let e_k be a trivialization of \mathcal{P}_{b_k} and write $\bar{p}_k = e_k p_k$. Then, $f(b_k) = e_k P$ and $\bar{g}_k = e_k p_k u_k^{-1} B$.

Thus, $f \in Z'_d = Z'_d(\tilde{\mathcal{E}})$, where $\tilde{\mathcal{E}} = (\mathcal{E}; \bar{g}_1, \dots, \bar{g}_k)$. By Subsection 3.7, the Zariski tangent space to Z'_d at f is equal to $H^0(\mathbb{P}^1, \mathcal{K}(\tilde{\mathcal{P}}))$, which is zero by assumption. By Remark 3.12, $H^1(\mathcal{P} \times^P T_{\tilde{e}}) = 0$.

Consider a one parameter family of deformations $\tilde{\mathcal{E}}_t = (\mathcal{E}_t; \bar{g}_1(t), \dots, \bar{g}_n(t))$ parameterized by a smooth curve S so that at a marked point $0 \in S$, $\tilde{\mathcal{E}}_0 = \tilde{\mathcal{E}}$ and the underlying bundle \mathcal{E}_t is trivial for general $t \in S$. We then have families $\pi : \mathcal{Z}_d \rightarrow S$ and $\pi' : \mathcal{Z}'_d \rightarrow S$ over S with fiber $Z_d(\mathcal{E}_t)$ and $Z'_d(\tilde{\mathcal{E}}_t)$ respectively. Thus, $f \in \mathcal{Z}'_d(0)$, the fiber of π' at 0. We claim that π' is a dominant morphism of relative dimension zero at f :

Observe first that π is smooth at f using the fact (noted above) that $H^1(\mathcal{P} \times^P T_{\tilde{e}}) = 0$. Since $\theta(\mathcal{K}(\tilde{\mathcal{P}})) \neq 0$, there exists a neighborhood of $\tilde{\mathcal{P}}$ in \mathcal{Z}_d such that for any $\tilde{\mathcal{Q}}$ in the

neighborhood, $\theta(\mathcal{K}(\tilde{Q})) \neq 0$. Moreover, restricted to this neighborhood, π is a smooth morphism and π' has finite fibers. Choose a lift g_k of \bar{g}_k , i.e., a section $g_k : S \rightarrow \cup_t \mathcal{E}_t$ such that $g_k(t) \in \mathcal{E}_t$ is a lift of $\bar{g}_k(t) \in \mathcal{E}_t/B$. This is, of course, possible replacing S (if needed) by a smaller étale neighborhood of $0 \in S$. Thus, there is a neighborhood \mathcal{Z}_d^o of f in \mathcal{Z}_d and a morphism

$$\beta : \mathcal{Z}_d^o \rightarrow (X^P)^n, \quad \beta(\tilde{Q}) = (h_k P)_{1 \leq k \leq n},$$

where $h_k P$ is the unique element such that $g_k(\pi(\tilde{Q})) \in \tilde{Q}_{b_k} h_k^{-1}$. Moreover, we can choose \mathcal{Z}_d^o small enough so that $\pi|_{\mathcal{Z}_d^o} : \mathcal{Z}_d^o \rightarrow S$ is a smooth morphism and $\pi'|_{\mathcal{Z}'_d \cap \mathcal{Z}_d^o} : \mathcal{Z}'_d \cap \mathcal{Z}_d^o \rightarrow S$ has finite fibers. From the definition of \mathcal{Z}'_d , it is clear that

$$\mathcal{Z}'_d \cap \mathcal{Z}_d^o = \beta^{-1}(C_{u_1}^P \times \cdots \times C_{u_n}^P).$$

Since X^P is smooth, $C_{u_1}^P \times \cdots \times C_{u_n}^P$ is locally defined by exactly r equations, where r is the codimension of $C_{u_1}^P \times \cdots \times C_{u_n}^P$ in $(X^P)^n$. Hence, by [20, Exercise 3.22, Chap. II],

$$\dim_f(\mathcal{Z}'_d) \geq \dim_f(\mathcal{Z}_d^o) - \sum_{k=1}^n \text{codim } C_{u_k}^P = 1,$$

where the last equality follows since ϕ_f (defined in Subsection 3.7) is an isomorphism and π is a smooth morphism. But, since $\pi'|_{\mathcal{Z}'_d \cap \mathcal{Z}_d^o}$ has only finite fibers, $\pi'|_{\mathcal{Z}'_d \cap \mathcal{Z}_d^o} : \mathcal{Z}'_d \cap \mathcal{Z}_d^o \rightarrow S$ is a dominant morphism. This proves the assertion that π' is a dominant morphism of relative dimension zero at f . Since the underlying bundle \mathcal{E}_t is trivial for general $t \in S$ and π' is dominant, by the definition of the Gromov–Witten numbers (cf. Definition 2.1), we get (a). \square

Remark 3.13. Even though we do not need, the map $\pi' : \mathcal{Z}'_d \rightarrow S$ is, in fact, a flat morphism in a neighborhood of f (with fiber dimension 0).

Similarly, let $\text{Parbun}_L = \text{Parbun}_L(d)$ be the moduli-stack of quasi-parabolic principal L -bundles on \mathbb{P}^1 of degree d , i.e., data $\tilde{\mathcal{L}} = (\mathcal{L}; \bar{l}_1, \dots, \bar{l}_n)$ such that

- \mathcal{L} is a principal L -bundle on \mathbb{P}^1 such that $\mathcal{L} \times^L \mathbb{C}_{-\omega_i}$ has degree a_i for each $\alpha_i \in S_P$.
- For $k = 1, \dots, n$, $\bar{l}_k \in \mathcal{L}_{b_k}/B_L$.

There is a canonical morphism of stacks $\phi : \text{Parbun}_L \rightarrow \text{Parbun}_P$. Similar to the definition of the theta section θ of the line bundle \mathcal{R} on Parbun_P as in Subsection 3.6, we can define the theta section θ' of the corresponding bundle \mathcal{R}' on Parbun_L . (Just as \mathcal{R} , the line bundle \mathcal{R}' depends on the choice of $u_1, \dots, u_n \in W^P$ satisfying the equation (9).) From the functoriality of the theta sections, it is easy to see that

$$\phi^*(\theta) = \theta'.$$

We have the following crucial definition.

Definition 3.14. We call $(u_1, \dots, u_n; d)$ *quantum Levi-movable* if θ' does not vanish identically on Parbun_L .

We have the following key proposition.

Proposition 3.15. Consider a quasi-parabolic L -bundle $\tilde{\mathcal{L}} = (\mathcal{L}; \bar{l}_1, \dots, \bar{l}_n)$ of degree d . Let $\phi(\tilde{\mathcal{L}}) = \tilde{\mathcal{P}} = (\mathcal{P}, \bar{l}_1, \dots, \bar{l}_n)$ be the corresponding point of Parbun_P . Then, the central one parameter subgroup $t^{\bar{x}_i}$ of L corresponding to \bar{x}_i , $\alpha_i \in S_P$, acts on $D(\mathcal{K}(\tilde{\mathcal{P}}))$ by multiplication by t^{μ_i} , where

$$\mu_i := (\chi_e - \sum_{k=1}^n \chi_{u_k})(\bar{x}_i) + \sum_{\alpha \in R^+ \setminus R_1^+} \alpha(\bar{x}_i) \alpha(\tilde{d}),$$

\bar{x}_i is defined by the equation (3), χ_u is defined by the equation (6), and, as in Theorem 3.3, $\tilde{d} = \sum_{\alpha_j \in S_P} a_j \alpha_j^\vee$.

Proof. Note that by the exact sequence (22),

$$D(\mathcal{K}) = D(\mathcal{P} \times^P T_{\dot{e}}) \otimes \prod_{k=1}^n D\left(i_{b_k^*} \frac{\mathcal{P}_{b_k} \times^P T_{\dot{e}}}{T(l_k, u_k, b_k)}\right)^*.$$

It is easy to see that $t^{\bar{x}_i}$ acts on $D\left(i_{b_k^*} \frac{\mathcal{P}_{b_k} \times^P T_{\dot{e}}}{T(l_k, u_k, b_k)}\right)^*$ by $t^{-\chi_{u_k}(\bar{x}_i)}$. We next calculate the action of $t^{\bar{x}_i}$ on $D(\mathcal{P} \times^P T_{\dot{e}})$:

Let \mathcal{V} be a vector bundle on \mathbb{P}^1 , and let $T : \mathcal{V} \rightarrow \mathcal{V}$ be the multiplication by the scalar c^{-1} on fibers. Then, clearly, T acts on $D(\mathcal{V})$ by the scale c raised to the exponent

$$\chi(\mathbb{P}^1, \mathcal{V}) = \text{rk } \mathcal{V} + \text{deg } \mathcal{V}.$$

Suppose the vector bundle $\mathcal{P} \times^P T_{\dot{e}}$ is filtered with the associated graded pieces being the vector bundles $\mathcal{V}_1, \dots, \mathcal{V}_s$, such that $t^{\bar{x}_i}$ acts on \mathcal{V}_r by the scale $t^{-\gamma_r}$. Then, $t^{\bar{x}_i}$ acts on $D(\mathcal{P} \times^P T_{\dot{e}})$ via

$$t^{\sum_r (\text{rk } \mathcal{V}_r + \text{deg } \mathcal{V}_r) \gamma_r}.$$

This allows us to reduce the calculation of the action of $t^{\bar{x}_i}$ on $D(\mathcal{P} \times^P T_{\dot{e}})$ using the filtration of $T_{\dot{e}}$ given in Subsection 3.2. The desired \mathcal{V}_r are the quotients $T_{i,r}/T_{i,r-1}$. Now, use the formula (17).

Finally, it is easy to see that $\sum_r \text{rk } \mathcal{V}_r = \chi_e(x_i)$. Combining these, we get the proposition. \square

Theorem 3.16. *The following are equivalent:*

- (a) $(u_1, \dots, u_n; d)$ is quantum Levi-movable.
- (b) $\langle \sigma_{u_1}^P, \sigma_{u_2}^P, \dots, \sigma_{u_n}^P \rangle_d \neq 0$ and

$$\forall \alpha_i \in S_P, \mu_i = 0, \tag{25}$$

where μ_i is defined in Proposition 3.15.

(c) $\langle \sigma_{u_1}^P, \sigma_{u_2}^P, \dots, \sigma_{u_n}^P \rangle_d^{\otimes 0} \neq 0$, where $\langle \sigma_{u_1}^P, \sigma_{u_2}^P, \dots, \sigma_{u_n}^P \rangle_d^{\otimes 0}$ is, by definition, equal to the coefficient of $q^d \sigma_{w_0 u_n w_0^P}$ in $\sigma_{u_1}^P \otimes \dots \otimes \sigma_{u_{n-1}}^P$.

3.8. Proofs of Theorems 3.3 and 3.16

Fix an element $x = \sum_{\alpha_i \in S_P} d_i \bar{x}_i \in \mathfrak{h}$ such that each d_i is a strictly positive integer. Then, t^x is a central one parameter subgroup of L ; in particular, it is contained in B_L . For any $t \in \mathbb{G}_m$, define the conjugation $\phi_t : P \rightarrow P, p \mapsto t^x p t^{-x}$. This extends to a group homomorphism $\phi_0 : P \rightarrow L \subset P$, giving rise to a regular map $\hat{\phi} : P \times \mathbb{A}^1 \rightarrow P$, extending the map $(p, t) \mapsto \phi_t(p)$, for $p \in P$ and $t \in \mathbb{G}_m$. Clearly, $\phi_t|_L = I_L$, for all $t \in \mathbb{A}^1$, where I_L is the identity map of L .

Let \mathcal{P} be a principal P -bundle. Define a family of principal P -bundles \mathcal{P}_t parameterized by $t \in \mathbb{A}^1$, where \mathcal{P}_t is the principal P -bundle induced by \mathcal{P} via ϕ_t , i.e., $\mathcal{P}_t = \mathcal{P} \times^{P, \phi_t} P$. Since, the image of ϕ_0 is contained in L , we get a principal L -bundle \mathcal{L} from \mathcal{P} via the homomorphism ϕ_0 . Clearly, $\mathcal{P}_0 = \mathcal{L} \times^L P$. We write $\text{Gr}(\mathcal{P}) = \mathcal{L} \times^L P$. We will refer to this as the *Levification process*. So, we have found a degeneration of \mathcal{P} to $\text{Gr}(\mathcal{P})$ parameterized by $t \in \mathbb{A}^1$.

If $\bar{p}_k \in \mathcal{P}_{b_k}/B_L$, we canonically have $\bar{p}_k(t) \in (\mathcal{P}_t)_{b_k}/B_L$, for any $t \in \mathbb{A}^1$, defined as $\bar{p}_k(t) = \phi_{t*}(\bar{p}_k)$. At $t = 0$, the image is in \mathcal{L}_{b_k}/B_L .

We therefore have a \mathbb{G}_m -equivariant line bundle $D(\mathcal{K}_t)$ on \mathbb{A}^1 . Furthermore, we have a \mathbb{G}_m -equivariant section $\theta(\mathcal{K}_t)$ of $D(\mathcal{K}_t)$. The following statement is immediate (cf. Proposition 10 in [12]).

Lemma 3.17. *Let \mathbb{G}_m act on \mathbb{A}^1 by multiplication. Let \mathcal{R} be a \mathbb{G}_m -equivariant line bundle on \mathbb{A}^1 with a \mathbb{G}_m -invariant section s . Suppose \mathbb{G}_m acts on the fiber \mathcal{R}_0 over 0 by multiplication by $t^\mu, \mu \in \mathbb{Z}$, i.e., the Mumford index $\mu^{\mathcal{R}}(t, \lambda_0) = \mu$, for any $t \in \mathbb{A}^1$, where λ_0 is the one parameter subgroup $z \mapsto z$. Then,*

- (a) If $s \neq 0$, then $\mu \geq 0$.
- (b) If $\mu = 0$ and $s \neq 0$, then $s(0) \neq 0 \in \mathcal{R}_0$.

3.8.1. Proof of Theorem 3.3

Assume that $\langle \sigma_{u_1}^P, \sigma_{u_2}^P, \dots, \sigma_{u_n}^P \rangle_d \neq 0$. Then, by Lemma 3.11, the section $\theta \in H^0(\text{Parbun}_P, \mathcal{R})$ is nonzero. Let $\mathcal{P} = (\mathcal{P}; \bar{p}_1, \dots, \bar{p}_n)$ be a quasi-parabolic P -bundle in Parbun_P such that $\theta(\tilde{\mathcal{P}}) \neq 0$. Considering the one parameter degeneration $\tilde{\mathcal{P}}_t$ as above

and using [Lemma 3.17](#) and [Proposition 3.15](#), we get that $\sum_{\alpha_i \in S_P} d_i \mu_i \geq 0$, for any strictly positive integers d_i , where

$$\mu_i = (\chi_e - \sum_{k=1}^n \chi_{u_k})(\bar{x}_i) + \sum_{\alpha \in R^+ \setminus R^+_i} \alpha(\bar{x}_i) \alpha(\tilde{d}).$$

From this we conclude that each $\mu_i \geq 0$. This proves [Theorem 3.3](#).

3.8.2. Proof of [Theorem 3.16](#)

We first prove (a) \implies (b):

Take a quasi-parabolic L -bundle $\tilde{\mathcal{L}} = (\mathcal{L}; \bar{l}_1, \dots, \bar{l}_n) \in \text{Parbun}_L(d)$ such that $\theta'(\tilde{\mathcal{L}}) \neq 0$. Let $\tilde{\mathcal{P}} = \phi(\tilde{\mathcal{L}}) = (\mathcal{P}; \bar{l}_1, \dots, \bar{l}_n)$ be the corresponding point of Parbun_P . Hence, θ does not vanish at the quasi-parabolic $\tilde{\mathcal{P}}$. The right multiplication by the L -central one parameter subgroups $t^{\bar{x}_i}$, for $\alpha_i \in S_P$, induces an automorphism of the quasi-parabolic L -bundle $\tilde{\mathcal{L}}$ and hence that of $\tilde{\mathcal{P}}$. These should act trivially on $\theta(\tilde{\mathcal{P}})$, and hence if $\theta(\tilde{\mathcal{P}}) \neq 0$, we get that $t^{\bar{x}_i}$ acts trivially on $D(\mathcal{K}(\tilde{\mathcal{P}}))$ (cf. [§3.5](#)). Hence, we obtain that (a) implies [\(25\)](#) by using [Proposition 3.15](#). Further, by [Lemma 3.11](#), we get that $\langle \sigma_{u_1}^P, \sigma_{u_2}^P, \dots, \sigma_{u_n}^P \rangle_d \neq 0$. This proves (b).

For the reverse direction, by [Lemma 3.11](#), assume that θ is non-vanishing on a quasi-parabolic $\tilde{\mathcal{P}}$. Performing the above degeneration, we find the desired conclusion using [Lemma 3.17](#) (b) and [Proposition 3.15](#). The equivalence of (b) and (c) follows from the definition of \otimes_0 . This proves [Theorem 3.16](#).

4. Determination of the multiplicative eigen polytope in terms of the deformed quantum cohomology

Let G be a simple, connected, simply-connected complex algebraic group.

As in Introduction, let $\mathcal{A} \subset \mathfrak{h}$ be the fundamental alcove:

$$\mathcal{A} = \{ \mu \in \mathfrak{h} : \alpha_i(\mu) \geq 0 \text{ for all the simple roots } \alpha_i \text{ and } \theta_o(\mu) \leq 1 \}.$$

Then, as in Introduction, \mathcal{A} parameterizes the K -conjugacy classes of K under the map $C : \mathcal{A} \rightarrow K/\text{Ad } K, \mu \mapsto c(\text{Exp}(2\pi i \mu))$. Fix a positive integer $n \geq 3$ and define the *multiplicative eigen polytope*

$$\mathcal{C}_n := \{ (\mu_1, \dots, \mu_n) \in \mathcal{A}^n : 1 \in C(\mu_1) \dots C(\mu_n) \}.$$

Then, it is known that \mathcal{C}_n is a rational convex polytope with nonempty interior in \mathfrak{h}^n (cf. [\[32, Corollary 4.13\]](#)). Our aim is to describe the facets (i.e., the codimension one faces) of \mathcal{C}_n .

The following theorem is one of our main results. In the case $G = \text{SL}_2$, it was proved by Biswas [\[14\]](#). For $G = \text{SL}_m$, it was proved by Belkale [\[5\]](#) (and a slightly weaker

result by Agnihotri–Woodward [1]). (Observe that for $G = \mathrm{SL}_m$, by Lemma 3.6, the deformed quantum cohomology coincides with the quantum cohomology of G/P for maximal P .)

Theorem 4.1. *Let $(\mu_1, \dots, \mu_n) \in \mathcal{A}^n$. Then, the following are equivalent:*

(a) $(\mu_1, \dots, \mu_n) \in \mathcal{C}_n$.

(b) *For any standard maximal parabolic subgroup P of G , any $u_1, \dots, u_n \in W^P$, and any $d \geq 0$ such that the deformed small quantum cohomology (Gromov–Witten) invariant (cf. Definition 3.5 and Theorem 3.16 (c))*

$$\langle \sigma_{u_1}^P, \dots, \sigma_{u_n}^P \rangle_d^{\otimes 0} = 1,$$

the following inequality is satisfied:

$$\mathcal{I}_{(u_1, \dots, u_n; d)}^P : \sum_{k=1}^n \omega_P(u_k^{-1} \mu_k) \leq d,$$

where ω_P is the fundamental weight ω_{i_P} such that α_{i_P} is the unique simple root in S_P .

Proof. (a) \Rightarrow (b): In fact, as proved in [39], for any $u_1, \dots, u_n \in W^P$ and any $d \geq 0$ such that the tuple $(P; u_1, \dots, u_n; d)$ is quantum non-null (see Definition 2.2), the inequality $\mathcal{I}_{(u_1, \dots, u_n; d)}^P$ is satisfied. We include a proof for completeness. Since \mathcal{C}_n is a rational polytope with nonempty interior in \mathfrak{h}^n , we can assume that each μ_k is regular (i.e., each $\alpha_i(\mu_k) > 0$), rational (i.e., each $\alpha_i(\mu_k) \in \mathbb{Q}$) and $\theta_o(\mu_k) < 1$. As in earlier sections, fix distinct points $b_1, \dots, b_n \in \mathbb{P}^1$ and let $\mathbb{M}_G(\vec{\mu})$ be the parabolic moduli space of parabolic semistable principal G -bundles over \mathbb{P}^1 with parabolic weights $\vec{\mu} = (\mu_1, \dots, \mu_n)$ associated to the points (b_1, \dots, b_n) respectively. We follow the version of $\mathbb{M}_G(\vec{\mu})$ from [39, §2.6].

By the generalization of the Mehta–Seshadri theorem [31] to arbitrary groups (cf. [39, Theorem 3.3]; [13]), the assumption (a) is equivalent to the assumption that the moduli space $\mathbb{M}_G(\vec{\mu})$ is nonempty. By [39, Proposition 4.2], $\mathbb{M}_G(\vec{\mu})$ is nonempty if and only if the trivial bundle $\epsilon_G : \mathbb{P}^1 \times G \rightarrow \mathbb{P}^1$ with general parabolic structures at the marked points b_1, \dots, b_n is parabolic semistable.

Let $u_1, \dots, u_n \in W^P$ and $d \geq 0$ be such that $(P; u_1, \dots, u_n; d)$ is quantum non-null. Hence, there exists a morphism $f : \mathbb{P}^1 \rightarrow X^P$ of degree d such that $f(b_k) \in g_k C_{u_k}^P$, for general $g_1, \dots, g_n \in G$, where $C_{u_k}^P := Bu_k P/P$ is the Schubert cell. Since $(g_1, \dots, g_n) \in G^n$ are general, we can assume that the trivial bundle ϵ_G with parabolic structure $g_k B$ at b_k and weights μ_k is parabolic semistable (we have used the assumption that each μ_k is regular). In particular, for the parabolic reduction $\sigma : \mathbb{P}^1 \rightarrow \epsilon_G/P$ induced from the morphism $f : \mathbb{P}^1 \rightarrow X^P$, we get (from the definition of the parabolic semistability):

$$\mathrm{deg}(\sigma^*(\epsilon_G(-\omega_P))) + \sum_{k=1}^n \omega_P(u_k^{-1} \mu_k) \leq 0,$$

since $u_k \in W^P \simeq W/W_P$ is the relative position of $\sigma(b_k) = f(b_k)$ and $g_k B$, where the line bundle $\epsilon_G(-\omega_P)$ over ϵ_G/P is defined in Subsection 3.1. But, by the definition of degree,

$$d := \text{deg}(\sigma^*(\epsilon_G(\omega_P))).$$

Hence, we get

$$\sum_{k=1}^n \omega_P(u_k^{-1} \mu_k) \leq d.$$

This proves the (b)-part. We prove the implication ‘(b) \Rightarrow (a)’ below. \square

Remark 4.2. As shown by [39, Proposition 4.4], the above argument shows that the inequalities $\mathcal{J}_{(u_1, \dots, u_n; d)}^P$, in fact, determine the polytope $\mathcal{C}_n \subset \mathcal{A}^n$ provided we run P through standard maximal parabolic subgroups and $(u_1, \dots, u_n; d) \in (W^P)^n \times \mathbb{Z}_+$ such that $(P; u_1, \dots, u_n; d)$ is quantum non-null.

Before we come to the proof of the implication ‘(b) \Rightarrow (a)’ in Theorem 4.1, we need to review the canonical reduction of parabolic G -bundles.

4.1. Canonical reduction of parabolic G -bundles

Let us fix parabolic weights $\vec{\mu} = (\mu_1, \dots, \mu_n) \in \mathcal{A}^n$ associated to the points $b_1, \dots, b_n \in \mathbb{P}^1$ respectively. We further assume that each μ_k is rational, regular and $\theta_o(\mu_k) < 1$. Fix a positive integer N such that $N\mu_k$ belongs to the coroot lattice of G for all k . Let $\Gamma := \mathbb{Z}/(N)$ be the cyclic group of order N . We fix a generator $\gamma_o \in \Gamma$. Now, take an irreducible smooth projective curve C with an action of Γ on C and a Γ -equivariant morphism $\pi : C \rightarrow \mathbb{P}^1$ (with the trivial action of Γ on \mathbb{P}^1) satisfying the following:

- (a) $\pi^{-1}(b_k)$ is a single point \tilde{b}_k , for all k ,
- (b) Γ acts freely on a nonempty open subset of C , and
- (c) the map π induces an isomorphism $C/\Gamma \simeq \mathbb{P}^1$.

Following Teleman–Woodward [39, Section 2.2], for any Γ -equivariant G -bundle E on C , such that, at the points \tilde{b}_k , the generator of Γ acts via the conjugacy class of $\text{Exp}(2\pi i \mu_k)$, we construct a quasi-parabolic G -bundle $\tilde{\mathcal{E}} = (\mathcal{E}; \bar{g}_1, \dots, \bar{g}_n)$ on \mathbb{P}^1 as follows.

For simplicity, we give the construction in the analytic category; the construction in the algebraic category is similar.

Choose a small enough analytic open neighborhood U_k of b_k in \mathbb{P}^1 and a coordinate z in $\tilde{U}_k := \pi^{-1}(U_k)$ such that the map $\pi : \tilde{U}_k \rightarrow U_k$ is given by $z \mapsto z^N$ and, moreover, the action of the generator $\gamma_o \in \Gamma$ on \tilde{U}_k is given by $z \mapsto e^{2\pi i/N} z$. Moreover, by [22, Section 11], we can choose U_k small enough so that there is a Γ -equivariant analytic isomorphism $\theta_k : E|_{\tilde{U}_k} \rightarrow \tilde{U}_k \times G$ such that γ_o acts on $\tilde{U}_k \times G$ via

$$\gamma_o(z, g) = (e^{2\pi i/N} z, \text{Exp}(2\pi i\mu_k)g).$$

Let $E^{-N\mu_k}$ denote the set of Γ -equivariant meromorphic sections $\sigma : \tilde{U}_k \rightarrow E|_{\tilde{U}_k}$ such that $z^{-N\mu_k} \cdot \sigma$ is regular on \tilde{U}_k . Then, $\sigma_k : \tilde{U}_k \rightarrow E|_{\tilde{U}_k}$, given by $\sigma_k(z) = (z, z^{N\mu_k})$ (under the above isomorphism θ_k) is a section contained in $E^{-N\mu_k}$.

As in [39, Section 2.2], there is a principal G -bundle \mathcal{E} over \mathbb{P}^1 isomorphic to $\Gamma \backslash E$ over $(C \setminus \{\tilde{b}_1, \dots, \tilde{b}_n\}) / \Gamma$ and such that $E^{-N\mu_k}$ is the set of sections of \mathcal{E} over U_k . Moreover, the section σ_k evaluated at b_k provides a parabolic reduction \bar{g}_k of the fiber \mathcal{E}_{b_k}/B . (We have used here the assumption that μ_k 's are regular.) Thus, for any Γ -equivariant G -bundle E on C , such that, at the points \tilde{b}_k , the generator of Γ acts via the conjugacy class of $\text{Exp}(2\pi i\mu_k)$, we have constructed a quasi-parabolic G -bundle $\tilde{\mathcal{E}} = (\mathcal{E}; \bar{g}_1, \dots, \bar{g}_n)$ on \mathbb{P}^1 .

Let P be a standard parabolic subgroup of G . From the above construction, it is clear that any Γ -equivariant principal P -subbundle of E canonically gives rise to a P -subbundle of \mathcal{E} .

Let $\text{Bun}_G^\Gamma(C) = \text{Bun}_{G, \bar{\mu}}^\Gamma(C)$ be the moduli stack of Γ -equivariant principal G -bundles on C , such that at the points \tilde{b}_k , the generator of Γ acts via the conjugacy class of $\text{Exp}(2\pi i\mu_k)$.

For a parabolic reduction E_P of $E \in \text{Bun}_G^\Gamma(C)$ to P , we have the notion of degree $\text{deg}(E_P) := (a_i)_{\alpha_i \in S_P} \in H_2(X^P, \mathbb{Z})$, where a_i is the degree of $E_P \times^P \mathbb{C}_{-\omega_i}$. Similarly, for a parabolic reduction $\tilde{\mathcal{E}}_P$ of a quasi-parabolic G -bundle $\tilde{\mathcal{E}} = (\mathcal{E}; \bar{g}_1, \dots, \bar{g}_n)$ on \mathbb{P}^1 , one defines the parabolic degree

$$\text{pardeg}(\tilde{\mathcal{E}}_P) := (b_i)_{\alpha_i \in S_P} \in H_2(X^P, \mathbb{Z}),$$

where $b_i := -\text{deg}(\mathcal{E}_P \times^P \mathbb{C}_{-\omega_i}) + \sum_{k=1}^n \omega_i(u_k^{-1}\mu_k)$ and $u_k \in W^P$ is the relative position of $\mathcal{E}_P(b_k)$ and \bar{g}_k .

We summarize this correspondence in the following result due to Teleman–Woodward [39, Theorem 2.3].

Theorem 4.3. *There is an isomorphism of stacks:*

$$\text{Bun}_G^\Gamma(C) \rightarrow \text{Parbun}_G$$

taking $E \mapsto \tilde{\mathcal{E}}$, where Parbun_G is as defined in Subsection 3.7.

Moreover, the Γ -equivariant reductions of any $E \in \text{Bun}_G^\Gamma(C)$ to a parabolic subgroup P of G correspond bijectively to the reductions of $\tilde{\mathcal{E}}$ to P .

Further, for any Γ -equivariant reduction E_P of E ,

$$\text{deg}(E_P) = N \text{pardeg}(\tilde{\mathcal{E}}_P), \tag{26}$$

where $\tilde{\mathcal{E}}_P$ is the corresponding reduction of $\tilde{\mathcal{E}}$.

Definition 4.4. We first recall the definition of *canonical reduction* of a principal G -bundle E over a smooth projective curve C (cf. [36]).

A reduction E_P of E to a parabolic subgroup P is called *canonical* if the following two conditions hold:

(a) Let L be the Levi of P realized as the quotient of P via its unipotent radical. Then, the principal L -bundle $E_P \times^P L$ obtained by the extension of the structure group is semistable.

(b) Fix a Borel subgroup $B \subset P$ of G . Then, for any nontrivial character λ of P which is a non-negative linear combination of the simple roots, the associated line bundle $E_P \times^P \mathbb{C}_\lambda$ is of positive degree.

Now, we come to the definition of *parabolic canonical reduction*:

Let $\tilde{\mathcal{E}}$ be a quasi-parabolic G -bundle on \mathbb{P}^1 with parabolic weights $\vec{\mu}$ assigned to the marked points b_1, \dots, b_n as in Subsection 4.1. Then, a reduction $\sigma_P(\tilde{\mathcal{E}})$ of $\tilde{\mathcal{E}}$ to a standard parabolic subgroup P is called *parabolic canonical* if the corresponding Γ -equivariant parabolic reduction σ_E of the corresponding (Γ -equivariant) G -bundle E over C is canonical. (Observe that, by the uniqueness of the canonical reduction of any G -bundle over C as in [15, Theorem 4.1], σ_E is unique; in particular, from the Γ -equivariance of E , we get that σ_E is Γ -equivariant.)

By the uniqueness of σ_E , we get that the parabolic canonical reduction $\sigma_P(\tilde{\mathcal{E}})$ of $\tilde{\mathcal{E}}$ is unique. In fact, $\sigma_P(\tilde{\mathcal{E}})$ does not admit any infinitesimal deformations either. More precisely, we have the following result:

Lemma 4.5. *Let $\tilde{\mathcal{E}} = (\mathcal{E}; \bar{g}_1, \dots, \bar{g}_n)$ be a quasi-parabolic G -bundle on \mathbb{P}^1 with parabolic weights $\vec{\mu}$ assigned to the marked points b_1, \dots, b_n and let $\sigma_P(\tilde{\mathcal{E}})$ be its canonical reduction (to the parabolic subgroup P) of degree d . Let $u_k \in W^P$ be the relative position of $\sigma_P(\tilde{\mathcal{E}})(b_k)$ and \bar{g}_k , for any $1 \leq k \leq n$. Recall that, by the definition of $Z'_d(\tilde{\mathcal{E}}) = Z'_d(\tilde{\mathcal{E}}; u_1, \dots, u_n)$ as in Subsection 3.7, $\sigma_P(\tilde{\mathcal{E}}) \in Z'_d(\tilde{\mathcal{E}})$. Then, $\sigma_P(\tilde{\mathcal{E}})$ is the unique point of $Z'_d(\tilde{\mathcal{E}})$ and, moreover, it is a reduced point, i.e., the Zariski tangent space*

$$T(Z'_d(\tilde{\mathcal{E}}))_{\sigma_P(\tilde{\mathcal{E}})} = 0.$$

Proof. To prove that the scheme $Z'_d(\tilde{\mathcal{E}})$ contains the unique closed point $\sigma_P(\tilde{\mathcal{E}})$, use the fact that the canonical reduction σ_E of a G -bundle on C is completely characterized by the degree (cf. [15, Proposition 3.1]). Now, use the correspondence as in Theorem 4.3 and the degree comparisons as in the equation (26).

We now prove that $\sigma_P(\tilde{\mathcal{E}})$ is a reduced point of $Z'_d(\tilde{\mathcal{E}})$. If $Z'_d(\tilde{\mathcal{E}})$ were to possess any infinitesimal deformations at $\sigma_P(\tilde{\mathcal{E}})$, then so would the canonical reduction σ_E on the curve C . But, according to a result of Heinloth [21, Theorem 1], the canonical reduction of a principal G -bundle on a smooth projective curve does not have any infinitesimal deformations. We are therefore done by the equivalence of stacks as in Theorem 4.3. \square

Now, we are ready to prove the other direction of [Theorem 4.1](#).

Proof. (b) \Rightarrow (a) in [Theorem 4.1](#): Let $\mathcal{S}_{(u_1, \dots, u_n; d)}^P$ be a facet of \mathcal{C}_n for some standard maximal parabolic subgroup P and $(u_1, \dots, u_n; d) \in (W^P)^n \times \mathbb{Z}_+$ such that $(P; u_1, \dots, u_n; d)$ is quantum non-null. Any facet of \mathcal{C}_n meeting the relative interior of \mathcal{S}^n is of this form by [Remark 4.2](#).

Take a rational regular element $\vec{\mu} = (\mu_1, \dots, \mu_n) \in \mathcal{S}^n$ (i.e., each μ_k is rational, regular and $\theta_o(\mu_k) < 1$) such that $\vec{\mu}$ does not satisfy the inequality $\mathcal{S}_{(u_1, \dots, u_n; d)}^P$,

$$\begin{aligned} &\text{but satisfies every other inequality } \mathcal{S}_{(v_1, \dots, v_n; e)}^Q \\ &\text{such that } (Q; v_1, \dots, v_n; e) \text{ is quantum non-null.} \end{aligned} \tag{*}$$

Since $\vec{\mu} \notin \mathcal{C}_n$, the corresponding parabolic moduli space $\mathbb{M}_G(\vec{\mu})$ is empty. Hence, the trivial bundle ϵ_G with any parabolic structure $\bar{g}_k = g_k B$ at b_k (and weights μ_k) is not semistable. We fix some general elements $\bar{g}_k \in G/B$ and consider the parabolic bundle (with weights $\vec{\mu}$) $\tilde{\mathcal{E}} = (\mathcal{E} = \epsilon_G, \bar{g})$, where $\bar{g} := (\bar{g}_1, \dots, \bar{g}_n)$. Since $\tilde{\mathcal{E}}$ is unstable (i.e., non-semistable), the canonical parabolic reduction $\sigma_Q(\tilde{\mathcal{E}})$ (to a standard parabolic subgroup Q) does not satisfy the semistability inequality for $\vec{\mu}$, i.e., for any maximal parabolic subgroup $Q' \supset Q$,

$$-\text{deg}(\sigma_Q(\tilde{\mathcal{E}})^*(\mathcal{E}(\omega_{Q'}))) + \sum_{k=1}^n \omega_{Q'}(v_k^{-1} \mu_k) > 0, \tag{27}$$

where $v_k \in W/W_{Q'}$ is the relative position of $\sigma_Q(\tilde{\mathcal{E}})(b_k)$ and \bar{g}_k . Thus, by the assumption [\(*\)](#), for each maximal parabolic subgroup $Q' \supset Q$, we get the following:

- (A₁) $Q' = P$ and hence $Q = P$,
- (A₂) each $v_k = u_k$, and
- (A₃) $d' = d$, where $d' := \text{deg}(\sigma_Q(\tilde{\mathcal{E}})^*(\mathcal{E}(\omega_{Q'})))$.

We next claim that

$$\langle \sigma_{u_1}^P, \dots, \sigma_{u_n}^P \rangle_d = 1. \tag{28}$$

By (A₁)–(A₃), the parabolic degree of $\sigma_P(\tilde{\mathcal{E}})$ is given by

$$\text{pardeg}(\sigma_P(\tilde{\mathcal{E}})) = -d + \sum_{k=1}^n \omega_P(u_k^{-1} \mu_k).$$

Since the canonical reduction $\sigma_P(\tilde{\mathcal{E}})$ is the only reduction of $\tilde{\mathcal{E}}$ to P with the parabolic degree that of $\sigma_P(\tilde{\mathcal{E}})$ (cf. [Lemma 4.5](#)), we get that $\langle \sigma_{u_1}^P, \dots, \sigma_{u_n}^P \rangle_d = 1$. This proves [\(28\)](#) (which was already proved in [\[39\]](#)). We now come to the new part which is that

$$\langle \sigma_{u_1}^P, \dots, \sigma_{u_n}^P \rangle_d^{\otimes 0} = 1, \tag{29}$$

i.e., in view of (28) and Theorem 3.16, $(u_1, \dots, u_n; d)$ is quantum Levi-movable:

Choose a trivialization \bar{p}_k of $\sigma_P(\tilde{\mathcal{E}})_{b_k}/B_L$ so that $\bar{g}_k B u_k P = \bar{p}_k \Lambda_{u_k}$. (Observe that \bar{p}_k is unique modulo the stabilizer of Λ_{u_k} .) Consider the family $\tilde{\sigma}_P(\tilde{\mathcal{E}})_t = (\sigma_P(\tilde{\mathcal{E}})_t; \bar{p}_1(t), \dots, \bar{p}_n(t))$ of quasi-parabolic P -bundles on \mathbb{P}^1 parameterized by $t \in \mathbb{A}^1$, defined in §3.8, where we take $x = \bar{x}_P$.

It is easy to see that, as quasi-parabolic P -bundles,

$$\tilde{\sigma}_P(\tilde{\mathcal{E}})_t \simeq \tilde{\sigma}_P(\tilde{\mathcal{E}}), \text{ for } t \neq 0. \tag{30}$$

Let $\tilde{\mathcal{E}}_0$ denote the parabolic G -bundle obtained from $\sigma_P(\tilde{\mathcal{E}})_0$ via the extension of the structure group $P \hookrightarrow G$ and the parabolic structures $\bar{p}_1(0)u_1^{-1}B, \dots, \bar{p}_n(0)u_n^{-1}B$ at the points b_1, \dots, b_n on \mathbb{P}^1 (and with the same parabolic weights μ_k at b_k). Then, we assert that the canonical reduction of $\tilde{\mathcal{E}}_0$ coincides with $\tilde{\sigma}_P(\tilde{\mathcal{E}})_0$.

To prove this, observe first that for any parabolic G -bundle $\tilde{\mathcal{E}}$ on \mathbb{P}^1 that corresponds to a Γ -equivariant principal G -bundle E on C (via the correspondence of Theorem 4.3) and any reduction E_P of E to a parabolic subgroup P and the corresponding reduction $\tilde{\mathcal{E}}_P$, the deformations $(E_P)_t$ and $(\tilde{\mathcal{E}}_P)_t$ over $t \in \mathbb{A}^1$ (defined in Subsection 3.8 for $x = \bar{x}_P$) correspond. Further, from the characterization of the canonical reduction of (non-parabolic) G -bundles on C as in [15, Proposition 3.1], we see that $\sigma_P(E)_0$ is the canonical reduction of E_0 , where E_0 is the G -bundle on C obtained from $\sigma_P(E)_0$ via the extension of the structure group $P \hookrightarrow G$. Combining these, we get that the canonical reduction of $\tilde{\mathcal{E}}_0$ coincides with $\tilde{\sigma}_P(\tilde{\mathcal{E}})_0$.

Finally, we come to the proof of the equation (29):

By the last assertion, the quasi-parabolic G -bundle $\tilde{\mathcal{E}}_0$ comes from a quasi-parabolic L -bundle $\tilde{\mathcal{L}}$ of degree d by extension of the structure group $L \hookrightarrow G$. Further, by Lemma 4.5, $Z'_d(\tilde{\mathcal{E}}_0)$ is a reduced scheme of dimension 0 at $\tilde{\mathcal{L}} \times^L G$. Thus, $\theta'(\tilde{\mathcal{L}}) \neq 0$. This proves the equation (29) and hence Theorem 4.1 is proved. \square

5. Eigenvalue problem and global sections of line bundles on moduli spaces

Recall the definition of the quasi-parabolic moduli stack Parbun_G from Subsection 3.7.

Definition 5.1. For any weight $\lambda \in X(H)$ and $1 \leq k \leq n$, define the line bundle $\mathcal{L}_k(\lambda)$ over Parbun_G , that assigns to $\tilde{\mathcal{E}} = (\mathcal{E}; \bar{g}_1, \dots, \bar{g}_n) \in \text{Parbun}_G$ the line which is the fiber of the line bundle $\mathcal{E}_{b_k} \times^B \mathbb{C}_{-\lambda} \rightarrow \mathcal{E}_{b_k}/B$ over the point \bar{g}_k .

Therefore, given weights $\vec{\lambda} = (\lambda_1, \dots, \lambda_n)$ and an integer m , we can form the line bundle $\mathcal{M}(\vec{\lambda}, m)$ over Parbun_G defined by

$$\mathcal{M}(\vec{\lambda}, m) := \mathbb{D}^m \otimes (\otimes_{k=1}^n \mathcal{L}_k(\lambda_k)),$$

where the determinant line bundle \mathbb{D} over Parbun_G assigns to $\tilde{\mathcal{E}}$ the determinant of cohomology line $D(\mathcal{E} \times^G \mathfrak{g})$.

Consider the product flag variety $\mathcal{F} = \mathcal{Q}_G \times (G/B)^{\times n}$, where \mathcal{Q}_G is the infinite Grassmannian associated to G . Then, \mathcal{F} provides a uniformization of the moduli stack Parbun_G (cf. [30, Theorem 8.5]). The line bundle $\mathcal{M}(\vec{\lambda}, m)$, under the above uniformization of Parbun_G , lies at level $2g^*m$, where g^* is the dual Coxeter number of \mathfrak{g} (defined in Lemma 3.4) (cf. [29, Theorem 5.4 and Lemma 5.2]).

A dominant integral weight λ is said to be of level a nonnegative integer d if $\lambda(\theta_o^\vee) \leq d$.

The following theorem is used in the proof of the irredundancy result Theorem 8.1 (see Subsection 8.3).

Theorem 5.2. *Let $m > 0$ and let $\vec{\lambda} = (\lambda_1, \dots, \lambda_n)$ be dominant regular integral weights each of level $< 2g^*m$. Then, the following are equivalent:*

(a) $H^0(\text{Parbun}_G, \mathcal{M}(\vec{\lambda}, m)^r) \neq 0$, for some integer $r > 0$.

(b) The point $(\frac{\lambda_1^*}{2g^*m}, \dots, \frac{\lambda_n^*}{2g^*m})$ lies in \mathcal{C}_n , where $\lambda_k^* := \kappa(\lambda_k)$ and κ is defined in Section 3.

Proof. We first prove (a) \Rightarrow (b):

Assume that there is a non-vanishing global section in $H^0(\text{Parbun}_G, \mathcal{M}(\vec{\lambda}, m)^r)$, non-vanishing at a point $\tilde{\mathcal{E}} = (\mathcal{E}; \bar{g}_1, \dots, \bar{g}_n)$ of Parbun_G . As in the proof of Lemma 3.11, consider a one parameter family of deformations $\tilde{\mathcal{E}}_t = (\mathcal{E}_t; \bar{g}_1(t), \dots, \bar{g}_n(t))$ in Parbun_G parameterized by a smooth curve S so that at a marked point $0 \in S$, $\tilde{\mathcal{E}}_0 = \tilde{\mathcal{E}}$ and the underlying bundle \mathcal{E}_t is trivial for general $t \in S$. Thus, we can assume that \mathcal{E} is the trivial bundle ϵ_G and $(\bar{g}_1, \dots, \bar{g}_n)$ are general points of $(G/B)^n$. To prove (b), by [39, Proposition 4.4], it suffices to show that for any reduction $\tilde{\mathcal{P}} = (\mathcal{P}; \bar{p}_1, \dots, \bar{p}_n) \in \text{Parbun}_P(d)$ of $(\epsilon_G; \bar{g}_1, \dots, \bar{g}_n)$ to a standard maximal parabolic subgroup P of degree d in relative position $u_1, \dots, u_n \in W^P$, the inequality

$$\sum_{k=1}^n \omega_P(u_k^{-1} \lambda_k^*) \leq 2g^*md \tag{31}$$

is satisfied.

As in Subsection 3.8, induced by the conjugation $\phi_t : P \rightarrow P, p \mapsto t^{\bar{x}_P} p t^{-\bar{x}_P}$, $\tilde{\mathcal{P}}$ admits a one parameter family of deformations $\tilde{\mathcal{P}}_t = (\mathcal{P}_t; \bar{p}_1(t), \dots, \bar{p}_n(t)) \in \text{Parbun}_P(d)$ ($t \in \mathbb{A}^1$), such that $\tilde{\mathcal{P}}_1 = \tilde{\mathcal{P}}$ and $\tilde{\mathcal{P}}_0$ comes from the extension of a parabolic L -bundle $\tilde{\mathcal{L}} = (\mathcal{L}; \bar{l}_1, \dots, \bar{l}_n) \in \text{Parbun}_L(d)$. This gives rise to a morphism

$$\beta : \mathbb{A}^1 \rightarrow \text{Parbun}_G, \quad t \mapsto \tilde{\mathcal{E}}_t = (\mathcal{E}_t := \mathcal{P}_t \times^P G; \bar{g}_1(t), \dots, \bar{g}_n(t)),$$

where $\bar{g}_k(t) := \bar{p}_k(t) u_k^{-1} B$. Pulling back the line bundle $\mathcal{M}(\vec{\lambda}, m)$ via β , we get a \mathbb{G}_m -equivariant line bundle (denoted) \mathcal{M} over \mathbb{A}^1 with a nonzero section, where \mathbb{G}_m

acts on \mathbb{A}^1 via multiplication. We now calculate the action of \mathbb{G}_m on the fiber of \mathcal{M} over 0:

We first calculate the \mathbb{G}_m -action on $D(\mathcal{L} \times^L \mathfrak{g})$. Decompose \mathfrak{g} into eigenspaces \mathfrak{g}_γ such that the L -central one parameter subgroup $t^{\bar{x}_P}$ acts on \mathfrak{g}_γ by multiplication by $t^{-\gamma}$. Then the desired action is by t raised to the exponent

$$\begin{aligned} \sum_{\gamma} (\dim \mathfrak{g}_{-\gamma} + \deg(\mathcal{L} \times^L \mathfrak{g}_{-\gamma}))\gamma &= \sum_{\gamma} \deg(\mathcal{L} \times^L \mathfrak{g}_{-\gamma})\gamma, \quad \text{since } \dim \mathfrak{g}_{-\gamma} = \dim \mathfrak{g}_{\gamma} \\ &= d \sum_{\alpha \in R} \alpha(\bar{x}_P)\alpha(\alpha_P^\vee), \quad \text{from the definition of } d \\ &= 2g^*d\langle \bar{x}_P, \alpha_P^\vee \rangle. \end{aligned} \tag{32}$$

The last equality follows from [Lemma 3.4](#).

It is easy to see that the action of $t^{\bar{x}_P}$ on the fiber $\mathcal{L}_k(\lambda_k)_{\tilde{\mathcal{L}}}$ of $\mathcal{L}_k(\lambda_k)$ over $\tilde{\mathcal{L}}$ is given by the exponent $-(u_k^{-1}\lambda_k)(\bar{x}_P)$. Thus, combining the equation (32) with the above expression of the action of $t^{\bar{x}_P}$ on the fiber $\mathcal{L}_k(\lambda_k)_{\tilde{\mathcal{L}}}$, we get that the action of $t^{\bar{x}_P}$ on the fiber of \mathcal{M} over 0 is given by the exponent

$$\mu := 2g^*md\langle \bar{x}_P, \alpha_P^\vee \rangle - \sum_{k=1}^n (u_k^{-1}\lambda_k)(\bar{x}_P). \tag{33}$$

But, by [Lemma 3.17](#), $\mu \geq 0$. From this and the equation (11), we get the equation (31). This proves the (b)-part of the proposition.

Proof of ‘(b) \Rightarrow (a)’: By assumption, each $\lambda_k^*/2g^*m$ is a rational, regular element of \mathcal{A} with $\theta_o(\lambda_k^*/2g^*m) < 1$. Let $\vec{\mu} = (\lambda_1^*/2g^*m, \dots, \lambda_n^*/2g^*m)$. Assign these weights to the parabolic points of \mathbb{P}^1 . By the assumption (b), the semistable parabolic moduli space of parabolic G -bundles on \mathbb{P}^1 (corresponding to the weights $\vec{\mu}$) is non-empty. Let $\tilde{\mathcal{E}}$ be a semistable parabolic bundle on \mathbb{P}^1 with the given weights. Under the ramified cover correspondence as in [Theorem 4.3](#), $\tilde{\mathcal{E}}$ corresponds to a semistable (Γ -equivariant, where $\Gamma = \mathbb{Z}/(N)$) principal G -bundle \mathcal{E}' on a (suitable) ramified cover $\pi : C \rightarrow \mathbb{P}^1$ under the map $f : \text{Parbun}_G \rightarrow \text{Bun}_G(C)$ (which is the inverse of the isomorphism of [Theorem 4.3](#)), where $\text{Bun}_G(C)$ is the moduli stack of (non-parabolic) principal G -bundles on C . Let \mathbb{D}_C be the line bundle over $\text{Bun}_G(C)$ which assigns to $\mathcal{F} \in \text{Bun}_G(C)$ the line $D(C, \mathcal{F} \times^G \mathfrak{g})$. We have the pull-back map

$$H^0(\text{Bun}_G(C), \mathbb{D}_C^r) \rightarrow H^0(\text{Parbun}_G, f^*\mathbb{D}_C^r).$$

Since our \mathcal{E}' is semistable, there is a global section $s \in H^0(\text{Bun}_G(C), \mathbb{D}_C^r)$ for a suitable $r > 0$ which is non-vanishing at \mathcal{E}' :

This follows from the isomorphism (cf. [\[30, §9.3\]](#)); by taking N large enough we can insure that C has genus ≥ 2)

$$H^0(\text{Bun}_G(C), \mathbb{D}_C^r) \simeq H^0(\mathfrak{M}_G(C), \Theta_{ad}^r),$$

where Θ_{ad} is the theta bundle on the moduli space $\mathfrak{M}_G(C)$ of semistable principal G -bundles on C associated to the adjoint representation of G . Further, since $\mathfrak{M}_G(C)$ is a projective variety and the Picard group $\text{Pic}(\mathfrak{M}_G(C)) \simeq \mathbb{Z}$ (cf. [28, Theorem 2.4]), we get that Θ_{ad} is ample and hence there exists $s' \in H^0(\mathfrak{M}_G(C), \Theta_{ad}^r)$, for large enough r , such that $s'(\mathcal{E}') \neq 0$.

Therefore, the pull-back of this section to Parbun_G is non-vanishing at $\tilde{\mathcal{E}}$. So, to finish the proof of (a), (replacing r by rm) we need to know that $f^*\mathbb{D}_C^m$ equals some power of $\mathcal{M}(\vec{\lambda}, m)$, which follows from Proposition 6.4. \square

Remark 5.3. (1) By [30, §8.9], the condition (a) in the above theorem is equivalent to the non-vanishing of the space of conformal blocks on \mathbb{P}^1 associated to the weights $r\vec{\lambda}$ at level $2rg^*m$.

(2) For the implication ‘(a) \Rightarrow (b)’ in the above theorem, we do not need to assume that each λ_k is regular in our above proof (since we verified all necessary inequalities). Also, the implication ‘(b) \Rightarrow (a)’ is true without this restriction using [2] as we explain below. The map $\tilde{f} : \text{Parbun}_G \rightarrow \text{Bun}_G^\Gamma(C)$ exists (as in Section 6 below) even without the assumption of regularity (and the level is allowed to equal $2g^*m$). It follows from [2] that \tilde{f} is surjective on objects [2, Theorem 4.1.6]. Therefore, it remains to show that there is a point of $\text{Bun}_G^\Gamma(C)$ which maps to a semistable point of $\text{Bun}_G(C)$. We can construct this point by considering the principal G -bundle corresponding to the local system (of K -spaces) in (b) pulled up to C and extended over the punctures (where the local monodromies are trivial). Since this Γ -equivariant bundle comes from a representation of the fundamental group of C in K , it is semistable by Ramanathan’s generalization [35, Theorem 7.1] of the Narasimhan–Seshadri theorem.

6. Comparison of determinants of cohomology under an elementary modification

Recall that the proof of Theorem 5.2 has used the following Proposition 6.4. Also, Lemma 6.5 will be used in the proof of the irredundance result Theorem 8.1 (cf. Subsection 8.3).

Let G be a connected reductive group in this section. The calculation will be applied to the simple groups of the earlier sections as well as their Levi subgroups. We analyze the effect of an “elementary modification” on determinants of cohomology.

Let \mathcal{E} be a principal G -bundle on a smooth irreducible projective curve C . Let $0 \in C$ with formal parameter z . Fix a section $s \in \mathcal{E}(D)$, i.e., a trivialization s of \mathcal{E} over the formal disc D at 0. For $h \in \mathfrak{h}$ such that $\exp(2\pi ih) = 1$, consider the map $\ell : D \rightarrow H$ given by $\ell(z) = z^h$, where \mathfrak{h} is the Lie algebra of a maximal torus H of G .

Consider a new principal G -bundle $\mathcal{E}' = \mathcal{E}_\ell$ which coincides with \mathcal{E} outside of 0. Sections of \mathcal{E}' over D are meromorphic sections $sa(z)$ of \mathcal{E} over D such that $\ell(z)a(z)$ is regular at 0. We have a section $s' = s\ell(z)^{-1}$ of \mathcal{E}' over D . Now, consider a representation $G \rightarrow \text{GL}(V)$. Decompose V into eigenspaces under the action of H :

$$V = \bigoplus_{\gamma \in A \subset X(H)} V_\gamma,$$

where $A \subset X(H)$ is defined to be the subset such that $V_\gamma \neq 0$. We assume that A is symmetric under taking negatives.

We want to compare the determinants of cohomologies of $\mathcal{E} \times^G V$ and $\mathcal{E}' \times^G V$. With our data, over D , we can write as a trivial vector bundle:

$$\mathcal{E} \times^G V = \bigoplus_{\gamma} (\mathcal{O}(D)s) \otimes V_\gamma = \bigoplus_{\gamma} L_\gamma, \quad \text{and} \quad \mathcal{E}' \times^G V = \bigoplus_{\gamma} (z^{-h} \mathcal{O}(D)s) \otimes V_\gamma = \bigoplus_{\gamma} L'_\gamma, \tag{34}$$

under the meromorphic identification of L_γ with L'_γ over D .

Definition 6.1. Let \mathcal{V} and \mathcal{V}' be vector bundles over C which are identified outside of 0. Then, as in [4], there is a well defined line $[\mathcal{V} : \mathcal{V}']$ (which formally stands for $\det(\mathcal{V}/\mathcal{V}')$) and a canonical isomorphism

$$D(\mathcal{V}') = D(\mathcal{V}) \otimes [\mathcal{V} : \mathcal{V}']. \tag{35}$$

To define it, find a large positive k so that $\mathcal{V}(k) \supset \mathcal{V}'$ and set

$$[\mathcal{V} : \mathcal{V}'] = \det(\mathcal{V}(k)/\mathcal{V})^{-1} \otimes \det(\mathcal{V}(k)/\mathcal{V}').$$

Note that $[\mathcal{V} : \mathcal{V}']$ is multiplicative.

Applying this to $\mathcal{E} \times^G V$ and $\mathcal{E}' \times^G V$, we get

$$D(\mathcal{E}' \times^G V) \otimes D(\mathcal{E} \times^G V)^{-1} = \bigotimes_{\gamma \in A} [L_\gamma : L'_\gamma]. \tag{36}$$

The following lemma follows easily from the equation (34) and the definition of $[- : -]$.

Lemma 6.2. *For any $\gamma \in A$ with $\gamma(h) < 0$,*

$$[L_\gamma : L'_\gamma] = \wedge^{top} \left(\frac{\mathcal{O}(D)s}{z^{-\gamma(h)} \mathcal{O}(D)s} \otimes V_\gamma \right),$$

and, for any $\gamma \in A$ with $\gamma(h) > 0$,

$$[L_\gamma : L'_\gamma] = \left(\wedge^{top} \left(\frac{z^{-\gamma(h)} \mathcal{O}(D)s}{\mathcal{O}(D)s} \otimes V_\gamma \right) \right)^{-1}.$$

Suppose $\ell(z)b\ell(z)^{-1}$ is regular at $z = 0$ for all $b \in B$, i.e., $h \in \mathfrak{h}_+$. Then, changing s to sb does not change \mathcal{E}' . Therefore, we get an action of B (in particular of H) on $D(\mathcal{E}' \times^G V) \otimes D(\mathcal{E} \times^G V)^{-1}$. The H -action respects $[L_\gamma : L'_\gamma]$ for any $\gamma \in A$. Combining the equation (36) with the above lemma, since A is symmetric, we get the following:

Proposition 6.3.

$$D(\mathcal{E}' \times^G V) = D(\mathcal{E} \times^G V) \otimes \left(\otimes_{\gamma \in A: \gamma(h) < 0} [\wedge^{\text{top}} \left(\frac{\mathcal{O}(D)_s}{z^{-\gamma(h)} \mathcal{O}(D)_s} \otimes V_\gamma \right) \otimes \wedge^{\text{top}} \left(\frac{z \mathcal{O}(D)_s}{z^{-\gamma(h)+1} \mathcal{O}(D)_s} \otimes V_\gamma \right)] \right).$$

Further, the weight of the H -action on $D(\mathcal{E}' \times^G V) \otimes D(\mathcal{E} \times^G V)^{-1}$ is given by

$$-2 \sum_{\gamma \in A: \gamma(h) > 0} \gamma(h) \dim V_\gamma \cdot \gamma.$$

We now come to the proof of the following result, which was used in the proof of [Theorem 5.2](#).

Proposition 6.4. *Let the notation and assumptions be as in the proof of [Theorem 5.2](#): ‘(b) \Rightarrow (a)’. Then,*

$$f^* \mathbb{D}_C^m = (\mathcal{M}(\vec{\lambda}, m))^N,$$

where N is the degree of the ramified cover $\pi : C \rightarrow \mathbb{P}^1$ of [Subsection 4.1](#).

Proof. Take $\tilde{\mathcal{F}} = (\mathcal{F}; \bar{g}_1, \dots, \bar{g}_n) \in \text{Parbun}_G$ and let $\mathcal{E} := \pi^* \mathcal{F}$. By the projection formula

$$D(C, \mathcal{E} \times^G \mathfrak{g}) = D(\mathbb{P}^1, \mathcal{F} \times^G \mathfrak{g})^N. \tag{37}$$

Let $\mathcal{E}' = f(\tilde{\mathcal{F}})$ be the new bundle over C obtained via modifying the bundle \mathcal{E} at the points \tilde{b}_k (by twisting \mathcal{E} at the points \tilde{b}_k via the morphism $\ell_k : \tilde{D}_k \rightarrow H, \tilde{z}_k \mapsto \tilde{z}_k^{N\lambda_k^*/2g^*m}$, where \tilde{z}_k is a local parameter for the formal disk \tilde{D}_k of C at \tilde{b}_k), where f is the inverse of the ramified cover correspondence as in [Theorem 4.3](#) for $\vec{\mu} = (\frac{\lambda_1^*}{2g^*m}, \dots, \frac{\lambda_n^*}{2g^*m})$. By [Definition 6.1](#),

$$D(\mathcal{E}' \times^G \mathfrak{g}) = D(\mathcal{E} \times^G \mathfrak{g}) \otimes \left(\otimes_{k=1}^n \left(\otimes_{\beta \in R \cup \{0\}} [L_\beta(k) : L'_\beta(k)] \right) \right), \tag{38}$$

where $L_\beta(k), L'_\beta(k)$ over \tilde{D}_k are defined as in the beginning of this section taking V to be the adjoint representation \mathfrak{g} of G . By [Proposition 6.3](#), the action of H on $\otimes_{\beta \in R \cup \{0\}} [L_\beta(k) : L'_\beta(k)]$ is via the linear form θ_k on \mathfrak{h} defined as follows:

$$\begin{aligned} \theta_k(h) &:= -2 \sum_{\beta \in R^+} \beta(N\lambda_k^*/2g^*m) \beta(h), \text{ for } h \in \mathfrak{h} \\ &= -\left\langle \frac{N\lambda_k^*}{m}, h \right\rangle, \text{ by [Lemma 3.4](#).} \end{aligned}$$

Thus,

$$\theta_k = -N\lambda_k/m. \tag{39}$$

Combining the equations (37), (38) and (39), we get

$$\begin{aligned} f^*(\mathbb{D}_C)^m &= \mathbb{D}^{Nm} \otimes (\otimes_{k=1}^n \mathcal{L}_k(N\lambda_k)) \\ &= \mathcal{M}(\vec{\lambda}, m)^N. \end{aligned} \tag{40}$$

This proves the proposition.

We give another proof of the proposition using the affine flag variety realization of Parbun_G .

Second Proof. Let \tilde{z}_k be a parameter for the curve C at the point \tilde{b}_k and let z_k be a parameter for \mathbb{P}^1 at the point b_k . Let \mathfrak{G}_{b_k} (resp. $\mathfrak{G}_{\tilde{b}_k}$) be the loop group $G(\mathbb{C}((z_k)))$ (resp. $G(\mathbb{C}((\tilde{z}_k)))$) and let \mathfrak{B}_{b_k} (resp. $\mathfrak{P}_{\tilde{b}_k}$) be the Iwahori (resp. parahoric) subgroup defined as $\{h \in G(\mathbb{C}[[z_k]]) : h(0) \in B\}$ (resp. $G(\mathbb{C}[[\tilde{z}_k]])$). Let $\Gamma_{\mathbb{P}^1}$ (resp. Γ_C) be the group of maps $\mathbb{P}^1 \setminus \{b_1, \dots, b_n\} \rightarrow G$ (resp. maps $C \setminus \{\tilde{b}_1, \dots, \tilde{b}_n\} \rightarrow G$). Then, there are canonical identifications (cf. [30, Theorem 8.5] for a similar result, the proof of which can be adapted to prove the following; also see [34]):

$$\text{Parbun}_G \simeq \Gamma_{\mathbb{P}^1} \backslash \left(\prod_{k=1}^n \mathfrak{G}_{b_k} / \mathfrak{B}_{b_k} \right), \tag{41}$$

and

$$\text{Bun}_G(C) \simeq \Gamma_C \backslash \left(\prod_{k=1}^n \mathfrak{G}_{\tilde{b}_k} / \mathfrak{P}_{\tilde{b}_k} \right), \tag{42}$$

where $\Gamma_{\mathbb{P}^1}$ acts on $\prod_{k=1}^n \mathfrak{G}_{b_k} / \mathfrak{B}_{b_k}$ diagonally via evaluation at b_k on the k -th factor and a similar action of Γ_C on $\prod_{k=1}^n \mathfrak{G}_{\tilde{b}_k} / \mathfrak{P}_{\tilde{b}_k}$. Now, the map $f : \text{Parbun}_G \rightarrow \text{Bun}_G(C)$ corresponds (under the above identifications) to the map

$$\hat{f} : \Gamma_{\mathbb{P}^1} \backslash \left(\prod_{k=1}^n \mathfrak{G}_{b_k} / \mathfrak{B}_{b_k} \right) \rightarrow \Gamma_C \backslash \left(\prod_{k=1}^n \mathfrak{G}_{\tilde{b}_k} / \mathfrak{P}_{\tilde{b}_k} \right),$$

taking

$$\Gamma_{\mathbb{P}^1}(g_1(z_1)\mathfrak{B}_{b_1}, \dots, g_n(z_n)\mathfrak{B}_{b_n}) \mapsto \Gamma_C(g_1(\tilde{z}_1^N)\tilde{z}_1^{-N\mu_1}\mathfrak{P}_{\tilde{b}_1}, \dots, g_n(\tilde{z}_n^N)\tilde{z}_n^{-N\mu_n}\mathfrak{P}_{\tilde{b}_n}).$$

Consider the commutative diagram

$$\begin{array}{ccc}
 \prod_{k=1}^n \mathfrak{G}_{b_k}/\mathfrak{B}_{b_k} & \xrightarrow{\tilde{f}} & \prod_{k=1}^n \mathfrak{G}_{\tilde{b}_k}/\mathfrak{P}_{\tilde{b}_k} \\
 \downarrow \pi & & \downarrow \tilde{\pi} \\
 \Gamma_{\mathbb{P}^1} \backslash \left(\prod_{k=1}^n \mathfrak{G}_{b_k}/\mathfrak{B}_{b_k} \right) & \xrightarrow{\hat{f}} & \Gamma_C \backslash \left(\prod_{k=1}^n \mathfrak{G}_{\tilde{b}_k}/\mathfrak{P}_{\tilde{b}_k} \right),
 \end{array}$$

where $\tilde{\pi}$ and π are the standard projection maps and \tilde{f} is defined similar to \hat{f} . Now, by [29, Theorem 5.4 and Lemma 5.2], the bundle $\tilde{\pi}^*(\mathbb{D}_C^m)$ over $\prod_{k=1}^n \mathfrak{G}_{\tilde{b}_k}/\mathfrak{P}_{\tilde{b}_k}$ is the homogeneous line bundle corresponding to the affine characters $(-2g^*m\omega_0, \dots, -2g^*m\omega_0)$ respectively on each factor, where ω_0 is the zeroth affine fundamental weight. Similarly, the bundle $\pi^*(\mathcal{M}(\vec{\lambda}, m))$ over $\prod_{k=1}^n \mathfrak{G}_{b_k}/\mathfrak{B}_{b_k}$ is the homogeneous line bundle corresponding to the affine characters $(-2g^*m\omega_0 - \lambda_1, \dots, -2g^*m\omega_0 - \lambda_n)$ respectively on each factor. It is easy to see that the homogeneous line bundle corresponding to the affine characters $(-2g^*m\omega_0, \dots, -2g^*m\omega_0)$ over $\prod_{k=1}^n \mathfrak{G}_{\tilde{b}_k}/\mathfrak{P}_{\tilde{b}_k}$ pulls back under \tilde{f} to the homogeneous line bundle corresponding to the affine characters $(-2g^*mN\omega_0 - 2g^*mN\mu_1^*, \dots, -2g^*mN\omega_0 - 2g^*mN\mu_n^*)$ over $\prod_{k=1}^n \mathfrak{G}_{b_k}/\mathfrak{B}_{b_k}$ (cf. [27, Definition 11.3.4] and the fact that $H^2(\mathfrak{G}_{b_k}/\mathfrak{B}_{b_k}, \mathbb{Z}) \simeq \text{Pic}(\mathfrak{G}_{b_k}/\mathfrak{B}_{b_k})$, where the latter can be proved by the same proof as that of [29, Proposition 2.3]). Since $\mu_k = \lambda_k^*/2g^*m$, and the map π induces an injective map of the corresponding Picard groups (cf. [30, Proof of Proposition 8.7]), we get the proposition. \square

Lemma 6.5. *For any (not necessarily dominant) integral weights $\vec{\lambda} = (\lambda_1, \dots, \lambda_n)$ and an integer m , the space $H^0(\text{Parbun}_G, \mathcal{M}(\vec{\lambda}, m)) = 0$ unless m is a non-negative integer and each λ_k is dominant of level $2g^*m$.*

Proof. First of all, from the affine analogue of the Borel–Weil–Bott theorem (cf. [27, Corollary 8.3.12]), the space (following the notation in the ‘Second Proof’ of Proposition 6.4) $H^0(\mathfrak{G}_{b_k}/\mathfrak{B}_{b_k}, \mathfrak{L}(\lambda_k + 2g^*m\omega_0))$ is zero unless m is a non-negative integer and λ_k is a dominant integral weight for G of level $2g^*m$, where $\mathfrak{L}(\lambda_k + 2g^*m\omega_0)$ is the homogeneous line bundle over $\mathfrak{G}_{b_k}/\mathfrak{B}_{b_k}$ corresponding to the affine character $\lambda_k + 2g^*m\omega_0$. Now, the lemma follows from the identification (41), since the line bundle $\mathcal{M}(\vec{\lambda}, m)$ pulls-back (under this identification) to the line bundle

$$\mathfrak{L}(\lambda_1 + 2g^*m\omega_0) \boxtimes \dots \boxtimes \mathfrak{L}(\lambda_n + 2g^*m\omega_0) \text{ over } \prod_{k=1}^n \mathfrak{G}_{b_k}/\mathfrak{B}_{b_k},$$

and the identification of the space of sections $H^0(\text{Parbun}_G, \mathcal{M}(\vec{\lambda}, m))$ with

$$H^0\left(\prod_{k=1}^n \mathfrak{G}_{b_k}/\mathfrak{B}_{b_k}, \mathfrak{L}(\lambda_1 + 2g^*m\omega_0) \boxtimes \dots \boxtimes \mathfrak{L}(\lambda_n + 2g^*m\omega_0)\right)^{\Gamma_{\mathbb{P}^1}}$$

(cf. [30, Section 8.9]).

We give another proof.

Second Proof. Assume that $H^0(\text{Parbun}_G, \mathcal{M}(\vec{\lambda}, m)) \neq 0$. By the first displayed identification given in [30, §8.10] induced from the uniformization of Parbun_G by the flag variety \mathcal{F} (cf. Definition 5.1), it follows immediately that λ_k are dominant. Suppose $\lambda_1(\theta_o^\vee) > k = 2g^*m$. Choose $e_{\theta_o} \in \mathfrak{g}_{\theta_o}$ and $f_{\theta_o} \in \mathfrak{g}_{-\theta_o}$ such that $[e_{\theta_o}, f_{\theta_o}] = \theta_o^\vee$, where \mathfrak{g}_{θ_o} (resp. $\mathfrak{g}_{-\theta_o}$) is the highest (resp. lowest) root space of \mathfrak{g} . Take distinct points $\{p_1, \dots, p_n\} \subset \mathbb{A}^1 \subset \mathbb{P}^1$ so that $p_1 = 0$. It is easy to see, using [3,30], that $H^0(\text{Parbun}_G, \mathcal{M}(\vec{\lambda}, m))$ is identified with the dual of a space of the form

$$\mathbb{V} = \frac{(V_{\lambda_1} \otimes \dots \otimes V_{\lambda_n})}{\mathfrak{g}(V_{\lambda_1} \otimes \dots \otimes V_{\lambda_n}) + \text{im } T^{k+1}},$$

where $k = 2g^*m$, V_{λ_k} is the finite dimensional irreducible representation of \mathfrak{g} with highest weight λ_k and $T = \sum_{k=2}^n p_k f_{\theta_o}^{(k)}$ ($f_{\theta_o}^{(k)}$ denotes the action of f_{θ_o} on the k -th factor). We show that $\mathbb{V} = 0$, which will complete the second proof.

We need only show that $\tau = v_1^+ \otimes v_2 \otimes \dots \otimes v_n = 0 \in \mathbb{V}$, where v_1^+ is the highest weight vector of V_{λ_1} and $v_k \in V_{\lambda_k}$ are arbitrary weight vectors (since these vectors generate $V_{\lambda_1} \otimes \dots \otimes V_{\lambda_n}$ as a \mathfrak{g} -module). To prove this, introduce the operator $S = \sum_{k=2}^n \frac{1}{p_k} e_{\theta_o}^{(k)}$ on $(V_{\lambda_1} \otimes \dots \otimes V_{\lambda_n})$. Let

$$W_0 = (\mathbb{C}v_1^+ \otimes V_{\lambda_2} \otimes \dots \otimes V_{\lambda_n}) \subseteq (V_{\lambda_1} \otimes \dots \otimes V_{\lambda_n}).$$

Clearly, S and T act on W_0 . Let $\theta_o^\vee(v_k) = \mu_k v_k$, $k = 2, \dots, n$. Then, we can see that $[S, T](v_1^+ \otimes v_2 \otimes \dots \otimes v_n) = (\sum_{k \geq 2} \mu_k)(v_1^+ \otimes v_2 \otimes \dots \otimes v_n)$.

This leads to a \mathfrak{sl}_2 -action on W_0 with the action of $H \in \mathfrak{sl}_2$ by

$$H(v_1^+ \otimes v_2 \otimes \dots \otimes v_n) = (\sum_{k \geq 2} \mu_k)(v_1^+ \otimes v_2 \otimes \dots \otimes v_n),$$

E by S and F by T , where $\{E, F, H\}$ is the standard basis of \mathfrak{sl}_2 . We can further assume that τ is of weight 0, for otherwise it is already in $\mathfrak{g}(V_{\lambda_1} \otimes \dots \otimes V_{\lambda_n})$ itself. Then, the action of H on τ is multiplication by $\sum_{k \geq 2} \mu_k = -\lambda_1(\theta_o^\vee) < -k$. Now, note that in any (not necessarily irreducible) finite dimensional \mathfrak{sl}_2 -representation W , if H acts on $\tau \in W$ by $-m$ with $m > 0$, then τ can be written as $F^m \tau'$, for some $\tau' \in W$. Applying this to our τ , we see that $\tau = T^{k+1} \tau'$ for some $\tau' \in W_0$. Hence, $\tau = 0$ in \mathbb{V} . \square

7. Levi twistings

Let G be a simple, simply-connected complex algebraic group and let P be a standard maximal parabolic subgroup. Let L be the Levi subgroup of P containing the maximal torus H .

This section is devoted to proving an isomorphism between the parabolic moduli stacks of L at degrees d and $d \pm 1$ (cf. Lemma 7.4). This *Levi twisting* allows us to use a parabolic analogue of Narasimhan–Seshadri theorem. This is used in the induction argument in the proof of the irredundance result (Theorem 8.1).

7.1. *The structure of Levi subgroups*

Note that

$$\pi_2(G/P) \simeq H_2(G/P) = \mathbb{Z} \quad \text{and} \quad \pi_2(G) = 0,$$

where the first identification is via the Hurewicz theorem and the second identification with \mathbb{Z} is by the Bruhat decomposition. (Of course $\pi_2(G) = 0$ for any Lie group G .) Hence, from the long exact homotopy sequence for the fibration $G \rightarrow G/P$, we get the isomorphism:

$$\beta : \mathbb{Z} = H_2(G/P) \rightarrow \pi_1(P) \simeq \pi_1(L), \tag{43}$$

where the last isomorphism of course follows since L is a deformation retract of P .

The fundamental character ω_P extends to a character (still) denoted by $\omega_P : L \rightarrow \mathbb{G}_m$ inducing a map $\pi_1(L) \rightarrow \pi_1(\mathbb{G}_m) = \mathbb{Z}$. The loop z^{α_P} goes over to $1 \in \mathbb{Z}$. Therefore, $\omega_P : L \rightarrow \mathbb{G}_m$ induces an isomorphism on fundamental groups.

Consider the maximal semisimple subgroup $L' = [L, L]$. Then, L' is a (connected) simply-connected group. To prove this, use the long exact homotopy sequence corresponding to the fibration $\pi : L \rightarrow L/L'$ together with the isomorphism (43) and the fact that L/L' is a one dimensional torus. Let Z^o be the identity component of the center Z of L . Then, the canonical map $i : Z^o \rightarrow L/L'$ is clearly an isogeny.

Consider the pull-back diagram:

$$\begin{array}{ccc} \tilde{L} & \xrightarrow{\tilde{i}} & L \\ \downarrow \tilde{\pi} & & \downarrow \pi \\ Z^o & \xrightarrow{i} & L/L'. \end{array}$$

Then, the central inclusion $Z^o \hookrightarrow L$ splits the left vertical projection, giving rise to a canonical isomorphism:

$$\tilde{L} \simeq L' \times Z^o.$$

As in equation (3), let N_P be the smallest positive integer such that $\bar{x}_P := N_P x_P$ belongs to the coroot lattice of G . Then, $z \mapsto z^{\bar{x}_P}$ gives an isomorphism $\mathbb{G}_m \rightarrow Z^o$. Let k_L be the order of the cokernel of $\pi_1(Z^o) \rightarrow \pi_1(L)$. Then, k_L is also the order of $\text{Ker } i$ (and $\text{Ker } \tilde{i}$).

7.2. *Degrees of principal L-bundles*

Recall that the degree of a principal L -bundle \mathcal{L} over \mathbb{P}^1 is the first Chern class of the line bundle $\mathcal{L} \times^L \mathbb{C}_{-\omega_P} \rightarrow \mathbb{P}^1$. Since topologically the principal L -bundles on \mathbb{P}^1 are classified by $\pi_1(L)$ via the clutching construction, one can view the degree as an element

in $\pi_1(L) \simeq \mathbb{Z}$ (under the canonical identification as in (43)). We will be interested in considering the k_L -degree

$$\text{deg}_{k_L}(\mathcal{L}) := \text{deg}(\mathcal{L}) \pmod{k_L}$$

of a principal L -bundle.

7.3. Moduli stacks

Consider the stack Parbun_L parameterizing the quasi-parabolic L -bundles $\tilde{\mathcal{L}} = (\mathcal{L}; \bar{l}_1, \dots, \bar{l}_n)$ on \mathbb{P}^1 consisting of a principal L -bundle \mathcal{L} on \mathbb{P}^1 with parabolic structure $\bar{l}_k \in \mathcal{L}_{b_k}/B_L$, $k = 1, \dots, n$. This breaks up according to the degrees $d \in \pi_1(L)$:

$$\text{Parbun}_L = \bigsqcup_d \text{Parbun}_L(d).$$

7.4. Line bundles on Parbun_L

Given the characters $\vec{\lambda} = (\lambda_1, \dots, \lambda_n)$ of H and an integer m , analogous to the definition of $\mathcal{M}(\vec{\lambda}, m)$ as in Section 5, we can form the line bundle $\mathcal{N}(\vec{\lambda}, m)$ on Parbun_L whose fiber over $\tilde{\mathcal{L}} = (\mathcal{L}, \bar{l}_1, \dots, \bar{l}_n)$ is the line

$$D(\mathcal{L} \times^L \mathfrak{g})^m \otimes (\otimes_{k=1}^n \bar{\mathcal{L}}_k(\lambda_k)),$$

where $\bar{\mathcal{L}}_k(\lambda_k)$ is the fiber of the line bundle $\mathcal{L}_{b_k} \times^{B_L} \mathbb{C}_{-\lambda_k} \rightarrow \mathcal{L}_{b_k}/B_L$ over \bar{l}_k .

7.5. Sections of $\mathcal{N}(\vec{\lambda}, m)$ over $\text{Parbun}_L(d)$

By a computation analogous to the equation (33), we see that the identity component Z^o of the center of L acts trivially on $\mathcal{N}(\vec{\lambda}, m)$ if and only if

$$\sum_{k=1}^n \lambda_k(x_P) = 4g^* dm / \langle \alpha_P, \alpha_P \rangle. \tag{44}$$

7.6. Existence of global sections of line bundles over $\text{Parbun}_L(d)$

Let \mathfrak{l}' be the Lie algebra of L' and let $\mathfrak{h}' := \mathfrak{h} \cap \mathfrak{l}'$ be its Cartan subalgebra. We can not apply Theorem 5.2 directly since L is not semisimple. We first consider the case when k_L divides d . Write $d = k_L d'$. Now, consider the map

$$\eta : \text{Parbun}_{L'} \rightarrow \text{Parbun}_L(d), \tag{45}$$

which sends a parabolic L' -bundle $\tilde{\mathcal{L}}'$ to the parabolic L -bundle $\tilde{\mathcal{L}}$ obtained as the image of the principal \tilde{L} -bundle $\mathcal{L}' \times A$ under \tilde{i} , where A is the principal \mathbb{G}_m -bundle corresponding to $\mathcal{O}(d')$. (The parabolic structure on \mathcal{L} is the canonical one coming from the parabolic structure on \mathcal{L}' .)

This gives rise to the pull-back map

$$\eta^* : H^0(\text{Parbun}_L(d), \mathcal{N}(\vec{\lambda}, m)) \rightarrow H^0(\text{Parbun}_{L'}(d), \mathcal{N}'(\vec{\lambda}', m)), \tag{46}$$

where $\vec{\lambda}' := (\lambda'_1, \dots, \lambda'_n)$, $\lambda'_k := \lambda_{k|_{\mathfrak{h}'}}$ and $\mathcal{N}'(\vec{\lambda}', m)$ is the line bundle which assigns to a parabolic L' -bundle $\tilde{\mathcal{L}}'$, the line

$$D(\mathcal{L}' \times^{L'} \mathfrak{g})^m \otimes (\otimes_{k=1}^n \tilde{\mathcal{L}}_k(\lambda'_k)).$$

Lemma 7.1. *The map η^* is an isomorphism for any $\mathcal{N}(\vec{\lambda}, m)$ such that the equation (44) is satisfied.*

Proof. The map η of stacks is surjective since the structure group of any principal L -bundle over \mathbb{P}^1 of degree divisible by k_L lifts to \tilde{L} and, moreover, $L'/B_{L'} = L/B_L$. Hence, η^* is injective.

To show that η^* is surjective, pick a $\phi' \in H^0(\text{Parbun}_{L'}(d), \mathcal{N}'(\vec{\lambda}', m))$. We need to lift the section ϕ' to a section $\phi \in H^0(\text{Parbun}_L(d), \mathcal{N}(\vec{\lambda}, m))$. Since $L'/B_{L'} = L/B_L$, it suffices to show that for any $\tilde{\mathcal{L}}_1, \tilde{\mathcal{L}}_2 \in \text{Parbun}_L(d)$, lifts $\tilde{\mathcal{L}}'_1, \tilde{\mathcal{L}}'_2 \in \text{Parbun}_{L'}$ (via η), and any L -bundle isomorphism $\phi : \mathcal{L}_1 \rightarrow \mathcal{L}_2$, there exists $z \in Z^\circ$ such that the composed L -bundle isomorphism $m_z \circ \phi : \mathcal{L}_1 \rightarrow \mathcal{L}_2$ (where $m_z : \mathcal{L}_2 \rightarrow \mathcal{L}_2$ is the isomorphism $e \mapsto ez^{-1}$) lifts to a L' -bundle isomorphism $\phi' : \mathcal{L}'_1 \rightarrow \mathcal{L}'_2$ (since, by assumption, Z° acts trivially on $\mathcal{N}(\vec{\lambda}, m)$). To prove this, take any elements $e_1(x) \in \mathcal{L}'_1(x)$, $e_2(x) \in \mathcal{L}'_2(x)$ in the fiber over $x \in \mathbb{P}^1$. Then, we can write (thinking of $e_1(x)$ and $e_2(x)$ as elements of $\mathcal{L}_1(x)$ and $\mathcal{L}_2(x)$ respectively)

$$\phi(e_1(x)) = e_2(x)l(x), \text{ for some } l(x) \in L.$$

It is easy to see that $l(x)L' \in L/L'$ does not depend upon the choices of $e_1(x)$ and $e_2(x)$. In particular, we get a function $l : \mathbb{P}^1 \rightarrow L/L'$, which must be constant say $l_o L'$. Moreover, since Z° maps surjectively onto L/L' (see Subsection 7.1), we can take $l_o \in Z^\circ$. This provides the lifting of $m_{l_o} \circ \phi$ to a L' -bundle isomorphism $\phi' : \mathcal{L}'_1 \rightarrow \mathcal{L}'_2$. \square

7.7. Changing the k_L -degree

Let G, P be as in the beginning of this section and let $Q^\vee \subset \mathfrak{h}$ be the coroot lattice of G .

Lemma 7.2. *There exists an element $\mu_P \in Q^\vee$ satisfying the following:*

(a) $0 \leq \alpha(\mu_P) \leq 1$, for all the roots $\alpha \in R_1^+$, where R_1^+ is the set of positive roots of \mathfrak{l} , and

(b) $|\omega_P(\mu_P)| = 1$.

Proof. If the maximal parabolic P is such that α_P is a long root (for simply-laced groups all the roots are considered long), then take $\mu_P = -\alpha_P^\vee$. By [16, Page 278], since $|\alpha(\mu_P)| \leq 1$, for all the roots $\alpha \neq \pm\alpha_P$, we get that (a) is satisfied for $\mu_P = -\alpha_P^\vee$ (since $\langle \alpha_i, \alpha_P^\vee \rangle \leq 0$ for any $\alpha_i \neq \alpha_P$). Of course, (b) is satisfied for $\mu_P = -\alpha_P^\vee$ (with $\omega_P(\mu_P) = -1$).

Following the Bourbaki [16, Planche I–IX] convention, this leaves us with the following cases to consider, where we give an explicit μ_P in each case. (In the following, we denote by P_i the maximal parabolic subgroup with $S_{P_i} = \{\alpha_i\}$ and θ_o denotes the highest root of G .)

- (1) $G = B_\ell, P = P_\ell$: Take $\mu_{P_\ell} = \theta_o^\vee$,
- (2) $G = C_\ell, P = P_i (1 \leq i < \ell)$: Take $\mu_{P_i} = \theta_o^\vee$,
- (3) $G = G_2, P = P_1$: Take $\mu_{P_1} = \theta_o^\vee$,
- (4) $G = F_4, P = P_3$: Take $\mu_{P_3} = \epsilon_1 - \epsilon_4$, and
- (5) $G = F_4, P = P_4$: Take $\mu_{P_4} = \theta_o^\vee$. \square

We refer to [35] for an analogue of the following construction. Choose a point $c \in \mathbb{P}^1$ distinct from b_1, \dots, b_n . For the convenience of the notation take $c = 0$. Let $\text{Parbun}_L^{[1]}(d)$ be the moduli stack of parabolic L -bundles of degree d with parabolic structures at b_1, \dots, b_n, c . There is the forgetful morphism $\text{Parbun}_L^{[1]}(d) \rightarrow \text{Parbun}_L(d)$ (forgetting the parabolic structure at c). It is clear that pulling back $\mathcal{N}(\vec{\lambda}, m)$ results in a similar line bundle (with $\lambda_{n+1} = 0$ and no changes in $\lambda_1, \dots, \lambda_n$) without any change in the space of global sections.

We describe an operation which results in a morphism $\text{Parbun}_L^{[1]}(d) \rightarrow \text{Parbun}_L^{[1]}(d \pm 1)$. Fix any $\mu \in Q^\vee$ and let $\ell_\mu : \mathbb{C}^* \rightarrow H \subset L$ be the one parameter subgroup $\ell_\mu(z) = z^\mu$. The operation is as follows: Fix a trivialization s of $\tilde{\mathcal{M}} = (\mathcal{M}; \bar{l}_1, \dots, \bar{l}_{n+1}) \in \text{Parbun}_L^{[1]}(d)$ in a formal neighborhood D of $c = 0$ so that $s(0)_{B_L} = \bar{l}_{n+1}$. The new principal L -bundle \mathcal{M}_μ , which coincides with \mathcal{M} outside of 0 , is defined as follows: Sections of \mathcal{M}_μ over D are meromorphic sections $sa(z)$ of \mathcal{M} so that $\ell_\mu(z)a(z)$ is regular at 0 . We have a section

$$s' = s\ell_\mu(z)^{-1} \text{ of } \mathcal{M}_\mu \text{ over } D. \tag{47}$$

The construction of the new bundle \mathcal{M}_μ depends on the choice of the trivialization s . In particular, it may not give a well defined bundle \mathcal{M}_μ for an arbitrary choice of μ . However, we have the following result:

Lemma 7.3. *Assume that $\mu \in Q^\vee$ satisfies the condition (a) of Lemma 7.2. Then, for any $\tilde{\mathcal{M}} = (\mathcal{M}; \bar{l}_1, \dots, \bar{l}_{n+1}) \in \text{Parbun}_L^{[1]}(d)$, the above operation gives a well-defined bun-*

dle \mathcal{M}_μ , i.e., the construction of \mathcal{M}_μ does not depend on the choice of the trivialization s satisfying $s(0)B_L = \bar{l}_{n+1}$.

Further, the degree d' of \mathcal{M}_μ is $d + \omega_P(\mu)$.

Proof. We first show that changing $s(z)$ by $s(z)c(z)$ with $c(z) \in L[[z]] := L(\mathbb{C}[[z]])$ and $c(0) \in B_L$ does not change \mathcal{M}_μ (in its meromorphic identification with \mathcal{M}). To prove this, it suffices to show that $\ell_\mu(z)c(z)\ell_\mu(z)^{-1}$ is regular at $z = 0$:

Considering the embedding $L[[z]] \subset G[[Z]]$, and the affine Kac–Moody Lie algebra $\hat{\mathcal{L}}(\mathfrak{g}) := \mathbb{C}[z, z^{-1}] \otimes_{\mathbb{C}} \mathfrak{g} \oplus \mathbb{C}c \oplus \mathbb{C}d$ (cf., [27, Section 13.1]), it suffices to show that

$$\text{Ad } \ell_\mu(z) \cdot (z\mathbb{C}[z] \otimes \mathfrak{g}_\beta) \subset \mathbb{C}[z] \otimes \mathfrak{l}, \text{ for any } \beta \in R_{\mathfrak{l}}, \tag{48}$$

and

$$\text{Ad } \ell_\mu(z) \cdot \mathfrak{g}_\beta \subset \mathbb{C}[z] \otimes \mathfrak{l}, \text{ for any } \beta \in R_{\mathfrak{l}}^+, \tag{49}$$

where \mathfrak{g}_β is the root space of \mathfrak{g} corresponding to the root β . But, as it is easy to see, for any $\mu \in Q^\vee$ and any $\beta \in R$,

$$\text{Ad}(z^\mu) \cdot (z^n \otimes \mathfrak{g}_\beta) \subset z^{n+\beta(\mu)} \otimes \mathfrak{g}_\beta. \tag{50}$$

From this equation, the equations (48) and (49) follow for any μ satisfying the condition (a) of Lemma 7.2.

By an easy calculation, the degree d' of \mathcal{M}_μ is $d + \omega_P(\mu)$. \square

We now describe the parabolic structure on \mathcal{M}_μ for any μ as in the above lemma. For any $k = 1, \dots, n$, the parabolic structure on \mathcal{M}_μ is the same as that of $\tilde{\mathcal{M}}$ at b_k . To describe the parabolic structure at c , following [6], consider the subgroup E_μ of L as the set of all limits

$$\lim_{z \rightarrow 0} \ell_\mu(z)c(z)\ell_\mu(z)^{-1}, \quad c(z) \in L[[z]] \text{ with } c(0) \in B_L.$$

Then, by the equation (50), it is easy to see that the Lie algebra

$$\text{Lie}(E_\mu) = \mathfrak{h} \oplus \left(\bigoplus_{\beta \in R_{\mathfrak{l}}^+ : \beta(\mu)=0} \mathfrak{g}_\beta \right) \oplus \left(\bigoplus_{\beta \in R_{\mathfrak{l}}^+ : \beta(\mu)=1} \mathfrak{g}_{-\beta} \right). \tag{51}$$

In particular, E_μ is a Borel subgroup of L containing H . Hence, there exists a unique Weyl group element w_μ of L such that $E_\mu = w_\mu B_L w_\mu^{-1}$. Thus, we get a well defined point in $(\mathcal{M}_\mu)_0/w_\mu B_L w_\mu^{-1}$ by taking $s'(0) \pmod{w_\mu B_L w_\mu^{-1}}$, for any section s' of \mathcal{M}_μ over D defined by the equation (47). By the definition of E_μ , it is easy to see that the element $s'(0) \pmod{w_\mu B_L w_\mu^{-1}}$ does not depend upon the choice of the section s of \mathcal{M} over D satisfying $s(0)B_L = \bar{l}_{n+1}$.

For any principal L -bundle \mathcal{E} over a scheme Y , any $y \in Y$, and w in the Weyl group of L , we can identify

$$\theta_w : \mathcal{E}_y / (wB_L w^{-1}) \rightarrow \mathcal{E}_y / B_L$$

induced by $e \mapsto ew$. This allows us to define an element \bar{l}_{n+1}^μ in the fiber $(\mathcal{M}_\mu)_c / B_L$ as the image of $s'(0) \pmod{w_\mu B_L w_\mu^{-1}}$ under θ_{w_μ} . So, we get the parabolic bundle $\tilde{\mathcal{M}}_\mu := (\mathcal{M}_\mu; \bar{l}_1, \dots, \bar{l}_n, \bar{l}_{n+1}^\mu) \in \text{Parbun}_L^{[1]}(d + \omega_P(\mu))$.

All in all, we therefore have completed our description of $\tau_\mu : \text{Parbun}_L^{[1]}(d) \rightarrow \text{Parbun}_L^{[1]}(d + \omega_P(\mu))$. Taking $\mu = \mu_P$ as in [Lemma 7.2](#), we get a map $\tau_{\mu_P} : \text{Parbun}_L^{[1]}(d) \rightarrow \text{Parbun}_L^{[1]}(d \pm 1)$.

Lemma 7.4. *The map $\tau_\mu : \text{Parbun}_L^{[1]}(d) \rightarrow \text{Parbun}_L^{[1]}(d + \omega_P(\mu))$, for any μ satisfying the condition (a) of [Lemma 7.2](#), is an isomorphism.*

Proof. Let $w_\mu \in W_L$ be the element such that $E_\mu = w_\mu B_L w_\mu^{-1}$, where $\text{Lie}(E_\mu)$ is defined by the equation (51). Then, it is easy to see that $\tau_{w_\mu^{-1}\mu} : \text{Parbun}_L^{[1]}(d + \omega_P(\mu)) \rightarrow \text{Parbun}_L^{[1]}(d)$ is a well defined morphism of stacks. Moreover, it is the inverse of τ_μ , proving that τ_μ is an isomorphism. (Observe that $w_{w_\mu^{-1}\mu} = w_\mu^{-1}$.) \square

Finally, we need to consider the pull-back of line bundles under the above map τ_μ .

Lemma 7.5. *For any μ satisfying the condition (a) of [Lemma 7.2](#), any characters $\vec{\lambda} = (\lambda_1, \dots, \lambda_n)$ of H and any integer m ,*

$$\tau_\mu^*(\mathcal{N}((\vec{\lambda}, 2g^* m \mu^*), m)) = \mathcal{N}((\vec{\lambda}, 0), m),$$

where $\mu^* = \kappa^{-1}(\mu)$ (κ being defined in [Section 3](#)).

Proof. By [Proposition 6.3](#), for any $\tilde{\mathcal{M}} \in \text{Parbun}_L^{[1]}(d)$, $D(\mathcal{M}' \times^L \mathfrak{g}) \otimes D(\mathcal{M} \times^L \mathfrak{g})^{-1}$ is the line $\mathcal{M}_0 \times^L \mathbb{C}_\delta$, where \mathcal{M}' is the underlying bundle of $\tau_\mu(\tilde{\mathcal{M}})$ and $\delta \in \mathfrak{h}^*$ is given by (for any $x \in \mathfrak{h}$)

$$\begin{aligned} \delta(x) &= -2 \sum_{\gamma \in R^+} \gamma(\mu) \gamma(x), \\ &= -2g^* \langle \mu, x \rangle, \text{ by [Lemma 3.4](#),} \end{aligned}$$

where R^+ is the set of positive roots of \mathfrak{g} . Thus, $\delta = -2g^* \mu^*$. Thus, the pull-back of $\mathcal{N}((\vec{\lambda}, 2g^* m \mu^*), m)$ under τ_μ is given by $\mathcal{N}((\vec{\lambda}, 0), m)$. This proves the lemma. \square

Let $\mu_P \in Q^\vee$ be any element satisfying [Lemma 7.2](#) and let d_o be the smallest positive integer such that $d + d_o \omega_P(\mu_P) \equiv 0 \pmod{k_L}$, where k_L is defined in [Subsection 7.1](#).

Choose points $c_1, \dots, c_{d_o} \in \mathbb{P}^1$ distinct from the points b_1, \dots, b_n . For any $0 \leq r \leq d_o$, let $\text{Parbun}_L^{[r]}(d)$ be the moduli stack of quasi-parabolic principal L -bundles on \mathbb{P}^1 of degree d with parabolic structures at $b_1, \dots, b_n, c_1, \dots, c_r$. Thus, $\text{Parbun}_L^{[0]}(d) = \text{Parbun}_L(d)$. There is a similar definition of $\text{Parbun}_L^{[r]}$, where we take $n + r$ parabolic points.

Combining [Lemmas 7.4 and 7.5](#), we get the following result, which is used in the proof of the irredundancy result [Theorem 8.1](#) (cf. Subsection [8.3](#)).

Corollary 7.6. *For any μ_P satisfying [Lemma 7.2](#), any characters $\vec{\lambda} = (\lambda_1, \dots, \lambda_n)$ of H and any integer $m \geq 0$ such that the equation [\(44\)](#) is satisfied, we have*

$$H^0(\text{Parbun}_L(d), \mathcal{N}(\vec{\lambda}, m)) \simeq H^0(\text{Parbun}_L^{[d_o]}(\vec{\lambda}', [2g^*m(\mu_P^*)_{|\mathfrak{h}'}]^{d_o}), m), \tag{52}$$

where $\vec{\lambda}' := (\lambda'_1, \dots, \lambda'_n)$, $\lambda'_k := \lambda_k|_{\mathfrak{h}'}$ and $[\lambda]^{d_o}$ denotes d_o copies of λ .

Proof. For any $1 \leq r \leq d_o$, the forgetful morphism $\text{Parbun}_L^{[r]}(d) \rightarrow \text{Parbun}_L^{[r-1]}(d)$ clearly induces an isomorphism

$$\begin{aligned} &H^0(\text{Parbun}_L^{[r-1]}(d + (r - 1)\omega_P(\mu_P)), \mathcal{N}(\vec{\lambda}, [2g^*m\mu_P^*]^{r-1}, m)) \simeq \\ &H^0(\text{Parbun}_L^{[r]}(d + (r - 1)\omega_P(\mu_P)), \mathcal{N}(\vec{\lambda}, [2g^*m\mu_P^*]^{r-1}, 0), m). \end{aligned}$$

Further, by [Lemma 7.5](#), the isomorphism $\tau_{\mu_P} : \text{Parbun}_L^{[r]}(d + (r - 1)\omega_P(\mu_P)) \rightarrow \text{Parbun}_L^{[r]}(d + r\omega_P(\mu_P))$ induces an isomorphism in cohomology:

$$\begin{aligned} &H^0(\text{Parbun}_L^{[r]}(d + r\omega_P(\mu_P)), \mathcal{N}(\vec{\lambda}, [2g^*m\mu_P^*]^r, m)) \simeq \\ &H^0(\text{Parbun}_L^{[r]}(d + (r - 1)\omega_P(\mu_P)), \mathcal{N}(\vec{\lambda}, [2g^*m\mu_P^*]^{r-1}, 0), m). \end{aligned}$$

Combining the above two isomorphisms over all $1 \leq r \leq d_o$, we get the isomorphism:

$$H^0(\text{Parbun}_L(d), \mathcal{N}(\vec{\lambda}, m)) \simeq H^0(\text{Parbun}_L^{[d_o]}(d + d_o\omega_P(\mu_P)), \mathcal{N}(\vec{\lambda}, [2g^*m\mu_P^*]^{d_o}), m).$$

Now, the corollary follows from [Lemma 7.1](#). (Observe that, by the identity [\(11\)](#), the equation [\(44\)](#) is satisfied for $(\vec{\lambda}, [2g^*m\mu_P^*]^{d_o})$ and degree $d + d_o\omega_P(\mu_P)$.) \square

Consider the decomposition $\mathfrak{h} = \mathfrak{h}' \oplus \mathfrak{z}^o$, where \mathfrak{z}^o (Lie algebra of Z^o) is the center of \mathfrak{l} and \mathfrak{h}' is the Cartan subalgebra of \mathfrak{l}' .

Lemma 7.7. *For any $\mu \in Q^\vee$ (where Q^\vee is the coroot lattice of \mathfrak{g}), write*

$$\mu = \mu' + \mu^o, \quad \mu' \in \mathfrak{h}', \mu^o \in \mathfrak{z}^o.$$

Then, $\text{Exp}(2\pi i\mu')$ lies in the center of L' .

Proof. Since $\mu \in Q^\vee$, $\text{Exp}(2\pi i\mu) = 1$ in G (and hence in L). Moreover,

$$\text{Exp}(2\pi i\mu) = \text{Exp}(2\pi i\mu') \cdot \text{Exp}(2\pi i\mu^\circ).$$

But, $\text{Exp}(2\pi i\mu^\circ) \in Z^\circ$ and hence $\text{Exp}(2\pi i\mu') \in Z^\circ \cap L'$. In particular, $\text{Exp}(2\pi i\mu')$ lies in the center of L' . \square

Remark 7.8. Let $f : \hat{L} \rightarrow L$ be the simply-connected cover. Then, $K := \text{Ker } f$ is canonically isomorphic with \mathbb{Z} by the identification (43). Moreover, there is a canonical decomposition of Lie groups $\hat{L} \simeq L' \times \mathfrak{z}^\circ$. Take the generator $1 \in K$ and consider its image x_o in L' under the above decomposition. Then, clearly, x_o is a central element of L' . Further, it can be seen, by using Lemma 7.2(b), that $\text{Exp}(\pm 2\pi i\mu'_P)$ coincides with x_o .

8. Irredundancy of the inequalities in Theorem 4.1

The aim of this section is to prove the following theorem, which is the multiplicative eigen polytope analogue of Ressayre’s result [37]. The following result for $G = \text{SL}_m$ was proved by Belkale combining the works [8,9] (see Remark 8.6).

Theorem 8.1. *Let $n \geq 3$. The inequalities*

$$\mathcal{I}_{(u_1, \dots, u_n; d)}^P : \sum_{k=1}^n \omega_P(u_k^{-1} \mu_k) \leq d,$$

given by part (b) of Theorem 4.1 (as we run through the standard maximal parabolic subgroups P , n -tuples $(u_1, \dots, u_n) \in (W^P)^n$ and non-negative integers d such that $\langle \sigma_{u_1}^P, \dots, \sigma_{u_n}^P \rangle_d^{\otimes 0} = 1$) are pairwise distinct (even up to scalar multiples) and form an irredundant system of inequalities defining the eigen polytope \mathcal{C}_n inside \mathcal{A}^n , i.e., the hyperplanes given by the equality in $\mathcal{I}_{(u_1, \dots, u_n; d)}^P$ are precisely the (codimension one) facets of the polytope \mathcal{C}_n which intersect the interior of \mathcal{A}^n .

We divide the proof into several parts.

8.1. First of all, the inequalities $\mathcal{I}_{(u_1, \dots, u_n; d)}^P$ are pairwise distinct, even up to scalar multiples:

The stabilizer of ω_P under the action of W is precisely equal to the subgroup W_P . Let $u_k \omega_P = z v_k \omega_P$, for all $1 \leq k \leq n$ and some real number z (independent of k) and elements $u_k, v_k \in W^P$. By considering the norms under the W -invariant form on \mathfrak{h}^* , we see that $z = \pm 1$. Further, $z \neq -1$, for otherwise \mathcal{C}_n would satisfy two inequalities with opposite signs contradicting the fact that \mathcal{C}_n has non-empty interior in \mathfrak{h}^n . Thus, $z = 1$ and each $u_k = v_k$.

Now, take two standard maximal parabolic subgroups $P \neq Q$ and assume that $u_k \omega_P = z v_k \omega_Q$, for all $1 \leq k \leq n$ and $d = z d'$, for some real number z (independent of k) and elements $u_k \in W^P, v_k \in W^Q$ and non-negative integers d, d' . For $z < 0$, again \mathcal{C}_n would satisfy two inequalities with opposite signs (which is not possible). So $z > 0$. Now, the only dominant element in the W -orbit of a dominant weight λ is λ itself. Hence, we get $\omega_P = z \omega_Q$, which is not possible since $P \neq Q$.

Finally, since none of $u_k \omega_P = 0$ and $n \geq 3$ (in particular $n \geq 2$), we get that the facet determined by any $\mathcal{J}_{(u_1, \dots, u_n; d)}^P$ cannot be a facet of the alcove \mathcal{A}^n (since the facets of \mathcal{A}^n are given by the vanishing of $(0, \dots, 0, \lambda_i, 0, \dots, 0)$ where λ_i placed in the i -th place, $1 \leq i \leq n$, is either a simple root or the highest root).

8.2. Now, we show that none of the inequalities $\mathcal{J}_{(u_1, \dots, u_n; d)}^P$ can be dropped. In the rest of the proof, we fix a standard maximal parabolic subgroup P of $G, u_1, \dots, u_n \in W^P$, and $d \geq 0$ such that

$$\langle \sigma_{u_1}^P, \dots, \sigma_{u_n}^P \rangle_d^{\otimes 0} = 1. \tag{53}$$

We wish to show that the inequality $\mathcal{J}_{(u_1, \dots, u_n; d)}^P$ can not be dropped. We will produce a collection of points of \mathcal{C}_n for which the above inequality is an equality, and such that their convex span has the dimension of a facet (i.e., $-1 + n \dim \mathfrak{h}$). Before we come to its proof, we need some preparatory material.

As in Subsection 3.7, for any quasi-parabolic principal G -bundle $\tilde{\mathcal{E}} = (\mathcal{E}; \bar{g}_1, \dots, \bar{g}_n)$ over \mathbb{P}^1 , define the subscheme

$$Z'_d(\tilde{\mathcal{E}}) = \{f \in Z_d(\mathcal{E}) : f(b_k) \text{ and } \bar{g}_k \text{ are in relative position } u_k \forall 1 \leq k \leq n\},$$

and its open subscheme:

$$Z_d^\circ(\tilde{\mathcal{E}}) = \{f \in Z'_d(\tilde{\mathcal{E}}) : \theta(\tilde{\mathcal{P}}(f)) \neq 0\},$$

where $Z_d(\mathcal{E})$ is the space of sections of \mathcal{E}/P of degree $d, \tilde{\mathcal{P}}(f)$ is the P -subbundle of \mathcal{E} associated to f with its parabolic structures $(\bar{p}_1, \dots, \bar{p}_n)$ as described in Subsection 3.7 and $\theta \in H^0(\text{Parbun}_P(d), \mathcal{R})$ is the section defined in Subsection 3.6. This gives rise to a stack \mathcal{Z}'_d and morphism

$$\pi' : \mathcal{Z}'_d \rightarrow \text{Parbun}_G \text{ with fiber } Z'_d(\tilde{\mathcal{E}}) \text{ over } \tilde{\mathcal{E}},$$

and an (non-empty) open substack \mathcal{Z}_d° of \mathcal{Z}'_d and morphism

$$\pi^\circ : \mathcal{Z}_d^\circ \rightarrow \text{Parbun}_G \text{ with fiber } Z_d^\circ(\tilde{\mathcal{E}}) \text{ over } \tilde{\mathcal{E}}.$$

Note that, by equation (53), π° is a smooth representable morphism with finite fibers with general fiber being a single reduced point (cf. the proof of Lemma 3.11). Now,

since Parbun_G is smooth, applying Zariski’s main theorem in its ‘original form’ (cf. [33, Chap. III, §9]) to the base change to a smooth atlas of Parbun_G , we obtain that the image of π^o is an open substack \mathcal{V} of Parbun_G , and that $\pi^o : \mathcal{Z}_d^o \rightarrow \mathcal{V}$ is an isomorphism of stacks.

Remark 8.2. Note that \mathcal{Z}_d^o is irreducible, otherwise the map π^o being smooth will have at least two points in a general fiber, which contradicts (53).

Define a morphism of stacks $i : \text{Parbun}_L(d) \rightarrow \text{Parbun}_G$, taking $\tilde{\mathcal{L}} = (\mathcal{L}; l_1 B_L, \dots, l_n B_L) \mapsto (\mathcal{L} \times^L G; l_1 u_1^{-1} B, \dots, l_n u_n^{-1} B)$. Also, the Levification process $\text{Gr } \mathcal{P}$ via the one parameter subgroup $t^{\tilde{x}^P}$, as in Subsection 3.8, gives a morphism of stacks $\xi : \mathcal{Z}'_d \rightarrow \text{Parbun}_L(d)$. (The ambiguity in the choice of \bar{p}_k as in Subsection 3.7 disappears in taking $\text{Gr } \mathcal{P}$.) Similarly, consider the morphism of stacks $j : \text{Parbun}_L(d) \rightarrow \mathcal{Z}'_d$, $\tilde{\mathcal{L}} = (\mathcal{L}; l_1 B_L, \dots, l_n B_L) \mapsto (\mathcal{L} \times^L P; l_1 B_L, \dots, l_n B_L)$. Then, clearly

$$i = \pi' \circ j.$$

These maps are organized in the following diagram:

$$\begin{array}{ccc}
 \mathcal{Z}_d^o \subseteq & & \mathcal{Z}'_d \\
 \downarrow \pi^o & & \downarrow \pi' \\
 \mathcal{V} \subseteq & & \text{Parbun}_G \xleftarrow{i} \text{Parbun}_L(d) \\
 & & \nearrow j \\
 & & \searrow \xi
 \end{array} \tag{54}$$

From the definition, it is easy to see that $i^*(\mathcal{M}(\vec{\lambda}_{\bar{u}}, m)) \simeq \mathcal{N}(\vec{\lambda}, m)$, where $\vec{\lambda}_{\bar{u}} := (u_1 \lambda_1, \dots, u_n \lambda_n)$.

Lemma 8.3. *For any $(\vec{\lambda}, m)$ such that the identity component Z^o of the center of L acts trivially on $\mathcal{N}(\vec{\lambda}, m)$, the line bundles $\xi^*(\mathcal{N}(\vec{\lambda}, m)) = \xi^* i^*(\mathcal{M}(\vec{\lambda}_{\bar{u}}, m))$ and $(\pi')^*(\mathcal{M}(\vec{\lambda}_{\bar{u}}, m))$ are isomorphic over the stack \mathcal{Z}'_d .*

In particular, the lemma applies to any $(\vec{\lambda}, m) \in S$, where S is defined below by the equation (55).

Proof. Since $t^{\tilde{x}^P} \subset Z^o$ acts trivially on $\mathcal{N}(\vec{\lambda}, m)$, the lemma follows from the definition of the Levification map ξ and the following simple lemma. \square

Let \mathcal{R} be a \mathbb{G}_m -equivariant line bundle on \mathbb{A}^1 such that the action of \mathbb{G}_m on \mathcal{R}_0 is trivial. Then, using the \mathbb{G}_m -action and taking limit as $t \rightarrow 0$, we get the following:

Lemma 8.4. *There is a canonical identification $\mathcal{R}_t \xrightarrow{\sim} \mathcal{R}_0$, for any $t \in \mathbb{A}^1$.*

Define the subset

$$S = \{(\vec{\lambda}, m) = (\lambda_1, \dots, \lambda_n; m) \in X(H)^n \times \mathbb{Z}_+ : H^0(\text{Parbun}_L(d), \mathcal{N}(\vec{\lambda}, m)^r) \neq 0, \text{ for some } r > 0\}. \tag{55}$$

In particular, for $(\vec{\lambda}, m) \in S$, Z^o acts trivially on $\mathcal{N}(\vec{\lambda}, m)$. By the equation (44), for any $(\vec{\lambda}, m) \in S$, $\sum_{k=1}^n \lambda_k(x_P) = 4g^*dm / \langle \alpha_P, \alpha_P \rangle$.

Define a map $S \rightarrow \mathfrak{h}^n$ by $(\vec{\lambda}, m) \mapsto \frac{1}{2g^*m} \vec{\lambda}$. We will show (in Subsection 8.3) that the convex span of the image of this map has dimension $-1 + n \dim \mathfrak{h}$.

Proposition 8.5. *For any $(\vec{\lambda}, m) \in S$, there exists a divisor $D_{(\vec{\lambda}, m)} \subset \text{Parbun}_G$ contained in the complement of \mathcal{V} such that $\mathcal{M}(\vec{\lambda}_{\vec{u}}, m)^r(D_{(\vec{\lambda}, m)})$ has a nonzero section over Parbun_G for some $r > 0$.*

Moreover, the line bundle $\mathcal{O}(D_{(\vec{\lambda}, m)})^p$ over Parbun_G (for some $p > 0$) is of the form $\mathcal{M}(\vec{\lambda}_o, m_o)$ for some $m_o \geq 0$ and some characters $\vec{\lambda}_o = (\lambda_1^o, \dots, \lambda_n^o)$ with each λ_k^o being a G -dominant character of level m_o . Further, $i^(\mathcal{O}(D_{(\vec{\lambda}, m)}))$ admits a nonzero section over $\text{Parbun}_L(d)$.*

For a finite subset F of S , we can choose a line bundle $\mathcal{O}(D_F)$ as above, which works for all $\vec{\lambda} \in F$.

Proof. By assumption, $\mathcal{N}(\vec{\lambda}, m)^r$ has a nonzero section over $\text{Parbun}_L(d)$. Since $(u_1, \dots, u_n; d)$ is quantum Levi-movable, $j^{-1}(\mathcal{Z}_d^o)$ is a non-empty open subset of $\text{Parbun}_L(d)$ and hence $\xi^*(\mathcal{N}(\vec{\lambda}, m)^r)$ has a nonzero section over \mathcal{Z}_d^o . Thus, by Lemma 8.3 and the isomorphism $\mathcal{Z}_d^o \simeq \mathcal{V}$, we get that $\mathcal{M}(\vec{\lambda}_{\vec{u}}, m)^r$ has a nonzero section σ over \mathcal{V} . Since Parbun_G is a smooth Artin stack (see [40] and the references therein), there is a divisor $D_{(\vec{\lambda}, m)} \subset \text{Parbun}_G \setminus \mathcal{V}$ such that σ extends as a nonzero section of $\mathcal{M}(\vec{\lambda}_{\vec{u}}, m)^r(D_{(\vec{\lambda}, m)})$ over the whole of Parbun_G (use a smooth atlas of Parbun_G).

To prove the second part, consider the identification of stacks obtained from the uniformization $L_C(G) \backslash \mathcal{F} \simeq \text{Parbun}_G$, where $\mathcal{F} := \mathcal{Q}_G \times (G/B)^{\times n}$, \mathcal{Q}_G is the infinite Grassmannian associated to G and $L_C(G)$ is the group of regular maps from $C \setminus \{*\}$ to G (cf. [30, Theorem 8.5]). This gives rise to the identification:

$$\text{Pic}(\text{Parbun}_G) \simeq \text{Pic}_{L_C(G)}(\mathcal{F}) \hookrightarrow \text{Pic}(\mathcal{F}),$$

where $\text{Pic}_{L_C(G)}(\mathcal{F})$ denotes the group of isomorphism classes of $L_C(G)$ -equivariant line bundles on \mathcal{F} . The injectivity of the second forgetful map follows immediately from the fact that $L_C(G)$ does not admit any nontrivial characters (cf. [30, Corollary 5.2]). From the description of $\text{Pic}(\mathcal{F})$ (cf., e.g., [30, Proposition 8.7]) and the above injective map $\text{Pic}(\text{Parbun}_G) \hookrightarrow \text{Pic}(\mathcal{F})$, $\mathcal{O}(D_{(\vec{\lambda}, m)})^p$ (for some $p > 0$) is of the form $\mathcal{M}(\vec{\lambda}_o, m_o)$ for suitable characters $\vec{\lambda}_o = (\lambda_1^o, \dots, \lambda_n^o)$ and integer m_o (use the definition of the line bundle $\mathcal{M}(\vec{\lambda}, m)$ on Parbun_G as in Definition 5.1). Since $\mathcal{O}(D_{(\vec{\lambda}, m)})$ has a nonzero global section, we get that $m_o \geq 0$ and moreover each λ_k^o is dominant of level $2g^*m_o$ (cf. Lemma 6.5).

Further, since the canonical section of $\mathcal{O}(D_{(\bar{\lambda}, m)})$ does not vanish on $\text{Parbun}_G \setminus D_{(\bar{\lambda}, m)} \supset \mathcal{V}$, and $i^{-1}(\mathcal{V})$ is non-empty, we get that $i^*(\mathcal{O}(D_{(\bar{\lambda}, m)}))$ admits a nonzero section over $\text{Parbun}_L(d)$. This proves the proposition. \square

8.3. We are now ready to complete the proof of [Theorem 8.1](#).

Proof of Theorem 8.1 (continued). Let $\mathcal{C}_n(L')$ denote the eigen polytope of L' (which is simply-connected, semisimple group). For any central element z of L' , consider the twisted eigen polytope

$$\mathcal{C}_n(L')_z := \left\{ \vec{\mu}' = (\mu'_1, \dots, \mu'_n) \in \mathcal{A}_{L'}^n : z \in C(\mu'_1) \dots C(\mu'_n) \right\},$$

where $\mathcal{A}_{L'} \subset \mathfrak{h}'$ is the fundamental alcove of L' (which is the product of the fundamental alcoves of the simple components of L') and $C(\mu'_k)$ denotes the conjugacy class of $\text{Exp}(2\pi i \mu'_k)$ under a maximal compact subgroup of L' .

Since z is central, it is easy to see that (just as $\mathcal{C}_n(L')$) $\mathcal{C}_n(L')_z$ is a rational convex polytope with nonempty interior in \mathfrak{h}^n .

Let μ'_P be the \mathfrak{h}' -component of $\mu_P \in \mathfrak{h} = \mathfrak{h}' \oplus \mathfrak{z}^o$, where $\mu_P \in Q^\vee$ is any element satisfying [Lemma 7.2](#). Then, by [Lemma 7.7](#), $\text{Exp}(2\pi i \mu'_P)$ is central in L' . Thus, taking $z = (\text{Exp}(2\pi i \mu'_P))^{-d_o}$ (where d_o is as in [Corollary 7.6](#)), for any $\vec{\mu}' \in \mathcal{C}_n(L')_z$, we get $(\vec{\mu}', [\mu'_P]^{d_o}) \in \mathcal{C}_{n+d_o}(L')$. (Observe that $\mu'_P \in \mathcal{A}_{L'}$.)

Thus, applying [Theorem 5.2](#) and [Remark 5.3\(b\)](#) for simple G replaced by semisimple L' and n replaced by $n + d_o$, we get that for any rational point $\vec{\mu}' \in \mathcal{C}_n(L')_z$, there exists a large enough positive integer r divisible by $2g^*$ such that

$$H^0(\text{Parbun}_{L'}^{[d_o]}, \mathcal{N}((r(\vec{\mu}')^*, [r(\mu'_P)^*]^{d_o}), r/2g^*)) \neq 0, \tag{56}$$

where $(\vec{\mu}')^* := (\kappa^{-1}(\mu'_1), \dots, \kappa^{-1}(\mu'_n))$. Thus, by [Corollary 7.6](#), we see (possibly by taking a multiple of r) that

$$H^0(\text{Parbun}_L(d), \mathcal{N}(r(\vec{\mu})^*, r/2g^*)) \neq 0,$$

for any rational $\vec{\mu} = (\mu_1, \dots, \mu_n) \in \mathfrak{h}^n$ satisfying the following two conditions:

- (a) The \mathfrak{h}' -component of μ_k coincides with μ'_k , for all $1 \leq k \leq n$, and
- (b) $r(\vec{\mu})^*$ satisfies the condition [\(44\)](#) for $m = r/2g^*$, i.e.,

$$\sum_{k=1}^n \langle \mu_k, x_P \rangle = 2d / \langle \alpha_P, \alpha_P \rangle. \tag{57}$$

Thus, for any rational point $\vec{\mu}' \in \mathcal{C}_n(L')_z$, we get that there exists a large enough positive integer r such that $(r(\vec{\mu})^*, r/2g^*) \in S$, for any rational $\vec{\mu}$ satisfying the above conditions (a) and (b). Take a finite collection \mathcal{F} of such $\vec{\mu}$ such that their convex span

is of dimension $-1 + n \dim \mathfrak{h}$. (This is possible since $\mathcal{C}_n(L')_z$ is of dimension $n \dim \mathfrak{h}'$ and, for any $\vec{\mu}'$, the extension of $\vec{\mu}'$ to $\vec{\mu}$ satisfying the above conditions (a) and (b) is a $(n - 1)$ -dimensional space.) Thus, we can find a uniform positive integer r_o such that $(r_o(\vec{\mu})^*, r_o/2g^*) \in S$ for any $\vec{\mu} \in \mathcal{F}$. Thus, by [Proposition 8.5](#), we get that there exists $(\vec{\lambda}_o, m_o)$ such that (replacing r_o by a positive multiple of r_o)

$$H^0(\text{Parbun}_G, \mathcal{M}(r_o(\vec{\mu})_{\vec{u}}^* + \vec{\lambda}_o, \frac{r_o}{2g^*} + m_o)) \neq 0,$$

and hence so is

$$H^0(\text{Parbun}_G, \mathcal{M}(r_o(\vec{\mu})_{\vec{u}}^* + r_o\vec{\lambda}_o, \frac{r_o}{2g^*} + r_o m_o)) \neq 0.$$

Hence, by [Theorem 5.2](#) and [Lemma 6.5](#),

$$\frac{\vec{\mu}_{\vec{u}} + \vec{\lambda}_o^*}{1 + 2g^*m_o} \in \mathcal{C}_n$$

and so is $\vec{\lambda}_o^*/2g^*m_o \in \mathcal{C}_n$ (by [Proposition 8.5](#)).

Now,

$$\begin{aligned} \sum_{k=1}^n \omega_P(u_k^{-1}(u_k\mu_k)) &= \sum_{k=1}^n \omega_P(\mu_k) = \frac{\langle \alpha_P, \alpha_P \rangle}{2} \sum_{k=1}^n \langle \mu_k, x_P \rangle, \text{ by the identity (11),} \\ &= d, \text{ by the identity (57).} \end{aligned}$$

Thus, for any $\vec{\mu} \in \mathcal{F}$, the element $\vec{\mu}_{\vec{u}} = (u_1\mu_1, \dots, u_n\mu_n)$ lies in the hyperplane H given by the equality of $\mathcal{S}_{(u_1, \dots, u_n; d)}^P$. Also, since $i^*(\mathcal{M}(\vec{\lambda}_o, m_o))$ admits a nonzero section over $\text{Parbun}_L(d)$ (by [Proposition 8.5](#)), we get that (by the equation (44)),

$$\sum_k (u_k^{-1}\lambda_k^o)(x_P) = \frac{4g^*dm_o}{\langle \alpha_P, \alpha_P \rangle},$$

i.e., $\vec{\lambda}_o^*/2g^*m_o$ lies in H . Hence, the convex combination

$$\frac{1}{1 + 2g^*m_o}\vec{\mu}_{\vec{u}} + \frac{2g^*m_o}{1 + 2g^*m_o} \frac{\vec{\lambda}_o^*}{2g^*m_o} = \frac{\vec{\mu}_{\vec{u}} + \vec{\lambda}_o^*}{1 + 2g^*m_o} \in H \cap \mathcal{C}_n.$$

From this we see that $H \cap \mathcal{C}_n$ is of dimension $-1 + n \dim \mathfrak{h}$, since, by assumption, convex span of $\vec{\mu} \in \mathcal{F}$ is of this dimension. This proves [Theorem 8.1](#) completely. \square

Remark 8.6. Let $G = \text{SL}_r$ and $\vec{\lambda} = (\lambda_1, \dots, \lambda_n)$ be a collection of dominant integral weights each of level $\leq k$ (which is arbitrary, and not necessarily a multiple of $g^* = r$) such that their sum lies in the root lattice of \mathfrak{g} , the Lie algebra of SL_r . Let $V_{\mathfrak{g}, \vec{\lambda}, k}$ be

the space of conformal blocks for $\vec{\lambda}$ at level k for the marked curve $(\mathbb{P}^1, b_1, \dots, b_n)$ (see, e.g., [3] for the definition of conformal blocks). The quantum generalization of Fulton’s conjecture (abbreviated as QFC) asserts that for any positive integer N , $V_{\mathfrak{g}, \vec{\lambda}, k}$ has rank 1 if and only if $V_{\mathfrak{g}, N\vec{\lambda}, Nk}$ has rank 1. As observed in [8, Page 50], QFC for SL_r is implied by the irredundance (i.e., Theorem 8.1) for SL_m for each $m > r$. Therefore, one obtains a new proof of QFC.

The article [8] shows also that Theorem 8.1 holds for SL_m assuming the validity of QFC. In [9], a geometric proof of the classical Fulton conjecture (proved earlier by Knutson–Tao–Woodward [24]) was given. It was noted there that the proof carries over to the quantum case as well (but full proofs were not given in the quantum case).

9. Example: determination of deformed product in quantum cohomology for rank-2 groups

We determine the deformed product in the quantum cohomology for G/P , where G is a rank-2 group and P is a maximal parabolic subgroup which is not cominusculc. The maximal parabolic P_i refers the one with $S_{P_i} = \{\alpha_i\}$. In the following examples, we follow the indexing convention as in Bourbaki [16, Planche II, IX], and the classes a_i, b_i, c_i refer to the unique classes $\sigma_{u_i}^P \in H^{2i}(X^P, \mathbb{Z})$, i.e., corresponding to the Schubert varieties $X_{u_i}^P$ of codimension i . As in Definition 3.5, the variable q (resp. τ) refers to the quantum (resp. deformed) variable.

The multiplication table for the small quantum cohomology of G/P , for maximal parabolic subgroups P , for G of type B_2 (resp. of type G_2) is taken from A. Buch’s homepage: www.math.rutgers.edu/~asbuch/qcalc/qcalc-manual.txt (resp. [39, §5.2]). This allows us to determine the product in the deformed quantum cohomology by using the identity (12).

Example 1. $G = B_2, P = P_2$:

$H^*(G/P_2)$	a_0	a_1	a_2	a_3
a_0	a_0	a_1	a_2	a_3
a_1		τa_2	a_3	$\tau q a_0$
a_2			$q a_0$	$q a_1$
a_3				$\tau q a_2$

Example 2. $G = G_2, P = P_1$:

$H^*(G/P_1)$	b_0	b_1	b_2	b_3	b_4	b_5
b_0	b_0	b_1	b_2	b_3	b_4	b_5
b_1		$\tau^2 b_2$	$5\tau b_3$	$\tau^2 b_4$	$b_5 + \tau^2 q b_0$	$\tau^2 q b_1$
b_2			$5\tau b_4$	$b_5 + \tau^2 q b_0$	$2q b_1$	$\tau^2 q b_2$
b_3				$\tau q b_1$	$\tau q b_2$	$\tau^2 q b_3$
b_4					$2q b_3$	$\tau^2 q b_4$
b_5						$\tau^4 q^2 b_0$

Example 3. $G = G_2$, $P = P_2$:

$H^*(G/P_2)$	c_0	c_1	c_2	c_3	c_4	c_5
c_0	c_0					
c_1		c_1				
c_2		$3c_2$	$2\tau c_3 + \tau q c_0$	$3c_4 + q c_1$	$c_5 + q c_2$	$\tau q c_3 + 2\tau q^2 c_0$
c_3			$2\tau c_4 + \tau q c_1$	$c_5 + 2q c_2$	$\tau q c_3 + \tau q^2 c_0$	$\tau q c_4 + \tau q^2 c_1$
c_4				$2q c_3 + 2q^2 c_0$	$q c_4 + q^2 c_1$	$2q^2 c_2$
c_5					$q^2 c_2$	$\tau q^2 c_3$
						$2\tau q^2 c_4$

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