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Hitchin's conjecture for simply-laced Lie algebras implies that for any simple Lie algebra

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ABSTRACT

Let $\mathfrak{g}$ be any simple Lie algebra over $\mathbb{C}$. Recall that there exists an embedding of $\mathfrak{sl}_2$ into $\mathfrak{g}$, called a principal TDS, passing through a principal nilpotent element of $\mathfrak{g}$ and uniquely determined up to conjugation. Moreover, $\wedge(\mathfrak{g}^*)^{\mathfrak{g}}$ is freely generated (in the super-graded sense) by primitive elements $\omega_1, \ldots, \omega_\ell$, where ℓ is the rank of $\mathfrak{g}$. N. Hitchin conjectured that for any primitive element $\omega \in \wedge^d(\mathfrak{g}^*)^{\mathfrak{g}}$, there exists an irreducible $\mathfrak{sl}_2$-submodule $V_\omega \subset \mathfrak{g}$ of dimension d such that ω is non-zero on the line $\wedge^d(V_\omega)$. We prove that the validity of this conjecture for simple simply-laced Lie algebras implies its validity for any simple Lie algebra.
Let G be a connected, simply-connected, simple, simply-laced algebraic group and let σ be a diagram automorphism of G with fixed subgroup K. Then, we show that the restriction map $R(G) \to R(K)$ is surjective, where R denotes the representation ring over $\mathbb{Z}$. As a corollary, we show that the restriction map in the singular cohomology $H^*(G) \to H^*(K)$ is surjective. Our proof of the reduction of Hitchin's conjecture to the simply-laced case relies on this cohomological surjectivity.

1. Introduction

Let $\mathfrak{g}$ be a finite dimensional simple Lie algebra over the complex numbers $\mathbb{C}$ with the associated connected, simply-connected complex algebraic group G. Recall that there is a unique (up to conjugation) embedding of $\mathfrak{sl}_2$ into $\mathfrak{g}$, called a principal TDS, such that the image passes through a principal nilpotent element of $\mathfrak{g}$. Under the adjoint action of a principal TDS, the Lie algebra $\mathfrak{g}$ decomposes as a direct sum of exactly ℓ irreducible $\mathfrak{sl}_2$-submodules $V_1, \ldots, V_\ell$ of dimensions $2m_1 + 1, \ldots, 2m_\ell + 1$ respectively, where ℓ is the rank of $\mathfrak{g}$ and $m_1, \ldots, m_\ell$ are the exponents of $\mathfrak{g}$.

Further, the singular cohomology $H^*(G) = H^*(G, \mathbb{C})$ with complex coefficients is a Hopf algebra. Let $P(\mathfrak{g}) \subset H^*(G)$ be the graded subspace of primitive elements. Then, $P(\mathfrak{g})$ has a basis in degrees $2m_1 + 1$, $\ldots, 2m_\ell + 1$. We identify $H^*(G)$ with $\wedge(\mathfrak{g}^*)^{\mathfrak{g}}$ and consider $P(\mathfrak{g})$ as a subspace of $\wedge(\mathfrak{g}^*)^{\mathfrak{g}}$.

Now, N. Hitchin made the following conjecture [6]:

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Conjecture 1.1. *Let $\mathfrak{g}$ be any simple Lie algebra. For any non-zero primitive element $\omega \in P_d \subset \wedge^d(\mathfrak{g}^*)^{\mathfrak{g}}$, there exists an irreducible subspace $V_\omega \subset \mathfrak{g}$ of dimension d with respect to the principal TDS action such that*

$$\omega|_{\wedge^d(V_\omega)} \neq 0.$$

The main motivation for Hitchin behind the above conjecture lies in its connection with the study of polyvector fields on the moduli space $M_G(\Sigma)$ of semistable principal G-bundles on a smooth projective curve Σ of any genus $g > 2$. Specifically, observe that the cotangent space at a smooth point E of $M_G(\Sigma)$ is isomorphic with $H^0(\Sigma, \mathfrak{g}(E) \otimes \Omega)$, where $\mathfrak{g}(E)$ denotes the associated adjoint bundle and Ω is the canonical bundle of the curve Σ. Given a bi-invariant differential form ω of degree k on G, i.e., $\omega \in \wedge^k(\mathfrak{g}^*)^{\mathfrak{g}}$, and elements $\Phi_j \in H^0(\Sigma, \mathfrak{g}(E) \otimes \Omega)$, $1 \leqslant j \leqslant k$, $\omega(\Phi_1, \ldots, \Phi_k)$ defines a skew form with values in the line bundle Ω^k. Dually, it defines a homomorphism

$$\Theta_\omega : H^1\big(\Sigma, \Omega^{1-k}\big) \to H^0\big(M_G(\Sigma), \wedge^k T\big),$$

where T is the tangent bundle of $M_G(\Sigma)$.

Now, as shown by Hitchin, the validity of the above conjecture would imply that the map Θ_ω is injective for any invariant form $\omega \in \wedge^k(\mathfrak{g}^*)^{\mathfrak{g}}$ (cf. [6]).

Any non-simply-laced simple Lie algebra $\mathfrak{k}$ can be realized as the fixed point subalgebra of a diagram automorphism of an appropriate simple simply-laced Lie algebra $\mathfrak{g}$. We prove that the validity of the conjecture for $\mathfrak{g}$ implies the validity for $\mathfrak{k}$. Thus, one needs to verify the conjecture only for the simple Lie algebras of types A, D and E. Specifically, we have the following result (cf. Theorem 2.5).

Theorem 1.2. *If Hitchin's conjecture is valid for any simply-laced simple Lie algebra $\mathfrak{g}$, then it is valid for any simple Lie algebra.*

More precisely, if Hitchin's conjecture is valid for $\mathfrak{g}$ of type $(A_{2\ell-1}; A_{2\ell}; D_4; E_6)$, then it is valid for $\mathfrak{g}$ of type $(C_\ell; B_\ell; G_2; F_4)$ respectively.

The proof relies on constructing a principal TDS in $\mathfrak{k}$ which remains a principal TDS in $\mathfrak{g}$. Moreover, we need to use the surjectivity of the space of primitive elements $P(\mathfrak{g}) \to P(\mathfrak{k})$, which allows us to lift primitive elements $\omega_d \in \wedge^d(\mathfrak{k}^*)^{\mathfrak{k}}$ to primitive elements $\widetilde{\omega}_d \in \wedge^d(\mathfrak{g}^*)^{\mathfrak{g}}$. This surjectivity is obtained as a consequence of the following theorem.

Let K be the algebraic subgroup of G with Lie algebra $\mathfrak{k}$, where $\mathfrak{k}$ is the fixed subalgebra under a diagram automorphism of a simple simply-laced Lie algebra $\mathfrak{g}$. Our next main result of the paper (cf. Theorem 3.1) asserts the following.

Theorem 1.3. *The canonical map $\phi : R(G) \to R(K)$ is surjective, where $R(G)$ denotes the representation ring of G (over $\mathbb{Z}$).*

In particular, the canonical restriction map $\psi : S^\bullet(\mathfrak{g}^*)^{\mathfrak{g}} \to S^\bullet(\mathfrak{k}^*)^{\mathfrak{k}}$ is surjective.

Finally, we use H. Cartan's transgression map and the surjectivity of ψ to obtain the desired surjectivity of $\gamma_o : P(\mathfrak{g}) \to P(\mathfrak{k})$ and thereby the surjectivity of $\gamma : H^*(G) \to H^*(K)$ (cf. Theorem 3.5). In our view, the surjectivity of ϕ, ψ, γ and γ_o is of independent interest.

2. Reduction of Hitchin's conjecture to simply-laced Lie algebras

Let $\mathfrak{g}$ be a simple Lie algebra over $\mathbb{C}$ with the associated connected, simply-connected complex algebraic group G (with Lie algebra $\mathfrak{g}$).

Definition 2.1. A Lie algebra embedding $\varphi : \mathfrak{sl}_2 \to \mathfrak{g}$ (or its image) is called a principal TDS if $\varphi(X)$ is a principal nilpotent element of $\mathfrak{g}$, i.e., $\operatorname{Ad} G \cdot \varphi(X)$ is the open orbit in the nilpotent cone $\mathcal{N}$ of $\mathfrak{g}$.

Here, $\mathfrak{sl}_2$ is the Lie algebra of traceless 2×2 matrices over $\mathbb{C}$ with the standard basis

$$X = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \qquad Y = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \quad \text{and} \quad H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Let $\varphi' : \mathfrak{sl}_2 \to \mathfrak{g}$ be another principal TDS. Then, by a result of Kostant [7, Corollary 3.7], φ' is conjugate to φ, i.e., there exists a $g \in G$ such that

$$\varphi' = \operatorname{Ad} g \cdot \varphi. \tag{1}$$

Decompose the adjoint representation of $\mathfrak{g}$ with respect to a principal TDS φ into irreducible components:

$$\mathfrak{g} = V_1 \oplus V_2 \oplus \cdots \oplus V_\ell,$$

labeling them so that

$$n_1 \leqslant \cdots \leqslant n_\ell, \quad \text{where } n_i = \dim V_i. \tag{2}$$

Then, it is known (cf. [7, Corollary 8.7]) that

(a) $\ell =$ rank of $\mathfrak{g}$.

(b) Each n_i is an odd integer $2m_i + 1$. Moreover,

$$m_1 \leqslant m_2 \leqslant \cdots \leqslant m_\ell$$

are the exponents of $\mathfrak{g}$. (The list of exponents for any $\mathfrak{g}$ can be found in [1, Planche I–IX].)

(c) Except when $\mathfrak{g}$ is of type D_ℓ (with $\ell \geqslant 4$ even), each V_i is an isotypic component (in particular, uniquely determined) for the principal TDS φ, i.e., $m_1 < m_2 < \cdots < m_\ell$.

When $\mathfrak{g}$ is of type D_ℓ (with $\ell \geqslant 4$ even), the exponents are:

$$1,\ 3,\ 5, \ldots,\ \ell-3,\ \ell-1,\ \ell-1,\ \ell+1, \ldots,\ 2\ell-3.$$

Hence, the isotypic component for the highest weight $2\ell - 2$ is a direct sum of two copies of the irreducible module $V_{\mathfrak{sl}_2}(2\ell - 2)$ with highest weight $2\ell - 2$.

By the identity (1), we see that the decomposition of $\mathfrak{g}$ with respect to another principal TDS φ' looks like

$$\mathfrak{g} = (\operatorname{Ad} g \cdot V_1) \oplus (\operatorname{Ad} g \cdot V_2) \oplus \cdots \oplus (\operatorname{Ad} g \cdot V_\ell). \tag{3}$$

Definition 2.2. Recall that the singular cohomology with complex coefficients $H^*(G) = H^*(G, \mathbb{C})$ is a Hopf algebra, where the product of course comes from the cup product, and the coproduct $\Delta : H^*(G) \to H^*(G) \otimes H^*(G)$ is induced from the multiplication map $\mu : G \times G \to G$.

Let $P = P(\mathfrak{g}) \subset H^*(G)$ be the subspace of primitive elements, i.e.,

$$P = \bigl\{ x \in H^*(G) \mid \Delta(x) = x \otimes 1 + 1 \otimes x \bigr\}.$$

(Observe that $H^*(G)$ does not depend upon the isogeny class of G and hence the notation $P(\mathfrak{g})$ is justified.)

Since Δ is a graded homomorphism, $P \subset H^*(G)$ is a graded linear subspace. It is well-known that, by a result of Hopf–Koszul–Samelson, P is concentrated in odd degrees and, moreover, the canonical map induced from the product

$$\theta : \wedge^{\bullet}(P) \to H^*(G)$$

is a graded algebra isomorphism. In particular, P generates $H^*(G)$ as an algebra over $\mathbb{C}$.

We can think of $\wedge(\mathfrak{g}^*)$ as the algebra of left invariant $\mathbb{C}$-valued forms on a maximal compact subgroup G_o of G. By a result of Chevalley–Eilenberg [3, Theorem 12.1], any $\omega \in \wedge(\mathfrak{g}^*)^{\mathfrak{g}}$ is a closed form and, moreover, the induced map (identifying $H^*(G_o)$ with the de Rham cohomology $H^*_{dR}(G_o, \mathbb{C})$)

$$\eta : \wedge(\mathfrak{g}^*)^{\mathfrak{g}} \xrightarrow{\sim} H^*(G_o) \cong H^*(G)$$

is a graded algebra isomorphism, where the restriction map $H^*(G) \to H^*(G_o)$ is an isomorphism since G_o is a deformation retract of G.

Via the isomorphism η, we identify the graded subspace $P \subset H^*(G)$ of primitive elements with a graded subspace (still denoted by) $P \subset \wedge(\mathfrak{g}^*)^{\mathfrak{g}}$.

For any $d \geqslant 1$, let P_d be the subspace of P of (homogeneous) degree d elements. Then, by [7, Corollary 8.7],

$$\dim P_d = \#\{1 \leqslant i \leqslant \ell \mid n_i = d\}, \tag{4}$$

where n_i's (given by (2)) are the dimensions of irreducible components of $\mathfrak{g}$ under the principal TDS action.

In particular, if $\mathfrak{g}$ is not of type D_ℓ (with $\ell \geqslant 4$ even), then

$$\dim P_d \leqslant 1 \tag{5}$$

and P_d is of dimension 1 if and only if d is equal to one of the n_i's. If $\mathfrak{g}$ is of type D_ℓ (with $\ell \geqslant 4$ even),

$$\dim P_d \leqslant 1, \quad \text{if } d \neq 2\ell - 1, \quad \text{and} \quad \dim P_{2\ell-1} = 2. \tag{6}$$

Fix a principal TDS. Hitchin made the following conjecture (cf. [6]).

Conjecture 2.3. *Let $\mathfrak{g}$ be any simple Lie algebra. For any non-zero primitive element $\omega \in P_d \subset \wedge^d(\mathfrak{g}^*)^{\mathfrak{g}}$, there exists an irreducible subspace $V_\omega \subset \mathfrak{g}$ of dimension d with respect to the principal TDS action such that*

$$\omega|_{\wedge^d(V_\omega)} \neq 0.$$

Remark 2.4. (a) Unless $\mathfrak{g}$ is of type D_ℓ (with $\ell \geqslant 4$ even), for any non-zero P_d, there exists a unique irreducible submodule V of dimension d in $\mathfrak{g}$ with respect to the principal TDS. Thus, V_ω is uniquely determined.

If $\mathfrak{g}$ is of type D_ℓ (with $\ell \geqslant 4$ even), unless $d = 2\ell - 1$, for any non-zero P_d, there is a unique irreducible submodule V of dimension d in $\mathfrak{g}$. Thus, again V_ω is uniquely determined (for $d \neq 2\ell - 1$).

(b) A different choice of principal TDS results in the irreducible submodules being equal to $\operatorname{Ad} g \cdot V$, for some $g \in G$, and some irreducible submodule V for the original principal TDS. But, since we are only considering forms $\omega \in \wedge^d(\mathfrak{g}^*)^{\mathfrak{g}}$ (which are, by definition, $\operatorname{Ad} G$-invariant), $\omega|_{\wedge^d(\operatorname{Ad} g \cdot V)} \neq 0$ if and only if $\omega|_{\wedge^d(V)} \neq 0$.

Now, we come to the main result of this section.

Theorem 2.5. *If Hitchin's conjecture is valid for any simply-laced simple Lie algebra $\mathfrak{g}$, then it is valid for any simple Lie algebra.*

More precisely, if Hitchin's conjecture is valid for $\mathfrak{g}$ of type $(A_{2\ell-1}; A_{2\ell}; D_4; E_6)$, $\ell \geqslant 2$, then it is valid for $\mathfrak{g}$ of type $(C_\ell; B_\ell; G_2; F_4)$ respectively.

Proof. Let $\mathfrak{k}$ be a non-simply-laced simple Lie algebra. Then, there exists a simply-laced simple Lie algebra $\mathfrak{g}$ together with a diagram automorphism σ (i.e., an automorphism σ of $\mathfrak{g}$ induced from a diagram automorphism of its Dynkin diagram) such that $\mathfrak{k}$ is the σ-fixed set $\mathfrak{g}^\sigma$ of $\mathfrak{g}$. Moreover, given $\mathfrak{k}$, we can choose $\mathfrak{g}$ to be of type given in the statement of the theorem. (For more details, see Section 3.1 on diagram folding.) In particular, we never need to take $\mathfrak{g}$ of type D_ℓ except D_4.

Choose a Borel subalgebra $\mathfrak{b}$ of $\mathfrak{g}$ and a Cartan subalgebra $\mathfrak{t} \subset \mathfrak{b}$ such that they both are stable under σ. Let $\Delta = \{\alpha_1, \dots, \alpha_\ell\} \subset \mathfrak{t}^*$ be the set of simple roots of $\mathfrak{g}$, where ℓ is the rank of $\mathfrak{g}$. Since σ keeps $\mathfrak{b}$ and $\mathfrak{t}$ stable, σ permutes the simple roots. Let $\{\widetilde{\beta}_1, \dots, \widetilde{\beta}_{\ell_\mathfrak{k}}\}$ be a set of simple roots taking exactly one simple root from each orbit of σ in Δ. Then, the fixed subalgebra $\mathfrak{b}_\mathfrak{k} := \mathfrak{b}^\sigma$ is a Borel subalgebra of $\mathfrak{k}$, $\mathfrak{t}_\mathfrak{k} := \mathfrak{t}^\sigma$ is a Cartan subalgebra of $\mathfrak{k}$ and $\{\beta_1, \dots, \beta_{\ell_\mathfrak{k}}\}$ is the set of simple roots of $\mathfrak{k}$, where $\beta_i := \widetilde{\beta}_i|_{\mathfrak{t}_\mathfrak{k}}$ (cf. [12]). In particular, $\ell_\mathfrak{k}$ is the rank of $\mathfrak{k}$.

For any $1 \leqslant n \leqslant \ell_\mathfrak{k}$, choose a non-zero element $x_n \in \mathfrak{g}_{\widetilde{\beta}_n}$, where $\mathfrak{g}_{\widetilde{\beta}_n}$ is the root space of $\mathfrak{g}$ corresponding to the root $\widetilde{\beta}_n$. Define

$$y_n = \sum_{k=1}^{\operatorname{ord}(\sigma)} \sigma^k(x_n),$$

where $\operatorname{ord}(\sigma)$ is the order of σ (which is 2 except when $\mathfrak{g}$ is of type D_4 and $\mathfrak{k}$ is of type G_2, in which case it is 3). If $\widetilde{\beta}_n$ is fixed by σ, then σ acts trivially on $\mathfrak{g}_{\widetilde{\beta}_n}$ (cf. [12]), hence y_n is never zero. Of course, $y_n \in \mathfrak{k}$ and, in fact, $y_n \in \mathfrak{k}_{\beta_n}$. Define the element $y \in \mathfrak{k}$ by

$$y = \sum_{n=1}^{\ell_\mathfrak{k}} y_n.$$

By [7, Theorem 5.3], y is a principal nilpotent element of $\mathfrak{k}$ and hence there exists a principal TDS in $\mathfrak{k}$:

$$\varphi : \mathfrak{sl}_2 \to \mathfrak{k} \quad \text{such that } \varphi(X) = y.$$

Moreover, since

$$y = \sum_{n=1}^{\ell_\mathfrak{k}} \sum_{k=1}^{\operatorname{ord}(\sigma)} \sigma^k(x_n),$$

again using [7, Theorem 5.3], we get that y is a principal nilpotent of $\mathfrak{g}$ as well. Hence, φ is a principal TDS of $\mathfrak{g}$ also. Decompose $\mathfrak{g}$ under the adjoint action of $\mathfrak{sl}_2$ via φ:

$$\mathfrak{g} = V_1 \oplus \dots \oplus V_{\ell_\mathfrak{k}} \oplus V_{\ell_\mathfrak{k}+1} \oplus \dots \oplus V_\ell,$$

where $V_1 \oplus \dots \oplus V_{\ell_\mathfrak{k}}$ is a decomposition of $\mathfrak{k}$.

Take a non-zero primitive element $\omega_d \in P_d(\mathfrak{k}) \subset \wedge^d(\mathfrak{k}^*)^\mathfrak{k}$, where $P_d(\mathfrak{k})$ is the space of primitive elements for $\mathfrak{k}$. By (subsequent) Theorem 3.5, the canonical restriction map $\wedge^d(\mathfrak{g}^*) \to \wedge^d(\mathfrak{k}^*)$ induces a surjection

$$P_d(\mathfrak{g}) \to P_d(\mathfrak{k}), \quad \text{for any } d > 0.$$

Take a preimage $\widetilde{\omega}_d \in P_d(\mathfrak{g})$ of ω_d. By (4), (5), there exists a unique irreducible $\mathfrak{sl}_2$-submodule V_{ω_d} of $\mathfrak{k}$ of dimension d. Further, by (4)–(6), there exists a unique irreducible $\mathfrak{sl}_2$-submodule $V_{\widetilde{\omega}_d} \subset \mathfrak{g}$ of dimension d. (For any $\mathfrak{k}$ not of type G_2, the uniqueness of $V_{\widetilde{\omega}_d}$ follows since we have chosen $\mathfrak{g}$ not of type D_ℓ; for $\mathfrak{k}$ of type G_2, $P_d(\mathfrak{k})$ is non-zero if and only if $d = 3, 11$ (cf. Definitions 2.1 and 2.2). Again, for these values of d, $\dim P_d(D_4) = 1$.) Hence, $V_{\omega_d} = V_{\widetilde{\omega}_d}$. Assuming the validity of Hitchin's conjecture for $\mathfrak{g}$, we get that $\widetilde{\omega}_d|_{\wedge^d(V_{\widetilde{\omega}_d})} \neq 0$. Hence,

$$\omega_d|_{\wedge^d(V_{\omega_d})} = \widetilde{\omega}_d|_{\wedge^d(V_{\widetilde{\omega}_d})} \neq 0.$$

This proves the theorem. □

3. GIT quotient $G//\operatorname{Ad} G$ and diagram automorphisms

Let $\mathfrak{g}$ be a simple, simply-laced Lie algebra over $\mathbb{C}$ and let G be the connected, simply-connected complex algebraic group with Lie algebra $\mathfrak{g}$. Let σ be a diagram automorphism of $\mathfrak{g}$ and let $\mathfrak{k} = \mathfrak{g}^\sigma$ be the fixed subalgebra. Then, $\mathfrak{k}$ is a simple Lie algebra again. Let K be the connected subgroup of G with Lie algebra $\mathfrak{k}$. In fact, $K = G^\sigma$ (cf. [12]). For the connection of the root datum of K with that of G, we refer, e.g., to [12].

With this notation, we have the following main result of this section.

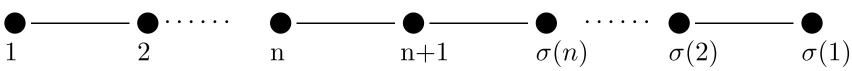
Theorem 3.1. *The canonical map $\phi : R(G) \to R(K)$ is surjective, where $R(G)$ denotes the representation ring of G (over $\mathbb{Z}$).*

In particular, the canonical map $K//\operatorname{Ad} K \to G//\operatorname{Ad} G$, between the GIT quotients, is a closed embedding.

Before we come to the proof of the theorem, we need some notational preliminaries on diagram automorphisms and 'diagram folding' (i.e., the process of getting $\mathfrak{k}$ from $\mathfrak{g}$). As in Section 2, fix a Borel subalgebra $\mathfrak{b}$ and a Cartan subalgebra $\mathfrak{t} \subset \mathfrak{b}$ of $\mathfrak{g}$ stable under σ. Then, $\mathfrak{b}_\mathfrak{k} := \mathfrak{b}^\sigma$ (resp. $\mathfrak{t}_\mathfrak{k} := \mathfrak{t}^\sigma$) is a Borel (resp. Cartan) subalgebra of $\mathfrak{k}$. Let $\Delta = \{\alpha_1, \ldots, \alpha_\ell\} \subset \mathfrak{t}^*$ be the simple roots of $\mathfrak{g}$ and let $\{\widetilde{\beta}_1, \ldots, \widetilde{\beta}_{\ell_\mathfrak{k}}\}$ be a set of simple roots taking exactly one simple root from each orbit of σ in Δ. Then, $\Delta_\mathfrak{k} := \{\beta_1, \ldots, \beta_{\ell_\mathfrak{k}}\} \subset \mathfrak{t}_\mathfrak{k}^*$ is the set of simple roots of $\mathfrak{k}$, where $\beta_i := \widetilde{\beta}_i|_{\mathfrak{t}_\mathfrak{k}}$. In the following diagrams, we will make a specific choice of indexing convention in each case of diagram folding.

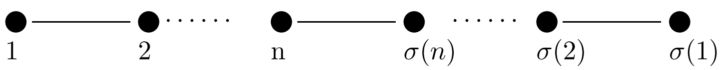
3.1. Diagram folding: Dynkin diagrams of $(\mathfrak{g}, \mathfrak{k})$

(A_{2n+1}, C_{n+1}), $n \geqslant 1$:



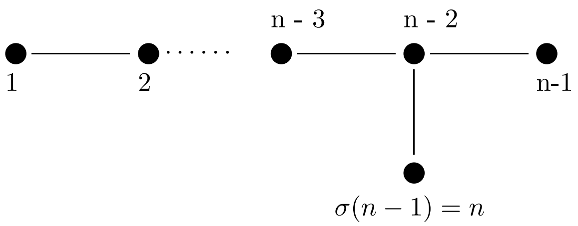
$\beta_i := \alpha_{i|\mathfrak{t}_\mathfrak{k}}$ for $i \leqslant n+1$ and β_{n+1} is a long root.

(A_{2n}, B_n), $n \geqslant 1$:



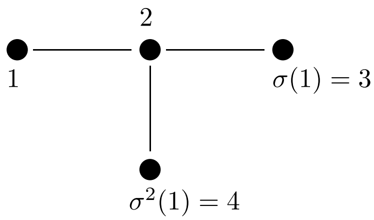
$\beta_i = \alpha_{i|\mathfrak{t}_\mathfrak{k}}$ for $1 \leqslant i \leqslant n$ and β_n is a short root.

$\underline{(D_n, B_{n-1}),\ n \geqslant 4:}$

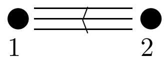


$\beta_i := \alpha_{i|\mathfrak{t}_\mathfrak{k}}$ for $1 \leqslant i \leqslant n-1$ and β_{n-1} is a short root.

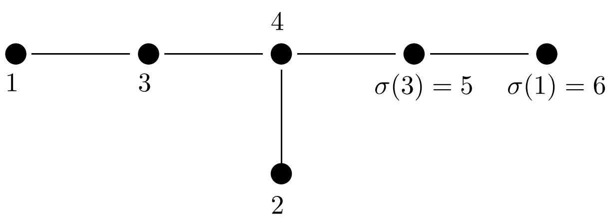
$\underline{(D_4, G_2):}$



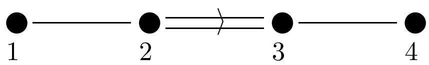
$\beta_1 := \alpha_{1|\mathfrak{t}_\mathfrak{k}}$, $\beta_2 := \alpha_{2|\mathfrak{t}_\mathfrak{k}}$, and β_2 is a long root.



$\underline{(E_6, F_4):}$



$\beta_1 = \alpha_{2|\mathfrak{t}_\mathfrak{k}}$, $\beta_2 = \alpha_{4|\mathfrak{t}_\mathfrak{k}}$, $\beta_3 = \alpha_{3|\mathfrak{t}_\mathfrak{k}}$ and $\beta_4 = \alpha_{1|\mathfrak{t}_\mathfrak{k}}$, with β_2 a long root.



Let $\{\varpi_1, \dots, \varpi_\ell\}$ (resp. $\{\nu_1, \dots, \nu_{\ell_\mathfrak{k}}\}$) be the fundamental weights for the root system of $\mathfrak{g}$ (resp. $\mathfrak{k}$). We next prove two facts special to our context. For any simple root α, we denote the corresponding coroot by $\alpha^\vee$. Let $\rho : \mathfrak{t}^* \to \mathfrak{t}_\mathfrak{k}^*$ be the canonical restriction. We follow the indexing convention as in Section 3.1.

Lemma 3.2. (a) *If G is not of type A_{2n} or E_6, then $\rho(\varpi_i) = \nu_i$ for $1 \leqslant i \leqslant \ell_{\mathfrak{k}} := \operatorname{rank}(\mathfrak{k})$.*

(b) *If G is of type A_{2n}, then $\rho(\varpi_i) = \rho(\varpi_{2n-i+1}) = \nu_i$ for $1 \leqslant i \leqslant n-1$, and $\rho(\varpi_n) = \rho(\varpi_{n+1}) = 2\nu_n$.*

(c) *If G is of type E_6, $\rho(\varpi_1) = \rho(\varpi_6) = \nu_4$; $\rho(\varpi_2) = \nu_1$; $\rho(\varpi_3) = \rho(\varpi_5) = \nu_3$; $\rho(\varpi_4) = \nu_2$.*

Proof. Observe first that, in each case, for

$$1 \leqslant j \neq k \leqslant \ell_{\mathfrak{k}}, \text{ no } \alpha_k \text{ is in the } \sigma\text{-orbit of } \alpha_j. \tag{7}$$

(a) It suffices to show

$$\langle \rho(\varpi_i), \beta_j^\vee \rangle = \delta_{i,j}, \quad \text{for } 1 \leqslant i, j \leqslant \ell_{\mathfrak{k}}. \tag{8}$$

In this case, we have ([12])

$$\beta_j^\vee = \sum \alpha_k^\vee,$$

where the summation runs over the orbit of α_j under σ. Thus, Eq. (8) follows from (7).

(b) When G is of type A_{2n}, by [12],

$$\beta_j^\vee = \begin{cases} \alpha_j^\vee + \alpha_{2n-j+1}^\vee, & \text{for } j \leqslant n-1, \\ 2\alpha_n^\vee + 2\alpha_{n+1}^\vee, & \text{for } j = n. \end{cases}$$

So, for $1 \leqslant i \leqslant 2n$,

$$\begin{aligned} \langle \rho(\varpi_i), \beta_j^\vee \rangle &= \begin{cases} \langle \varpi_i, \alpha_j^\vee \rangle + \langle \varpi_i, \alpha_{2n-j+1}^\vee \rangle, & \text{for } j \leqslant n-1, \\ 2\langle \varpi_i, \alpha_n^\vee \rangle + 2\langle \varpi_i, \alpha_{n+1}^\vee \rangle, & \text{for } j = n \end{cases} \\ &= \begin{cases} \delta_{i,j} + \delta_{i,2n-j+1}, & \text{for } j \leqslant n-1, \\ 2\delta_{i,n} + 2\delta_{i,n+1}, & \text{for } j = n. \end{cases} \end{aligned}$$

From this (b) follows.

(c) By [12], following the indexing convention as in Section 3.1, we get that

$$\beta_1^\vee = \alpha_2^\vee, \qquad \beta_2^\vee = \alpha_4^\vee, \qquad \beta_3^\vee = \alpha_3^\vee + \alpha_5^\vee, \qquad \beta_4^\vee = \alpha_1^\vee + \alpha_6^\vee.$$

Thus,

$$\begin{aligned} &\rho(\varpi_1) = \rho(\varpi_6) = \nu_4, \\ &\rho(\varpi_2) = \nu_1, \\ &\rho(\varpi_3) = \rho(\varpi_5) = \nu_3, \\ &\rho(\varpi_4) = \nu_2. \qquad \square \end{aligned}$$

Let $\Lambda^+(\mathfrak{g}) \subset \mathfrak{t}^*$ (resp. $\Lambda^+(\mathfrak{k}) \subset \mathfrak{t}_{\mathfrak{k}}^*$) be the set of dominant integral weights for the root system of $\mathfrak{g}$ (resp. $\mathfrak{k}$) and let $\Lambda^+(K) \subset \Lambda^+(\mathfrak{k})$ be the submonoid of dominant characters for the group K, i.e., $\Lambda^+(K)$ is the set of characters of the maximal torus T_K (with Lie algebra $\mathfrak{t}_{\mathfrak{k}}$) of K which are dominant with respect to the group K. Observe that since G is simply-connected, $\Lambda^+(G) = \Lambda^+(\mathfrak{g})$. Moreover, under the restriction map $\rho : \mathfrak{t}^* \to \mathfrak{t}_{\mathfrak{k}}^*$,

$$\rho\bigl(\Lambda^+(\mathfrak{g})\bigr) = \Lambda^+(K). \tag{9}$$

To see this, let $\Lambda(K)$ be the character lattice of T_K (similarly for $\Lambda(G) = \Lambda(\mathfrak{g})$). Then, by Springer's original construction of $\Lambda(K)$ [12], the restriction $\rho : \Lambda(\mathfrak{g}) \to \Lambda(K)$ is surjective. Further, from the description of the coroots of $\mathfrak{k}$ as in [12], $\rho(\Lambda^+(\mathfrak{g})) \subset \Lambda^+(\mathfrak{k})$. Thus, we have

$$\rho\bigl(\Lambda^+(\mathfrak{g})\bigr) \subset \Lambda^+(\mathfrak{k}) \cap \Lambda(K) = \Lambda^+(K).$$

Conversely, in all cases except for $\mathfrak{g}$ of type A_{2n}, by Lemma 3.2, $\rho(\Lambda^+(\mathfrak{g})) = \Lambda^+(\mathfrak{k}) \supset \Lambda^+(K)$, so Eq. (9) holds in these cases. When $\mathfrak{g}$ is of type A_{2n}, again by Lemma 3.2,

$$\rho\bigl(\Lambda^+(\mathfrak{g})\bigr) = \left(\bigoplus_{i=1}^{n-1} \mathbb{Z}_+\nu_i\right) \oplus 2\mathbb{Z}_+\nu_n,$$

and

$$\Lambda(K) = \rho\bigl(\Lambda(\mathfrak{g})\bigr) = \left(\bigoplus_{i=1}^{n-1} \mathbb{Z}\nu_i\right) \oplus 2\mathbb{Z}\nu_n.$$

From this again, we see that (9) is satisfied. This proves (9) in all cases.

For any $\lambda \in \Lambda^+(\mathfrak{g})$, let $V(\lambda)$ be the irreducible G-module with highest weight λ. Similarly, for $\mu \in \Lambda^+(K)$, let $W(\mu)$ be the irreducible K-module with highest weight μ. We denote the fundamental representations $V(\varpi_i)$ of $\mathfrak{g}$ by V_i and $W(\nu_j)$ of $\mathfrak{k}$ by W_j.

Lemma 3.3. *For any $\lambda \in \Lambda^+(\mathfrak{g})$, $W(\rho(\lambda))$ has multiplicity one in $V(\lambda)$ as a $\mathfrak{k}$-module.* (*Observe that by* (9), $\rho(\lambda) \in \Lambda^+(K)$.)

Proof. Note that the Borel subalgebra $\mathfrak{b}_{\mathfrak{k}}$ of $\mathfrak{k}$ is contained in the Borel subalgebra $\mathfrak{b}$ of $\mathfrak{g}$. So, if v_λ is the highest weight vector of $V(\lambda)$ (of weight λ), then v_λ remains a highest weight vector of weight $\rho(\lambda)$ in $V(\lambda)$ for the action of $\mathfrak{k}$. Hence, $W(\rho(\lambda)) \subset V(\lambda)$.

Multiplicity one is clear from the weight consideration. □

3.2. Proof of Theorem 3.1

Let $\{\mu_1, \ldots, \mu_N\} \subset \Lambda^+(K)$ be a set of semigroup generators of $\Lambda^+(K)$. Then, the classes $\{[W(\mu_j)]\}_{1\leqslant j\leqslant N}$ generate the $\mathbb{Z}$-algebra $R(K)$, where $[W(\mu_j)] \in R(K)$ denotes the class of the irreducible K-module $W(\mu_j)$ (cf. [10, Theorem 3.12]).

We proceed separately for each of the five cases depending on the type of $(\mathfrak{g}, \mathfrak{k})$.

Case I. (A_{2n+1}, C_{n+1})**:** By Lemmas 3.2 and 3.3, for $1 \leqslant j \leqslant n+1$, $W_j \subset V_j$ (as $\mathfrak{k}$-modules). Recall that $V_1 \simeq W_1 \simeq \mathbb{C}^{2n+2}$ (so $W_1 = V_1$) and $V_j = \wedge^j V_1$ for all $1 \leqslant j \leqslant 2n+1$. Also, for $2 \leqslant j \leqslant n+1$, W_j is given as the kernel of the surjective $\mathfrak{k}$-equivariant contraction map $\wedge^j W_1 \to \wedge^{j-2} W_1$ (cf. [5, Theorem 17.5]). Hence, for $2 \leqslant j \leqslant n+1$, in $R(\mathfrak{k})$ (where $R(\mathfrak{k})$ is the representation ring of $\mathfrak{k}$),

$$[W_j] + \bigl[\wedge^{j-2} W_1\bigr] = \bigl[\wedge^j W_1\bigr].$$

Thus,

$$\phi\bigl([V_1]\bigr) = [W_1], \quad \text{and} \quad \phi\bigl([V_j]\bigr) - \phi\bigl([V_{j-2}]\bigr) = [W_j], \quad \text{for } 2 \leqslant j \leqslant n+1,$$

where V_0 is interpreted as the trivial one dimensional module $\mathbb{C}$. Thus, the class $[W_j]$ of each fundamental representation lies in the image of ϕ, and hence ϕ is surjective.

Case II. (A_{2n}, B_n)**:** By Lemmas 3.2 and 3.3, for $1 \leqslant j \leqslant n-1$, $W_j \subset V_j$ and $W(2\nu_n) \subset V_n$ (as $\mathfrak{k}$-modules). Recall that $V_1 \simeq W_1 \simeq \mathbb{C}^{2n+1}$ (so $W_1 = V_1$), and $V_j = \wedge^j V_1$ for all $1 \leqslant j \leqslant 2n$. Also, $W_j = \wedge^j W_1$ for $1 \leqslant j \leqslant n-1$ and $W(2\nu_n) = \wedge^n W_1$ (see, e.g., [5, Theorem 19.14]). Thus, as $\mathfrak{k}$-modules,

$$W_j = V_j, \quad j \leqslant n-1; \qquad W(2\nu_n) = V_n.$$

Thus,

$$[W_1], \dots, [W_{n-1}], \big[W(2\nu_n)\big] \in \text{Image } \phi.$$

By Lemma 3.2(b) and the identity (9), $\Lambda^+(K)$ is generated (as a semigroup) by $\{\nu_1, \dots, \nu_{n-1}, 2\nu_n\}$. Hence, ϕ is surjective in this case.

Case III. (D_n, B_{n-1})**:** Recall that $V_1 \simeq \mathbb{C}^{2n}$ and $W_1 \simeq \mathbb{C}^{2n-1}$. By Lemmas 3.2 and 3.3, for $1 \leqslant j \leqslant n-1$, $W_j \subset V_j$ (as $\mathfrak{k}$-modules). Since $W_1 \subset V_1$ (as $\mathfrak{k}$-modules), we get (as $\mathfrak{k}$-modules):

$$V_1 = W_1 \oplus \mathbb{C}.$$

Thus, for $1 \leqslant k \leqslant n-2$, as $\mathfrak{k}$-modules,

$$V_k = \wedge^k V_1 = \wedge^k (W_1 \oplus \mathbb{C}) \simeq \big(\wedge^k W_1\big) \oplus \big(\wedge^{k-1} W_1\big) = W_k \oplus W_{k-1},$$

where the first equality is by [5, Theorem 19.2]; W_0 is interpreted as the one dimensional trivial module and the last equality is from the proof of Case II.

Since $W_{n-1} \subset V_{n-1}$ as $\mathfrak{k}$-modules, and both being spin representations with the same dimension 2^{n-1} (see, e.g., [4, Section 6.2.2]), we get $V_{n-1} = W_{n-1}$. Therefore,

$$\phi\big([V_k]\big) = [W_k] + [W_{k-1}] \quad \text{for } 1 \leqslant k \leqslant n-2, \quad \text{and} \quad \phi\big([V_{n-1}]\big) = [W_{n-1}].$$

In particular, each of $[W_1], \dots, [W_{n-1}]$ lies in the image of ϕ, proving the surjectivity of ϕ in this case.

Case IV. (D_4, G_2)**:** The two fundamental representations W_1 and W_2 have respective dimensions 7 and 14 [5, Section 22.3]. On the other hand, V_1 is eight dimensional and $V_2 = \wedge^2 V_1$. Since $\rho(\varpi_1) = \nu_1$ (by Lemma 3.2), by Lemma 3.3 we get $W_1 \subset V_1$ (as $\mathfrak{k}$-modules). So, we have the decomposition (as $\mathfrak{k}$-modules):

$$V_1 = W_1 \oplus \mathbb{C}.$$

Thus, as $\mathfrak{k}$-modules,

$$V_2 = \wedge^2 V_1 = \wedge^2 (W_1 \oplus \mathbb{C}) \simeq \big(\wedge^2 W_1\big) \oplus W_1.$$

But, $\wedge^2 W_1 \simeq W_2 \oplus W_1$ ([5, Section 22.3]). Hence, as $\mathfrak{k}$-modules,

$$V_2 = W_2 \oplus W_1^{\oplus 2}.$$

This gives

$$\phi\big([V_1]\big) = [W_1] + 1 \quad \text{and} \quad \phi\big([V_2]\big) = [W_2] + 2[W_1],$$

which proves the surjectivity of ϕ in this case.

Case V. (E_6, F_4): By Lemma 3.2(c), we see that ρ is surjective with kernel given by $\{a\varpi_1 + b\varpi_3 - b\varpi_5 - a\varpi_6 \mid a, b \in \mathbb{C}\}$. Considering the images of ϖ_i under ρ, we have as $\mathfrak{k}$-modules (by Lemmas 3.2(c) and 3.3),

$$\begin{aligned} W_1 &\subset V_2, \\ W_2 &\subset V_4, \\ W_3 &\subset V_3, V_5, \\ W_4 &\subset V_1, V_6. \end{aligned}$$

Using [11, Tables 44 and 47] or [9], we obtain

$$\begin{aligned} &\dim(W_1) = 52, && \dim(V_2) = 78, \\ &\dim(W_2) = 1274, && \dim(V_4) = 2925, \\ &\dim(W_3) = 273, && \dim(V_3) = \dim(V_5) = 351, \\ &\dim(W_4) = 26, && \dim(V_1) = \dim(V_6) = 27. \end{aligned}$$

Along with the fundamental $\mathfrak{k}$-modules, there are only three other irreducible $\mathfrak{k}$-modules of dimensions at most 1651. These are $\dim(W(2\nu_4)) = 324$, $\dim(W(\nu_1 + \nu_4)) = 1053$, and $\dim(W(2\nu_1)) = 1053$ ([11, Table 44], or [9]).

Let U^k denote an arbitrary $\mathfrak{k}$-module of dimension k. Considering the dimensions, we get (as $\mathfrak{k}$-modules):

$$\begin{aligned} V_1 = V_6 &= W_4 \oplus \mathbb{C}, \\ V_2 &= W_1 \oplus U^{26}, \\ V_3 = V_5 &= W_3 \oplus U^{78}, \\ V_4 &= W_2 \oplus U^{1651}. \end{aligned}$$

Now, U^{26} must be either W_4 or the trivial module $\mathbb{C}^{26}$, and U^{78} must be some combination of W_4, W_1, and $\mathbb{C}$. Since $\phi([V_1]) - 1 = [W_4]$, this implies that $[W_4]$, $[W_1]$ and $[W_3]$ are in the image of ϕ. (We remark that [11] gives $F_4 \subset E_6$ branching, but we continue without these results for clarity and completeness.)

Using appropriate tensor product decompositions in [9], we get

$$\big[W(2\nu_4)\big] = [W_4]^2 - [W_3] - [W_1] - [W_4] - 1, \tag{10}$$

$$\big[W(\nu_1 + \nu_4)\big] = [W_1][W_4] - [W_3] - [W_4], \tag{11}$$

$$\big[W(2\nu_1)\big] = [W_1]^2 - [W_2] - \big[W(2\nu_4)\big] - [W_1] - 1. \tag{12}$$

Since W_2 appears in V_4 as a $\mathfrak{k}$-submodule exactly once by Lemma 3.3, from the above identities, we get that $[W_2]$ lies in the image of ϕ if $W(2\nu_1)$ is not a component of V_4. In fact, we prove below that $2\nu_1$ is not a $\mathfrak{k}$-weight of V_4 at all.

In order that $2\nu_1$ be a $\mathfrak{k}$-weight of V_4, we should have $2\nu_1 = \mu|_{\mathfrak{t}_\mathfrak{k}}$, where μ is a weight of V_4. This is only possible if there exists a weight of V_4 of the form $\mu = a\varpi_1 + 2\varpi_2 + b\varpi_3 - b\varpi_5 - a\varpi_6$, for some $a, b \in \mathbb{Z}$. We claim this is impossible. Indeed, all the weights of V_4 are of the form $\varpi_4 - \sum_{i=1}^6 d_i\alpha_i$, where $d_i \in \mathbb{Z}^+$. If such μ existed, then by [1, Planche V],

$$\begin{aligned} \sum_{i=1}^{6} d_i\alpha_i &= \varpi_4 - \mu \\ &= \varpi_4 + a(\varpi_6 - \varpi_1) - 2\varpi_2 + b(\varpi_5 - \varpi_3) \end{aligned}$$

$$= (2\alpha_1 + 3\alpha_2 + 4\alpha_3 + 6\alpha_4 + 4\alpha_5 + 2\alpha_6) + (a/3)(-2\alpha_1 - \alpha_3 + \alpha_5 + 2\alpha_6)$$
$$- 2(\alpha_1 + 2\alpha_2 + 2\alpha_3 + 3\alpha_4 + 2\alpha_5 + \alpha_6) + (b/3)(-\alpha_1 - 2\alpha_3 + 2\alpha_5 + \alpha_6),$$

from which we immediately see a contradiction since the α_2 coefficient is -1.

This completes the proof in this last case and hence the proof of the first part of Theorem 3.1 is completed.

To prove that $\eta : K// \operatorname{Ad} K \to G// \operatorname{Ad} G$ is a closed embedding, it suffices to show that the induced map between the affine coordinate rings $\eta^* : \mathbb{C}[G// \operatorname{Ad} G] \to \mathbb{C}[K// \operatorname{Ad} K]$ is surjective. But, by [10, Theorem 3.5], there is a functorial isomorphism

$$\mathbb{C} \otimes_{\mathbb{Z}} R(G) \to \mathbb{C}[G// \operatorname{Ad} G],$$

and similarly we have an isomorphism

$$\mathbb{C} \otimes_{\mathbb{Z}} R(K) \to \mathbb{C}[K// \operatorname{Ad} K].$$

From this, the surjectivity of η^* follows from the surjectivity of $R(G) \to R(K)$. This proves the theorem. □

We give the following Lie algebra analogue as a corollary.

Corollary 3.4. *With the notation and assumptions as in Theorem 3.1, the canonical restriction map*

$$S(\mathfrak{g}^*)^{\mathfrak{g}} \to S(\mathfrak{k}^*)^{\mathfrak{k}}$$

is surjective.

Proof. By [13, Corollary 6.4], for any connected semisimple algebraic group H over $\mathbb{C}$, the restriction map

$$r : \mathbb{C}[H// \operatorname{Ad} H] \simeq \mathbb{C}[H]^H \to \mathbb{C}[T_H]^{W_H} \tag{13}$$

is an isomorphism of $\mathbb{C}$-algebras, where $T_H \subset H$ is a maximal torus and W_H is the Weyl group of H.

Similarly, the restriction map

$$r_o : \mathbb{C}[\mathfrak{h}]^H \to \mathbb{C}[\mathfrak{t}_{\mathfrak{h}}]^{W_H} \tag{14}$$

is a graded algebra isomorphism, where $\mathfrak{h}$ (resp. $\mathfrak{t}_{\mathfrak{h}}$) is the Lie algebra of H (resp. T_H). Thus, to prove the corollary, it suffices to show that the canonical restriction map

$$\beta_o^* : \mathbb{C}[\mathfrak{t}]^W \to \mathbb{C}[\mathfrak{t}_{\mathfrak{k}}]^{W_K}$$

is surjective, where W (resp. W_K) is the Weyl group of G (resp. K). (Recall that W_K is the subgroup W^σ of W keeping T_K and $\mathfrak{t}_K$ stable under the standard action of W on T and $\mathfrak{t}$ respectively, where $T_K \subset K$ is the maximal torus with Lie algebra $\mathfrak{t}_{\mathfrak{k}}$ and $T \subset G$ is the maximal torus with Lie algebra $\mathfrak{t}$.) Since β_o^* is a graded algebra homomorphism induced from the $\mathbb{C}^*$-equivariant map $\beta_o : \mathfrak{t}_{\mathfrak{k}}/W_K \to \mathfrak{t}/W$ (where the $\mathbb{C}^*$-action is the standard homothety action), it suffices to show that the tangent map between the Zariski tangent spaces at 0:

$$(d\beta_o)_0 : T_0(\mathfrak{t}_{\mathfrak{k}}/W_K) \to T_0(\mathfrak{t}/W)$$

is injective. Let T^{anal} denote the analytic tangent space. Then, the canonical map

$$T_x^{anal}(X) \to T_x(X)$$

is an isomorphism for any algebraic variety X and any point $x \in X$.

Consider the commutative diagram:

$$\begin{array}{ccc} \mathfrak{t}_{\mathfrak{k}}/W_K & \xrightarrow{\beta_o} & \mathfrak{t}/W \\ \downarrow{\scriptstyle \mathrm{Exp}} & & \downarrow{\scriptstyle \mathrm{Exp}} \\ T_K/W_K & \xrightarrow[\beta]{} & T/W, \end{array}$$

where $\beta : T_K/W_K \to T/W$ is the canonical map. Since T_K, T are tori, Exp is a local isomorphism in the analytic category. In particular, there exist open subsets (in the analytic topology) $0 \in U_{\mathfrak{k}} \subset \mathfrak{t}_{\mathfrak{k}}/W_K$, $0 \in U \subset \mathfrak{t}/W$, $1 \in V_K \subset T_K/W_K$ and $1 \in V \subset T/W$ such that $\beta_o(U_K) \subset U$ and $\mathrm{Exp}_{|U_{\mathfrak{k}}} : U_{\mathfrak{k}} \to V_K$ is an analytic isomorphism and so is $\mathrm{Exp}_{|U} : U \to V$. Since, by Theorem 3.1 and the isomorphism (13), β is a closed embedding,

$$(d\beta)_1 : T_1^{anal}(T_K/W_K) \simeq T_1(T_K/W_K) \to T_1^{anal}(T/W) \simeq T_1(T/W)$$

is injective and hence so is $T_0(\mathfrak{t}_{\mathfrak{k}}/W_K) \to T_0(\mathfrak{t}/W)$. This proves the corollary. □

As a consequence of Corollary 3.4, we get the following.

Theorem 3.5. *With the notation and assumptions as in Theorem 3.1, the canonical restriction map $\gamma : H^*(G) \to H^*(K)$ is surjective. Moreover, this induces a surjective (graded) map*

$$\gamma_o : P(\mathfrak{g}) \to P(\mathfrak{k}),$$

where $P(\mathfrak{g}) \subset H^*(G)$ is the subspace of primitive elements.

Proof. From the definition of coproduct, it is easy to see that the following diagram is commutative:

$$\begin{array}{ccc} H^*(G) & \xrightarrow{\Delta_G} & H^*(G) \otimes H^*(G) \\ \downarrow{\scriptstyle \gamma} & & \downarrow{\scriptstyle \gamma\otimes\gamma} \\ H^*(K) & \xrightarrow[\Delta_K]{} & H^*(K) \otimes H^*(K). \end{array}$$

Thus, γ takes $P(\mathfrak{g})$ to $P(\mathfrak{k})$.

Let $\mathfrak{h}$ be a reductive Lie algebra. For any $v \in \mathfrak{h}$, define the derivation $i(v) : S(\mathfrak{h}^*) \to S(\mathfrak{h}^*)$ induced by $i(v)(f) = f(v)$, for $f \in \mathfrak{h}^*$. Further, define an algebra homomorphism $\lambda : S(\mathfrak{h}^*) \to \wedge^{\mathrm{even}}(\mathfrak{h}^*)$ by $\lambda(f) = df$, for $f \in \mathfrak{h}^* = S^1(\mathfrak{h}^*)$, where $d : \wedge^1(\mathfrak{h}^*) = \mathfrak{h}^* \to \wedge^2(\mathfrak{h}^*)$ is the standard differential in the Lie algebra cochain complex $\wedge^\bullet(\mathfrak{h}^*)$. Let

$$S^+(\mathfrak{h}^*) := \bigoplus_{n\geqslant 1} S^n(\mathfrak{h}^*) \quad \text{and} \quad \wedge^+(\mathfrak{h}^*) := \bigoplus_{n\geqslant 1} \wedge^n(\mathfrak{h}^*).$$

Now, define the *transgression map*

$$\tau = \tau_{\mathfrak{h}} : S^+(\mathfrak{h}^*)^{\mathfrak{h}} \to \wedge^+(\mathfrak{h}^*)^{\mathfrak{h}}, \qquad \tau(p) = \sum_j e_j^* \wedge \lambda(i(e_j)p),$$

for $p \in S^+(\mathfrak{h}^*)^{\mathfrak{h}}$, where $\{e_j\}$ is a basis of $\mathfrak{h}$ and $\{e_j^*\}$ is the dual basis of $\mathfrak{h}^*$.

By a result of Cartan (cf. [2, Théorème 2]; also see [8]), τ factors through

$$S^+(\mathfrak{h}^*)^{\mathfrak{h}}/\big(S^+(\mathfrak{h}^*)^{\mathfrak{h}}\big)\cdot\big(S^+(\mathfrak{h}^*)^{\mathfrak{h}}\big)$$

to give an injective map

$$\bar{\tau}: S^+(\mathfrak{h}^*)^{\mathfrak{h}}/\big(S^+(\mathfrak{h}^*)^{\mathfrak{h}}\big)\cdot\big(S^+(\mathfrak{h}^*)^{\mathfrak{h}}\big)\to \wedge^+(\mathfrak{h}^*)^{\mathfrak{h}}$$

with image precisely equal to the space of primitive elements $P(\mathfrak{h})$. From the definition of τ, it is easy to see that the following diagram is commutative:

$$\begin{array}{ccc} S^+(\mathfrak{g}^*)^{\mathfrak{g}} & \xrightarrow{\tau_{\mathfrak{g}}} & \wedge^+(\mathfrak{g}^*)^{\mathfrak{g}} \\ \downarrow & & \downarrow \\ S^+(\mathfrak{k}^*)^{\mathfrak{k}} & \xrightarrow{\tau_{\mathfrak{k}}} & \wedge^+(\mathfrak{k}^*)^{\mathfrak{k}}, \end{array}$$

where the vertical maps are the canonical restriction maps. By using Corollary 3.4, this proves that $P(\mathfrak{g})$ surjects onto $P(\mathfrak{k})$. Since $P(\mathfrak{k})$ generates $\wedge^*(\mathfrak{k}^*)^{\mathfrak{k}} \simeq H^*(K)$ as an algebra, we get that γ is surjective. This proves the theorem. □

Remark 3.6. As a consequence of the above theorem, we see that the Leray–Serre homology (or cohomology) spectral sequence with coefficients in $\mathbb{C}$ for the fibration

$$K \to G \to G/K$$

degenerates at the E^2-term.

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