

A study of saturated tensor cone for symmetrizable Kac–Moody algebras

Merrick Brown · Shrawan Kumar

Received: 31 May 2013 / Revised: 11 February 2014 / Published online: 27 May 2014
© Springer-Verlag Berlin Heidelberg 2014

Abstract Let \mathfrak{g} be a symmetrizable Kac–Moody Lie algebra with the standard Cartan subalgebra \mathfrak{h} and the Weyl group W . Let P_+ be the set of dominant integral weights. For $\lambda \in P_+$, let $L(\lambda)$ be the integrable, highest weight (irreducible) representation of \mathfrak{g} with highest weight λ . For a positive integer s , define the *saturated tensor semigroup* as

$$\Gamma_s := \{(\lambda_1, \dots, \lambda_s, \mu) \in P_+^{s+1} : \exists N \geq 1 \text{ with } L(N\mu) \subset L(N\lambda_1) \otimes \cdots \otimes L(N\lambda_s)\}.$$

The aim of this paper is to begin a systematic study of Γ_s in the infinite dimensional symmetrizable Kac–Moody case. In this paper, we produce a set of necessary inequalities satisfied by Γ_s . These inequalities are indexed by products in $H^*(G^{\min}/B; \mathbb{Z})$ for B the standard Borel subgroup, where G^{\min} is the ‘minimal’ Kac–Moody group with Lie algebra \mathfrak{g} . The proof relies on the Kac–Moody analogue of the Borel–Weil theorem and Geometric Invariant Theory (specifically the Hilbert–Mumford index). In the case that \mathfrak{g} is affine of rank 2, we show that these inequalities are necessary and sufficient. We further prove that any integer $d > 0$ is a saturation factor for $A_1^{(1)}$ and 4 is a saturation factor for $A_2^{(2)}$.

Mathematics Subject Classification (2000) 22E65 · 17B67 · 17B68

M. Brown · S. Kumar (✉)

Department of Mathematics, University of North Carolina, Chapel Hill, NC 27599-3250, USA
e-mail: shrawan@email.unc.edu

M. Brown
e-mail: merrickb@email.unc.edu

1 Introduction

Let \mathfrak{g} be a symmetrizable Kac–Moody Lie algebra with the standard Cartan subalgebra \mathfrak{h} . Let $P_+ \subset \mathfrak{h}^*$ be the set of dominant integral weights. For $\lambda \in P_+$, let $L(\lambda)$ denote the irreducible, integrable, highest weight representation of \mathfrak{g} with highest weight λ . In this way, P_+ parameterizes all the integrable, highest weight \mathfrak{g} -modules. Moreover, the tensor product of integrable, highest weight \mathfrak{g} -modules decomposes as a (possibly infinite) direct sum of irreducibles.

For a positive integer s , define the *saturated tensor semigroup* as

$$\Gamma_s := \left\{ (\lambda_1, \dots, \lambda_s, \mu) \in P_+^{s+1} \mid \begin{array}{l} L(N\mu) \subset L(N\lambda_1) \otimes \dots \otimes L(N\lambda_s) \\ \text{for some } N > 0 \end{array} \right\}.$$

In the case that \mathfrak{g} is a semisimple complex Lie algebra, Γ_s is given by an explicit set of inequalities coming from the products in the cohomology ring of the flag varieties for the corresponding algebraic group.

The aim of this paper is to begin a systematic study of Γ_s in the infinite dimensional symmetrizable Kac–Moody case. Specifically, we produce a set of necessary inequalities satisfied by Γ_s , which we describe now. Let $X = G^{\min}/B$ be the standard full Kac–Moody flag variety associated to \mathfrak{g} (cf. [13, §7.1]), where G^{\min} is the ‘minimal’ Kac–Moody group as in [13, §7.4] with Lie algebra \mathfrak{g} , and B is the standard Borel subgroup of G^{\min} . For w in the Weyl group W , let $X_w = \overline{BwB}/B \subset X$ be the corresponding Schubert variety. Let $\{\varepsilon^w\}_{w \in W} \subset H^*(X, \mathbb{Z})$ be the (Schubert) basis dual (with respect to the standard pairing) to the basis of the singular homology of X given by the fundamental classes of X_w . The following result is our first main theorem valid for any symmetrizable \mathfrak{g} (cf. Theorem 3.2).

Theorem 1.1 *Let $(\lambda_1, \dots, \lambda_s, \mu) \in \Gamma_s$. Then, for any $u_1, \dots, u_s, v \in W$ such that $n_{u_1, \dots, u_s}^v \neq 0$, where*

$$\varepsilon^{u_1} \dots \varepsilon^{u_s} = \sum_w n_{u_1, \dots, u_s}^w \varepsilon^w,$$

we have

$$\left(\sum_{j=1}^s \lambda_j(u_j x_i) \right) - \mu(v x_i) \geq 0, \text{ for any } x_i,$$

where $x_i \in \mathfrak{h}$ is dual to the simple roots of \mathfrak{g} .

The proof of the theorem relies on the Kac–Moody analogue of the Borel–Weil theorem and the Geometric Invariant Theory (specifically the Hilbert–Mumford index). We conjecture that the above inequalities are sufficient as well to describe Γ_s . In fact, we conjecture a much sharper result, where many fewer inequalities suffice to describe the semigroup Γ_s . To explain our conjecture, we need some more notation.

Let $P \supset B$ be a (standard) parabolic subgroup and let $X_P := G^{\min}/P$ be the corresponding partial flag variety. Let W_P be the Weyl group of P (which is, by definition, the Weyl group of the Levi subgroup L of P) and let W^P be the set of minimal length coset representatives of cosets in W/W_P . The projection map $X \rightarrow X_P$ induces an injective homomorphism $H^*(X_P, \mathbb{Z}) \rightarrow H^*(X, \mathbb{Z})$ and $H^*(X_P, \mathbb{Z})$ has the Schubert basis $\{\varepsilon_P^w\}_{w \in W^P}$ such that ε_P^w goes to ε^w for any $w \in W^P$. As defined by Belkale–Kumar [1, §6] in the finite dimensional case (and extended here in Sect. 7 for any symmetrizable Kac–Moody case), there is a new deformed product \odot_0 in $H^*(X_P, \mathbb{Z})$, which is commutative and associative. Now, we are ready to state our conjecture (see Conjecture 2 in Sec. 7).

Conjecture 1 Let \mathfrak{g} be any indecomposable symmetrizable Kac–Moody Lie algebra and let $(\lambda_1, \dots, \lambda_s, \mu) \in P_+^{s+1}$. Assume further that none of λ_j is W -invariant and $\mu - \sum_{j=1}^s \lambda_j \in Q$, where Q is the root lattice of G . Then, the following are equivalent:

- (a) $(\lambda_1, \dots, \lambda_s, \mu) \in \Gamma_s$.
- (b) For every standard maximal parabolic subgroup P in G^{\min} and every choice of $s + 1$ -tuples $(w_1, \dots, w_s, v) \in (W^P)^{s+1}$ such that ε_P^v occurs with coefficient 1 in the deformed product

$$\varepsilon_P^{w_1} \odot_0 \cdots \odot_0 \varepsilon_P^{w_s} \in (H^*(X_P, \mathbb{Z}), \odot_0),$$

the following inequality holds:

$$\left(\sum_{j=1}^s \lambda_j(w_j x_P) \right) - \mu(v x_P) \geq 0, \tag{I_{(w_1, \dots, w_s, v)}^P}$$

where α_{i_P} is the (unique) simple root not in the Levi of P and $x_P := x_{i_P}$.

This conjecture is motivated from its validity in the finite case due to Belkale–Kumar [1, Theorem 22]. (For a survey of these results in the finite case, see [15].) So far, the only evidence of its validity in the infinite dimensional case is shown for $s = 2$ and \mathfrak{g} of types $A_1^{(1)}$ and $A_2^{(2)}$ (cf. Theorems 7.3 and 8.6). In these cases, we explicitly determine Γ_2 and thereby show the validity of the conjecture.

A positive integer d_o is called a *saturation factor* for \mathfrak{g} if for any $\Lambda, \Lambda', \Lambda'' \in P_+$ such that $\Lambda - \Lambda' - \Lambda'' \in Q$ and $L(N\Lambda)$ is a submodule of $L(N\Lambda') \otimes L(N\Lambda'')$, for some $N \in \mathbb{Z}_{>0}$, then $L(d_o\Lambda)$ is a submodule of $L(d_o\Lambda') \otimes L(d_o\Lambda'')$.

We prove the following result on saturation factors (cf. Corollaries 6.3 and 8.8).

Theorem 1.2 For $A_1^{(1)}$, any integer $d_o > 1$ is a saturation factor. For $A_2^{(2)}$, 4 is a saturation factor.

The proof in these affine rank-2 cases makes use of basic representation theory of the Virasoro algebra (in particular, Lemma 4.1). Let δ be the smallest positive imaginary root of \mathfrak{g} . To determine the saturated tensor semigroup for $s = 2$, we show that it is enough to know the components of $L(\lambda_1) \otimes L(\lambda_2)$ which are δ -maximal, i.e., the components $L(\mu) \subset L(\lambda_1) \otimes L(\lambda_2)$ such that $L(\mu + n\delta) \not\subset L(\lambda_1) \otimes L(\lambda_2)$ for

any $n > 0$. Let $m_{\lambda_1, \lambda_2}^\mu$ be the multiplicity of $L(\mu)$ in $L(\lambda_1) \otimes L(\lambda_2)$. If $L(\mu)$ is a δ -maximal component of $L(\lambda_1) \otimes L(\lambda_2)$, then $\sum_{n \in \mathbb{Z}_{\leq 0}} L(\mu + n\delta)^{\oplus m_{\lambda_1, \lambda_2}^{\mu+n\delta}}$ is a unitarizable coset module for the Virasoro algebra arising from the Sugawara construction for the diagonal embedding $\mathfrak{g} \hookrightarrow \mathfrak{g} \oplus \mathfrak{g}$. Proposition 5.5 for $A_1^{(1)}$ (and the analogous Proposition 8.2 for $A_2^{(2)}$) determining the maximal δ -components plays a crucial role in the proofs.

2 Notation

We take the base field to be the field of complex numbers \mathbb{C} . By a variety, we mean an algebraic variety over \mathbb{C} , which is reduced but not necessarily irreducible.

Let G be any symmetrizable Kac–Moody group over \mathbb{C} completed along the negative roots (as opposed to completed along the positive roots as in [13, Chapter 6]) and $G^{\min} \subset G$ be the ‘minimal’ Kac–Moody group as in [13, §7.4]. Let B be the standard (positive) Borel subgroup, B^- the standard negative Borel subgroup, $H = B \cap B^-$ the standard maximal torus and W the Weyl group (cf. [13, Chapter 6]). Let U (resp. U^-) be the unipotent radical $[B, B]$ (resp. $[B^-, B^-]$) of B (resp. B^-). Let

$$\bar{X} = G/B$$

be the ‘thick’ flag variety which contains the standard Kac–Moody flag variety

$$X = G^{\min}/B.$$

If G is not of finite type, \bar{X} is an infinite dimensional non quasi-compact scheme (cf. [7, §4]) and X is an ind-projective variety (cf. [13, §7.1]). The group G^{\min} acts on \bar{X} and X .

More generally, for any standard parabolic subgroup $G^{\min} \supset P \supset B$, define the partial flag variety

$$X_P = G^{\min}/P,$$

and

$$\bar{X}_P = G/P.$$

Recall that if W_P is the Weyl group of P (which is, by definition, the Weyl Group W_L of its Levi subgroup L), then in each coset of W/W_P we have a unique member w of minimal length. Let W^P be the set of the minimal length representatives in the cosets of W/W_P .

For any $w \in W^P$, define the Schubert cell:

$$C_w^P := BwP/P \subset X_P$$

endowed with the reduced subscheme structure. Then, it is a locally closed subvariety of the ind-variety X_P isomorphic with the affine space $\mathbb{A}^{\ell(w)}$, $\ell(w)$ being the length of w (cf. [13, §7.1]). Its closure is denoted by X_w^P , which is an irreducible (projective) subvariety of X_P of dimension $\ell(w)$. We denote the point $wP \in C_w^P$ by \dot{w} . We abbreviate C_w^B, X_w^B by C_w, X_w respectively.

Similarly, define the opposite Schubert cell

$$C_P^w := B^- wP/P \subset \bar{X}_P,$$

and the opposite Schubert variety

$$X_P^w := \overline{C_P^w} \subset \bar{X}_P,$$

both endowed with the reduced subscheme structures. Then, X_P^w is a finite codimensional irreducible subscheme of \bar{X}_P (cf. [13, Section 7.1] and [7, §4]). As above, we abbreviate C_B^w, X_B^w by C^w, X^w respectively.

For any integral weight λ (i.e., any character e^λ of H), we have a G^{\min} -equivariant line bundle $\mathcal{L}_B(\lambda)$ on X associated to the character $e^{-\lambda}$ of H . Similarly, we have a G -equivariant line bundle $\mathcal{L}_{B^-}(\lambda)$ on $X^- := G/B^-$ associated to the character e^λ of H .

By the Bruhat decomposition

$$X_P = \sqcup_{w \in W^P} C_w^P,$$

the singular homology $H_*(X_P, \mathbb{Z})$ of X_P with integral coefficients has a basis $\{\mu(X_w^P)\}_{w \in W^P}$, where $\mu(X_w^P) \in H_{2\ell(w)}(X_P, \mathbb{Z})$ denotes the fundamental class of X_w^P . Let $\{\epsilon_P^w\}_{w \in W^P}$ be the dual basis of the singular cohomology $H^*(X_P, \mathbb{Z})$ under the standard pairing of cohomology with homology, i.e.,

$$\epsilon_P^u(\mu(X_v^P)) = \delta_{u,v}, \quad \text{for any } u, v \in W^P.$$

Thus, $\epsilon_P^w \in H^{2\ell(w)}(X_P, \mathbb{Z})$. If $P = B$, we abbreviate ϵ_P^u by ϵ^u .

Let $\Delta = \{\alpha_1, \dots, \alpha_r\} \subset \mathfrak{h}^*$ be the set of simple roots, $\{\alpha_i^\vee, \dots, \alpha_r^\vee\} \subset \mathfrak{h}$ the set of simple coroots and $\{s_1, \dots, s_r\} \subset W$ the corresponding simple reflections, where $\mathfrak{h} := \text{Lie } H$. Let $\rho \in X(H)$ be any weight satisfying

$$\rho(\alpha_i^\vee) = 1, \quad \text{for all } 1 \leq i \leq r,$$

where $X(H)$ is the character group of H (identified as a subgroup of \mathfrak{h}^* via the derivative). When G is a finite dimensional semisimple group, ρ is unique, but for a general Kac–Moody group G , it may not be unique.

Choose elements $x_i \in \mathfrak{h}$ such that

$$\alpha_j(x_i) = \delta_{i,j}, \quad \text{for any } 1 \leq i, j \leq r. \tag{1}$$

Observe that x_i may not be unique.

Define the set of *dominant integral weights*

$$P_+ := \{\lambda \in X(H) : \lambda(\alpha_i^\vee) \in \mathbb{Z}_+ \forall 1 \leq i \leq r\},$$

and the set of *dominant integral regular weights*

$$P_{++} := \{\lambda \in X(H) : \lambda(\alpha_i^\vee) \in \mathbb{Z}_{\geq 1} \forall 1 \leq i \leq r\},$$

where \mathbb{Z}_+ is the set of non-negative integers. The integrable highest weight (irreducible) modules of G^{\min} are parameterized by P_+ . For $\lambda \in P_+$, let $L(\lambda)$ be the corresponding integrable highest weight (irreducible) G -module with highest weight λ .

3 Necessary inequalities for the saturated tensor semigroup

Fix a positive integer s and define the *saturated tensor semigroup* $\Gamma_s = \Gamma_s(G)$:

$$\Gamma_s := \left\{ (\lambda_1, \dots, \lambda_s, \mu) \in P_+^{s+1} \mid \begin{array}{l} L(N\mu) \subset L(N\lambda_1) \otimes \dots \otimes L(N\lambda_s) \\ \text{for some } N > 0 \end{array} \right\}.$$

It is indeed a semigroup by the analogue of the Borel–Weil theorem for the Kac–Moody case [see the identity (2) in the proof of Theorem 3.2]. We give a certain set of inequalities satisfied by Γ_s . But, we first recall some basic results about the Hilbert–Mumford index.

Definition *Let S be any (not necessarily reductive) algebraic group acting on a (not necessarily projective) variety \mathbb{X} and let \mathbb{L} be an S -equivariant line bundle on \mathbb{X} . Let $O(S)$ be the set of all one parameter subgroups (for short OPS) in S . Take any $x \in \mathbb{X}$ and $\delta \in O(S)$ such that the limit $\lim_{t \rightarrow 0} \delta(t)x$ exists in \mathbb{X} (i.e., the morphism $\delta_x : \mathbb{G}_m \rightarrow \mathbb{X}$ given by $t \mapsto \delta(t)x$ extends to a morphism $\tilde{\delta}_x : \mathbb{A}^1 \rightarrow \mathbb{X}$). Then, following Mumford, define a number $\mu^{\mathbb{L}}(x, \delta)$ as follows: Let $x_o \in \mathbb{X}$ be the point $\tilde{\delta}_x(0)$. Since x_o is \mathbb{G}_m -invariant via δ , the fiber of \mathbb{L} over x_o is a \mathbb{G}_m -module; in particular, it is given by a character of \mathbb{G}_m . This integer is defined as $\mu^{\mathbb{L}}(x, \delta)$.*

We record the following standard properties of $\mu^{\mathbb{L}}(x, \delta)$ (cf. [16, Chap. 2, §1]):

Proposition 3.1 *For any $x \in \mathbb{X}$ and $\delta \in O(S)$ such that $\lim_{t \rightarrow 0} \delta(t)x$ exists in \mathbb{X} , we have the following (for any S -equivariant line bundles $\mathbb{L}, \mathbb{L}_1, \mathbb{L}_2$):*

- (a) $\mu^{\mathbb{L}_1 \otimes \mathbb{L}_2}(x, \delta) = \mu^{\mathbb{L}_1}(x, \delta) + \mu^{\mathbb{L}_2}(x, \delta)$.
- (b) If there exists $\sigma \in H^0(\mathbb{X}, \mathbb{L})^S$ such that $\sigma(x) \neq 0$, then $\mu^{\mathbb{L}}(x, \delta) \geq 0$.
- (c) If $\mu^{\mathbb{L}}(x, \delta) = 0$, then any element of $H^0(\mathbb{X}, \mathbb{L})^S$ which does not vanish at x does not vanish at $\lim_{t \rightarrow 0} \delta(t)x$ as well.
- (d) For any S -variety \mathbb{X}' together with an S -equivariant morphism $f : \mathbb{X}' \rightarrow \mathbb{X}$ and any $x' \in \mathbb{X}'$ such that $\lim_{t \rightarrow 0} \delta(t)x'$ exists in \mathbb{X}' , we have $\mu^{f^*\mathbb{L}}(x', \delta) = \mu^{\mathbb{L}}(f(x'), \delta)$.

(e) (Hilbert–Mumford criterion) Assume that \mathbb{X} is projective, S is connected and reductive and \mathbb{L} is ample. Then, $x \in \mathbb{X}$ is semistable (with respect to \mathbb{L}) if and only if $\mu^{\mathbb{L}}(x, \delta) \geq 0$, for all $\delta \in O(S)$.

In particular, if $x \in \mathbb{X}$ is semistable and δ -fixed, then $\mu^{\mathbb{L}}(x, \delta) = 0$.

The following theorem is one of our main results giving a collection of necessary inequalities defining the semigroup Γ_s .

Theorem 3.2 Let G be any symmetrizable Kac–Moody group and let $(\lambda_1, \dots, \lambda_s, \mu) \in \Gamma_s$. Then, for any $u_1, \dots, u_s, v \in W$ such that $n_{u_1, \dots, u_s}^v \neq 0$, where

$$\varepsilon^{u_1} \dots \varepsilon^{u_s} = \sum_w n_{u_1, \dots, u_s}^w \varepsilon^w \in H^*(X, \mathbb{Z}),$$

we have

$$\left(\sum_{j=1}^s \lambda_j(u_j x_i) \right) - \mu(v x_i) \geq 0, \quad \text{for any } x_i,$$

where x_i is defined by the equation (1).

Proof Let

$$Z := \{(\bar{g}_1, \dots, \bar{g}_s) \in (X^-)^s : g_1 X^{u_1} \cap \dots \cap g_s X^{u_s} \cap X_v \neq \emptyset\},$$

where $X^- := G/B^-$ and $\bar{g}_j = g_j B^-$. Then, Z contains a nonempty open set by Proposition 3.5. (In fact, by Proposition 3.5, $Z = (X^-)^s$, but we do not need this stronger result.)

Take a nonzero $\sigma \in H^0((X^-)^s \times X, \mathcal{L}^N)^{G^{\min}}$, where

$$\mathcal{L} := \mathcal{L}_{B^-(\lambda_1)} \boxtimes \dots \boxtimes \mathcal{L}_{B^-(\lambda_s)} \boxtimes \mathcal{L}_B(\mu).$$

Such a nonzero σ exists, for some $N > 0$, since by [13, Corollary 8.3.12(a) and Lemma 8.3.9],

$$\begin{aligned} H^0\left((X^-)^s \times X, \mathcal{L}^N\right)^{G^{\min}} &\simeq \text{Hom}_{G^{\min}}\left(L(N\lambda_1)^\vee \otimes \dots \otimes L(N\lambda_s)^\vee \otimes L(N\mu), \mathbb{C}\right) \\ &\simeq \text{Hom}_{G^{\min}}\left(L(N\mu), [L(N\lambda_1)^\vee \otimes \dots \otimes L(N\lambda_s)^\vee]^*\right) \\ &\simeq \text{Hom}_{G^{\min}}\left(L(N\mu), [L(N\lambda_1)^\vee \otimes \dots \otimes L(N\lambda_s)^\vee]^\vee\right) \\ &\simeq \text{Hom}_{G^{\min}}\left(L(N\mu), L(N\lambda_1) \otimes \dots \otimes L(N\lambda_s)\right) \\ &\neq 0, \end{aligned} \tag{2}$$

since $(\lambda_1, \dots, \lambda_s, \mu) \in \Gamma_s$, where, for a G^{\min} -module M , M^\vee denotes the direct sum of the H -weight spaces of the full dual module M^* .

Pick $(\bar{g}_1, \dots, \bar{g}_s) \in Z$ such that $\sigma(\bar{g}_1, \dots, \bar{g}_s, \bar{1}) \neq 0$, where $\bar{1} = 1.B$. Since $(\bar{g}_1, \dots, \bar{g}_s) \in Z$, there exists $u'_1 \geq u_1, \dots, u'_s \geq u_s$ and $v' \leq v$ such that

$g_1 C^{u'_1} \cap \dots \cap g_s C^{u'_s} \cap C_{v'}$ is nonempty (cf. [13, Proposition 7.1.15] and [7, Proposition 4.5.11]). Now, pick $g \in G^{\min}$ such that

$$gB \in g_1 C^{u'_1} \cap \dots \cap g_s C^{u'_s} \cap C_{v'}. \tag{3}$$

By Proposition 3.1, for any $\delta \in O(G^{\min})$, $\mu^{\mathcal{L}}(\bar{x}, \delta(t)) \geq 0$, where $\bar{x} = (\bar{g}_1, \dots, \bar{g}_s, \bar{1})$ (since $\sigma(\bar{x}) \neq 0$). By the following Lemma 3.3, applied to the OPS $\delta(t) = gt^{x_i}g^{-1}$, we get

$$\left(\sum_{j=1}^s \lambda_j(u'_j x_i) \right) - \mu(v' x_i) \geq 0. \tag{4}$$

But, by [13, Lemma 8.3.3],

$$(u'_j)^{-1} \lambda_j \leq u_j^{-1}(\lambda_j).$$

Thus,

$$\lambda_j(u'_j x_i) \leq \lambda_j(u_j x_i).$$

Similarly,

$$\mu(v' x_i) \geq \mu(v x_i).$$

Thus, from (4), we get

$$\left(\sum_{j=1}^s \lambda_j(u_j x_i) \right) - \mu(v x_i) \geq 0.$$

This proves the theorem. □

Lemma 3.3 *Let $g \in G^{\min}$ be as in the equation (3). Consider the one parameter subgroup $\delta(t) = gt^{x_i}g^{-1} \in O(G^{\min})$. Then,*

- (a) $\mu^{\mathcal{L}_{B^{-}(\lambda_j)}}(g_j B^{-}, \delta(t)) = \lambda_j(u'_j x_i)$.
- (b) $\mu^{\mathcal{L}_B(\mu)}(1 \cdot B, \delta(t)) = -\mu(v' x_i)$.

Proof (a) $\mu^{\mathcal{L}_{B^{-}(\lambda_j)}}(g_j B^{-}, \delta(t)) = \mu^{\mathcal{L}_{B^{-}(\lambda_j)}}(g^{-1} g_j B^{-}, t^{x_i})$.

By assumption, $g_j^{-1} g \in U^{-} u'_j B$. Write

$$g_j^{-1} g = b_j^{-} u'_j p_j, \quad \text{for some } b_j^{-} \in U^{-}, p_j \in B.$$

Thus,

$$1 = g^{-1} g_j b_j^{-} u'_j p_j.$$

Let

$$b_j(t) = b_j^- u'_j t^{-x_i} (u'_j)^{-1} (b_j^-)^{-1} \in B^-.$$

Then,

$$t^{x_i} g^{-1} g_j b_j(t) = t^{x_i} p_j^{-1} t^{-x_i} (u'_j)^{-1} (b_j^-)^{-1}. \tag{5}$$

Consider the G_m -invariant section (via t^{x_i}) of $\mathcal{L}_{B^-}(\lambda_j)$:

$$\begin{aligned} \hat{\sigma}(t) &= \left(t^{x_i} g^{-1} g_j, 1 \right) \pmod{B^-} \\ &= \left(t^{x_i} g^{-1} g_j b_j(t), \lambda_j(b_j(t)^{-1}) \right) \pmod{B^-}. \end{aligned}$$

Clearly, $\lim_{t \rightarrow 0} t^{x_i} g^{-1} g_j b_j(t)$ exists in G by (5).

Now,

$$\begin{aligned} \lambda_j \left(b_j(t)^{-1} \right) &= \lambda_j \left(b_j^- u'_j t^{x_i} (u'_j)^{-1} (b_j^-)^{-1} \right) \\ &= \lambda_j \left(t^{u'_j x_i} \right). \end{aligned}$$

This gives

$$\mu^{\mathcal{L}_{B^-}(\lambda_j)}(g_j B^-, \delta(t)) = \lambda_j(u'_j(x_i)).$$

This proves the (a) part of the lemma.

(b) $\mu^{\mathcal{L}_B(\mu)}(1 \cdot B, \delta(t)) = \mu^{\mathcal{L}_B(\mu)}(g^{-1} B, t^{x_i})$. By assumption,

$$g \in Bv' \cdot B.$$

Write

$$g = bv'p, \quad \text{for } b \in U, p \in B.$$

Thus,

$$1 = g^{-1}bv'p.$$

Let

$$b(t) = bv't^{-x_i} (v')^{-1} b^{-1} \in B.$$

Now,

$$t^{x_i} g^{-1} b(t) = t^{x_i} p^{-1} t^{-x_i} (v')^{-1} b^{-1}.$$

Thus,

$$\lim_{t \rightarrow 0} t^{x_i} g^{-1} b(t) \text{ exists in } G^{\min}.$$

Consider the G_m -invariant section (via t^{x_i})

$$\begin{aligned} \hat{\sigma}(t) &= (t^{x_i} g^{-1}, 1) \pmod{B} \\ &= \left(t^{x_i} g^{-1} b(t), \mu(b(t)) \right) \pmod{B}. \end{aligned}$$

Now,

$$\begin{aligned} \mu(b(t)) &= \mu(bv' t^{-x_i} (v')^{-1} b^{-1}) \\ &= \mu(t^{-v' x_i}). \end{aligned}$$

This gives

$$\mu^{\mathcal{L}_B(\mu)}(1 \cdot B, \delta(t)) = -\mu(v'(x_i)).$$

This proves the (b)-part and hence the lemma is proved.

Alternatively, we can use [1, Equation 13] to prove the lemma. □

Definition For a quasi-compact scheme Y , an \mathcal{O}_Y -module \mathcal{S} is called coherent if it is finitely presented as an \mathcal{O}_Y -module and any \mathcal{O}_Y -submodule of finite type admits a finite presentation.

An $\mathcal{O}_{\tilde{X}}$ -module \mathcal{S} is called coherent if $\mathcal{S}_{|V^S}$ is a coherent \mathcal{O}_{V^S} -module for any finite ideal $S \subset W$ (where a subset $S \subset W$ is called an ideal if for $x \in S$ and $y \leq x \Rightarrow y \in S$), where V^S is the quasi-compact open subset of \tilde{X} defined by

$$V^S = \bigcup_{w \in S} wU^-B/B.$$

Let $K^0(\tilde{X})$ denote the Grothendieck group of coherent $\mathcal{O}_{\tilde{X}}$ -modules \mathcal{S} .

Similarly, define $K_0(X) := \lim_{n \rightarrow \infty} K_0(X_n)$, where $\{X_n\}_{n \geq 1}$ is the filtration of X giving the ind-projective variety structure (i.e., $X_n = \bigcup_{\ell(w) \leq n} C_w$) and $K_0(X_n)$ is the Grothendieck group of coherent sheaves on the projective variety X_n .

We also define

$$K^{\text{top}}(X) := \text{Inv} \lim_{n \rightarrow \infty} K^{\text{top}}(X_n),$$

where $K^{\text{top}}(X_n)$ is the topological K -group of the projective variety X_n .

Let $*$: $K^{\text{top}}(X_n) \rightarrow K^{\text{top}}(X_n)$ be the involution induced from the operation which takes a vector bundle to its dual. This, of course, induces the involution $*$ on $K^{\text{top}}(X)$.

For any $w \in W$,

$$[\mathcal{O}_{X_w}] \in K_0(X).$$

Lemma 3.4 $\{[\mathcal{O}_{X_w}]\}_{w \in W}$ forms a basis of $K_0(X)$ as a \mathbb{Z} -module.

Proof By [2, §5.2.14 and Theorem 5.4.17], the result follows. □

For $u \in W$, by [8, §2], \mathcal{O}_{X^u} is a coherent $\mathcal{O}_{\bar{X}}$ -module. In particular, $\mathcal{O}_{\bar{X}}$ is a coherent $\mathcal{O}_{\bar{X}}$ -module.

Define a pairing

$$\begin{aligned} \langle \cdot, \cdot \rangle &: K^0(\bar{X}) \otimes K_0(X) \rightarrow \mathbb{Z}, \\ \langle [\mathcal{S}], [\mathcal{F}] \rangle &= \sum_i (-1)^i \chi \left(X_n, \text{Tor}_i^{\mathcal{O}_{\bar{X}}}(\mathcal{S}, \mathcal{F}) \right), \end{aligned}$$

if \mathcal{S} is a coherent sheaf on \bar{X} and \mathcal{F} is a coherent sheaf on X supported in X_n (for some n), where χ denotes the Euler–Poincaré characteristic. Then, as in [14, Lemma 3.4], the above pairing is well defined.

By [8, Proof of Proposition 3.4], for any $u \in W$,

$$\text{Ext}_{\mathcal{O}_{\bar{X}}}^k(\mathcal{O}_{X^u}, \mathcal{O}_{\bar{X}}) = 0 \quad \forall k \neq \ell(u). \tag{6}$$

Define the sheaf

$$\omega_{X^u} := \text{Ext}_{\mathcal{O}_{\bar{X}}}^{\ell(u)}(\mathcal{O}_{X^u}, \mathcal{O}_{\bar{X}}) \otimes \mathcal{L}(-2\rho),$$

which, by the analogy with the Cohen–Macaulay (for short CM) schemes of finite type, will be called the *dualizing sheaf* of X^u .

Now, set the sheaf on \bar{X}

$$\begin{aligned} \xi^u &:= \mathcal{L}(\rho)\omega_{X^u} \\ &= \mathcal{L}(-\rho)\text{Ext}_{\mathcal{O}_{\bar{X}}}^{\ell(u)}(\mathcal{O}_{X^u}, \mathcal{O}_{\bar{X}}). \end{aligned}$$

Then, as proved in [14, Proposition 3.5], for any $u, w \in W$,

$$\langle [\xi^u], [\mathcal{O}_{X_w}] \rangle = \delta_{u,w}. \tag{7}$$

With these preliminaries, we are ready to prove the following result.

Proposition 3.5 *With the notation as in the proof of Theorem 3.2, $Z = (X^-)^s$, if ε^v occurs in $\varepsilon^{u_1} \dots \varepsilon^{u_s}$ with nonzero coefficient.*

Proof We give the proof in the case $s = 2$. The proof for general s is similar.

For $u, v \in W$, express

$$\varepsilon^u \varepsilon^v = \sum_{\substack{w \\ \ell(w)=\ell(u)+\ell(v)}} n_{u,v}^w \varepsilon^w.$$

Express the product in topological K -theory $K^{\text{top}}(X)$ of $X = G^{\text{min}}/B$:

$$\psi_o^u \psi_o^v = \sum_{\ell(w) \geq \ell(u) + \ell(v)} m_{u,v}^w \psi_o^w,$$

where $\psi^w := * \tau^{w^{-1}}$ (τ^w being the Kostant–Kumar ‘basis’ of $K_H^{\text{top}}(X)$ as in [10, Remark 3.14]) and $\{\psi_o^w\}_{w \in W}$ is the corresponding ‘basis’ of $K^{\text{top}}(X) \simeq \mathbb{Z} \otimes_{R(H)} K_H^{\text{top}}(X)$ (cf. [10, Proposition 3.25]).

Then, by [10, Proposition 2.30],

$$n_{u,v}^w = m_{u,v}^w, \quad \text{if } \ell(w) = \ell(u) + \ell(v). \tag{8}$$

Let $\Delta : X \rightarrow X \times X$ be the diagonal map. Then, by [14, Proposition 4.1] and the identity (7), for any $u, v, w \in W, g_1, g_2 \in G^{\text{min}}$,

$$\begin{aligned} m_{u,v}^w &= \langle [\xi^u \boxtimes \xi^v], [\Delta_* \mathcal{O}_{X_w}] \rangle \\ &= \langle [\xi^u \boxtimes \xi^v], [(g_1^{-1}, g_2^{-1}) \cdot (\Delta_* \mathcal{O}_{X_w})] \rangle, \end{aligned}$$

since $[(g_1^{-1}, g_2^{-1}) \cdot \Delta_* \mathcal{O}_{X_w}] = [\Delta_* \mathcal{O}_{X_w}]$ as elements of $K_0(X \times X)$. Thus, G^{min} being connected,

$$\begin{aligned} m_{u,v}^w &= \langle [\xi^u \boxtimes \xi^v], [(g_1^{-1}, g_2^{-1}) \cdot (\Delta_* \mathcal{O}_{X_w})] \rangle \\ &:= \sum_i (-1)^i \chi(\bar{X} \times \bar{X}, \mathcal{T}or_i^{\mathcal{O}_{\bar{X} \times \bar{X}}}(\xi^u \boxtimes \xi^v, (g_1^{-1}, g_2^{-1}) \cdot (\Delta_* \mathcal{O}_{X_w}))). \end{aligned} \tag{9}$$

Now, by definition, the support of ξ^u is contained in X^u and hence the support of the sheaf

$$\mathcal{S}_i := \mathcal{T}or_i^{\mathcal{O}_{\bar{X} \times \bar{X}}}(\xi^u \boxtimes \xi^v, (g_1^{-1}, g_2^{-1}) \cdot \Delta_* \mathcal{O}_{X_w})$$

is contained in

$$X^u \times X^v \cap \left((g_1^{-1}, g_2^{-1}) \cdot \Delta(X_w) \right), \tag{10}$$

which is empty if

$$(g_1 X^u) \cap (g_2 X^v) \cap X_w = \emptyset. \tag{11}$$

Thus, if the equation (11) is true, then the Tor sheaf $\mathcal{S}_i = 0 \forall i \geq 0$. Thus, if the equation (11) is satisfied, by the equation (9),

$$m_{u,v}^w = 0.$$

Now, assume that $\ell(w) = \ell(u) + \ell(v)$. Then, by the equation (8),

$$n_{u,v}^w = 0, \quad \text{if the equation (11) is satisfied.}$$

But, since by assumption, $n_{u,v}^w \neq 0$, we see that

$$(g_1 X^u) \cap (g_2 X^v) \cap X_w \neq \emptyset, \quad \text{for any } g_1, g_2 \in G^{\min}.$$

But since $G^{\min}/(G^{\min} \cap B^-) \xrightarrow{\sim} X^-$, we get the proposition. □

4 Tensor product decomposition for affine Kac–Moody Lie algebras

4.1 The Virasoro algebra

We recall the definition of the Virasoro algebra and its basic representation theory, which we need. For more details, we refer to [5, Lectures 2,3] and also [3]. The Virasoro algebra Vir has a basis $\{C, L_n : n \in \mathbb{Z}\}$ over \mathbb{C} and the Lie bracket is given by

$$[L_m, L_n] = (m - n)L_{m+n} + \frac{1}{12}(m^3 - m)\delta_{m,-n}C \text{ and } [\text{Vir}, C] = 0.$$

Let $\text{Vir}_0 := \mathbb{C}L_0 \oplus \mathbb{C}C$. Then, a Vir module V is said to be a *highest weight representation* if there exists a Vir_0 -eigenvector $v_o \in V$ such that $L_n v_o = 0$ for $n \in \mathbb{Z}_{>0}$ and $U(\bigoplus_{n<0} \mathbb{C}L_n)v_o = V$. Such a V is said to have *highest weight* $\lambda \in \text{Vir}_0^*$ if $X v_o = \lambda(X)v_o$, for all $X \in \text{Vir}_0$. (It is easy to see that such a v_o is unique up to a scalar multiple and hence λ is unique.) The irreducible highest weight representations of Vir are in 1–1 correspondence with elements of Vir_0^* given by the highest weight. Denote the basis of Vir_0^* dual to the basis $\{L_0, C\}$ of Vir_0 as $\{h, z\}$. For any $\mu \in \text{Vir}_0^*$, denote the μ -th weight space of V by V_μ , i.e.,

$$V_\mu := \{v \in V : X \cdot v = \mu(X)v \ \forall X \in \text{Vir}_0\}.$$

Define a Vir module V to be *unitarizable* if there exists a positive definite Hermitian form (\cdot, \cdot) on V so that $(L_n v, w) = (v, L_{-n} w)$ for all $n \in \mathbb{Z}$ and $(Cv, w) = (v, Cw)$. It is easy to see that if M is a Vir -submodule of V , then M^\perp is also a submodule. Hence, any unitarizable representation of Vir is completely reducible. Note that for a unitarizable highest weight Vir -representation V with highest weight λ , if v_o is a highest weight vector, then

$$\begin{aligned} 0 &\leq (L_{-n} v_o, L_{-n} v_o) \\ &= (L_n L_{-n} v_o, v_o) \\ &= (2n\lambda(L_0) + \frac{1}{12}(n^3 - n)\lambda(C))(v_o, v_o) \end{aligned} \tag{12}$$

for all $n > 0$. Therefore, both $\lambda(L_0)$ and $\lambda(C)$ must be nonnegative real numbers.

Lemma 4.1 *Let V be a unitarizable, highest weight (irreducible) representation of Vir with highest weight λ .*

- (a) *If $\lambda(L_0) \neq 0$, then $V_{\lambda+nh} \neq 0$, for any $n \in \mathbb{Z}_+$.*
- (b) *If $\lambda(L_0) = 0$ and $\lambda(C) \neq 0$, then $V_{\lambda+nh} \neq 0$, for any $n \in \mathbb{Z}_{>1}$ and $V_{\lambda+h} = 0$.*
- (c) *If $\lambda(L_0) = \lambda(C) = 0$, then V is one dimensional.*

Proof If $\lambda(L_0) \neq 0$, then by the equation (12) (since both of $\lambda(L_0)$ and $\lambda(C) \in \mathbb{R}_+$), $L_{-n}v_o \neq 0$, for any $n \in \mathbb{Z}_+$.

If $\lambda(L_0) = 0$ and $\lambda(C) \neq 0$, then again by the equation (12), $L_{-n}v_o \neq 0$, for any $n \in \mathbb{Z}_{>1}$. Also, $L_{-1}v_o = 0$.

If $\lambda(L_0) = \lambda(C) = 0$, then [by the equation (12) again], $L_{-n}v_o = 0$, for any $n \in \mathbb{Z}_{\geq 1}$. This shows that V is one dimensional. □

In the following, we recall some basic results (we need in the paper) from the Representation Theory of affine Kac–Moody Lie algebras. For a more exhaustive treatment, we refer to [4, Chapter 12].

4.2 Tensor product decomposition: a general method

Let \mathfrak{g} be the untwisted affine Kac–Moody Lie algebra associated to a finite dimensional simple Lie algebra $\mathring{\mathfrak{g}}$, i.e.,

$$\mathfrak{g} = \left(\mathring{\mathfrak{g}} \otimes \mathbb{C}[t, t^{-1}] \right) \oplus \mathbb{C}c \oplus \mathbb{C}d.$$

Let \mathfrak{h} be a Cartan subalgebra of $\mathring{\mathfrak{g}}$. Then,

$$\mathfrak{h} := \mathring{\mathfrak{h}} \otimes 1 \oplus \mathbb{C}c \oplus \mathbb{C}d$$

is the standard Cartan subalgebra of \mathfrak{g} . Let $\delta \in \mathfrak{h}^*$ be the smallest positive imaginary root of \mathfrak{g} (so that the positive imaginary roots of \mathfrak{g} are precisely $\{n\delta, n \in \mathbb{Z}_{\geq 1}\}$). Then, δ is given by $\delta \circ_{\mathfrak{h} \oplus \mathbb{C}c} \equiv 0$ and $\delta(d) = 1$. For any $\Lambda \in P_+$, let $P(\Lambda)$ be the set of weights of $L(\Lambda)$ and let $P^o(\Lambda)$ be the set of δ -maximal weights of $L(\Lambda)$, i.e.,

$$P^o(\Lambda) = \{ \lambda \in \mathfrak{h}^* : \lambda \in P(\Lambda) \text{ but } \lambda + n\delta \notin P(\Lambda) \text{ for any } n > 0 \}.$$

For any $\lambda \in X(H)$, define the δ -character of $L(\Lambda)$ through λ by

$$c_{\Lambda, \lambda} = \sum_{n \in \mathbb{Z}} \dim L(\Lambda)_{\lambda+n\delta} e^{n\delta}.$$

Since δ is W -invariant,

$$c_{\Lambda, \lambda} = c_{\Lambda, w\lambda}, \quad \text{for any } w \in W. \tag{13}$$

Moreover, $P^o(\Lambda)$ is W -stable. It is obvious that

$$ch L(\Lambda) = \sum_{\lambda \in P^o(\Lambda)} c_{\Lambda, \lambda} e^{\lambda}. \tag{14}$$

By [13, Exercise 13.1.E.8], for any $\lambda \in P(\Lambda')$ and $\Lambda'' \in P_+$, $\Lambda'' + \lambda + \rho$ belongs to the Tits cone, where the Tits cone is defined, e.g., in [13, Definition 1.4.1]. Hence, there exists $v \in W$ such that $v^{-1}(\Lambda'' + \lambda + \rho) \in P_+$. Moreover, if $\Lambda'' + \lambda + \rho$ has nontrivial W -isotropy, then its isotropy group must contain a reflection (cf. [13, Proposition 1.4.2(a)]). Thus, for such a $\lambda \in P(\Lambda')$, i.e., if $\Lambda'' + \lambda + \rho$ has nontrivial W -isotropy,

$$\sum_{w \in W} \varepsilon(w) e^{w(\Lambda'' + \lambda + \rho)} = 0. \tag{15}$$

Define

$$\bar{P}_+ := \{\Lambda \in P_+ : \Lambda(d) = 0\}.$$

For any $m \in \mathbb{Z}_+$, let

$$P_+^{(m)} := \{\Lambda \in P_+ : \Lambda(c) = m\},$$

and let

$$\bar{P}_+^{(m)} := \bar{P}_+ \cap P_+^{(m)}.$$

Then, $\bar{P}_+^{(m)}$ provides a set of representatives in $P_+^{(m)} \bmod (P_+ \cap \mathbb{C}\delta)$.

For any $\Lambda, \Lambda', \Lambda'' \in P_+$, define

$$T_{\Lambda}^{\Lambda', \Lambda''} = \{\lambda \in P^o(\Lambda') : \exists v_{\Lambda, \Lambda'', \lambda} \in W \text{ and } S_{\Lambda, \Lambda'', \lambda} \in \mathbb{Z} \text{ with } \lambda + \Lambda'' + \rho = v_{\Lambda, \Lambda'', \lambda}(\Lambda + \rho) + S_{\Lambda, \Lambda'', \lambda} \delta\}.$$

Observe that since $\Lambda + \rho + n\delta \in P_{++}$ for any $n \in \mathbb{Z}$, such a $v_{\Lambda, \Lambda'', \lambda}$ and $S_{\Lambda, \Lambda'', \lambda}$ are unique by [13, Proposition 1.4.2 (a), (b)] (if they exist). Also, observe that

$$T_{\Lambda}^{\Lambda', \Lambda''} = \emptyset, \text{ unless } \Lambda(c) = \Lambda'(c) + \Lambda''(c) \text{ and } \Lambda' + \Lambda'' - \Lambda \in Q, \tag{16}$$

where Q is the root lattice of \mathfrak{g} .

Proposition 4.2 For any Λ' and $\Lambda'' \in P_+$,

$$ch (L(\Lambda') \otimes L(\Lambda'')) = \sum_{\Lambda \in \bar{P}_+^{(m)}} ch L(\Lambda) \sum_{\lambda \in T_{\Lambda}^{\Lambda', \Lambda''}} \varepsilon(v_{\Lambda, \Lambda'', \lambda}) c_{\Lambda', \lambda} e^{S_{\Lambda, \Lambda'', \lambda} \delta},$$

where $m := \Lambda'(c) + \Lambda''(c)$.

Moreover, $\sum_{\lambda \in T_{\Lambda}^{\Lambda', \Lambda''}} \varepsilon(v_{\Lambda, \Lambda'', \lambda}) c_{\Lambda', \lambda} e^{S_{\Lambda, \Lambda'', \lambda} \delta}$ is the ‘shifted’ character of a unitary representation (though, in general, not irreducible) of the Virasoro algebra Vir with central charge

$$\dim \mathfrak{g} \cdot \left(\frac{m'}{m' + g} + \frac{m''}{m'' + g} - \frac{m}{m + g} \right),$$

where $m' := \Lambda'(c)$, $m'' := \Lambda''(c)$ and g is the dual Coxeter number of \mathfrak{g} .

In fact, the multiplicity space

$$M(\Lambda; \Lambda', \Lambda'') := \text{Hom}_{\mathfrak{g}'} (L(\Lambda') \otimes L(\Lambda''), L(\Lambda))$$

is a representation of the Virasoro algebra (cf. [5, Proposition 10.3]), where \mathfrak{g}' is the derived subalgebra of \mathfrak{g} .

Proof By the Weyl–Kac character formula (cf. [13, Theorem 2.2.1 and Corollary 3.2.10]) and the identity (14), for any $\Lambda', \Lambda'' \in P_+$,

$$\begin{aligned} & \left(\sum_{w \in W} \varepsilon(w) e^{w\rho} \right) \cdot \text{ch } L(\Lambda') \cdot \text{ch } L(\Lambda'') \\ &= \left(\sum_{\lambda \in P^o(\Lambda')} c_{\Lambda', \lambda} e^{\lambda} \right) \cdot \left(\sum_{w \in W} \varepsilon(w) e^{w(\Lambda'' + \rho)} \right) \\ &= \sum_{\lambda \in P^o(\Lambda')} c_{\Lambda', \lambda} \sum_{w \in W} \varepsilon(w) e^{w(\Lambda'' + \lambda + \rho)}, \text{ by (13)} \\ &= \sum_{\Lambda \in \bar{P}_+^{(m)}} \sum_{\lambda \in T_{\Lambda}^{\Lambda', \Lambda''}} c_{\Lambda', \lambda} \sum_{w \in W} \varepsilon(w) e^{w(v_{\Lambda, \Lambda'', \lambda}(\Lambda + \rho)) + S_{\Lambda, \Lambda'', \lambda} \delta}, \text{ by (15)} \\ &= \sum_{\Lambda \in \bar{P}_+^{(m)}} \sum_{\lambda \in T_{\Lambda}^{\Lambda', \Lambda''}} c_{\Lambda', \lambda} \sum_{w \in W} \varepsilon(w) \varepsilon(v_{\Lambda, \Lambda'', \lambda}) e^{w(\Lambda + \rho)} e^{S_{\Lambda, \Lambda'', \lambda} \delta} \\ &= \sum_{\Lambda \in \bar{P}_+^{(m)}} \sum_{w \in W} \varepsilon(w) e^{w(\Lambda + \rho)} \sum_{\lambda \in T_{\Lambda}^{\Lambda', \Lambda''}} \varepsilon(v_{\Lambda, \Lambda'', \lambda}) c_{\Lambda', \lambda} e^{S_{\Lambda, \Lambda'', \lambda} \delta}. \end{aligned}$$

Thus,

$$\text{ch} (L(\Lambda') \otimes L(\Lambda'')) = \sum_{\Lambda \in \bar{P}_+^{(m)}} \text{ch } L(\Lambda) \sum_{\lambda \in T_{\Lambda}^{\Lambda', \Lambda''}} \varepsilon(v_{\Lambda, \Lambda'', \lambda}) c_{\Lambda', \lambda} e^{S_{\Lambda, \Lambda'', \lambda} \delta}.$$

To prove the second part of the proposition, use [5, Proposition 10.3]. This proves the proposition. □

Remark 4.3 For an affine Kac–Moody Lie algebra \mathfrak{g} , if we consider the tensor product decomposition of $L(\Lambda') \otimes L(\Lambda'')$ with respect to the derived subalgebra \mathfrak{g}' (i.e.,

without the d -action), then the components $L(\Lambda)$ are precisely of the form $\Lambda \in (\Lambda' + \Lambda'' + \overset{\circ}{Q}) \cap P_+$, where $\overset{\circ}{Q}$ is the root lattice of $\overset{\circ}{\mathfrak{g}}$ (cf. [6]). Thus, the determination of the tensor semigroup and the saturated tensor semigroup is fairly easy for \mathfrak{g}' .

Let $\theta = \sum_{i=1}^{\ell} h_i \alpha_i$ be the highest root of $\overset{\circ}{\mathfrak{g}}$ (with respect to a choice of the positive roots), written as a linear combination of the simple roots $\{\alpha_1, \dots, \alpha_{\ell}\}$ of $\overset{\circ}{\mathfrak{g}}$. Let

$$S := \left\{ \sum_{i=0}^{\ell} n_i \alpha_i : n_i \in \mathbb{Z}_+ \text{ for any } i \text{ and } 0 \leq n_i < h_i \text{ for some } 0 \leq i \leq \ell \right\},$$

where $h_0 := 1$.

Proposition 4.4 *Let \mathfrak{g} be an untwisted affine Kac–Moody Lie algebra as above. Then, for any integer $\Lambda \in P_+$ with $\Lambda(c) > 0$,*

$$P^o(\Lambda)_+ = S(\Lambda) \cap P_+,$$

where $P^o(\Lambda)_+ := P^o(\Lambda) \cap P_+$ and $S(\Lambda) = \{\Lambda - \beta : \beta \in S\}$.

Proof Take $\lambda \in S(\Lambda)$. Then, for any integer $n \geq 1$,

$$\Lambda - (\lambda + n\delta) = \left(\sum_{i=0}^{\ell} n_i \alpha_i \right) - n\delta = (n_0 - n)\alpha_0 + \sum_{i=1}^{\ell} (n_i - nh_i)\alpha_i,$$

since $\alpha_0 := \delta - \theta$. Now, the coefficient of some α_i in the above sum is negative, for any positive n , since $\lambda \in S(\Lambda)$. Thus, $\lambda + n\delta$ could not be a weight of $L(\Lambda)$ for any positive n . Therefore, if $\lambda \in P(\Lambda) \cap S(\Lambda)$, then it is δ -maximal.

By [4, Proposition 12.5(a)], if $\Lambda(c) \neq 0$, then $S(\Lambda) \cap P_+ \subset P(\Lambda)$. Therefore, $S(\Lambda) \cap P_+ \subset P^o(\Lambda)_+$.

Conversely, take $\lambda \in P^o(\Lambda)_+$. Then, $\lambda \in P(\Lambda) \cap P_+$ and $\lambda + \delta \notin P(\Lambda)$. Express $\lambda = \Lambda - n_0\alpha_0 - \sum_{i=1}^{\ell} n_i \alpha_i$, for some $n_i \in \mathbb{Z}_+$. Then,

$$\lambda + \delta = \Lambda - (n_0 - 1)\alpha_0 - \sum_{i=1}^{\ell} (n_i - h_i)\alpha_i.$$

Again applying [4, Proposition 12.5(a)], $\lambda + \delta \notin P(\Lambda)$ if and only if $\lambda + \delta \not\leq \Lambda$, i.e., for some $0 \leq i \leq \ell$, $n_i < h_i$. Thus, $\lambda \in S(\Lambda)$. This proves the proposition. \square

5 $A_1^{(1)}$ Case

In this section, we consider $\mathfrak{g} = \widehat{\mathfrak{sl}}_2 = \left(\bigoplus_{n \in \mathbb{Z}} \mathfrak{sl}_2 \otimes t^n \right) \oplus \mathbb{C}c \oplus \mathbb{C}d$. In this case $\mathfrak{h}^* = \mathbb{C}\alpha \oplus \mathbb{C}\delta \oplus \mathbb{C}\Lambda_0$, where α is the simple root of \mathfrak{sl}_2 and $\Lambda_0 \Big|_{\mathfrak{h} \oplus \mathbb{C}d} \equiv 0$ and $\Lambda_0(c) = 1$. Then, Λ_0 is a zeroeth fundamental weight. The simple roots of $\widehat{\mathfrak{sl}}_2$ are

$\alpha_0 := \delta - \alpha$ and $\alpha_1 := \alpha$. The simple coroots are $\alpha_0^\vee := c - \alpha^\vee$ and $\alpha_1^\vee := \alpha^\vee$. It is easy to see that an element of \mathfrak{h}^* of the form $m\Lambda_0 + \frac{j}{2}\alpha$ belongs to P_+ if and only if $m, j \in \mathbb{Z}_+$ and $m \geq j$.

Specializing Proposition 4.4 to the case of $\mathfrak{g} = \widehat{\mathfrak{sl}}_2$, we get the following.

Corollary 5.1 For $\mathfrak{g} = \widehat{\mathfrak{sl}}_2$ and $\Lambda = m\Lambda_0 + \frac{j}{2}\alpha \in P_+$,

$$P^o(\Lambda)_+ = \left\{ \Lambda - k\alpha, \Lambda - l(\delta - \alpha) \mid \begin{array}{l} k, l \in \mathbb{Z}_+ \text{ and} \\ k \leq \frac{j}{2}, l \leq \frac{m-j}{2} \end{array} \right\}. \tag{17}$$

Proof The corollary follows from Proposition 4.4 since $m_1\Lambda_0 + \frac{m_2}{2}\alpha + m_3\delta$ belongs to P_+ if and only if $m_1, m_2 \in \mathbb{Z}_+$ and $m_1 \geq m_2$. \square

Let π be the projection $\mathfrak{h}^* = \mathbb{C}\Lambda_0 \oplus \mathbb{C}\alpha \oplus \mathbb{C}\delta \rightarrow \mathbb{C}\Lambda_0 \oplus \mathbb{C}\alpha$.

Lemma 5.2 Let $\mathfrak{g} = \widehat{\mathfrak{sl}}_2$. For $\Lambda = m\Lambda_0 + \frac{j}{2}\alpha \in P_+$ (i.e., $m, j \in \mathbb{Z}_+$ and $m \geq j$) such that $m > 0$,

$$\pi(P^o(\Lambda)) = \{ \Lambda + k\alpha : k \in \mathbb{Z} \}. \tag{18}$$

Moreover, for any $k \in \mathbb{Z}$, let n_k be the unique integer such that $\Lambda + k\alpha + n_k\delta \in P^o(\Lambda)$. Then, writing $k = qm + r, 0 \leq r < m$, we have:

$$n_k = n_r - q(k + r + j). \tag{19}$$

Proof The assertion (18) follows from the identity (17) together with the action of the affine Weyl group $W \simeq \overset{\circ}{W} \times (\mathbb{Z}\alpha^\vee)$ on \mathfrak{h}^* , where $\overset{\circ}{W}$ is the Weyl group of \mathfrak{sl}_2 and $\mathbb{Z}\alpha^\vee$ acts on \mathfrak{h}^* via:

$$T_{n\alpha^\vee}(\mu) = \mu + n\mu(c)\alpha - [n\mu(\alpha^\vee) + n^2\mu(c)]\delta, \text{ for } n \in \mathbb{Z}, \mu \in \mathfrak{h}^*. \tag{20}$$

Since $P^o(\Lambda)$ is W -stable, the identity (19) can be established from the action of the affine Weyl group element $T_{-q\alpha^\vee}$ on $\Lambda + k\alpha + n_k\delta$. \square

The value of n_r for $0 \leq r < m$ can be determined from the identity (17) by applying $T_{\alpha^\vee}, T_{\alpha^\vee} \cdot s_1$ to $\Lambda - k\alpha$ and applying $1, T_{\alpha^\vee} \cdot s_1$ to $\Lambda - l(\delta - \alpha)$, where s_1 is the nontrivial element of $\overset{\circ}{W}$. We record the result in the following lemma.

Lemma 5.3 With the notation as in the above lemma, the value of n_r for any integer $0 \leq r < m$ is given by

$$n_r = \begin{cases} -r, & \text{for } 0 \leq r \leq m - j \\ m - j - 2r & \text{for } m - j \leq r < m. \end{cases}$$

Lemma 5.4 *Take the following elements in P_+ :*

$$\Lambda = m\Lambda_0 + \frac{j}{2}\alpha, \quad \Lambda' = m'\Lambda_0 + \frac{j'}{2}\alpha, \quad \Lambda'' = m''\Lambda_0 + \frac{j''}{2}\alpha,$$

where $m := m' + m''$ and we assume that $m' > 0$. Then,

$$\pi \left(T_{\Lambda}^{\Lambda', \Lambda''} \right) = \left\{ \Lambda' + k\alpha \mid k \in \mathbb{Z}, \begin{array}{l} k \equiv \frac{1}{2}(j - j' - j'') \pmod{M} \text{ or} \\ k \equiv -\frac{1}{2}(j + j' + j'') - 1 \pmod{M} \end{array} \right\},$$

where $M := m + 2$. In particular, by the equation (16), $T_{\Lambda}^{\Lambda', \Lambda''}$ is nonempty if and only if $\frac{j-j'-j''}{2} \in \mathbb{Z}$.

Moreover, for $\lambda = \Lambda' + k\alpha + n_k\delta \in T_{\Lambda}^{\Lambda', \Lambda''}$,

$$v_{\Lambda, \Lambda'', \lambda} = \begin{cases} T_{\frac{k-\frac{1}{2}(j-j'-j'')}{M}\alpha^\vee}, & \text{if } k \equiv \frac{1}{2}(j - j' - j'') \pmod{M} \\ s_1 T_{-\frac{k+\frac{1}{2}(j+j'+j'')+1}{M}\alpha^\vee}, & \text{if } k \equiv -\frac{1}{2}(j + j' + j'') - 1 \pmod{M}, \end{cases}$$

where $T_{n\alpha^\vee}$ is defined by the equation (20). Further,

$$S_{\Lambda, \Lambda'', \lambda} = n_k + \frac{(k - \frac{1}{2}(j - j' - j''))(k + \frac{1}{2}(j + j' + j'') + 1)}{M}.$$

Proof Follows from the fact that $W = \overset{\circ}{W} \rtimes \mathbb{Z}\alpha^\vee$ and that $\rho = 2\Lambda_0 + \frac{1}{2}\alpha$. □

We have the following very crucial result.

Proposition 5.5 *Fix Λ, Λ' and Λ'' as in Lemma 5.4 and assume that $\frac{j-j'-j''}{2} \in \mathbb{Z}$ and both of $m', m'' > 0$. Then, the maximum of*

$$\left\{ S_{\Lambda, \Lambda'', \lambda} : \lambda \in T_{\Lambda}^{\Lambda', \Lambda''} \text{ and } \varepsilon(v_{\Lambda, \Lambda'', \lambda}) = 1 \right\}$$

is achieved precisely when $\pi(\lambda) = \Lambda' + \frac{1}{2}(j - j' - j'')\alpha$.

Proof By Lemma 5.4 and the explicit description of the length function of $T_{n\alpha^\vee}$ (cf. [13, Exercise 13.1.E.3]),

$$\pi \{ \lambda \in T_{\Lambda}^{\Lambda', \Lambda''} : \varepsilon(v_{\Lambda, \Lambda'', \lambda}) = 1 \} = \{ \Lambda' + k_l\alpha : l \in \mathbb{Z} \},$$

where $M := m + 2$ and $k_l := \frac{j-j'-j''}{2} + lM$. Take $\lambda = \Lambda' + k_l\alpha \in \pi \left(T_{\Lambda}^{\Lambda', \Lambda''} \right)$ for $l \in \mathbb{Z}$. Write $k_l = q_l m' + r_l$ for $q_l \in \mathbb{Z}$ and $0 \leq r_l < m'$. Then, by Lemmas 5.2, 5.3

and 5.4, for $\lambda = \Lambda' + k_l\alpha$ (setting $J := \frac{j-j'-j''}{2}$),

$$\begin{aligned} S_{\Lambda, \Lambda'', \lambda} &= n_{r_l} - \frac{(J + j' + lM + r_l)(J + lM - r_l)}{m'} + l(lM + 1 + j) \\ &= l^2M \left(1 - \frac{M}{m'}\right) + l \left(1 + j - \frac{M(j - j'')}{m'}\right) - \frac{(j - j'')^2 - j'^2}{4m'} \\ &\quad + \frac{r_l^2}{m'} + \frac{r_l j'}{m'} + n_{r_l} \\ &= l^2M \left(1 - \frac{M}{m'}\right) + l \left(1 + j - \frac{M}{m'}(j - j'')\right) - \frac{(j - j'')^2 - j'^2}{4m'} \\ &\quad + p(k_l), \end{aligned}$$

where

$$p(k_l) := \frac{r_l^2}{m'} + \frac{r_l}{m'}j' + n_{k_l}.$$

Let $P = P_{m', j'} : \mathbb{R} \rightarrow \mathbb{R}$ be the following function:

$$P(s) := \begin{cases} \frac{(s - \frac{m'}{2}k)^2}{m'} - \frac{(j')^2}{4m'}, & \text{if } \left|s - \frac{m'}{2}k\right| \leq \frac{j'}{2} \text{ for some } k \in 2\mathbb{Z} \\ \frac{(s - \frac{m'}{2}k)^2}{m'} - \frac{(m' - j')^2}{4m'}, & \text{if } \left|s - \frac{m'}{2}k\right| \leq \frac{m' - j'}{2} \text{ for some } k \in 2\mathbb{Z} + 1. \end{cases}$$

Let $k_s \in \mathbb{Z}$ be such a k . (Of course, k_s depends upon m' and j' .)

Claim $P(s) = p\left(s - \frac{j'}{2}\right)$ for $s \in \frac{j'}{2} + \mathbb{Z}$.

Proof Clearly, both of P and p are periodic with period m' . So, it is enough to show that $P(s) = p\left(s - \frac{j'}{2}\right)$, for $s - \frac{j'}{2}$ equal to any of the integral points of the interval $[-j', m' - j']$. By Lemma 5.3 and the identity (19), for any integer $-j' \leq r \leq 0$,

$$p(r) = \frac{1}{m'}r(r + j'),$$

and for any integer $0 \leq r \leq m' - j'$,

$$p(r) = \frac{r(r + j')}{m'} - r.$$

From this, the claim follows immediately. □

Fix $m' > 0$. Let

$$I := \left\{ (t, j', m'', j'', j) \in \mathbb{R}^5 \mid \begin{array}{l} 0 \leq j' \leq m', 1 \leq m'', \\ 0 \leq j'' \leq m'', 0 \leq j \leq m' + m'' \end{array} \right\}.$$

Define $F : I \rightarrow \mathbb{R}$ by

$$F : (t, j', m'', j'') \mapsto t^2 M \left(1 - \frac{M}{m'} \right) + t \left[j \left(1 - \frac{M}{m'} \right) + 1 + \frac{M}{m'} j'' \right] + \frac{(j')^2 - (j - j'')^2}{4m'} + P \left(\frac{1}{2} (j - j'') + tM \right).$$

Thus, F is a continuous, piecewise smooth function with failure of differentiability along the set

$$\left\{ (t, j', m'', j'') \in I : \frac{1}{2}(j \pm j' - j'') + tM \in m'\mathbb{Z} \right\}.$$

Claim Let $\Delta(t) = \Delta(t, j', m'', j'') := F(t+1, j', m'', j'') - F(t, j', m'', j'')$. Then, on I ,

- a. Δ is a nonincreasing function of t
- b. Δ is increasing with respect to j''
- c. Δ is nonincreasing in j
- d. $\Delta(0)$ is decreasing in m''
- e. $\Delta(-1)$ is nondecreasing in m'' .

Proof We compute and give bounds for the partial derivatives of Δ , where they exist.

$$\begin{aligned} \Delta(t) &= 2tM \left(1 - \frac{M}{m'} \right) + \left((j + M) \left(1 - \frac{M}{m'} \right) + 1 + \frac{M}{m'} j'' \right) \\ &\quad + P \left(tM + M + \frac{1}{2}(j - j'') \right) - P \left(tM + \frac{1}{2}(j - j'') \right). \end{aligned}$$

Hence,

$$\begin{aligned} \partial_t \Delta(t) &= 2M \left(1 - \frac{M}{m'} \right) + M \cdot P' \left(tM + M + \frac{1}{2}(j - j'') \right) \\ &\quad - M \cdot P' \left(tM + \frac{1}{2}(j - j'') \right) \\ &= 2M \left(1 - \frac{M}{m'} \right) + 2 \frac{M}{m'} \left(M - \frac{m'}{2} k_1 + \frac{m'}{2} k_0 \right) \\ &= 2M \left(1 - \frac{k_1 - k_0}{2} \right), \end{aligned}$$

where $k_1 := k_{(t+1)M + \frac{1}{2}(j - j'')}$ and $k_0 := k_{tM + \frac{1}{2}(j - j'')}$. Since $2 \leq k_1 - k_0$, we see that $\partial_t \Delta \leq 0$, wherever $\partial_t \Delta$ exists. Since Δ is continuous everywhere and differentiable on all but a discrete set, Δ is nonincreasing in t .

$$\partial_{j''} \Delta(t) = \frac{M}{m'} - \frac{1}{2} \left[P' \left(tM + M + \frac{1}{2}(j - j'') \right) - P' \left(tM + \frac{1}{2}(j - j'') \right) \right].$$

Now, $|P'| \leq 1$, so $\frac{M}{m'} + 1 \geq \partial_{j''} \Delta \geq \frac{M}{m'} - 1 = \frac{m''+2}{m'} > 0$.

For (c):

$$\begin{aligned} \partial_j \Delta(t) &= 1 - \frac{M}{m'} + \frac{1}{2} \left[P' \left(tM + M + \frac{1}{2}(j - j'') \right) - P' \left(tM + \frac{1}{2}(j - j'') \right) \right] \\ &= 1 - \frac{M}{m'} + \frac{1}{m'} \left(M - \frac{m'}{2}k_1 + \frac{m'}{2}k_0 \right) \\ &= 1 - \frac{k_1 - k_0}{2} \leq 0. \end{aligned}$$

(d) and (e) follow from the following calculation:

$$\begin{aligned} \partial_{m''} \Delta &= 2t \left(1 - 2\frac{M}{m'} \right) + \left(1 - 2\frac{M}{m'} + \frac{1}{m'}(j'' - j) \right) \\ &\quad + (t + 1)P' \left(tM + M + \frac{1}{2}(j - j'') \right) \\ &\quad - tP' \left(tM + \frac{1}{2}(j - j'') \right). \end{aligned}$$

Hence,

$$\begin{aligned} \partial_{m''} \Delta(0) &= 1 - 2\frac{M}{m'} + \frac{1}{m'}(j'' - j) + P' \left(M + \frac{1}{2}(j - j'') \right) \\ &\leq 1 - 2\frac{M}{m'} + \frac{m''}{m'} + 1 \\ &= \frac{-m'' - 4}{m'} < 0, \end{aligned}$$

and

$$\begin{aligned} \partial_{m''} \Delta(-1) &= -2 \left(1 - 2\frac{M}{m'} \right) + \left(1 - 2\frac{M}{m'} + \frac{1}{m'}(j'' - j) \right) \\ &\quad + P' \left(-M + \frac{1}{2}(j - j'') \right) \\ &= -1 + 2\frac{M}{m'} + \frac{1}{m'}(j'' - j) + P' \left(-M + \frac{1}{2}(j - j'') \right) \\ &= -1 + 2\frac{M}{m'} + \frac{1}{m'}(j'' - j) - 2\frac{M}{m'} + \frac{1}{m'}(j - j'') - k_0 \\ &= -1 - k_0. \end{aligned}$$

Note that $k_0 \leq -1$ since $-\frac{(j-j'')}{2} - M < -\frac{m'}{2}$. Thus, $\partial_{m''} \Delta(-1) \geq 0$. □

Claim The maximum of $F = F(-, j', m'', j'', j) : \mathbb{Z} \rightarrow \mathbb{R}$ occurs at 0.

Proof We show that $\Delta(-1) > 0 > \Delta(0)$. Since Δ is nonincreasing in t , it would follow that $F(0) > F(t)$ for all $t \in \mathbb{Z}_{\neq 0}$.

Let us begin with $\Delta(-1)$. By the previous Claim e, $\Delta(-1)$ is as small as possible when $m'' = 1, j'' = 0$, and $j = m' + 1$. So, let us compute with these values:

$$\begin{aligned} \Delta(-1) &\geq \frac{6}{m'} + 1 + P\left(\frac{1}{2}m' + \frac{1}{2}\right) - P\left(-2 - \frac{1}{2}m' - \frac{1}{2}\right) \\ &= \frac{6}{m'} + 1 + \frac{\left(\frac{1}{2}m' + \frac{1}{2} - \frac{1}{2}m'k_1\right)^2}{m'} - \frac{\left(2 + \frac{1}{2}m' + \frac{1}{2} + \frac{1}{2}m'k_0\right)^2}{m'} \\ &\quad + \begin{cases} \frac{m'}{4} - \frac{j'}{2} & \text{if } k_0 \text{ odd, } k_1 \text{ even} \\ 0 & \text{if } k_1 - k_0 \text{ even} \\ \frac{j'}{2} - \frac{m'}{4} & \text{if } k_1 \text{ odd, } k_0 \text{ even.} \end{cases} \end{aligned}$$

Note that for $m' \geq 5$, the possible values of (k_1, k_0) are $(1, -1); (1, -2);$ or $(2, -2)$. So, the result, that $\Delta(-1) > 0$, is established by considering such pairs directly and by cases for smaller m' .

For $\Delta(0)$, we take $m'' = 1, j'' = 1$, and $j = 0$.

$$\begin{aligned} \Delta(0) &= \left(\frac{-3(3+m')}{m'} + 1 + \frac{3+m'}{m'}\right) + P\left(\frac{1}{2} + 2 + m'\right) - P\left(-\frac{1}{2}\right) \\ &= 1 - \frac{2(3+m')}{m'} + P\left(\frac{1}{2} + 2 + m'\right) - P\left(-\frac{1}{2}\right) \\ &= 1 - \frac{2(3+m')}{m'} + \frac{\left(\frac{1}{2} + 2 + m' - \frac{1}{2}m'k_1\right)^2}{m'} - \frac{\left(\frac{1}{2} + \frac{1}{2}m'k_0\right)^2}{m'} \\ &\quad + \begin{cases} \frac{m'}{4} - \frac{j'}{2} & \text{if } k_0 \text{ odd, } k_1 \text{ even} \\ 0 & \text{if } k_1 - k_0 \text{ even} \\ \frac{j'}{2} - \frac{m'}{4} & \text{if } k_1 \text{ odd, } k_0 \text{ even.} \end{cases} \end{aligned}$$

For $m' \geq 5$, the possible values of (k_1, k_0) are $(3, -1); (3, 0);$ or $(2, 0)$. So, again the result, that $\Delta(0) < 0$, is established by considering such pairs directly and by cases for smaller m' .

This completes the proof of the proposition. □

Remark 5.6 We have shown that $F(l, j', m'', j'', j) = S_{\Lambda, \Lambda'', \lambda}$ for integral values of l . If l is not an integer, then $\lambda_l := \Lambda' + (lM + J)\alpha$ may not be in $\pi(T_{\Lambda}^{\Lambda', \Lambda''})$, in which case $S_{\Lambda, \Lambda'', \lambda_l}$ is not defined. On the other hand, if $\lambda_l \in \pi(T_{\Lambda}^{\Lambda', \Lambda''})$, we note that the equality $F(l, j', m'', j'', j) = S_{\Lambda, \Lambda'', \lambda_l}$ holds, as can be seen by letting $k_l = lM - \frac{1}{2}(j + j' + j'') - 1$ in the above proof.

Now, let us apply the same analysis to the case that $\varepsilon(v_{\Lambda, \Lambda'', \lambda}) = -1$. By Lemma 5.4, this corresponds to $k_l = -\frac{1}{2}(j + j' + j'') - 1 + lM$. For $\pi(\lambda) = \Lambda' + k_l\alpha$, let us denote the function $S_{\Lambda, \Lambda'', \lambda}$ by $G_{\mathbb{Z}}(l) = G_{\mathbb{Z}}(l, j', m'', j'', j)$. Thus, $G_{\mathbb{Z}} : \mathbb{Z} \rightarrow \mathbb{Z}$.

Lemma 5.7 Define the function $G = G(-, j', m'', j'', j) : \mathbb{R} \rightarrow \mathbb{R}$ by

$$G(t, j', m'', j'', j) = F\left(t - \frac{j+1}{M}, j', m'', j'', j\right).$$

Then, $G_{|\mathbb{Z}} = G_{\mathbb{Z}}$.

Hence, $S_{\Lambda, \Lambda'', \lambda}$ has a maximum when $l = 0$ or $l = 1$.

Proof By the proof of Proposition 5.5 and Remark 5.6, $S_{\Lambda, \Lambda'', \lambda + (j+1)\alpha} = F(l)$, for $\lambda = \Lambda' + k\alpha$. Since $\lambda = \Lambda' + (-\frac{1}{2}(j + j' + j'') - 1 + lM)\alpha$, by Proposition 5.5, $S_{\Lambda, \Lambda'', \lambda} = F\left(l - \frac{j+1}{M}\right)$. This proves the lemma. \square

Lemma 5.8 Suppose

$$\lambda_1 := \Lambda' - \left(\frac{1}{2}(j + j' + j'') + 1\right)\alpha + n_1\delta$$

and

$$\lambda_2 := \Lambda' + \frac{1}{2}(j - j' - j'')\alpha + n_2\delta$$

are δ -maximal weights of $L(\Lambda')$. Then $n_1 = n_2$ is possible only if $n_1 = n_2 = 0$ and in this case

$$(*) \quad \frac{1}{2}(j + j'') + 1 \leq \frac{j'}{2}.$$

Conversely, $\Lambda' - \ell\alpha \in P^o(\Lambda')$ for any $0 \leq \ell \leq j'$. In particular, under the condition $(*)$, $\lambda_1, \lambda_2 \in P^o(\Lambda')$ with $n_1 = n_2 = 0$.

Proof Fix an integer n and consider the set $P_n = \{v \in P(\Lambda') : \Lambda' - v = k\alpha + n\delta, k \in \mathbb{Z}\}$. We give a description of $P_n \cap P^o(\Lambda')$. Clearly, $P_n = \{\lambda, \lambda - \alpha, \dots, \lambda - \langle \lambda, \alpha^\vee \rangle \alpha\}$ for some $\lambda = \lambda_n$ and that this λ is uniquely determined by n (cf. [13, Exercise 2.3.E.2]). Suppose that some $\mu \in P_n$ is not δ -maximal, then none of $\{\mu, \dots, \mu - \langle \mu, \alpha^\vee \rangle \alpha\}$ are δ -maximal, since if $\mu + k\delta \in P(\Lambda')$, then the whole string $\{\mu + k\delta, \dots, \mu + k\delta - \langle \mu, \alpha^\vee \rangle \alpha\} \subset P(\Lambda')$. In particular, if $\lambda - \alpha$ is δ -maximal, then so is λ . Hence, $\mathfrak{g}_{\delta-\alpha}L(\Lambda')_\lambda = 0$ and $\mathfrak{g}_\alpha L(\Lambda')_\lambda = 0$. Therefore, λ is the highest weight Λ' . Thus, $P_n \cap P^o(\Lambda')$ is either empty, or $\lambda = \Lambda'$ (in the case that $n = 0$), or the set $\{\lambda, s_1\lambda\}$.

The last case, i.e., $P_n \cap P^o(\Lambda') = \{\lambda, s_1\lambda\}$ (for $n = -n_1 = -n_2$) is not possible under the assumption of the lemma, since this would force $j'' = -1$ or $j = -1$, which is a contradiction since $j, j'' \in \mathbb{Z}_+$. Thus, we get $n_1 = n_2 = 0$ and $\{\lambda_1, \lambda_2\} \subset P_0$. From this it is easy to see that $(*)$ is satisfied. By Corollary 5.1, $\Lambda' - k\alpha \in P^o(\Lambda')$, for any $0 \leq k \leq \frac{j'}{2}$. Applying s_1 to this, we get the converse statement. \square

From Lemma 5.7 and the definition of F , it is easy to see that

$$G(t, j', m'', j'') = G(1 - t, m' - j', m'', m'' - j'', m' + m'' - j) + \frac{1}{2}(j' + j'' - j), \tag{21}$$

for any $t \in \mathbb{R}$. Hence, if the maximum of $G_{\mathbb{Z}}$ occurs at 1, it is equal to

$$G(0, m' - j', m'', m'' - j'', m' + m'' - j) + \frac{1}{2}(j' + j'' - j). \tag{22}$$

We also record the following identity, which is easy to prove from the definition of F .

$$F(0, j', m'', j'') = F(0, m' - j', m'', m'' - j'', m' + m'' - j) + \frac{1}{2}(j' + j'' - j). \tag{23}$$

As a corollary of Proposition 5.5 and Lemmas 5.7 and 5.8, we get the following ‘Non-Cancellation Lemma’.

Corollary 5.9 *Let $\Lambda, \Lambda', \Lambda''$ be as in Proposition 5.5 and let*

$$\begin{aligned} \mu_{\Lambda}^{\Lambda', \Lambda''} &:= \max \left\{ S_{\Lambda, \Lambda'', \lambda} : \lambda \in T_{\Lambda}^{\Lambda', \Lambda''} \text{ and } \varepsilon(v_{\Lambda, \Lambda'', \lambda}) = 1 \right\}, \\ \bar{\mu}_{\Lambda}^{\Lambda', \Lambda''} &:= \max \left\{ S_{\Lambda, \Lambda'', \lambda} : \lambda \in T_{\Lambda}^{\Lambda', \Lambda''} \text{ and } \varepsilon(v_{\Lambda, \Lambda'', \lambda}) = -1 \right\}. \end{aligned}$$

Assume that $\mu_{\Lambda}^{\Lambda', \Lambda''} = \bar{\mu}_{\Lambda}^{\Lambda', \Lambda''}$. Then,

$$\mu_{\Lambda}^{\Lambda'', \Lambda'} \neq \bar{\mu}_{\Lambda}^{\Lambda'', \Lambda'}.$$

Proof We proceed in two cases:

Case I. Suppose the maximum $\bar{\mu}_{\Lambda}^{\Lambda', \Lambda''}$ occurs when $\pi(\lambda) = \Lambda' - (\frac{1}{2}(j + j' + j'') + 1)\alpha$ (cf. Lemma 5.7). This means that the δ -maximal weights of $L(\Lambda')$ through $\Lambda' - (\frac{1}{2}(j + j' + j'') + 1)\alpha$ and through $\Lambda' + \frac{1}{2}(j - j' - j'')\alpha$ have the same δ coordinate (cf. Proposition 5.5 and Lemma 5.4). By Lemma 5.8, we know that this occurs if and only if the following condition is satisfied:

$$(*) \quad \frac{1}{2}(j + j'') + 1 \leq \frac{j'}{2}.$$

(In this case, these two δ -maximal weights have their δ -coordinates equal to 0.)

So, for the equality $\mu_{\Lambda}^{\Lambda', \Lambda''} = \bar{\mu}_{\Lambda}^{\Lambda', \Lambda''}$ in this case, the necessary and sufficient condition is:

$$\frac{1}{2}(j + j'') + 1 \leq \frac{j'}{2}. \tag{24}$$

Case II. Suppose the maximum $\bar{\mu}_\Lambda^{\Lambda', \Lambda''}$ occurs when $\pi(\lambda) = \Lambda' - (\frac{1}{2}(j + j' + j'') + 1 - M)\alpha$ (cf. Lemma 5.7). Then, by the identities (22) and (23), we get

$$\begin{aligned} G(0, m' - j', m'', m'' - j'', m' + m'' - j) \\ = F(0, m' - j', m'', m'' - j'', m' + m'' - j). \end{aligned} \tag{25}$$

So, from the case I, we get in this case II, $\mu_\Lambda^{\Lambda', \Lambda''} = \bar{\mu}_\Lambda^{\Lambda', \Lambda''}$ if and only if

$$\frac{1}{2}((m' + m'' - j) + (m'' - j'')) + 1 \leq \frac{1}{2}(m' - j'). \tag{26}$$

So, if either of the inequalities (24) or (26) is satisfied, then none of them can be satisfied for the triple $(\Lambda, \Lambda', \Lambda'')$ replaced by $(\Lambda, \Lambda'', \Lambda')$. This proves the corollary. □

6 Saturation factor for the $A_1^{(1)}$ case

We assume that $\mathfrak{g} = \widehat{\mathfrak{sl}}_2$ in this section.

Definition Let $\Lambda' \in P_+^{(m')}$, $\Lambda'' \in P_+^{(m'')}$ and $\Lambda \in P_+^{(m'+m'')}$. Then, we call $L(\Lambda + n\delta)$ the δ -maximal component of $L(\Lambda') \otimes L(\Lambda'')$ through Λ if $L(\Lambda + n\delta)$ is a submodule of $L(\Lambda') \otimes L(\Lambda'')$ but $L(\Lambda + m\delta)$ is not a submodule for any $m > n$.

Theorem 6.1 Let $\Lambda', \Lambda'', \Lambda$ be as in Proposition 5.5. Then, $L(\Lambda + n\delta)$ is a δ -maximal component of $L(\Lambda') \otimes L(\Lambda'')$ if $n = \min(n_1, n_2)$, where n_1 is such that $\Lambda - \Lambda'' + n_1\delta \in P^o(\Lambda')$ and n_2 is such that $\Lambda - \Lambda' + n_2\delta \in P^o(\Lambda'')$. (Observe that, by Lemma 5.2, n_1 and n_2 do exist.)

Proof This follows immediately by combining Propositions 4.2, 5.5, Corollary 5.9 and Lemma 5.4. □

Lemma 6.2 Fix a positive integer N . Let $\Lambda \in \bar{P}_+$ and let $\lambda \in \Lambda + Q$, where Q is the root lattice $\mathbb{Z}\alpha \oplus \mathbb{Z}\delta$ of $\widehat{\mathfrak{sl}}_2$. Then, $N\lambda \in P^o(N\Lambda)$ if and only if $\lambda \in P^o(\Lambda)$.

Proof The validity of the lemma is clear for $\lambda \in P^o(\Lambda)_+$ from Corollary 5.1. But since $P^o(\Lambda) = W \cdot (P^o(\Lambda)_+)$, and the action of W on \mathfrak{h}^* is linear, the lemma follows for any $\lambda \in P^o(\Lambda)$. (Observe that $P(\Lambda) \subset$ Tits cone by [13, Exercise 13.1.E.8].) □

Definition A positive integer d_o is called a saturation factor for \mathfrak{g} if for any $\Lambda, \Lambda', \Lambda'' \in P_+$ such that $\Lambda - \Lambda' - \Lambda'' \in Q$ and $L(N\Lambda)$ is a submodule of $L(N\Lambda') \otimes L(N\Lambda'')$, for some $N \in \mathbb{Z}_{>0}$, then $L(d_o\Lambda)$ is a submodule of $L(d_o\Lambda') \otimes L(d_o\Lambda'')$.

Corollary 6.3 Any $d_o \in \mathbb{Z}_{>1}$ is a saturation factor for $\widehat{\mathfrak{sl}}_2$.

Proof If $\Lambda'(c) = 0$ or $\Lambda''(c) = 0$, then

$$L(N\Lambda') \otimes L(N\Lambda'') \simeq L(N(\Lambda' + \Lambda'')),$$

for any $N \geq 1$. Thus, the corollary is clearly true in this case. So, let us assume that both of $A'(c) > 0$ and $A''(c) > 0$. Let $L(N\Lambda + n\delta)$ be the δ -maximal component of $L(N\Lambda') \otimes L(N\Lambda'')$ through $L(N\Lambda)$, for some $n \geq 0$. For any $\Psi \in P_+$, let $\bar{\Psi} \in \bar{P}_+$ be the projection $\pi(\Psi)$ defined just before Lemma 5.2. Applying Theorem 6.1 to $\bar{\Lambda}', \bar{\Lambda}'', \bar{\Lambda}$, and observing that

$$L(\bar{\Psi} + k\delta) \simeq L(\bar{\Psi}) \otimes L(k\delta) \tag{27}$$

and $L(k\delta)$ is one dimensional, we get that there is a δ -maximal component $L(\Lambda + \tilde{n}\delta)$ of $L(\Lambda') \otimes L(\Lambda'')$ through $L(\Lambda)$, for some (unique) $\tilde{n} \in \mathbb{Z}$.

Again applying Theorem 6.1 to $N\bar{\Lambda}', N\bar{\Lambda}'', N\bar{\Lambda}$, and observing (using Lemma 6.2) that

$$P^o(N\bar{\Psi}) \supset NP^o(\bar{\Psi}), \tag{28}$$

we get that $L(N\Lambda + N\tilde{n}\delta)$ is the δ -maximal component of $L(N\Lambda') \otimes L(N\Lambda'')$ through $L(N\Lambda)$. Thus, $n = N\tilde{n}$. In particular,

$$\tilde{n} \geq 0. \tag{29}$$

Let

$$\sum_{\lambda \in T_{\bar{\Lambda}}^{\Lambda', \Lambda''}} \varepsilon(v_{\bar{\Lambda}, \Lambda'', \lambda}) c_{\Lambda', \lambda} e^{S_{\bar{\Lambda}, \Lambda'', \lambda} \delta} = \sum_{k \in \mathbb{Z}_+} c_k e^{(\Lambda(d) + \tilde{n} - k)\delta}, \tag{30}$$

for some $c_k \in \mathbb{Z}_+$ with c_0 nonzero. By Proposition 4.2, this is the character of a unitarizable Virasoro representation with each irreducible component having the same nonzero central charge. Thus, by Lemma 4.1, for any $k > 1$, we get $c_k \neq 0$. (Here we have used that $L_0 = -d + p$ on any \mathfrak{g} -isotypical component of $L(\Lambda') \otimes L(\Lambda'')$ with highest weight in $\Lambda + \mathbb{Z}\delta$, for a number p depending only upon $\bar{\Lambda}, \Lambda'$ and Λ'' , cf. [5, Identity 10.25 in Proposition 10.3].)

By the above, $L(d_o\Lambda + d_o\tilde{n}\delta)$ is the δ -maximal component of $L(d_o\Lambda') \otimes L(d_o\Lambda'')$ through $L(d_o\Lambda)$. If $\tilde{n} = 0$, we get that

$$L(d_o\Lambda) \subset L(d_o\Lambda') \otimes L(d_o\Lambda'').$$

If $\tilde{n} > 0$, then $d_o\tilde{n}$ being > 1 , by the analogue of (30) for $d_o\Lambda', d_o\Lambda''$ and $d_o\Lambda$, $L(d_o\Lambda) \subset L(d_o\Lambda') \otimes L(d_o\Lambda'')$. This proves the corollary. □

Remark 6.4 We note that $L(2\Lambda_0 - \delta)$ is not a component of $L(\Lambda_0) \otimes L(\Lambda_0)$ (cf. [4, Exercise 12.16]). But, of course, $L(2\Lambda_0)$ is a δ -maximal component. By the identity (30), we know that $L(2d_o\Lambda_0 - d_o\delta)$ must be a component of $L(d_o\Lambda_0) \otimes L(d_o\Lambda_0)$, for any $d_o > 1$. So d_o can not be taken to be 1 in Corollary 6.3.

7 A conjecture

In this section, G is any symmetrizable Kac–Moody group. We recall the following definition of the deformed product due to Belkale–Kumar [1]. (Even though they gave the definition in the finite case, the same definition works in the symmetrizable Kac–Moody case, though with only one parameter.)

Definition *Let P be any standard parabolic subgroup of G^{\min} . Recall from Sect. 2 that $\{\epsilon_P^w\}_{w \in W^P}$ is a basis of the singular cohomology $H^*(X_P, \mathbb{Z})$. Write the standard cup product in $H^*(X_P, \mathbb{Z})$ in this basis as follows:*

$$\epsilon_P^u \cdot \epsilon_P^v = \sum_{w \in W^P} n_{u,v}^w \epsilon_P^w, \quad \text{for some (unique) } n_{u,v}^w \in \mathbb{Z}. \tag{31}$$

Introduce the indeterminate τ and define a deformed cup product \odot as follows:

$$\epsilon_P^u \odot \epsilon_P^v = \sum_{w \in W^P} \tau^{(u^{-1}\rho + v^{-1}\rho - w^{-1}\rho - \rho)(x_P)} n_{u,v}^w \epsilon_P^w, \tag{32}$$

where $x_P := \sum_{\alpha_i \in \Delta \setminus \Delta(P)} x_i$, $\Delta(P)$ is the set of simple roots of the Levi L of P and, as in Sect. 2, Δ is the set of simple roots of G .

The following lemma is a generalization of the corresponding result in the finite case (cf. [1, Proposition 17]).

Proposition 7.1 (a) *The product \odot is associative and clearly commutative.*
 (b) *Whenever $n_{u,v}^w$ is nonzero, the exponent of τ in the above is a nonnegative integer.*

Proof The proof of the associativity of \odot is identical to the proof given in [1, Proof of Proposition 17 (b)].

(b) We give the proof in the case when P is of finite type, i.e., its Levi subgroup is finite dimensional. The proof for general P is along similar lines. The proof of this part follows the proof of [1, Theorem 43]. Consider the decreasing filtration $\mathcal{A} = \{\mathcal{A}_m\}_{m \geq 0}$ of $H^*(X_P, \mathbb{C})$ defined as follows:

$$\mathcal{A}_m := \bigoplus_{w \in W^P : (\rho - w^{-1}\rho)(x_P) \geq m} \mathbb{C}\epsilon_P^w.$$

A priori $\{\mathcal{A}_m\}_{m \geq 0}$ may not be a multiplicative filtration.

We next introduce another filtration $\{\tilde{\mathcal{F}}_m\}_{m \geq 0}$ of $H^*(X_P, \mathbb{C})$ in terms of the Lie algebra cohomology. Recall that $H^*(X_P, \mathbb{C})$ can be identified canonically with the Lie algebra cohomology $H^*(\mathfrak{g}, \mathfrak{l})$, where \mathfrak{l} is the Lie algebra of the Levi subgroup L of P (cf. [12, Theorem 1.6]). The underlying cochain complex $C^\bullet = C^\bullet(\mathfrak{g}, \mathfrak{l})$ for $H^*(\mathfrak{g}, \mathfrak{l})$ can be written as

$$C^\bullet := [\wedge^\bullet(\mathfrak{g}/\mathfrak{l})^*]^{\mathfrak{l}} = \text{Hom}_{\mathfrak{l}}(\wedge^\bullet(\mathfrak{u}_P) \otimes \wedge^\bullet(\mathfrak{u}_P^-), \mathbb{C}),$$

where u_P (resp. $u_{\bar{P}}$) is the nil-radical of the Lie algebra of P (resp. the opposite parabolic subgroup P^-). Define a decreasing multiplicative filtration $\mathcal{F} = \{\mathcal{F}_m\}_{m \geq 0}$ of the cochain complex C^\bullet by subcomplexes:

$$\mathcal{F}_m := \text{Hom}_\mathbb{I} \left(\frac{\wedge^\bullet(u_P) \otimes \wedge^\bullet(u_{\bar{P}})}{\bigoplus_{s+t \leq m-1} \wedge^\bullet_{(s)}(u_P) \otimes \wedge^\bullet_{(t)}(u_{\bar{P}})}, \mathbb{C} \right),$$

where $\wedge^\bullet_{(s)}(u_P)$ (resp. $\wedge^\bullet_{(s)}(u_{\bar{P}})$) denotes the subspace of $\wedge^\bullet(u_P)$ (resp. $\wedge^\bullet(u_{\bar{P}})$) spanned by the \mathfrak{h} -weight vectors of weight β with P -relative height

$$\text{ht}_P(\beta) := |\beta(x_P)| = s.$$

Now, define the filtration $\bar{\mathcal{F}} = \{\bar{\mathcal{F}}_m\}_{m \geq 0}$ of $H^*(\mathfrak{g}, \mathbb{I}) \simeq H^*(X_P, \mathbb{C})$ by

$$\bar{\mathcal{F}}_m := \text{Image of } H^*(\mathcal{F}_m) \rightarrow H^*(C^\bullet).$$

The filtration \mathcal{F} of C^\bullet gives rise to the cohomology spectral sequence $\{E_r\}_{r \geq 1}$ converging to $H^*(C^\bullet) = H^*(X_P, \mathbb{C})$. By [13, Proof of Proposition 3.2.11], for any $m \geq 0$,

$$E_1^m = \bigoplus_{s+t=m} \left[H^\bullet_{(s)}(u_P) \otimes H^\bullet_{(t)}(u_{\bar{P}}) \right]^\mathbb{I},$$

where $H^\bullet_{(s)}(u_P)$ denotes the cohomology of the subcomplex $(\wedge^\bullet_{(s)}(u_P))^*$ of the standard cochain complex $\wedge^\bullet(u_P)^*$ associated to the Lie algebra u_P and similarly for $H^\bullet_{(t)}(u_{\bar{P}})$. Moreover, by loc. cit., the spectral sequence degenerates at the E_1 term, i.e.,

$$E_1^m = E_\infty^m. \tag{33}$$

Further, by the definition of P -relative height,

$$[H^\bullet_{(s)}(u_P) \otimes H^\bullet_{(t)}(u_{\bar{P}})]^\mathbb{I} \neq 0 \Rightarrow s = t.$$

Thus,

$$\begin{aligned} E_1^m &= 0, \quad \text{unless } m \text{ is even and} \\ E_1^{2m} &= [H^\bullet_{(m)}(u_P) \otimes H^\bullet_{(m)}(u_{\bar{P}})]^\mathbb{I}. \end{aligned}$$

In particular, from (33) and the general properties of spectral sequences (cf. [13, Theorem E.9]), we have a canonical algebra isomorphism:

$$\text{gr}(\bar{\mathcal{F}}) \simeq \bigoplus_{m \geq 0} \left[H^\bullet_{(m)}(u_P) \otimes H^\bullet_{(m)}(u_{\bar{P}}) \right]^\mathbb{I}, \tag{34}$$

where $[H_{(m)}^\bullet(u_P) \otimes H_{(m)}^\bullet(u_P^-)]^{\uparrow}$ sits inside $\text{gr}(\bar{\mathcal{F}})$ precisely as the homogeneous part of degree $2m$; homogeneous parts of $\text{gr}(\bar{\mathcal{F}})$ of odd degree being zero.

Finally, we claim that, for any $m \geq 0$,

$$\mathcal{A}_m = \bar{\mathcal{F}}_{2m} : \tag{35}$$

Following Kumar [11], take the d - ∂ harmonic representative \hat{s}^w in C^\bullet for the cohomology class ϵ_P^w . An explicit expression is given in [11, Proposition 3.17]. From this explicit expression, we easily see that

$$\mathcal{A}_m \subset \bar{\mathcal{F}}_{2m}, \text{ for all } m \geq 0. \tag{36}$$

Moreover, from the definition of \mathcal{A} , for any $m \geq 0$,

$$\dim \frac{\mathcal{A}_m}{\mathcal{A}_{m+1}} = \# \left\{ w \in W^P : (\rho - w^{-1}\rho)(x_P) = m \right\}.$$

Also, by the isomorphism (34) and [13, Theorem 3.2.7],

$$\dim \frac{\bar{\mathcal{F}}_{2m}}{\bar{\mathcal{F}}_{2m+1}} = \# \left\{ w \in W^P : (\rho - w^{-1}\rho)(x_P) = m \right\}.$$

Thus,

$$\dim \frac{\mathcal{A}_m}{\mathcal{A}_{m+1}} = \dim \frac{\bar{\mathcal{F}}_{2m}}{\bar{\mathcal{F}}_{2m+1}}. \tag{37}$$

Of course,

$$\mathcal{A}_0 = \bar{\mathcal{F}}_0. \tag{38}$$

Thus, combining the equations (36), (37) and (38), we get (35). It is easy to see that the filtration $\{\bar{\mathcal{F}}_{2m}\}_{m \geq 0}$ is multiplicative and hence so is $\{\mathcal{A}_m\}_{m \geq 0}$. This proves the (b) part of the proposition. \square

The cohomology of X_P obtained by setting $\tau = 0$ in $(H^*(X_P, \mathbb{Z}) \otimes \mathbb{Z}[\tau], \odot)$ is denoted by $(H^*(X_P, \mathbb{Z}), \odot_0)$. Thus, as a \mathbb{Z} -module, it is the same as the singular cohomology $H^*(X_P, \mathbb{Z})$ and under the product \odot_0 it is associative (and commutative).

The following conjecture is a generalization of the corresponding result in the finite case due to Belkale–Kumar [1, Theorem 22].

Conjecture 2 Fix $s > 1$. Let G be any indecomposable symmetrizable Kac–Moody group (i.e., its generalized Cartan matrix is indecomposable, cf. [13, §1.1]) and let $(\lambda_1, \dots, \lambda_s, \mu) \in P_+^{s+1}$. Assume further that none of λ_j is W -invariant and $\mu - \sum_{j=1}^s \lambda_j \in Q$, where Q is the root lattice of G . Then, the following are equivalent:

- (a) $(\lambda_1, \dots, \lambda_s, \mu) \in \Gamma_s$.

- (b) For every standard maximal parabolic subgroup P in G and every choice of $s + 1$ -tuples $(w_1, \dots, w_s, v) \in (W^P)^{s+1}$ such that ϵ_P^v occurs with coefficient 1 in the deformed product

$$\epsilon_P^{w_1} \odot_0 \cdots \odot_0 \epsilon_P^{w_s} \in (H^*(X_P, \mathbb{Z}), \odot_0),$$

the following inequality holds:

$$\left(\sum_{j=1}^s \lambda_j(w_j x_P) \right) - \mu(v x_P) \geq 0, \quad (I_{(w_1, \dots, w_s, v)}^P)$$

where α_{i_P} is the (unique) simple root in $\Delta \setminus \Delta(P)$ and $x_P := x_{i_P}$.

Remark 7.2 (a) By Theorem 3.2, the above inequalities $I_{(w_1, \dots, w_s, v)}^P$ are indeed satisfied for any $(\lambda_1, \dots, \lambda_s, \mu) \in \Gamma_s$.

(b) If G is an affine Kac–Moody group, then the condition that $\lambda \in P_+$ is W -invariant is equivalent to the condition that $\lambda(c) = 0$.

Theorem 7.3 *Let $\mathfrak{g} = \widehat{\mathfrak{sl}}_2$. Let $\lambda, \mu, \nu \in P_+$ be such that $\lambda + \mu - \nu \in Q$ and both of $\lambda(c)$ and $\mu(c)$ are nonzero. Then, the following are equivalent:*

- (a) $(\lambda, \mu, \nu) \in \Gamma_2$.
- (b) The following set of inequalities is satisfied for all $w \in W$ and $i = 0, 1$.

$$\begin{aligned} \lambda(x_i) + \mu(w x_i) - \nu(w x_i) &\geq 0, \quad \text{and} \\ \lambda(w x_i) + \mu(x_i) - \nu(w x_i) &\geq 0. \end{aligned}$$

In particular, Conjecture 2 is true for $\mathfrak{g} = \widehat{\mathfrak{sl}}_2$ and $s = 2$.

Proof By Lemma 5.2, there exist (unique) $n_1, n_2 \in \mathbb{Z}$ such that

$$\nu - \mu + n_1 \delta \in P^o(\lambda), \quad \text{and} \quad \nu - \lambda + n_2 \delta \in P^o(\mu).$$

Let $n := \min(n_1, n_2)$. By our description of the δ -maximal components as in Theorem 6.1 applied to $\bar{\lambda}, \bar{\mu}, \bar{\nu}$ and using the identity (27), we see that $L(\nu + n\delta)$ is a δ -maximal component of $L(\lambda) \otimes L(\mu)$. Thus, by the equation (28), for any $N \geq 1$, $L(N\nu + Nn\delta)$ is a δ -maximal component of $L(N\lambda) \otimes L(N\mu)$. In particular, by Proposition 4.2 and Lemma 4.1,

$$L(N\nu) \subset L(N\lambda) \otimes L(N\mu) \quad \text{for some } N > 1 \quad \text{if and only if } n \geq 0. \quad (39)$$

By [4, Proposition 12.5 (a)], if a weight $\gamma + k\delta \in P(\lambda)$ (for some $k \in \mathbb{Z}_+$), then $\gamma \in P(\lambda)$. Thus,

$$n \geq 0 \quad \text{if and only if } \nu \in (P(\lambda) + \mu) \cap (P(\mu) + \lambda). \quad (40)$$

We next show that

$$\tilde{P}(\lambda) = (\lambda + Q) \cap C_\lambda, \tag{41}$$

where $C_\lambda := \{\gamma \in \mathfrak{h}^* : \lambda(x_i) - \gamma(wx_i) \geq 0 \text{ for all } w \in W \text{ and all } x_i\}$. Clearly,

$$P(\lambda) \subset (\lambda + Q) \cap C_\lambda.$$

Since $\lambda + Q$ and C_λ are W -stable, and $\lambda + Q$ is contained in the Tits cone (by [13, Exercise 13.1.E.8(a)]), $(\lambda + Q) \cap C_\lambda = W \cdot ((\lambda + Q) \cap C_\lambda \cap P_+)$.

Conversely, take $\gamma \in (\lambda + Q) \cap C_\lambda \cap P_+$. Then, $(\lambda - \gamma)(x_i) \geq 0$ and $(\lambda - \gamma)(c) = 0$ and hence $\lambda - \gamma \in \oplus_i \mathbb{Z}_+ \alpha_i$, i.e., $\lambda \geq \gamma$. Thus, by [4, Proposition 12.5(a)], $\gamma \in P(\lambda)$. This proves (41). Now, combining (39), (40) and (41), we get $L(N\nu) \subset L(N\lambda) \otimes L(N\mu)$ for some $N > 1$ if and only if for all $w \in W$ and $i = 0, 1$,

$$\lambda(x_i) - (v - \mu)(wx_i) \geq 0, \quad \text{and} \quad \mu(x_i) - (v - \lambda)(wx_i) \geq 0.$$

This proves the equivalence of (a) and (b) in the theorem.

To prove the ‘In particular’ statement of the theorem, let P_0 (resp. P_1) be the maximal parabolic subgroup of $G^{\min} = \widehat{SL}_2$ with $\Delta(P_0) = \{\alpha_1\}$ (resp. $\Delta(P_1) = \{\alpha_0\}$). For any $n \geq 0$, let

$$w_n := \cdots s_0 s_1 s_0 \quad (n\text{-factors}) \quad \text{and} \quad v_n := \cdots s_1 s_0 s_1 \quad (n\text{-factors}).$$

Then, by [13, Exercise 11.3.E.4], $H^*(G/P_0)$ has a \mathbb{Z} -basis $\{\epsilon_{P_0}^n\}_{n \geq 0}$, where $\epsilon_{P_0}^n := \epsilon_{P_0}^{w_n}$. Moreover,

$$\epsilon_{P_0}^n \cdot \epsilon_{P_0}^m = \binom{n+m}{n} \epsilon_{P_0}^{n+m}.$$

In particular, $\epsilon_{P_0}^{n+m}$ appears with coefficient one in $\epsilon_{P_0}^n \cdot \epsilon_{P_0}^m$ if and only if at least one of n or m is 0.

By using the diagram automorphism of \widehat{SL}_2 , one similarly gets that $H^*(G/P_1)$ has a \mathbb{Z} -basis $\{\epsilon_{P_1}^n\}_{n \geq 0}$, where $\epsilon_{P_1}^n := \epsilon_{P_1}^{v_n}$, with the product given by

$$\epsilon_{P_1}^n \cdot \epsilon_{P_1}^m = \binom{n+m}{n} \epsilon_{P_1}^{n+m}.$$

Moreover, from the definition of the deformed product \odot_0 , clearly

$$\epsilon_{P_0}^0 \odot_0 \epsilon_{P_0}^m = \epsilon_{P_0}^0 \cdot \epsilon_{P_0}^m,$$

and similarly for P_1 . From this the ‘In particular’ statement of the theorem follows. \square

Remark 7.4 (1) It is easy to see that if $\lambda = m\delta$ for some $m \in \mathbb{Z}$, then the equivalence of (a) and (b) in the above theorem breaks down.

(2) Though we have proved Conjecture 2 for $\widehat{\text{SL}}_2$ only for $s = 2$, it is quite likely that a similar proof will prove it for any $s > 2$ (for $\widehat{\text{SL}}_2$). One can generalize the construction of the action of the Virasoro algebra on tensor products for $s > 2$, hence Γ_s can be computed by determining only δ -maximal components. Furthermore, Proposition 4.2 can be generalized by writing the characters of the first $s - 1$ factors in terms of their “string functions” while using the Kac–Weyl character formula for the last factor. For $\widehat{\text{SL}}_2$, this results in a computation of the δ -maximal components that is similar to the $s = 2$ case, albeit with extra parameters (many of the bounds used in the proof of Proposition 5.5 remain valid still).

8 The $A_2^{(2)}$ case

By a method similar to that of $A_1^{(1)}$, we handle the $A_2^{(2)}$ case, with minor modifications where necessary. Write $\mathfrak{h} = \mathbb{C}c \oplus \mathbb{C}\alpha^\vee \oplus \mathbb{C}d$ and $\mathfrak{h}^* = \mathbb{C}\omega_0 \oplus \mathbb{C}\alpha \oplus \mathbb{C}\delta$, where $\alpha(\alpha^\vee) = 2, \delta(d) = 1, \omega_0(c) = 1$, and all other values are 0. Then $(\mathfrak{h}, \{\alpha_0 := \delta - 2\alpha, \alpha_1 := \alpha\}, \{\alpha_0^\vee := c - \frac{1}{2}\alpha^\vee, \alpha_1^\vee := \alpha^\vee\})$ is a realization of the Generalized Cartan Matrix

$$\begin{pmatrix} 2 & -1 \\ -4 & 2 \end{pmatrix}$$

of $A_2^{(2)}$. The fundamental weights are ω_0 and $\omega_1 = \frac{1}{2}\omega_0 + \frac{1}{2}\alpha$. This easily allows one to compute the dominant δ -maximal weights. Analogous to Corollary 5.1, we have the following:

Lemma 8.1 *Let λ be a dominant integral weight. Then, the dominant δ -maximal weights of $L(\lambda)$ are the dominant weights of the form*

$$P_+ \cap \{\lambda - j\alpha, \lambda + k(2\alpha - \delta), \lambda + \alpha - \delta + l(2\alpha - \delta) : j, k, l \in \mathbb{Z}_{\geq 0}\}.$$

Moreover, $P^o(\lambda)$ is the W -orbit of the above.

Again, to determine the saturated tensor cone, it is enough to describe the δ -maximal components. Thus, to determine the δ -maximal components, by virtue of proposition 4.2, we must find the highest δ -degree term in $\sum_{\lambda \in T_A^{\Lambda', \Lambda''}} \varepsilon(v_{\Lambda, \Lambda'', \lambda}) c_{\Lambda', \lambda} e^{S_{\Lambda, \Lambda'', \lambda} \delta}$.

This computation is done in a somewhat similar manner as in the $A_1^{(1)}$ case, but there are some important modifications. First, we need to use two different piecewise smooth functions to describe the δ -maximal weights of $L(\lambda)$. An upper function A^+ interpolates the δ -maximal weights which are in the W -orbit of the dominant weights of the form

$$\{\lambda - j\alpha, \lambda + k(2\alpha - \delta) : j, k \in \mathbb{Z}_{\geq 0}\},$$

while another function A^- interpolates the δ -maximal weights in the W -orbit of the dominant weights of the form

$$\{\lambda - j\alpha, \lambda + \alpha - \delta + l(2\alpha - \delta) : j, l \in \mathbb{Z}_{\geq 0}\}.$$

Now, all of the arguments made in the $\widehat{\mathfrak{sl}}_2$ case must be made for two extensions of $S_{\Lambda, \Lambda'', \lambda}$ to non-integral values, using A^+ and A^- respectively. Let $\Lambda := m_0\omega_0 + m_1\omega_1$, $\Lambda' := m'_0\omega_0 + m'_1\omega_1$, and $\Lambda'' := m''_0\omega_0 + m''_1\omega_1$. The following result is an analogue of Proposition 5.5 and Lemma 5.7 for the $A_2^{(2)}$ case.

Proposition 8.2 *Let $\Lambda, \Lambda', \Lambda''$ be as above. Assume that both of $\Lambda'(c)$ and $\Lambda''(c) > 0$ and $\Lambda - \Lambda' - \Lambda'' \in Q$, where $Q = \mathbb{Z}\alpha + \mathbb{Z}\delta$ is the root lattice of $A_2^{(2)}$. Then, the maximum $\mu_{\Lambda}^{\Lambda', \Lambda''}$ of the set*

$$\{S_{\Lambda, \Lambda'', \lambda} : \lambda \in T_{\Lambda}^{\Lambda', \Lambda''}, \varepsilon(v_{\Lambda, \Lambda'', \lambda}) = 1\}$$

occurs when $\lambda \equiv \Lambda' + \frac{1}{2}(m_1 - m'_1 - m''_1)\alpha \pmod{\mathbb{C}\delta}$. The maximum $\bar{\mu}_{\Lambda}^{\Lambda', \Lambda''}$ of the set

$$\{S_{\Lambda, \Lambda'', \lambda} : \lambda \in T_{\Lambda}^{\Lambda', \Lambda''}, \varepsilon(v_{\Lambda, \Lambda'', \lambda}) = -1\}$$

occurs when $\lambda \equiv \Lambda' - (\frac{1}{2}(m'_1 + m''_1 + m_1) + 1)\alpha \pmod{\mathbb{C}\delta}$ or when $\lambda \equiv \Lambda' - (\frac{1}{2}(m'_1 + m''_1 + m_1) - 2(\Lambda'(c) + \Lambda''(c) + 1))\alpha \pmod{\mathbb{C}\delta}$.

Corollary 8.3 *Let $\Lambda, \Lambda', \Lambda''$ be as in Proposition 8.2. Assume further that $\Lambda'(c) \geq 2$, $\Lambda''(c) \geq 2$, $m'_1, m''_1 \neq 1$. Then, if $\mu_{\Lambda}^{\Lambda', \Lambda''} = \bar{\mu}_{\Lambda}^{\Lambda', \Lambda''}$, we have*

$$\mu_{\Lambda}^{\Lambda'', \Lambda'} \neq \bar{\mu}_{\Lambda}^{\Lambda'', \Lambda'}.$$

The proof of Corollary 8.3 requires a description of the situations in which $\mu_{\Lambda}^{\Lambda', \Lambda''} = \bar{\mu}_{\Lambda}^{\Lambda', \Lambda''}$. We reduce these situations to certain cases, and show that in most of these cases, if the roles of Λ' and Λ'' are interchanged, then (as in the $\widehat{\mathfrak{sl}}_2$ case) the equality does not occur. In the remaining cases, we show that $\Lambda'(c) < 2$, $\Lambda''(c) < 2$, $m'_1 = 1$, or $m''_1 = 1$.

Theorem 8.4 *Let $\Lambda, \Lambda', \Lambda''$ be as in Proposition 8.2. Then, $L(\Lambda + n\delta)$ is a δ -maximal component of $L(\Lambda') \otimes L(\Lambda'')$ if $n = \min(n_1, n_2)$, where n_1 is such that $\Lambda - \Lambda'' + n_1\delta \in P^o(\Lambda')$ and n_2 is such that $\Lambda - \Lambda' + n_2\delta \in P^o(\Lambda'')$.*

Lemma 8.5 *Fix a positive integer N . Let $\Lambda \in \bar{P}_+$ and let $\lambda \in \Lambda + Q$. Then, $N\lambda \in P^o(N\Lambda)$ if and only if $\lambda \in P^o(\Lambda)$.*

Combining the above results, we get a description of Γ_2 , which is identical to that of $\widehat{\mathfrak{sl}}_2$ (cf. Theorem 7.3).

Theorem 8.6 *Let $\mathfrak{g} = A_2^{(2)}$. Let $\lambda, \mu, \nu \in P_+$ be such that $\lambda + \mu - \nu \in Q$ and both of $\lambda(c)$ and $\mu(c)$ are nonzero. Then, the following are equivalent:*

- (a) $(\lambda, \mu, \nu) \in \Gamma_2$.
- (b) *The following set of inequalities is satisfied for all $w \in W$ and $i = 0, 1$.*

$$\begin{aligned} \lambda(x_i) + \mu(wx_i) - \nu(wx_i) &\geq 0, \quad \text{and} \\ \lambda(wx_i) + \mu(x_i) - \nu(wx_i) &\geq 0. \end{aligned}$$

In particular, Conjecture 2 is true for this case as well for $s = 2$.

The ‘In particular’ statement of the above theorem follows by using the description of the cup product in the cohomology of the two partial flag varieties of $A_2^{(2)}$ (given below) corresponding to the two maximal parabolic subgroups P_0 and P_1 , where the Levi subgroup of P_0 (resp. P_1) has α_1 (resp. α_0) for its simple root.

Observe that

$$W^{P_0} = \{w_n\}_{n \geq 0}, \quad \text{where } w_n := \cdots s_0 s_1 s_0 \text{ (} n\text{-factors),}$$

and similarly

$$W^{P_1} = \{v_n\}_{n \geq 0}, \quad \text{where } v_n := \cdots s_1 s_0 s_1 \text{ (} n\text{-factors).}$$

Let

$$\tau_n := \epsilon_{P_0}^{w_n} \in H^{2n}(X_{P_0}, \mathbb{Z}); \quad \delta_n := \epsilon_{P_1}^{v_n} \in H^{2n}(X_{P_1}, \mathbb{Z}).$$

Then, the following result is due to Kitchloo [9].

Theorem 8.7 *For any $n, m \geq 1$,*

$$\tau_n \cdot \tau_m = D(n, m)\tau_{n+m},$$

and

$$\delta_n \cdot \delta_m = C(n, m)\delta_{n+m},$$

where

$$C(n, m) = D(n, m) = \binom{m+n}{n}, \quad \text{if at least one of } m, n \text{ is even.}$$

If both of m, n is odd, then

$$D(n, m) = 2 \binom{m+n}{n} \quad \text{and} \quad C(n, m) = 2^{-1} \binom{m+n}{n}.$$

It is clear that if the level of $L(\Lambda')$ or $L(\Lambda'')$ is zero, then the tensor product has a single component. Thus, it is already saturated. Assume now that the levels of both of $L(\Lambda')$ and $L(\Lambda'')$ are > 0 . Then, since there are representations of level $\frac{1}{2}$, the conditions of Corollary 8.3 are satisfied for any $N\Lambda, N\Lambda', N\Lambda''$ with $\Lambda - \Lambda' - \Lambda'' \in Q$, provided $N \geq 4$. Hence:

Corollary 8.8 For $A_2^{(2)}$, 4 is a saturation factor.

Remark 8.9 When the Kac–Moody Lie algebra \mathfrak{g} is infinite dimensional, then the saturated tensor semigroup Γ_s is *not* finitely generated, for any $s \geq 2$. Thus, it is not clear a priori that there exists a saturation factor for such a \mathfrak{g} .

Acknowledgments We thank Evgeny Feigin and Victor Kac for some helpful correspondences. We also thank the referee for some helpful suggestions. Both the authors were partially supported by the NSF grant number DMS-1201310.

References

1. Belkale, P., Kumar, S.: Eigenvalue problem and a new product in cohomology of flag varieties. *Inventiones Math.* **166**, 185–228 (2006)
2. Chriss, N., Ginzburg, V.: *Representation Theory and Complex Geometry*. Birkhäuser, Basel (1997)
3. Iohara, K., Koga, Y.: *Representation Theory of the Virasoro Algebra*. Springer, Berlin (2011)
4. Kac, V.: *Infinite Dimensional Lie Algebras*, 3rd edn. Cambridge University Press, Cambridge (1990)
5. Kac, V., Raina, A., Rozhkovskaya, N.: *Bombay lectures on highest weight representations of infinite dimensional Lie algebras*. In: *Advanced Series in Mathematical Physics*, vol. 29, 2nd edn. World Scientific, Singapore (2013)
6. Kac, V., Wakimoto, M.: Modular and conformal invariance constraints in representation theory of affine algebras. *Adv. Math.* **70**, 156–236 (1988)
7. Kashiwara, M.: The flag manifold of Kac–Moody Lie algebra. In: Igusa, J.-I. (ed.) *Algebraic Analysis, Geometry, and Number Theory*, pp. 161–190. The Johns Hopkins University Press, Baltimore (1989)
8. Kashiwara, M., Shimozono, M.: Equivariant K -theory of affine flag manifolds and affine Grothendieck polynomials. *Duke Math. J.* **148**, 501–538 (2009)
9. Kitchloo, N.: On the topology of Kac–Moody groups. Preprint (2013)
10. Kostant, B., Kumar, S.: T -equivariant K -theory of generalized flag varieties. *J. Differ. Geom.* **32**, 549–603 (1990)
11. Kumar, S.: Geometry of Schubert cells and cohomology of Kac–Moody Lie algebras. *J. Differ. Geom.* **20**, 389–431 (1984)
12. Kumar, S.: Rational homotopy theory of flag varieties associated to Kac–Moody groups. In: *Infinite Dimensional Groups with Applications*, vol. 4, pp. 233–273. MSRI Publications, Springer-Verlag (1985)
13. Kumar, S.: Kac–Moody groups, their flag varieties and representation theory. In: *Progress in Mathematics*, vol. 204. Birkhäuser, Basel (2002)
14. Kumar, S.: Positivity in T -equivariant K -theory of flag varieties associated to Kac–Moody groups. Preprint (2012)
15. Kumar, S.: Additive eigenvalue problem (a survey). Preprint (2013)
16. Mumford, D., Fogarty, J., Kirwan, F.: *Geometric Invariant Theory*. In: *Ergebnisse der Mathematik und ihrer Grenzgebiete*, vol. 34, 3rd edn. Springer, Berlin (1994)