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Dimension of zero weight space: An algebro-geometric approach

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Let G be a connected, adjoint, simple algebraic group over the complex numbers with a maximal torus T and a Borel subgroup B containing T. The study of zero weight spaces in irreducible representations of G has been a topic of considerable interest; there are many works which study the zero weight space as a representation space for the Weyl group. In this paper, we study the variation on the dimension of the zero weight space as the highest weight of the irreducible representation varies over the set of dominant integral weights of T, which are lattice points in a certain polyhedral cone. The theorem proved here asserts that the zero weight spaces have dimensions which are piecewise quasi-polynomial functions on the polyhedral cone of dominant integral weights.The main tool we use are the Geometric Invariant Theory and the Riemann–Roch theorem.

1. Introduction

Let G be a connected, adjoint, simple algebraic group over the complex numbers $\mathbb{C}$ with a maximal torus T and a Borel subgroup $B \supset T$. The study of zero weight spaces in irreducible representations of G has been a topic of considerable interest; there are many works which study the zero weight space as a representation space for the Weyl group. In this paper, we study the variation of the dimension of the zero weight space

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as the irreducible representation varies over the set of dominant integral weights for T, which are lattice points in a certain polyhedral cone (using algebro-geometric methods).

The theorem proved here asserts that the zero weight spaces have dimensions which are piecewise polynomial functions on the polyhedral cone generated by dominant integral weights. The precise statement of the theorem is given below.

Let $\Lambda = \Lambda(T)$ be the character group of T and let $\Lambda^+ \subset \Lambda$ (resp. Λ^{++}) be the semigroup of dominant (resp. dominant regular) weights. Then, by taking derivatives, we can identify Λ with Q, where Q is the root lattice (since G is an adjoint group). For $\lambda \in \Lambda^+$, let $V(\lambda)$ be the irreducible G-module with highest weight λ. Let $\mu_0 : \Lambda^+ \to \mathbb{Z}_+$ be the function: $\mu_0 = \dim V(\lambda)_0$, where $V(\lambda)_0$ is the 0-weight space of $V(\lambda)$.

Let $\Gamma = \Gamma_G \subset Q$ be the sublattice as in Theorem 3.1.

Also, let $\Lambda(\mathbb{R}) := \Lambda \otimes_{\mathbb{Z}} \mathbb{R}$ and let $\Lambda^+(\mathbb{R})$ be the cone inside $\Lambda(\mathbb{R})$ generated by Λ^+. Further, let $\Lambda^{++}(\mathbb{R})$ be the relative interior of $\Lambda^+(\mathbb{R})$. Let $C_1, \dots, C_N \subset \Lambda^{++}(\mathbb{R})$ be the chambers (i.e., the GIT classes in $\Lambda^{++}(\mathbb{R})$ of maximal dimension: equal to the dimension of $\Lambda(\mathbb{R})$, with respect to the T-action) (see Section 2).

For any $w \in W$ and $1 \leqslant i \leqslant \ell$, define the hyperplane

$$H_{w,i} := \bigl\{\lambda \in \Lambda(\mathbb{R}):\ \lambda(wx_i) = 0\bigr\},$$

where W is the Weyl group of G and $\{x_1, \dots, x_\ell\}$ is the basis of $\mathfrak{t}$ dual to the basis of $\mathfrak{t}^*$ given by the simple roots. Then, by virtue of Corollary 3.6, $C_1, \dots, C_N$ are the connected components of

$$\Lambda^{++}(\mathbb{R}) \setminus \Biggl(\bigcup_{w \in W,\ 1 \leqslant i \leqslant \ell} H_{w,i}\Biggr).$$

With this notation, we have the following main result of our paper (cf. Theorem 4.1).

Theorem 1.1. *Let $\bar{\mu} = \mu + \Gamma$ be a coset of Γ in Q. Then, for any GIT class C_k, $1 \leqslant k \leqslant N$, there exists a polynomial $f_{\bar{\mu},k} : \Lambda(\mathbb{R}) \to \mathbb{R}$ with rational coefficients of degree $\leqslant \dim_{\mathbb{C}} X - \ell$, such that*

$$f_{\bar{\mu},k}(\lambda) = \mu_0(\lambda), \quad \text{for all } \lambda \in \bar{C}_k \cap \bar{\mu}, \tag{1}$$

where $\bar{C}_k$ is the closure of C_k inside $\Lambda(\mathbb{R})$ and X is the full flag variety G/B. Further, $f_{\Gamma,k}$ has constant term 1.

The proof of the above theorem relies on Geometric Invariant Theory (GIT). Specifically, we realize the function μ_0 restricted to $\bar{C}_k \cap \Lambda$ as an Euler–Poincaré characteristic of a reflexive sheaf on a certain GIT quotient (depending on C_k) of $X = G/B$ via the maximal torus T. Then, one can use the Riemann–Roch theorem for singular varieties to calculate this Euler–Poincaré characteristic. From this calculation, we conclude that the function μ_0 restricted to $\bar{C}_k \cap (\mu + \Gamma)$ is a polynomial function. The result on descent of the homogeneous line bundles on X to the GIT quotient plays a crucial role (cf. Lemma 3.7).

We end the paper by determining this piecewise polynomial for the group of type A_3 (in Section 6). We also state the result for all the rank 2 simple groups in Section 5. Even though these are known (see below), but we have included them for illustration.

The results of the paper can easily be extended to show the piecewise polynomial behavior of the dimension of any weight space (of a fixed weight μ) in any finite dimensional irreducible representation $V(\lambda)$.

By a similar proof, we can also obtain a piecewise polynomial behavior of the dimension of H-invariant subspace in any finite dimensional irreducible representation $V(\lambda)$ of G, where $H \subset G$ is a reductive subgroup. However, the results in this general case are not as precise (cf. Remark 4.2).

It should be mentioned that Meinrenken and Sjamaar [16] have obtained a result similar to our above result Theorem 1.1 (also in the generality of H-invariants) by using techniques from Symplectic Geometry. But, their result in the case of T-invariants is less precise than our Theorem 1.1, in that our result determines the 'periods' of quasi-polynomials more precisely by virtue of Theorem 3.1.

Billey–Guillemin–Rassart have determined the weight multiplicity functions for $\mathrm{SL}_n(\mathbb{C})$ in terms of a vector partition function (cf. [4, Theorem 2.1]).

As pointed out by M. Vergne, using Kostant's formula for the weight multiplicities, one can prove that the function μ_0 restricted to any connected component C_j of $\Lambda^{++}(\mathbb{R}) \setminus (\bigcup_{w \in W,\, 1 \leqslant i \leqslant \ell} H_{w,i})$ is a quasi-polynomial; its polynomial extension to the boundary is more delicate.

Baldoni-Silva and Vergne have computed the Kostant partition function for A_2, A_3 (cf. [3, Appendix 2]), from which the function μ_0 can be computed for them. (In Section 6, we have followed a different approach to compute μ_0 for the case of A_3 by using the branching from $\mathrm{GL}(n+1)$ to $\mathrm{GL}(n)$.) We also refer to the papers [9,10] and [2] for some programs to calculate the Kostant's partition function and the weight multiplicity functions for the simple Lie algebras of classical type (the underlying maple program is available at the homepage of M. Vergne stationed at: http://www.math.jussieu.fr/~vergne/). For some other explicit zero weight multiplicity formulae, we also refer to [1], [13, §2].

The piecewise quasi-polynomiality of the weight multiplicity functions is also obtained by Bliem by a different method using 'chopped and sliced cone' (cf. [5, Theorem 3][1]). The tables of weight multiplicities for the groups of types A_2 and B_2 are given in [5]. (The computation for the weight multiplicity functions for B_2 is also done by Cagliero and Tirao [8].)

It may be mentioned that general results from Geometric Invariant Theory/Symplectic Geometry give piecewise polynomiality in a wide range of problems. But, we have considered only the case of T-quotients of G/B because of its importance in representation

[1] We thank A. Khare for pointing out this reference.

theory to the problem of determining the behavior of the dimensions of zero weight spaces in irreducible representations of G, and also because we have more precise results in this case.

2. Notation

Let G be a connected, adjoint, semisimple algebraic group over the complex numbers $\mathbb{C}$. Fix a Borel subgroup B and a maximal torus $T \subset B$. We denote their Lie algebras by the corresponding Gothic characters: $\mathfrak{g}$, $\mathfrak{b}$ and $\mathfrak{t}$ respectively. Let $R^+ \subset \mathfrak{t}^*$ be the set of positive roots (i.e., the roots of B) and let $\Delta = \{\alpha_1, \dots, \alpha_\ell\} \subset R^+$ be the set of simple roots. Let $Q = \bigoplus_{i=1}^{\ell} \mathbb{Z}\alpha_i$ be the root lattice. Then, the group of characters Λ of T can be identified with Q (since G is adjoint) by taking the derivative. We will often make this identification. Let Λ^+ (resp. Λ^{++}) be the semigroup of dominant (resp. dominant regular) weights, i.e.,

$$\Lambda^+ := \{\lambda \in \Lambda\colon\ \lambda(\alpha_i^\vee) \in \mathbb{Z}_+,\ \text{for all the simple coroots } \alpha_i^\vee\},$$

and

$$\Lambda^{++} := \{\lambda \in \Lambda^+\colon\ \lambda(\alpha_i^\vee) \geqslant 1 \text{ for all } \alpha_i^\vee\}.$$

Then, Λ^+ bijectively parameterizes the isomorphism classes of finite dimensional irreducible G-modules. For $\lambda \in \Lambda^+$, let $V(\lambda)$ be the corresponding irreducible G-module (with highest weight λ).

Let $W := N(T)/T$ be the Weyl group of G, where $N(T)$ is the normalizer of T in G. Let $\Lambda(\mathbb{R}) := \Lambda \otimes_{\mathbb{Z}} \mathbb{R}$ and let $\Lambda^+(\mathbb{R})$ be the cone inside $\Lambda(\mathbb{R})$ generated by Λ^+. Further, let $\Lambda^{++}(\mathbb{R})$ be the relative interior of $\Lambda^+(\mathbb{R})$. Any element $\lambda \in \Lambda(\mathbb{R})$ can uniquely be written as

$$\lambda = \sum_{i=1}^{\ell} z_i \omega_i, \quad z_i \in \mathbb{R}, \tag{2}$$

where $\omega_i \in \Lambda^+(\mathbb{R})$ is the i-th fundamental weight:

$$\omega_i(\alpha_j^\vee) = \delta_{i,j}.$$

Then,

$$\Lambda^+(\mathbb{R}) = \bigoplus_{i=1}^{\ell} \mathbb{R}_{\geqslant 0}\omega_i, \qquad \Lambda^{++}(\mathbb{R}) = \bigoplus_{i=1}^{\ell} \mathbb{R}_{>0}\omega_i,$$

where $\mathbb{R}_{\geqslant 0}$ (resp. $\mathbb{R}_{>0}$) is the set of non-negative (resp. strictly positive) real numbers. We will denote any $\lambda \in \Lambda(\mathbb{R})$ in the coordinates $z_\lambda = (z_i)_{1 \leqslant i \leqslant \ell}$ as in (2).

A function $f : S \subset \Lambda^+ \to \mathbb{Q}$ defined on a subset S of Λ^+ is called a *polynomial function* if there exists a polynomial $\hat{f}(z) \in \mathbb{Q}[z_i]_{1\leqslant i\leqslant \ell}$ such that $f(\lambda) = \hat{f}(z_\lambda)$, for all $\lambda \in S$.

For any $\lambda \in \Lambda$, we have the G-equivariant line bundle $\mathcal{L}(\lambda)$ on $X := G/B$ associated to the principal B-bundle $G \to G/B$ via the character λ^{-1} of B, i.e.,

$$\mathcal{L}(\lambda) = G \overset{B}{\times} \mathbb{C}_{-\lambda} \to G/B,$$

where $\mathbb{C}_{-\lambda}$ denotes the one dimensional T-module with weight $-\lambda$. (Observe that for any $\lambda \in \Lambda$, the T-module structure on $\mathbb{C}_{-\lambda}$ extends uniquely to a B-module structure.) The line bundle $\mathcal{L}(\lambda)$ is ample if and only if $\lambda \in \Lambda^{++}$.

Following Dolgachev and Hu [11], λ, $\mu \in \Lambda^{++}(\mathbb{R})$ are said to be *GIT equivalent* if $X^{ss}(\lambda) = X^{ss}(\mu)$, where $X^{ss}(\lambda)$ denotes the set of semistable points in X with respect to the element $\lambda \in \Lambda^{++}(\mathbb{R})$. Recall that if $\lambda \in \Lambda^{++}(\mathbb{Q}) := \bigoplus_{i=1}^{\ell} \mathbb{Q}_{>0}\omega_i$, then $X^{ss}(\lambda)$ is the set of T-semistable points of X with respect to the T-equivariant line bundle $\mathcal{L}(d\lambda)$, for any positive integer d such that $d\lambda \in \Lambda^{++}$.

Definition 2.1. By a *rational polyhedral cone* C in $\Lambda^{++}(\mathbb{R})$, one means a subset of $\Lambda^{++}(\mathbb{R})$ defined by a finite number of linear inequalities with rational coefficients.

For a $\mathbb{R}$-linear form f on $\Lambda(\mathbb{R})$ which is non-negative on C, the set of points $c \in C$ such that $f(c) = 0$ is called a *face* of C.

By [11] or [18, Proposition 7], any GIT equivalence class in $\Lambda^{++}(\mathbb{R})$ is the relative interior of a rational polyhedral cone in $\Lambda^{++}(\mathbb{R})$ and, moreover, there are only finitely many GIT classes (cf. [11, Theorem 1.3.9] or [18, Theorem 3]). Let $C_1, \dots, C_N$ be the GIT classes of maximal dimension, i.e., of dimension equal to that of $\Lambda(\mathbb{R})$. These are called *chambers*. Let $X_T(C_k)$ denote the GIT quotient $X^{ss}(\lambda)//T$ for any $\lambda \in C_k$.

Since, for any $\lambda \in \Lambda^+$, the irreducible module $V(\lambda)$ has its zero weight space $V(\lambda)_0$ nonzero, we have $X^{ss}(\lambda) \neq \emptyset$ for any $\lambda \in \Lambda^{++}(\mathbb{R})$.

Let $\mathfrak{t}_+ := \{x \in \mathfrak{t}\colon\ \alpha_i(x) \geqslant 0$, for all the simple roots $\alpha_i\}$ be the dominant chamber. Clearly,

$$\mathfrak{t}_+ = \bigoplus_{i=1}^{\ell} \mathbb{R}_+ x_i, \tag{3}$$

where $\{x_i\}$ is the basis of $\mathfrak{t}$ dual to the basis of $\mathfrak{t}^*$ consisting of the simple roots, i.e.,

$$\alpha_i(x_j) = \delta_{i,j}. \tag{4}$$

3. Descent of line bundles to GIT quotients and determination of chambers

There exists the largest lattice $\Gamma \subset Q$ such that for any $\lambda \in \Lambda^{++} \cap \Gamma$, the homogeneous line bundle $\mathcal{L}(\lambda)$ descends as a line bundle $\widehat{\mathcal{L}}(\lambda)$ on the GIT quotient $X_T(\lambda)$. In fact, Γ is determined precisely in [15, Theorem 3.10] for any simple G, which we recall below.

Theorem 3.1. *For any simple G, $\Gamma = \Gamma_G$ is the following lattice (following the indexing in [6, Planche I–IX]).*

(1) *G of type $A_\ell (\ell \geqslant 1) : Q$*
(2) *G of type $B_\ell (\ell \geqslant 3) : 2Q$*
(3) *G of type $C_\ell (\ell \geqslant 2) : \mathbb{Z}2\alpha_1 + \cdots + \mathbb{Z}2\alpha_{\ell-1} + \mathbb{Z}\alpha_\ell$*
(4) *G of type D_4: $\{n_1\alpha_1 + 2n_2\alpha_2 + n_3\alpha_3 + n_4\alpha_4\colon\ n_i \in \mathbb{Z}$ and $n_1 + n_3 + n_4$ is even$\}$*
(5) *G of type $D_\ell (\ell \geqslant 5)$: $\{2n_1\alpha_1 + 2n_2\alpha_2 + \cdots + 2n_{\ell-2}\alpha_{\ell-2} + n_{\ell-1}\alpha_{\ell-1} + n_\ell\alpha_\ell\colon\ n_i \in \mathbb{Z}$ and $n_{\ell-1} + n_\ell$ is even$\}$*
(6) *G of type G_2: $\mathbb{Z}6\alpha_1 + \mathbb{Z}2\alpha_2$*
(7) *G of type F_4: $\mathbb{Z}6\alpha_1 + \mathbb{Z}6\alpha_2 + \mathbb{Z}12\alpha_3 + \mathbb{Z}12\alpha_4$*
(8) *G of type E_6: $6\widetilde{\Lambda}$*
(9) *G of type E_7: $12\widetilde{\Lambda}$*
(10) *G of type E_8: $60Q$,*

where $\widetilde{\Lambda}$ is the lattice generated by the fundamental weights.

Definition 3.2. Let S be any connected reductive algebraic group acting on a projective variety $\mathbb{X}$ and let $\mathbb{L}$ be an S-equivariant line bundle on $\mathbb{X}$. Let $O(S)$ be the set of all one parameter subgroups (for short OPS) in S. Take any $x \in \mathbb{X}$ and $\delta \in O(S)$. Then, $\mathbb{X}$ being projective, the morphism $\delta_x : \mathbb{G}_m \to \mathbb{X}$ given by $t \mapsto \delta(t)x$ extends to a morphism $\widetilde{\delta}_x : \mathbb{A}^1 \to \mathbb{X}$. Following Mumford, define a number $\mu^{\mathbb{L}}(x, \delta)$ as follows: Let $x_o \in \mathbb{X}$ be the point $\widetilde{\delta}_x(0)$. Since x_o is $\mathbb{G}_m$-invariant via δ, the fiber of $\mathbb{L}$ over x_o is a $\mathbb{G}_m$-module; in particular, it is given by a character of $\mathbb{G}_m$. This integer is defined as $\mu^{\mathbb{L}}(x, \delta)$.

Let V be a finite dimensional representation of S and let $i : \mathbb{X} \hookrightarrow \mathbb{P}(V)$ be an S-equivariant embedding. Take $\mathbb{L} := i^*(\mathcal{O}(1))$. Let $\lambda \in O(S)$ and let $\{e_1, \ldots, e_n\}$ be a basis of V consisting of eigenvectors, i.e., $\lambda(t) \cdot e_l = t^{\lambda_l} e_l$, for $l = 1, \ldots, n$. For any $x \in \mathbb{X}$, write $i(x) = [\sum_{l=1}^n x_l e_l]$. Then, as it is easy to see, we have [17, Proposition 2.3, p. 51]

$$\mu^{\mathbb{L}}(x, \lambda) = \max_{l : x_l \neq 0} (-\lambda_l). \tag{5}$$

We record the following standard properties of $\mu^{\mathbb{L}}(x, \delta)$ (cf. [17, Chapter 2, §1]):

Proposition 3.3. *For any $x \in \mathbb{X}$ and $\delta \in O(S)$, we have the following (for any S-equivariant line bundles $\mathbb{L}, \mathbb{L}_1, \mathbb{L}_2$):*

(a) *$\mu^{\mathbb{L}_1 \otimes \mathbb{L}_2}(x, \delta) = \mu^{\mathbb{L}_1}(x, \delta) + \mu^{\mathbb{L}_2}(x, \delta)$.*
(b) *If $\mu^{\mathbb{L}}(x, \delta) = 0$, then any element of $H^0(\mathbb{X}, \mathbb{L})^S$ which does not vanish at x does not vanish at $\lim_{t\to 0} \delta(t)x$ as well.*
(c) *For any projective S-variety $\mathbb{X}'$ together with an S-equivariant morphism $f : \mathbb{X}' \to \mathbb{X}$ and any $x' \in \mathbb{X}'$, we have $\mu^{f^*\mathbb{L}}(x', \delta) = \mu^{\mathbb{L}}(f(x'), \delta)$.*

(d) (*Hilbert–Mumford criterion*) *Assume that* $\mathbb{L}$ *is ample. Then,* $x \in \mathbb{X}$ *is semistable* (*resp. stable*) *with respect to* $\mathbb{L}$ *if and only if* $\mu^{\mathbb{L}}(x,\delta) \geqslant 0$ (*resp.* $\mu^{\mathbb{L}}(x,\delta) > 0$), *for all non-constant* $\delta \in O(S)$.

Lemma 3.4. *For any* $\lambda \in \Lambda^{++}$, *the set* $X^s(\lambda)$ *of stable points* (*in* $X = G/B$) *is nonempty.*

Proof. Consider the embedding

$$i_\lambda : X \hookrightarrow \mathbb{P}\big(V(\lambda)\big), \qquad gB \mapsto [gv_\lambda],$$

where v_λ is a highest weight vector in $V(\lambda)$. Then, the line bundle $\mathcal{O}(1)$ over $\mathbb{P}(V(\lambda))$ restricts to the line bundle $\mathcal{L}(\lambda)$ on X via i_λ (as can be easily seen).

Consider the open subset $U_\lambda \subset X$ defined by $U_\lambda = \{gB \in X\colon\ gv_\lambda$ has a nonzero component in each of the weight spaces $V(\lambda)_{w\lambda}$ of weight $w\lambda$, for all $w \in W\}$.

Since $V(\lambda)$ is an irreducible G-module, it is easy to see that U_λ is nonempty. We claim that

$$U_\lambda \subset X^s(\lambda). \tag{6}$$

By the Hilbert–Mumford criterion (cf. Proposition 3.3(d)), it suffices to prove that for any $gB \in U_\lambda$, the Mumford index

$$\mu^{\mathcal{L}(\lambda)}(gB,\sigma) > 0, \tag{7}$$

for any non-constant one parameter subgroup $\sigma : \mathbb{G}_m \to T$. Express

$$gv_\lambda = \sum_{\mu \in X(T)} v_\mu,$$

as a sum of weight vectors. Let $\dot\sigma$ be the derivative of σ considered as an element of $\mathfrak{t}$. Then, by the identity (5),

$$\mu^{\mathcal{L}(\lambda)}(gB,\sigma) = \max_{\substack{\mu \in X(T):\\ v_\mu \neq 0}} \big\{-\mu(\dot\sigma)\big\} \geqslant \max_{w \in W}\big\{\lambda(-w\dot\sigma)\big\}, \quad \text{since } gB \in U_\lambda. \tag{8}$$

Choose $w' \in W$ such that $-w'\dot\sigma \in \mathfrak{t}_+$. Since σ is non-constant, $-w'\dot\sigma \neq 0$. We next claim that

$$\lambda\big(-w'\dot\sigma\big) > 0. \tag{9}$$

To prove this, first observe that any fundamental weight ω_j belongs to $\bigoplus_{i=1}^{\ell} \mathbb{Q}_{>0}\alpha_i$. (One could check this case by case for any simple group from [6, Planche I–IX]. Alternatively, one can give a uniform proof as well.) Thus, by the decomposition (3), since $-\omega'\dot\sigma \neq 0 \in \mathfrak{t}_+$, we get (9). In particular, by (8), $\mu^{\mathcal{L}(\lambda)}(gB,\sigma) > 0$, proving (7). This proves the lemma. □

Proposition 3.5. *For $\lambda \in \Lambda^{++}$, $X^s(\lambda) \neq X^{ss}(\lambda)$ if and only if there exists $w \in W$ and x_j such that $\lambda(wx_j) = 0$, where $x_i \in \mathfrak{t}$ is defined by (4).*

Proof. Assume first that $X^s(\lambda) \neq X^{ss}(\lambda)$. Take $x \in X^{ss}(\lambda) \setminus X^s(\lambda)$. Then, by the Hilbert–Mumford criterion Proposition 3.3(d), there exists a non-constant one parameter subgroup δ in T such that $\mu^{\mathcal{L}(\lambda)}(x, \delta) = 0$. Since both of $X^s(\lambda)$ and $X^{ss}(\lambda)$ are $N(T)$-stable under the left multiplication on X by $N(T)$ (by Proposition 3.3(d)), we can assume that δ is G-dominant, i.e., the derivative $\dot{\delta} \in \mathfrak{t}_+$. Thus, by Proposition 3.3(b), $x_o := \lim_{t\to 0} \delta(t)x \in X^{ss}(\lambda)$, since x is semistable. Let G^δ be the fixed point subgroup of G under the conjugation action by δ. Then, G^δ is a (connected) Levi subgroup of G. Let W^{G^δ} be the set of minimal length coset representatives in the cosets W/W_{G^δ}, where $W_{G^\delta} \subset W$ is the Weyl group of G^δ. The fixed point set of X under the left multiplication by δ is given by $X^\delta = \bigsqcup_{v\in W^{G^\delta}} G^\delta v^{-1}B/B$. Let $w \in W^{G^\delta}$ be such that $x_o \in G^\delta w^{-1}B/B$. Thus, by [15, Lemma 3.4],

$$w^{-1}\lambda \in \sum_{\alpha_i \in \Delta(G^\delta)} \mathbb{Z}\alpha_i, \tag{10}$$

where $\Delta(G^\delta) \subset \Delta$ is the set of simple roots of G^δ. Since δ is non-constant, G^δ is a proper Levi subgroup. Take $\alpha_j \in \Delta \setminus \Delta(G^\delta)$. Then, by (10), $\lambda(wx_j) = 0$.

Conversely, assume that

$$\lambda(wx_j) = 0, \tag{11}$$

for some $w \in W$ and some x_j. For any $1 \leqslant i \leqslant \ell$, let L_i be the Levi subgroup containing T such that $\Delta(L_i) = \Delta \setminus \{\alpha_i\}$. By the assumption (11), $w^{-1}\lambda \in \sum_{\alpha_i \in \Delta(L_j)} \mathbb{Z}\alpha_i$. Moreover, we can choose $w \in W^{L_j}$ and hence $w^{-1}\lambda$ is a dominant weight for L_j. In particular, $v_{w^{-1}\lambda}$ is a highest weight vector for L_j, where $v_{w^{-1}\lambda}$ is a nonzero vector of (extremal) weight $w^{-1}\lambda$ in $V(\lambda)$. (To prove this, observe that $|w^{-1}\lambda + \alpha_i| > |\lambda|$ for any $\alpha_i \in \Delta(L_j)$, and hence $w^{-1}\lambda + \alpha_i$ cannot be a weight of $V(\lambda)$.) Thus, the L_j-submodule $V_{L_j}(w^{-1}\lambda)$ of $V(\lambda)$ generated by $v_{w^{-1}\lambda}$ is an irreducible L_j-module. By [15, Lemma 3.1], applied to the L_j-module $V_{L_j}(w^{-1}\lambda)$, we get that $V_{L_j}(w^{-1}\lambda)$ contains the zero weight space. Hence, by [15, Lemma 3.4], there exists a $g \in L_j$ such that $gw^{-1}B \in X^{ss}(\lambda)$. Define the one parameter subgroup $\delta_j := \operatorname{Exp}(zx_j)$. Then, $\mu^{\mathcal{L}(\lambda)}(gw^{-1}B, \delta_j) = \mu^{\mathcal{L}(\lambda)}(w^{-1}B, \delta_j)$, since g fixes δ_j. But, $\mu^{\mathcal{L}(\lambda)}(w^{-1}B, \delta_j) = 0$, by (5) (due to the assumption (11)). Thus, $gw^{-1}B \notin X^s(\lambda)$ by Proposition 3.3(d). $\square$

For any $w \in W$ and $1 \leqslant i \leqslant \ell$, define the hyperplane

$$H_{w,i} := \bigl\{\lambda \in \Lambda(\mathbb{R}):\ \lambda(wx_i) = 0\bigr\}.$$

Decompose into connected components:

$$\Lambda^{++}(\mathbb{R}) \setminus \left(\bigcup_{w \in W,\, 1 \leqslant i \leqslant \ell} H_{w,i} \right) = \bigsqcup_{k=1}^{N} C_k.$$

The following corollary follows immediately from Proposition 3.5 and [11, Theorems 3.3.2 and 3.4.2].

Corollary 3.6. *With the notation as above, $\{C_1, \ldots, C_N\}$ are precisely the GIT classes of maximal dimension (equal to dim $\mathfrak{t}$).*

Lemma 3.7. *For any GIT class C_k (of maximal dimension) and any $\lambda \in \Gamma$, the line bundle $\mathcal{L}(\lambda)$ descends as a line bundle on the GIT quotient $X_T(C_k)$. We denote this line bundle by $\widehat{\mathcal{L}}_{C_k}(\lambda)$.*

Proof. By Theorem 3.1, for any $\lambda \in \Lambda^{++} \cap \Gamma$, the line bundle $\mathcal{L}(\lambda)$ on X descends to a line bundle on $X_T(\lambda)$. Hence, for any $\lambda \in \Gamma \cap C_k$, the line bundle $\mathcal{L}(\lambda)$ descends to a line bundle $\widehat{\mathcal{L}}_{C_k}(\lambda)$ on $X_T(C_k)$.

Let $\mathbb{Z}(\Gamma \cap C_k)$ denote the subgroup of Γ generated by the semigroup $\Gamma \cap C_k$. For any $\lambda = \lambda_1 - \lambda_2 \in \mathbb{Z}(\Gamma \cap C_k)$ (for $\lambda_1, \lambda_2 \in \Gamma \cap C_k$), define

$$\widehat{\mathcal{L}}_{C_k}(\lambda) = \widehat{\mathcal{L}}_{C_k}(\lambda_1) \otimes \widehat{\mathcal{L}}_{C_k}(\lambda_2)^*.$$

We now show that $\widehat{\mathcal{L}}_{C_k}(\lambda)$ is well defined, i.e., it does not depend upon the choice of the decomposition $\lambda = \lambda_1 - \lambda_2$ as above. Take another decomposition $\lambda = \lambda_1' - \lambda_2'$, with $\lambda_1', \lambda_2' \in \Gamma \cap C_k$. Thus, $\lambda_1 + \lambda_2' = \lambda_1' + \lambda_2 \in \Gamma \cap C_k$ (since $\Gamma \cap C_k$ is a semigroup). In particular, $\widehat{\mathcal{L}}_{\mathbb{C}_k}(\lambda_1 + \lambda_2') \simeq \widehat{\mathcal{L}}_{C_k}(\lambda_1' + \lambda_2)$.

But, from the uniqueness of $\widehat{L}_{C_k}(\lambda)$ (cf. [19, §3]), we have $\widehat{\mathcal{L}}_{C_k}(\lambda_1 + \lambda_2') \simeq \widehat{\mathcal{L}}_{C_k}(\lambda_1) \otimes \widehat{\mathcal{L}}_{C_k}(\lambda_2')$. This proves the assertion that $\mathcal{L}_{C_k}(\lambda)$ is well defined.

Observe that, by definition, C_k is an open convex cone in $\Lambda(\mathbb{R})$. We next claim that

$$\mathbb{Z}(\Gamma \cap C_k) = \Gamma. \tag{12}$$

Take a $\mathbb{Z}$-basis $\{\gamma_1, \ldots, \gamma_\ell\}$ of Γ and let $d := \max_i \|\gamma_i\|$, with respect to a norm $\|\cdot\|$ on $\Lambda(\mathbb{R})$. Take a 'large enough' $\gamma \in \Gamma \cap C_k$ such that the closed ball $B(\gamma, d)$ of radius d centered at γ is contained in C_k. Then, for any $1 \leqslant i \leqslant \ell$, $\gamma + \gamma_i \in B(\gamma, d)$ and hence γ, $\gamma + \gamma_i \in \Gamma \cap C_k$ for any i. Thus, each $\gamma_i \in \mathbb{Z}(\Gamma \cap C_k)$ and hence $\Gamma = \mathbb{Z}(\Gamma \cap C_k)$, proving the assertion (12). Thus, the lemma is proved. □

4. The main result and its proof

Let $\mu_0 : \Lambda^+ \to \mathbb{Z}_+$ be the function: $\mu_0 = \dim V(\lambda)_0$, where $V(\lambda)_0$ is the 0-weight space of $V(\lambda)$. Following the notation from Sections 2 and 3, the following is our main result (cf. Section 1 for a brief discussion of similar results obtained by others by different methods).

Theorem 4.1. *Let G be a connected, adjoint, simple algebraic group. Let $\bar{\mu} = \mu + \Gamma$ be a coset of Γ in Q, where Γ is as in Theorem 3.1. Then, for any GIT class C_k (of maximal dimension), $1 \leqslant k \leqslant N$, there exists a polynomial $f_{\bar{\mu},k} : \Lambda(\mathbb{R}) \to \mathbb{R}$ with rational coefficients of degree $\leqslant \dim_{\mathbb{C}} X - \ell$, such that*

$$f_{\bar{\mu},k}(\lambda) = \mu_0(\lambda), \quad \text{for all } \lambda \in \bar{C}_k \cap \bar{\mu}, \tag{13}$$

where $\bar{C}_k$ is the closure of C_k inside $\Lambda(\mathbb{R})$. Further, $f_{\Gamma,k}$ has constant term 1.

Proof. By the Borel–Weil theorem, for any $\lambda \in \Lambda^+$,

$$\mu_0(\lambda) = \dim\bigl(H^0\bigl(X, \mathcal{L}(\lambda)\bigr)^T\bigr), \tag{14}$$

since

$$\dim\bigl(V(\lambda)_0\bigr) = \dim\bigl(\bigl(V(\lambda)^*\bigr)_0\bigr).$$

Moreover, by the Borel–Weil–Bott theorem, for $\lambda \in \Lambda^+$,

$$H^p\bigl(X, \mathcal{L}(\lambda)\bigr) = 0, \quad \text{for all } p > 0. \tag{15}$$

We first prove the identity (13) for $\lambda \in C_k \cap \bar{\mu}$:

Take $\lambda \in C_k \cap \bar{\mu}$. Let $\pi : X^{ss}(C_k) \to X_T(C_k)$ be the standard quotient map. For any T-equivariant sheaf $\mathcal{S}$ on $X^{ss}(C_k)$, define the T-invariant direct image sheaf $\pi_*^T(\mathcal{S})$ as the sheaf on $X_T(C_k)$ with sections $U \mapsto \Gamma(\pi^{-1}(U), \mathcal{S})^T$. Then, by Lemma 3.7, and the projection formula for π_*^T,

$$\pi_*^T\bigl(\mathcal{L}(\lambda)\bigr) \simeq \pi_*^T\bigl(\mathcal{L}(\mu)\bigr) \otimes \widehat{\mathcal{L}}_{C_k}(\lambda - \mu). \tag{16}$$

By [19, Remark 3.3(i)] and (15), we get

$$\begin{aligned} H^p\bigl(X_T(C_k), \pi_*^T\bigl(\mathcal{L}(\lambda)\bigr)\bigr) &\simeq H^0\bigl(X, \mathcal{L}(\lambda)\bigr)^T, \quad \text{for } p = 0 \\ &= 0, \quad \text{otherwise}. \end{aligned} \tag{17}$$

Thus, for $\lambda \in C_k \cap \bar{\mu}$, by (14),

$$\mu_0(\lambda) = \chi\bigl(X_T(C_k), \pi_*^T\bigl(\mathcal{L}(\lambda)\bigr)\bigr), \tag{18}$$

where for any projective variety Y and a coherent sheaf $\mathcal{S}$ on Y, we define the Euler–Poincaré characteristic

$$\chi(Y, \mathcal{S}) := \sum_{i \geqslant 0} (-1)^i \dim H^i(Y, \mathcal{S}).$$

Now, take a basis (as a $\mathbb{Z}$-module) $\{\gamma_1, \dots, \gamma_\ell\}$ of the lattice $\Gamma \subset \Lambda(\mathbb{R})$. Then, for any $\lambda = \mu + \sum_{i=1}^{\ell} a_i\gamma_i \in \bar{\mu}$, with $a_i \in \mathbb{Z}$, we have by (16),

$$\pi_*^T\big(\mathcal{L}(\lambda)\big) \simeq \pi_*^T\big(\mathcal{L}(\mu)\big) \otimes \widehat{\mathcal{L}}_{C_k}\Big(\sum a_i\gamma_i\Big). \tag{19}$$

Thus, by the Riemann–Roch theorem for singular varieties (cf. [12, Theorem 18.3]) applied to the sheaf $\pi_*^T(\mathcal{L}(\lambda))$, we get for any $\lambda = \mu + \sum a_i\gamma_i \in \bar{\mu}$,

$$\chi\big(X_T(C_k), \pi_*^T\big(\mathcal{L}(\lambda)\big)\big) = \sum_{n\geqslant 0} \int_{X_T(C_k)} \frac{(a_1c_1(\gamma_1) + \cdots + a_\ell c_1(\gamma_\ell))^n}{n!} \cap \tau\big(\pi_*^T\big(\mathcal{L}(\mu)\big)\big), \tag{20}$$

where $\tau(\pi_*^T(\mathcal{L}(\mu)))$ is a certain class in the chow group $A_*(X_T(C_k)) \otimes_{\mathbb{Z}} \mathbb{Q}$ and $c_1(\gamma_i)$ is the first Chern class of the line bundle $\widehat{\mathcal{L}}_{C_k}(\gamma_i)$. Combining (18) and (20), we get that for any $\lambda \in C_k \cap \bar{\mu}$, $\mu_0(\lambda)$ is a polynomial $f_{\bar{\mu},k}$ with rational coefficients in the variables $\{a_i\} : \lambda = \mu + \sum_{i=1}^{\ell} a_i\gamma_i$.

Since $X^s(C_k) \neq \emptyset$ by Lemma 3.4, $\dim(X_T(C_k)) = \dim X - \ell$. Thus, $\deg f_{\bar{\mu},k} \leqslant \dim X - \ell$. This proves the identity (13) for $\lambda \in C_k \cap \bar{\mu}$.

We now come to the proof of the identity (13) for any $\lambda \in \bar{C}_k \cap \bar{\mu}$:

Let $P = P_\lambda \supset B$ be the unique parabolic subgroup such that the line bundle $\mathcal{L}(\lambda)$ descends as an ample line bundle (denoted $\mathcal{L}^P(\lambda)$) on $X^P := G/P$ via the standard projection $q : G/B \to G/P$. Fix any $\mu' \in C_k \cap \Lambda$. By [19, §1.2], applied to $q : G/B \to G/P$, we get that $q^*(\mathcal{L}^P(\lambda))$ is adapted to the stratification on X induced from $q^*(\mathcal{L}^P(\lambda)) + \epsilon\mathcal{L}(\mu')$, for any small rational $\epsilon > 0$ (cf. [19] for the terminology). Thus, by [19, Theorem 3.2.a and Remarks 3.3], since $\lambda + \epsilon\mu' \in C_k$, we get that (for any $\lambda \in \bar{C}_k \cap \bar{\mu}$)

$$\mu_0(\lambda) = \chi\big(X_T(C_k), \pi_*^T q^*\big(\mathcal{L}^P(\lambda)\big)\big) = \chi\big(X_T(C_k), \pi_*^T\big(\mathcal{L}(\lambda)\big)\big). \tag{21}$$

Hence, the identity (18) is established for any $\lambda \in \bar{C}_k \cap \bar{\mu}$. Thus, by the above proof, $\mu_0(\lambda) = f_{\bar{\mu},k}$, where $f_{\bar{\mu},k}$ is the polynomial given above. This proves the identity (13) for all $\lambda \in \bar{C}_k \cap \bar{\mu}$.

By the formula (20), the constant term of $f_{\Gamma,k}$ is equal to

$$\chi\big(X_T(C_k), \pi_*^T\big(\mathcal{L}(0)\big)\big),$$

which is 1 by the identity (21), since $\mu_0(0) = 1$. This completes the proof of the theorem. □

Remark 4.2. (a) By a similar proof, we can obtain a piecewise polynomial behavior of the dimension of any weight space (of a fixed weight μ) in any finite dimensional irreducible representation $V(\lambda)$, by considering the GIT theory associated to the T-equivariant line bundle $\mathcal{L}(\lambda)$ twisted by the character μ^{-1}.

(b) By a similar proof, we can also obtain a piecewise polynomial behavior of the dimension of H-invariant subspace in any finite dimensional irreducible representation $V(\lambda)$ of G, where $H \subset G$ is a reductive subgroup. In this case, we will need to apply the GIT theory to the same line bundle $\mathcal{L}(\lambda)$ on X but with respect to the group H. However, in this general case, we do not have a precise description of the lattice Γ as in Theorem 3.1, nor do we have an explicit description of the GIT classes of maximal dimension as in Corollary 3.6.

(c) As pointed out by Kapil Paranjape, we can obtain the polynomial behavior of $\chi(X_T(C_k), \pi_*^T(\mathcal{L}(\lambda)))$ as in the above proof (by using the Riemann–Roch theorem) more simply by applying Snapper's theorem (cf. [14, Theorem in Section 1 of Chap. 1]). However, the use of Riemann–Roch theorem gives a more precise result.

5. Examples of rank 2 groups

In this section, we explicitly write down the quasi-polynomial which gives the dimensions of the T-invariant subspaces in irreducible representations of any rank 2 group. We follow the indexing convention as in [6, Planche I, II, IX]. The result for G_2 was communicated to us by M. Vergne, who used the Kostant formula for the multiplicity of a weight and the Szenes–Vergne formula via iterated residues for computation of the individual Kostant partition function. The results for A_2 and B_2 can be obtained similarly (and is also available in [7, Exercises 9, 10 of §9]). Alternatively, for A_2 (resp. B_2) we can use the branching of the $\mathrm{GL}_3(\mathbb{C})$-reprsentations restricted to $\mathrm{GL}_2(\mathbb{C})$ (resp. the branching of $\mathrm{SO}_5(\mathbb{C})$-representations restricted to $\mathrm{SO}_4(\mathbb{C})$).

In the following, as earlier, ω_i (resp. α_i) denotes the i-th fundamental weight (resp. simple root). As in Section 1, $\mu_0 : \Lambda^+ \to \mathbb{Z}$ denotes the function which takes λ to the dimension of the zero weight space of $V(\lambda)$. The notation Γ and $\widetilde{\Lambda}$ is as in Theorem 3.1.

$$A_2\colon\quad \Gamma = Q = \{m\omega_1 + n\omega_2\colon\ 3|m+2n\},$$
$$\Lambda^+ = \{m\omega_1 + n\omega_2\colon\ m,n \geqslant 0 \text{ and } 3|m+2n\}.$$

The polynomials are

$$\mu_0(m\omega_1 + n\omega_2) = \begin{cases} m+1, & \text{if } 0 \leqslant m \leqslant n \text{ and } 3|m+2n, \\ n+1, & \text{if } 0 \leqslant n \leqslant m \text{ and } 3|m+2n. \end{cases}$$

$$B_2\colon\quad \Gamma = \mathbb{Z}\alpha_1 + \mathbb{Z}2\alpha_2 = 2\widetilde{\Lambda}, \Lambda^+ \cap \Gamma = \{2m\omega_1 + 2n\omega_2\colon\ m,n \geqslant 0\},$$
$$Q = \mathbb{Z}\omega_1 + \mathbb{Z}2\omega_2.$$

The polynomials are

$$\mu_0(2m\omega_1 + 2n\omega_2) = 1 + m + n + 2mn, \quad m,n \geqslant 0,$$
$$\mu_0\big((2m+1)\omega_1 + 2n\omega_2\big) = (2n+1)(m+1), \quad m,n \geqslant 0,$$
$$G_2\colon\quad \Gamma = \mathbb{Z}6\alpha_1 + \mathbb{Z}2\alpha_2.$$

For $k = (k_1, k_2) \in \{0, \dots, 5\} \times \{0, 1\}$, let

$$\mu_k := k_1\alpha_1 + k_2\alpha_2.$$

Then, $\{\mu_k\}_k$ provides a set of coset representatives for Q/Γ. Let $\bar{\mu}_k$ be the coset $\mu_k + \Gamma$. We express a weight $\lambda \in Q = \Lambda$ in terms of the simple roots: $\lambda = m\alpha_1 + n\alpha_2$. Consider the polynomial

$$\theta(m,n) = \frac{1}{9}m^4 - \frac{29}{36}m^3n + \frac{17}{8}m^2n^2 - \frac{29}{12}mn^3 + n^4 - \frac{7}{36}m^3 + \frac{2}{3}m^2n - \frac{1}{2}mn^2 - \frac{1}{12}n^3 - \frac{19}{24}m^2 + 3mn - \frac{21}{8}n^2.$$

Define the polynomials:

$$f^o_{(0,0)}(m,n) = \frac{1}{4}m + \frac{1}{12}n + 1,$$
$$f^o_{(1,0)}(m,n) = f^o_{(1,1)}(m,n) = f^o_{(4,1)}(m,n) = \frac{17}{36}m - \frac{13}{36}n + \frac{29}{72},$$
$$f^o_{(2,0)}(m,n) = \frac{25}{36}m - \frac{29}{36}n + 5/9,$$
$$f^o_{(3,0)}(m,n) = f^o_{(3,1)}(m,n) = f^o_{(0,1)}(m,n) = 1/4m + 1/12n + 5/8,$$
$$f^o_{(4,0)}(m,n) = \frac{17}{36}m - \frac{13}{36}n + \frac{7}{9},$$
$$f^o_{(5,0)}(m,n) = f^o_{(2,1)}(m,n) = f^o_{(5,1)}(m,n) = \frac{25}{36}m - \frac{29}{36}n + \frac{13}{72}.$$

Finally, define

$$f_k(m,n) = \theta(m,n) + f^o_k(m,n).$$

Then, for any $k \in \{0, \dots, 5\} \times \{0, 1\}$,

$$\mu_0(\lambda) = f_k(m,n), \quad \text{for } \lambda = m\alpha_1 + n\alpha_2 \in \bar{\mu}_k \cap \Lambda^+.$$

Remark 5.1. In the case of A_2, there are exactly two chambers (i.e., GIT classes of maximal dimension) and $\Gamma = Q$ (by Theorem 3.1). Similarly, for B_2, there is only one chamber and Q/Γ is of order 2 (by Theorem 3.1). Thus, in these cases, our Theorem 4.1 gives an optimal result in terms of the number of distinct polynomials needed to fully describe the function μ_0 on Λ^+. For G_2, there is only one chamber and Q/Γ is of order 12 (by Theorem 3.1). Thus, in this case, our Theorem 4.1 gives 12 polynomials, whereas we only need 6 polynomials to describe μ_0.

6. The example of PGL_4

In this section, we compute the dimension of the zero weight space of any irreducible representation of $G = \mathrm{PGL}_4(\mathbb{C})$.

Theorem 6.1. *For an irreducible representation of* $\mathrm{GL}_4(\mathbb{C})$ *with highest weight* $(\lambda_1 \geqslant \lambda_2 \geqslant \lambda_3 \geqslant \lambda_4)$, *and with trivial central character, i.e.,* $\lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 = 0$, *the dimension* $d(\lambda_1, \lambda_2, \lambda_3, \lambda_4)$ *of the zero weight space is given as a piecewise polynomial in the domain* $\lambda_1 \geqslant \lambda_2 \geqslant \lambda_3 \geqslant \lambda_4$ *as follows*:

(1) $\lambda_2 \leqslant 0$, *where it is given by the polynomial*

$$p_1(\lambda_1, \lambda_2, \lambda_3, \lambda_4) = \frac{1}{2}(\lambda_2 - \lambda_3 + 1)(\lambda_3 - \lambda_4 + 1)(\lambda_2 - \lambda_4 + 2).$$

(2) $\lambda_3 \geqslant 0$, *where it is given by the polynomial*

$$p_2(\lambda_1, \lambda_2, \lambda_3, \lambda_4) = \frac{1}{2}(\lambda_1 - \lambda_2 + 1)(\lambda_2 - \lambda_3 + 1)(\lambda_1 - \lambda_3 + 2).$$

(3) $\lambda_2 \geqslant 0$, $\lambda_3 \leqslant 0$, $\lambda_1 + \lambda_4 \geqslant 0$, *where it is given by the polynomial*

$$p_3(\lambda_1, \lambda_2, \lambda_3, \lambda_4) = \frac{1}{2}(\lambda_4 - \lambda_3 - 1)\bigl(2\lambda_2^2 - \lambda_1\lambda_2 - \lambda_3\lambda_4 - \lambda_1 + \lambda_3 - 2\bigr),$$

(4) $\lambda_2 \geqslant 0$, $\lambda_3 \leqslant 0$, $\lambda_1 + \lambda_4 \leqslant 0$, *where it is given by the polynomial*

$$p_4(\lambda_1, \lambda_2, \lambda_3, \lambda_4) = \frac{1}{2}(\lambda_2 - \lambda_1 - 1)\bigl(2\lambda_3^2 - \lambda_1\lambda_2 - \lambda_3\lambda_4 + \lambda_4 - \lambda_2 - 2\bigr).$$

The automorphism $(\lambda_1, \lambda_2, \lambda_3, \lambda_4) \to (-\lambda_4, -\lambda_3, -\lambda_2, -\lambda_1)$, *which corresponds to taking a representation to its dual, interchanges the regions* (1) *and* (2), *and their polynomials, and similarly the regions* (3) *and* (4) *and their polynomials. Further, we have*

$$p_3 - p_4 = (\lambda_2 + \lambda_3) - (\lambda_2 + \lambda_3)^3.$$

Proof. The method we follow to prove this theorem is based on the restriction of a $\mathrm{GL}_4(\mathbb{C})$ representation to $\mathrm{GL}_3(\mathbb{C})$. We start with an irreducible representation of $\mathrm{GL}_4(\mathbb{C})$ with highest weight $(\lambda_1 \geqslant \lambda_2 \geqslant \lambda_3 \geqslant \lambda_4)$, and with trivial central character, i.e., $\lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 = 0$.

We look at irreducible representations of $\mathrm{GL}_3(\mathbb{C})$ with highest weight $(\mu_1 \geqslant \mu_2 \geqslant \mu_3)$ appearing in this representation of $\mathrm{GL}_4(\mathbb{C})$. Thus, we have

$$\lambda_1 \geqslant \mu_1 \geqslant \lambda_2 \geqslant \mu_2 \geqslant \lambda_3 \geqslant \mu_3 \geqslant \lambda_4.$$

For analyzing the zero weight space, it suffices to consider only those representations of $\mathrm{GL}_3(\mathbb{C})$ with highest weight (μ_1, μ_2, μ_3) satisfying $\mu_1+\mu_2+\mu_3 = 0$; it is actually keeping track of this central character condition (on $\mathrm{GL}_3(\mathbb{C})$) that complicates our analysis.

Denote the dimension of the zero weight space in the irreducible representation of $\mathrm{GL}_4(\mathbb{C})$ with highest weight $(\lambda_1 \geqslant \lambda_2 \geqslant \lambda_3 \geqslant \lambda_4)$ by $d(\lambda_1, \lambda_2, \lambda_3, \lambda_4)$. Similarly, denote the dimension of the zero weight space in the irreducible representation of $\mathrm{GL}_3(\mathbb{C})$ with highest weight (μ_1, μ_2, μ_3) by $d(\mu_1, \mu_2, \mu_3)$; we will always assume that the central character of this representation of $\mathrm{GL}_3(\mathbb{C})$ is trivial, and so $d(\mu_1, \mu_2, \mu_3)$ is a positive integer, explicitly given by the following well-known Lemma 6.2. We remind the reader (cf. Section 5) that the value of $d(\mu_1, \mu_2, \mu_3)$ is a polynomial in (μ_1, μ_2, μ_3) (of degree 1) which depends on whether μ_2 is non-negative or non-positive.

Lemma 6.2. *An irreducible representation of* $\mathrm{GL}_3(\mathbb{C})$ *with highest weight* $(\lambda_1 \geqslant \lambda_2 \geqslant \lambda_3)$, *and with trivial central character, i.e.,* $\lambda_1+\lambda_2+\lambda_3 = 0$, *has zero weight space of dimension*

(1) $\lambda_1 - \lambda_2 + 1$, *if* $\lambda_2 \geqslant 0$, *and*
(2) $\lambda_2 - \lambda_3 + 1$, *if* $\lambda_2 \leqslant 0$.

Denote the interval $[\lambda_1, \lambda_2]$ by I_1 (we abuse the notation $[\lambda_1, \lambda_2]$ which is customarily denoted by $[\lambda_2, \lambda_1]$), the interval $[\lambda_2, \lambda_3]$ by I_2, and the interval $[\lambda_3, \lambda_4]$ by I_3. Our problem lies in choosing integers $\mu_i \in I_i$ such that $\mu_1 + \mu_2 + \mu_3 = 0$.

One lucky situation is when the set $I_j + I_k \subset -I_\ell$, for a triple $\{j, k, \ell\} = \{1, 2, 3\}$, in which case, one can choose $\mu_j \in I_j$, $\mu_k \in I_k$ arbitrarily, and then $\mu_\ell := -(\mu_j + \mu_k)$ automatically belongs to I_ℓ. This is what happens in cases I and II below; but the other cases that we deal with in III, ..., VI, the analysis is considerably more complicated.

Case I: $\lambda_2 \leqslant 0$, and therefore $\lambda_1 \geqslant 0 \geqslant \lambda_2 \geqslant \lambda_3 \geqslant \lambda_4$.

This implies that $\mu_1 \geqslant 0 \geqslant \mu_2 \geqslant \mu_3$; in particular, in this case μ_2 is always $\leqslant 0$. Further, $I_2 + I_3 = [\lambda_2 + \lambda_3, \lambda_3 + \lambda_4]$ is contained in $-I_1 = [-\lambda_2, -\lambda_1]$.

Therefore, by Lemma 6.2,

$$\begin{aligned} d(\lambda_1, \lambda_2, \lambda_3, \lambda_4) &= \sum_{\mu_i \in I_i} d(\mu_1, \mu_2, \mu_3) \\ &= \sum_{\mu_i \in I_i} (\mu_2 - \mu_3 + 1) \\ &= \sum_{\substack{\lambda_2 \geqslant \mu_2 \geqslant \lambda_3 \\ \lambda_3 \geqslant \mu_3 \geqslant \lambda_4}} (\mu_2 - \mu_3 + 1) \\ &= \sum_{\lambda_3 \geqslant \mu_3 \geqslant \lambda_4} \left[\frac{(\lambda_2 + \lambda_3)(\lambda_2 - \lambda_3 + 1) - (\mu_3 - 1)(\lambda_2 - \lambda_3 + 1)}{2} \right] \\ &= \frac{(\lambda_2 + \lambda_3)(\lambda_2 - \lambda_3 + 1)(\lambda_3 - \lambda_4 + 1)}{2} \end{aligned}$$

$$- \frac{(\lambda_2 - \lambda_3 + 1)(\lambda_3 + \lambda_4 - 2)(\lambda_3 - \lambda_4 + 1)}{2}$$
$$= \frac{1}{2}(\lambda_2 - \lambda_3 + 1)(\lambda_3 - \lambda_4 + 1)(\lambda_2 - \lambda_4 + 2).$$

Case II: $\lambda_3 \geqslant 0$, and therefore $\lambda_1 \geqslant \lambda_2 \geqslant \lambda_3 \geqslant 0 \geqslant \lambda_4$.

This implies that $\mu_1 \geqslant \mu_2 \geqslant 0$; in particular, in this case μ_2 is always $\geqslant 0$. Further, $I_1 + I_2 = [\lambda_1 + \lambda_2, \lambda_2 + \lambda_3]$ is contained in $-I_3 = [-\lambda_4, -\lambda_3]$.

Therefore, by Lemma 6.2,

$$\begin{aligned} d(\lambda_1, \lambda_2, \lambda_3, \lambda_4) &= \sum_{\mu_i \in I_i} d(\mu_1, \mu_2, \mu_3) \\ &= \sum_{\mu_i \in I_i} (\mu_1 - \mu_2 + 1) \\ &= \sum_{\substack{\lambda_1 \geqslant \mu_1 \geqslant \lambda_2 \\ \lambda_2 \geqslant \mu_2 \geqslant \lambda_3}} (\mu_1 - \mu_2 + 1) \\ &= \frac{1}{2}(\lambda_1 - \lambda_2 + 1)(\lambda_2 - \lambda_3 + 1)(\lambda_1 - \lambda_3 + 2). \end{aligned}$$

Rest of the cases: $\lambda_2 > 0 > \lambda_3$, and therefore $\lambda_1 \geqslant \lambda_2 > 0 > \lambda_3 \geqslant \lambda_4$.

Given that $\mu_i \in I_i$ with $\mu_1 + \mu_2 + \mu_3 = 0$, we find that

$$\lambda_3 \geqslant -(\mu_1 + \mu_2) \geqslant \lambda_4,$$

and therefore,

$$-\lambda_4 - \mu_2 \geqslant \mu_1 \geqslant -\lambda_3 - \mu_2.$$

Since we already have

$$\lambda_1 \geqslant \mu_1 \geqslant \lambda_2,$$

μ_1 is in the intersection of the two intervals $-\lambda_4 - \mu_2 \geqslant \mu_1 \geqslant -\lambda_3 - \mu_2$ and $\lambda_1 \geqslant \mu_1 \geqslant \lambda_2$. Therefore, μ_1 must belong to the interval

$$I(\mu_2) := \left[\min(\lambda_1, -\lambda_4 - \mu_2), \max(-\lambda_3 - \mu_2, \lambda_2)\right].$$

Conversely, it is clear that if $\mu_2 \in I_2$, $\mu_1 \in I(\mu_2)$, and $\mu_3 = -(\mu_1 + \mu_2)$, then each of the μ_i belongs to I_i.

Thus, we can start the calculation of the dimension of the zero weight space in the representation of $\mathrm{GL}_4(\mathbb{C})$ with highest weight $(\lambda_1 \geqslant \lambda_2 \geqslant \lambda_3 \geqslant \lambda_4)$, and with trivial central character as

$$d(\lambda_1,\lambda_2,\lambda_3,\lambda_4)=\sum_{\mu_i\in I_i} d(\mu_1,\mu_2,\mu_3)$$
$$=\sum_{\substack{\lambda_2\geqslant\mu_2>0\\ \mu_1\in I(\mu_2)}}(\mu_1-\mu_2+1)+\sum_{\substack{0\geqslant\mu_2\geqslant\lambda_3\\ \mu_1\in I(\mu_2)}}(\mu_2-\mu_3+1).$$

At this point, we assume that $\lambda_1+\lambda_4\geqslant 0$. In this case, if $\mu_2\geqslant 0$, then $\lambda_1\geqslant -\lambda_4-\mu_2$. On the other hand, under the same condition (i.e., $\lambda_1+\lambda_4\geqslant 0$), if $\mu_2\leqslant 0$, then $-\lambda_3-\mu_2\geqslant\lambda_2$. This means that for $\mu_2\geqslant 0$, $I(\mu_2):=[\min(\lambda_1,-\lambda_4-\mu_2),\max(-\lambda_3-\mu_2,\lambda_2)]=[-\lambda_4-\mu_2,\max(-\lambda_3-\mu_2,\lambda_2)]$, and for $\mu_2\leqslant 0$, $I(\mu_2)=[\min(\lambda_1,-\lambda_4-\mu_2),-\lambda_3-\mu_2]$. Therefore, we get

$$d(\lambda_1,\lambda_2,\lambda_3,\lambda_4)=\sum_{\mu_i\in I_i} d(\mu_1,\mu_2,\mu_3)$$
$$=\sum_{\substack{\lambda_2\geqslant\mu_2>0\\ -\lambda_4-\mu_2\geqslant\mu_1\geqslant\max(-\lambda_3-\mu_2,\lambda_2)}}(\mu_1-\mu_2+1)$$
$$+\sum_{\substack{0\geqslant\mu_2\geqslant\lambda_3\\ \min(\lambda_1,-\lambda_4-\mu_2)\geqslant\mu_1\geqslant-\lambda_3-\mu_2}}(\mu_1+2\mu_2+1).$$

At this point, we assume that besides $\lambda_1+\lambda_4\geqslant 0$, we also have $2\lambda_2+\lambda_3\geqslant 0$; this latter condition has the effect that the region $[\lambda_2,0]$ where μ_2 is supposed to belong, splits into two regions where $\max(\lambda_3-\mu_2,\lambda_2)$ takes the two possible options. Similarly, the region $[0,\lambda_3]$ where μ_2 belongs in the second sum gets divided into two regions.

Case III: $\lambda_1+\lambda_4\geqslant 0$ and $2\lambda_2+\lambda_3\geqslant 0$.

$$d(\lambda_1,\lambda_2,\lambda_3,\lambda_4)=\sum_{\mu_i\in I_i} d(\mu_1,\mu_2,\mu_3)$$
$$=\sum_{\substack{\lambda_2\geqslant\mu_2\geqslant-(\lambda_2+\lambda_3)\\ -\lambda_4-\mu_2\geqslant\mu_1\geqslant\lambda_2}}(\mu_1-\mu_2+1)+\sum_{\substack{-(\lambda_2+\lambda_3)>\mu_2>0\\ -\lambda_4-\mu_2\geqslant\mu_1\geqslant-\lambda_3-\mu_2}}(\mu_1-\mu_2+1)$$
$$+\sum_{\substack{0\geqslant\mu_2\geqslant-(\lambda_1+\lambda_4)\\ -\lambda_4-\mu_2\geqslant\mu_1\geqslant-\lambda_3-\mu_2}}(\mu_1+2\mu_2+1)$$
$$+\sum_{\substack{-(\lambda_1+\lambda_4)>\mu_2\geqslant\lambda_3\\ \lambda_1\geqslant\mu_1\geqslant-\lambda_3-\mu_2}}(\mu_1+2\mu_2+1)$$
$$=\frac{1}{4}\big[-4\lambda_1^3-2\lambda_1^2\lambda_2-2\lambda_2^3-6\lambda_1^2\lambda_3+4\lambda_1\lambda_2\lambda_3-4\lambda_2^2\lambda_3-4\lambda_1\lambda_3^2$$
$$-2\lambda_3^3-6\lambda_1^2\lambda_4-4\lambda_1\lambda_2\lambda_4-8\lambda_1\lambda_3\lambda_4-4\lambda_2\lambda_3\lambda_4-4\lambda_3^2\lambda_4$$
$$+4\lambda_1\lambda_4^2+4\lambda_2\lambda_4^2+6\lambda_4^3-12\lambda_1^2-5\lambda_1\lambda_2+\lambda_2^2-7\lambda_1\lambda_3+8\lambda_2\lambda_3+\lambda_3^2$$

$$
\begin{aligned}
&- 34\lambda_1\lambda_4 - 15\lambda_2\lambda_4 - 13\lambda_3\lambda_4 - 20\lambda_4^2 + 5\lambda_1 + 3\lambda_2 + 5\lambda_3 - \lambda_4 + 4\big] \\
&= \frac{1}{4}\big[2\lambda_1^2\lambda_2 - 2\lambda_1\lambda_2^2 - 4\lambda_2^3 - 2\lambda_1^2\lambda_3 - 10\lambda_2^2\lambda_3 - 6\lambda_1\lambda_3^2 - 6\lambda_2\lambda_3^2 \\
&\quad - 4\lambda_3^3 + 2\lambda_1^2 + 4\lambda_1\lambda_2 - 4\lambda_2^2 - 4\lambda_2\lambda_3 - 6\lambda_3^2 + 6\lambda_1 + 4\lambda_2 + 6\lambda_3 + 4\big] \\
&= -\frac{1}{2}(\lambda_1 + \lambda_2 + 2\lambda_3 + 1) \\
&\quad \times \big(-\lambda_1\lambda_2 + 2\lambda_2^2 + \lambda_1\lambda_3 + \lambda_2\lambda_3 + \lambda_3^2 - \lambda_1 + \lambda_3 - 2\big) \\
&= \frac{1}{2}(\lambda_4 - \lambda_3 - 1)\big(2\lambda_2^2 - \lambda_1\lambda_2 - \lambda_3\lambda_4 - \lambda_1 + \lambda_3 - 2\big),
\end{aligned}
$$

where in the third last equality, we have used the equation $\lambda_4 = -(\lambda_1 + \lambda_2 + \lambda_3)$ to write the polynomial in only $\lambda_1, \lambda_2, \lambda_3$.

Case IV: $\lambda_1 + \lambda_4 \geqslant 0$ and $2\lambda_2 + \lambda_3 < 0$.

$$
\begin{aligned}
d(\lambda_1, \lambda_2, \lambda_3, \lambda_4) &= \sum_{\mu_i \in I_i} d(\mu_1, \mu_2, \mu_3) \\
&= \sum_{\substack{\lambda_2 \geqslant \mu_2 > 0 \\ -\lambda_4 - \mu_2 \geqslant \mu_1 \geqslant -\lambda_3 - \mu_2}} (\mu_1 - \mu_2 + 1) \\
&\quad + \sum_{\substack{0 \geqslant \mu_2 \geqslant -(\lambda_1 + \lambda_4) \\ -\lambda_4 - \mu_2 \geqslant \mu_1 \geqslant -\lambda_3 - \mu_2}} (\mu_1 + 2\mu_2 + 1) \\
&\quad + \sum_{\substack{-(\lambda_1 + \lambda_4) > \mu_2 \geqslant \lambda_3 \\ \lambda_1 \geqslant \mu_1 \geqslant -\lambda_3 - \mu_2}} (\mu_1 + 2\mu_2 + 1) \\
&= \frac{1}{4}\big[-2\lambda_1^3 - 2\lambda_1^2\lambda_2 - 2\lambda_1^2\lambda_3 + 2\lambda_1\lambda_2\lambda_3 - 4\lambda_2^2\lambda_3 - 2\lambda_1\lambda_3^2 - 2\lambda_2\lambda_3^2 \\
&\quad - 2\lambda_3^3 - 4\lambda_1^2\lambda_4 - 4\lambda_1\lambda_2\lambda_4 + 4\lambda_2^2\lambda_4 - 4\lambda_1\lambda_3\lambda_4 - 2\lambda_2\lambda_3\lambda_4 - 2\lambda_3^2\lambda_4 \\
&\quad + 2\lambda_2\lambda_4^2 - 2\lambda_3\lambda_4^2 + 2\lambda_4^3 - 5\lambda_1^2 - 3\lambda_1\lambda_2 - 4\lambda_2^2 + 3\lambda_2\lambda_3 + \lambda_3^2 \\
&\quad - 14\lambda_1\lambda_4 - 7\lambda_2\lambda_4 - 7\lambda_4^2 + \lambda_1 - \lambda_2 + \lambda_3 - 5\lambda_4 + 4\big] \\
&= \frac{1}{4}\big[2\lambda_1^2\lambda_2 - 2\lambda_1\lambda_2^2 - 4\lambda_2^3 - 2\lambda_1^2\lambda_3 - 10\lambda_2^2\lambda_3 - 6\lambda_1\lambda_3^2 - 6\lambda_2\lambda_3^2 \\
&\quad - 4\lambda_3^3 + 2\lambda_1^2 + 4\lambda_1\lambda_2 - 4\lambda_2^2 - 4\lambda_2\lambda_3 - 6\lambda_3^2 + 6\lambda_1 + 4\lambda_2 + 6\lambda_3 + 4\big] \\
&= -\frac{1}{2}(\lambda_1 + \lambda_2 + 2\lambda_3 + 1) \\
&\quad \times \big(-\lambda_1\lambda_2 + 2\lambda_2^2 + \lambda_1\lambda_3 + \lambda_2\lambda_3 + \lambda_3^2 - \lambda_1 + \lambda_3 - 2\big) \\
&= \frac{1}{2}(\lambda_4 - \lambda_3 - 1)\big(2\lambda_2^2 - \lambda_1\lambda_2 - \lambda_3\lambda_4 - \lambda_1 + \lambda_3 - 2\big),
\end{aligned}
$$

where again in the third last equality, we have used the equation $\lambda_4 = -(\lambda_1 + \lambda_2 + \lambda_3)$ to write the polynomial in only $\lambda_1, \lambda_2, \lambda_3$.

Case V: $\lambda_1 + \lambda_4 < 0$ and $\lambda_2 + 2\lambda_3 \leqslant 0$.

$$\begin{aligned}
d(\lambda_1,\lambda_2,\lambda_3,\lambda_4) &= \sum_{\mu_i\in I_i} d(\mu_1,\mu_2,\mu_3)\\
&= \sum_{\substack{\lambda_2\geqslant\mu_2>(\lambda_2+\lambda_3)\\ -\lambda_4-\mu_2\geqslant\mu_1\geqslant\lambda_2}} (\mu_1-\mu_2+1) + \sum_{\substack{(\lambda_2+\lambda_3)\geqslant\mu_2\geqslant 0\\ \lambda_1\geqslant\mu_1\geqslant\lambda_2}} (\mu_1-\mu_2+1)\\
&\quad + \sum_{\substack{0>\mu_2>(\lambda_1+\lambda_4)\\ \lambda_1\geqslant\mu_1\geqslant\lambda_2}} (\mu_1+2\mu_2+1) + \sum_{\substack{\lambda_1+\lambda_4\geqslant\mu_2\geqslant\lambda_3\\ \lambda_1\geqslant\mu_1\geqslant-\lambda_3-\mu_2}} (\mu_1+2\mu_2+1)\\
&= \frac{1}{4}\big[-2\lambda_1^3 + 2\lambda_1^2\lambda_2 + 2\lambda_1\lambda_2^2 + 2\lambda_2^3 - 2\lambda_1^2\lambda_3 + 2\lambda_1\lambda_2\lambda_3 + 2\lambda_2^2\lambda_3\\
&\quad - 4\lambda_1\lambda_3^2 + 4\lambda_2\lambda_3^2 + 4\lambda_1\lambda_2\lambda_4 + 2\lambda_2^2\lambda_4 + 4\lambda_1\lambda_3\lambda_4 - 2\lambda_2\lambda_3\lambda_4\\
&\quad + 4\lambda_1\lambda_4^2 + 2\lambda_2\lambda_4^2 + 2\lambda_3\lambda_4^2 + 2\lambda_4^3 - 7\lambda_1^2 + \lambda_2^2 - 7\lambda_1\lambda_3 + 3\lambda_2\lambda_3\\
&\quad - 4\lambda_3^2 - 14\lambda_1\lambda_4 - 3\lambda_3\lambda_4 - 5\lambda_4^2 + 5\lambda_1 - \lambda_2 + \lambda_3 - \lambda_4 + 4\big]\\
&= \frac{1}{4}\big[2\lambda_1^2\lambda_2 - 2\lambda_1\lambda_2^2 - 2\lambda_1^2\lambda_3 + 2\lambda_2^2\lambda_3 - 6\lambda_1\lambda_3^2 + 6\lambda_2\lambda_3^2 + 2\lambda_1^2\\
&\quad + 4\lambda_1\lambda_2 - 4\lambda_2^2 - 4\lambda_2\lambda_3 - 6\lambda_3^2 + 6\lambda_1 + 2\lambda_3 + 4\big]\\
&= \frac{1}{2}(-\lambda_1+\lambda_2-1)\big(-\lambda_1\lambda_2 + \lambda_1\lambda_3 + \lambda_2\lambda_3 + 3\lambda_3^2 - \lambda_1 - 2\lambda_2 - \lambda_3 - 2\big)\\
&= \frac{1}{2}(\lambda_2-\lambda_1-1)\big(2\lambda_3^2 - \lambda_1\lambda_2 - \lambda_3\lambda_4 + \lambda_4 - \lambda_2 - 2\big).
\end{aligned}$$

Case VI: $\lambda_1 + \lambda_4 < 0$ and $\lambda_2 + 2\lambda_3 \geqslant 0$.

$$\begin{aligned}
d(\lambda_1,\lambda_2,\lambda_3,\lambda_4) &= \sum_{\mu_i\in I_i} d(\mu_1,\mu_2,\mu_3)\\
&= \sum_{\substack{\lambda_2\geqslant\mu_2>(\lambda_2+\lambda_3)\\ -\lambda_4-\mu_2\geqslant\mu_1\geqslant\lambda_2}} (\mu_1-\mu_2+1) + \sum_{\substack{(\lambda_2+\lambda_3)\geqslant\mu_2\geqslant 0\\ \lambda_1\geqslant\mu_1\geqslant\lambda_2}} (\mu_1-\mu_2+1)\\
&\quad + \sum_{\substack{0>\mu_2\geqslant\lambda_3\\ \lambda_1\geqslant\mu_1\geqslant\lambda_2}} (\mu_1+2\mu_2+1)\\
&= \frac{1}{4}\big[-2\lambda_1^3 + 2\lambda_1^2\lambda_2 + 2\lambda_1\lambda_2^2 + 2\lambda_2^3 - 2\lambda_1^2\lambda_3 + 2\lambda_1\lambda_2\lambda_3 + 2\lambda_2^2\lambda_3\\
&\quad - 4\lambda_1\lambda_3^2 + 4\lambda_2\lambda_3^2 + 4\lambda_1\lambda_2\lambda_4 + 2\lambda_2^2\lambda_4 + 4\lambda_1\lambda_3\lambda_4 - 2\lambda_2\lambda_3\lambda_4\\
&\quad + 4\lambda_1\lambda_4^2 + 2\lambda_2\lambda_4^2 + 2\lambda_3\lambda_4^2 + 2\lambda_4^3 - 7\lambda_1^2 + \lambda_2^2 - 7\lambda_1\lambda_3 + 3\lambda_2\lambda_3\\
&\quad - 4\lambda_3^2 - 14\lambda_1\lambda_4 - 3\lambda_3\lambda_4 - 5\lambda_4^2 + 5\lambda_1 - \lambda_2 + \lambda_3 - \lambda_4 + 4\big]\\
&= \frac{1}{4}\big[2\lambda_1^2\lambda_2 - 2\lambda_1\lambda_2^2 - 2\lambda_1^2\lambda_3 + 2\lambda_2^2\lambda_3 - 6\lambda_1\lambda_3^2 + 6\lambda_2\lambda_3^2 + 2\lambda_1^2
\end{aligned}$$

$$+ 4\lambda_1\lambda_2 - 4\lambda_2^2 - 4\lambda_2\lambda_3 - 6\lambda_3^2 + 6\lambda_1 + 2\lambda_3 + 4\big]$$
$$= \frac{1}{2}(-\lambda_1 + \lambda_2 - 1)\big(-\lambda_1\lambda_2 + \lambda_1\lambda_3 + \lambda_2\lambda_3 + 3\lambda_3^2 - \lambda_1 - 2\lambda_2 - \lambda_3 - 2\big)$$
$$= \frac{1}{2}(\lambda_2 - \lambda_1 - 1)\big(2\lambda_3^2 - \lambda_1\lambda_2 - \lambda_3\lambda_4 + \lambda_4 - \lambda_2 - 2\big). \qquad \square$$

Remark 6.3. In the case of A_3, there are exactly four chambers (and $\Gamma = Q$ by Theorem 3.1). To identify the chambers given by Corollary 3.6, note that the chambers are defined using the hyperplanes $\lambda_i = 0$ for some i, and hyperplanes $\lambda_i + \lambda_j = 0$ for $i \neq j$ in the convex cone $\lambda_1 \geqslant \lambda_2 \geqslant \lambda_3 \geqslant \lambda_4$ with $\lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 = 0$. But, it is easy to see that the hyperplanes $\lambda_1 = 0$, $\lambda_4 = 0$, $\lambda_1 + \lambda_2 = 0$, $\lambda_1 + \lambda_3 = 0$ all lie on the boundary of the cone $\lambda_1 \geqslant \lambda_2 \geqslant \lambda_3 \geqslant \lambda_4$ with $\lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 = 0$. Therefore, we are left with only the hyperplanes defined by $\lambda_2 = 0$, $\lambda_3 = 0$, and $\lambda_2 + \lambda_3 = 0$. These define the four chambers considered in Theorem 6.1.

Thus, in this case as well, our Theorem 4.1 gives an optimal result in terms of the number of distinct polynomials needed to fully describe the function μ_0 on Λ^+.

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