

An approach towards the Kollár-Peskin problem via the Instanton Moduli Space

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1. Introduction

Kollár and Peskin (cf. [BC, page 278]) asked the following question on complete intersections over the field \mathbb{C} of complex numbers. In this note, the field \mathbb{C} is taken as the base field. By a variety, we mean a complex quasiprojective (reduced) (but not necessarily irreducible) variety.

QUESTION 1.1. *Let $C_t \subset \mathbb{P}^3$ be a family of smooth curves parameterized by the formal disc $D := \text{spec } R$, where R is the formal power series ring $\mathbb{C}[[t]]$ in one variable. Assume that the general member of the family is a complete intersection. Then, is the special member C_0 also a complete intersection?*

By using a construction due to Serre, the above problem is equivalent to the following (cf. [Ku]).

QUESTION 1.2. *Let V_t be a family of rank two vector bundles on \mathbb{P}^3 . Assume that the general member of the family is a direct sum of line bundles. Then, is the special member V_0 also a direct sum of line bundles?*

Let us consider the following slightly weaker version of the above question.

QUESTION 1.3. *Let V_t be a family of rank two vector bundles on \mathbb{P}^3 . Assume that the general member of the family is a trivial vector bundle. Then, is the special member V_0 also a trivial vector bundle?*

In the next section, we show that the above question is equivalent to a question on the nonexistence of algebraic maps from \mathbb{P}^3 to the infinite Grassmannian \mathcal{X} associated to the affine $\text{SL}(2)$. Specifically, we have the following result (cf. Theorem 2.5):

THEOREM 1.4. *Let X be any irreducible projective variety. Then, the following two conditions are equivalent:*

- (a) *Any rank-2 vector bundle \mathcal{F} on $X \times D$ with trivial determinant, such that $\mathcal{F}|_{X \times D^*}$ is trivial, is itself trivial.*
- (b) *There exists no nonconstant morphism $X \rightarrow \mathcal{X}$.*

Thus, Question 1.3 is equivalent to the following question (cf. Question 2.6):

QUESTION 1.5. *Does there exist no nonconstant morphism $\mathbb{P}^3 \rightarrow \mathcal{X}$?*

Let $\text{Mor}_*^d(\mathbb{P}^1, \mathcal{X})$ denote the set of base point preserving morphisms from $\mathbb{P}^1 \rightarrow \mathcal{X}$ of degree d . It is a complex algebraic variety. As we show in Section 3, any morphism $\phi : \mathbb{P}^3 \rightarrow \mathcal{X}$ of degree d , preserving the base points, canonically induces a morphism

$$\hat{\phi} : \mathbb{C}^3 \setminus \{0\} \rightarrow \text{Mor}_*^d(\mathbb{P}^1, \mathcal{X}).$$

Let \mathcal{M}_d be the set of isomorphism classes of rank two vector bundles \mathcal{V} over \mathbb{P}^2 with trivial determinant and with second Chern class d together with a trivialization of $\mathcal{V}|_{\mathbb{P}^1}$. Then, \mathcal{M}_d has a natural variety structure, which will be referred to by the *Donaldson moduli space*. Donaldson showed that there is a natural diffeomorphism between \mathcal{M}_d and the instanton moduli space \mathcal{I}_d of Yang-Mills d -instantons over the flat \mathbb{R}^4 with group $\text{SU}(2)$ modulo based gauge equivalence. As shown by Atiyah, there is a natural embedding

$$i : \text{Mor}_*^d(\mathbb{P}^1, \mathcal{X}) \hookrightarrow \mathcal{M}_d$$

as an open subset (cf. Proposition 3.1). Thus, the morphism $\hat{\phi}$ gives rise to a morphism (still denoted by) $\hat{\phi} : \mathbb{C}^3 \setminus \{0\} \rightarrow \mathcal{M}_d$. Define an action of \mathbb{C}^* on $\mathbb{C}^3 \setminus \{0\}$ by homothety and on \mathbb{P}^2 via:

$$z \cdot [\lambda, \mu, \nu] = [z^{-1}\lambda, \mu, \nu].$$

This gives rise to an action of \mathbb{C}^* on \mathcal{M}_d via the pull-back of bundles. Then, the embedding $\hat{\phi}$ is \mathbb{C}^* -equivariant (cf. Theorem 3.2).

We would like to make the following conjecture (cf. Conjecture 3.3).

CONJECTURE 1.6. *For any $d > 0$, there does not exist any \mathbb{C}^* -equivariant morphism $\hat{f} : \mathbb{C}^3 \setminus \{0\} \rightarrow \mathcal{M}_d$.*

Assuming the validity of the above conjecture 1.6, we get that there is no nonconstant morphism $\phi : \mathbb{P}^3 \rightarrow \mathcal{X}$.

Thus, by Theorem 1.4, assuming the validity of the above Conjecture 1.6, any rank-2 bundle \mathcal{F} on $\mathbb{P}^3 \times D$ with trivial determinant, such that $\mathcal{F}|_{\mathbb{P}^3 \times D^*}$ is trivial, is itself trivial (cf. Corollary 3.4).

As a generalization of the above, we would like to make the following conjecture (cf. Conjecture 3.5).

CONJECTURE 1.7. *For any $n \geq 2$, let \mathcal{X}_n be the infinite Grassmannian associated to the group $G = \text{SL}(n)$, i.e., $\mathcal{X}_n := \text{SL}(n, K)/\text{SL}(n, R)$, where K is the quotient field of R . Then, there does not exist any nonconstant morphism $\phi : \mathbb{P}^{n+1} \rightarrow \mathcal{X}_n$.*

Finally, in Section 4, we recall an explicit construction of the moduli space \mathcal{M}_d via the *monad* construction and show that the \mathbb{C}^* -action on \mathcal{M}_d takes a relatively simple form (cf. Lemma 4.3).

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2. Kollár-Peskine problem and infinite Grassmannian

For more details on the following construction of the infinite Grassmannians, see [K, Chapter 13].

Set $\mathcal{G} = \mathrm{SL}(2, K), \mathcal{P} = \mathrm{SL}(2, R)$, where $K := \mathbb{C}[[t]][t^{-1}]$ denotes the ring of Laurent series in one variable and R is the subring $\mathbb{C}[[t]]$ of power series. The ring homomorphism $R \rightarrow \mathbb{C}, t \mapsto 0$, gives rise to a group homomorphism $\pi : \mathcal{P} \rightarrow \mathrm{SL}(2, \mathbb{C})$. Define $\mathcal{B} = \pi^{-1}(B)$, where $B \subset \mathrm{SL}(2, \mathbb{C})$ is the Borel subgroup consisting of the upper triangular matrices. For any $d \geq 0$, define

$$X_d = \bigcup_{n=0}^d \mathcal{B} \begin{pmatrix} t^n & 0 \\ 0 & t^{-n} \end{pmatrix} \mathcal{P} / \mathcal{P} \subset \mathcal{G} / \mathcal{P}.$$

Then, X_d admits a natural structure of a projective variety and $\cup_{d \geq 0} X_d = \mathcal{G} / \mathcal{P}$. Moreover, X_d is irreducible (of dimension d), and $X_d \hookrightarrow X_{d+1}$ is a closed embedding. In particular, $\mathcal{X} := \mathcal{G} / \mathcal{P}$ is a projective ind-variety.

For any integer $d \geq 0$, consider the set \mathcal{L}_d of R -submodules $L \subset K \otimes_{\mathbb{C}} V$ such that

$$t^d L_o \subset L \subset t^{-d} L_o, \text{ and } \dim_{\mathbb{C}}(L/t^d L_o) = 2d,$$

where $V := \mathbb{C}^2$ and $L_o := R \otimes V$. Let

$$\mathcal{L} := \cup_{d \geq 0} \mathcal{L}_d.$$

Any element of \mathcal{L} is called an R -lattice in $K \otimes_{\mathbb{C}} V$.

The group $\mathrm{SL}(2, K)$ acts canonically on $K \otimes_{\mathbb{C}} V$. Recall the following from [K, Lemma 13.2.14].

LEMMA 2.1. *The map $g \mathrm{SL}(2, R) \mapsto gL_o$ (for $g \in \mathrm{SL}(2, K)$) induces a bijection $\beta : \mathcal{X} \rightarrow \mathcal{L}$.*

Let X be any irreducible projective variety and let \mathcal{F} be a rank two vector bundle on $X \times D$ with trivial determinant, where $D := \mathrm{spec} R$. Fix a trivialization of the determinant of \mathcal{F} . Assume that $\mathcal{F}|_{X \times D^*}$ is trivial, D^* being the punctured formal disc $D^* := \mathrm{spec} K$. Fix a compatible trivialization τ of $\mathcal{F}|_{X \times D^*}$ (compatible with the trivialization of the determinant of \mathcal{F}). For any $x \in X$,

$$H^0(x \times D, \mathcal{F}) \hookrightarrow H^0(x \times D^*, \mathcal{F}) \simeq K \otimes_{\mathbb{C}} V.$$

Thus,

$$L_x := H^0(x \times D, \mathcal{F}) \hookrightarrow K \otimes_{\mathbb{C}} V.$$

It can be seen that L_x is an R -lattice in $K \otimes_{\mathbb{C}} V$. Moreover, the map $x \mapsto L_x$ provides a morphism $\phi_{\mathcal{F}}(\tau) : X \rightarrow \mathcal{X}$ under the identification of Lemma 2.1 (depending upon the trivialization τ). If we choose a different compatible trivialization τ' of the bundle $\mathcal{F}|_{X \times D^*}$, it is easy to see that the morphism $\phi_{\mathcal{F}}(\tau')$ differs from $\phi_{\mathcal{F}}(\tau)$ by the left multiplication of an element $g \in \mathcal{G}$, i.e.,

$$\phi_{\mathcal{F}}(\tau')(x) = g\phi_{\mathcal{F}}(\tau)(x), \text{ for all } x \in X.$$

(To prove this, observe that any morphism $X \times D^* \rightarrow \mathrm{SL}(2, \mathbb{C})$ is constant in the X -variable since X is an irreducible projective variety by assumption.)

Set $[\phi_{\mathcal{F}}]$ as the equivalence class of the map $\phi_{\mathcal{F}}(\tau) : X \rightarrow \mathcal{X}$ (for some compatible trivialization τ), where two maps $X \rightarrow \mathcal{X}$ are called equivalent if they differ by left multiplication by an element of \mathcal{G} . Thus, $[\phi_{\mathcal{F}}]$ does not depend upon the choice of the compatible trivialization τ of $\mathcal{F}|_{X \times D^*}$.

LEMMA 2.2. *The bundle \mathcal{F} is trivial on $X \times D$ if and only if the map $[\phi_{\mathcal{F}}]$ is a constant map.*

PROOF. If \mathcal{F} is trivial on $X \times D$, then $[\phi_{\mathcal{F}}]$ is clearly a constant map. Conversely, assume that $[\phi_{\mathcal{F}}]$ is a constant map. Choose a compatible trivialization τ of $\mathcal{F}|_{X \times D^*}$ so that

$$L_x = L_o, \quad \forall x \in X.$$

Let $\phi := \phi_{\mathcal{F}}(\tau)$. Take a basis $\{e_1, e_2\}$ of V . This gives rise to unique sections $\sigma_1(x), \sigma_2(x) \in H^0(x \times D, \mathcal{F})$ corresponding to the elements $1 \otimes e_1$ and $1 \otimes e_2$ respectively under the map ϕ . Let $s_1, s_2 \in H^0(X \times D^*, \mathcal{F})$ be everywhere linearly independent sections such that $\sigma_i(x)|_{x \times D^*} = s_i|_{x \times D^*}$, for all $x \in X$.

It suffices to show that $\sigma_1(x), \sigma_2(x)$ are linearly independent at 0 as well. Take a small open subset $U \subset X$ so that the bundle $\mathcal{F}|_{U \times D}$ is trivial. Fix a compatible trivialization τ' of $\mathcal{F}|_{U \times D}$. Then, the sections σ_i can be thought of as maps $U \times D \xrightarrow{\widehat{\sigma}_i} V$ which are linearly independent over any point of $U \times D^*$. From this it is easy to see that $\widehat{\sigma}_i$ are linearly independent over any point of $U \times D$ since the transition matrix over $U \times D^*$ with respect to the two trivializations τ and τ' of $\mathcal{F}|_{U \times D^*}$ has determinant 1. Covering X by such small open subsets U , the lemma is proved. \square

As above, a bundle \mathcal{F} gives rise to a morphism $\phi_{\mathcal{F}} : X \rightarrow \mathcal{X}$ (unique up to the left multiplication by an element of \mathcal{G}). Conversely, any morphism $\phi : X \rightarrow \mathcal{X}$ gives rise to a bundle \mathcal{F} . Before we can prove this, we need the following result.

Let $\mathfrak{V} := \mathbb{P}^1 \times V \rightarrow \mathbb{P}^1$ be the trivial rank-2 vector bundle over \mathbb{P}^1 , where V is the two dimensional complex vector space \mathbb{C}^2 . For any $g \in \mathcal{G}$, define a rank-2 locally free sheaf \mathfrak{V}_g on \mathbb{P}^1 as the sheaf associated to the following presheaf:

For any Zariski open subset $U \subset \mathbb{P}^1$, set

$$\mathfrak{V}_g(U) = H^0(U, \mathfrak{V}), \text{ if } 0 \notin U, \text{ and}$$

$$\mathfrak{V}_g(U) = \{\sigma \in H^0(U \setminus \{0\}, \mathfrak{V}) : (\sigma)_0 \in g(R \otimes_{\mathbb{C}} V)\}, \text{ if } 0 \in U,$$

where $(\sigma)_0$ denotes the germ of the rational section σ at 0 viewed canonically as an element of $K \otimes_{\mathbb{C}} V$.

With this notation, we have the following result from [KNR, Proposition 2.8]. (In fact, we only give a particular case of loc. cit. for $G = \text{SL}(2, \mathbb{C})$ and for the curve $C = \mathbb{P}^1$, which is sufficient for our purposes.)

PROPOSITION 2.3. *There is a rank-2 algebraic vector bundle \mathcal{U} on $\mathcal{X} \times \mathbb{P}^1$ satisfying the following:*

- (1) *The bundle \mathcal{U} is of trivial determinant,*
- (2) *The bundle \mathcal{U} is trivial restricted to $\mathcal{X} \times (\mathbb{P}^1 \setminus \{0\})$,*
- (3) *For any $x = g\mathcal{P} \in \mathcal{X}$ (for $g \in \mathcal{G}$), the restriction $\mathcal{U}|_{x \times \mathbb{P}^1}$ is isomorphic with the locally free sheaf \mathfrak{V}_g as above.*

LEMMA 2.4. *For any morphism $\phi : X \rightarrow \mathcal{X}$, there exists a rank two vector bundle \mathcal{F}_{ϕ} on $X \times D$ with trivial determinant (explicitly constructed in the proof) such that $\mathcal{F}_{\phi}|_{X \times D^*}$ is trivial and such that the associated morphism $[\phi_{\mathcal{F}_{\phi}}] = [\phi]$.*

PROOF. As in Proposition 2.3, consider the vector bundle \mathcal{U} on $\mathcal{X} \times \mathbb{P}^1$ of rank two. Let \mathcal{U}_{ϕ} be the pull-back of the family \mathcal{U} to $X \times \mathbb{P}^1$ via the morphism $\phi \times \text{Id}$. Let \mathcal{F}_{ϕ} be the restriction of \mathcal{U}_{ϕ} to $X \times D$, where D is the formal disc around 0.

Then, by the properties (1)-(2) of Proposition 2.3, the bundle \mathcal{F}_ϕ satisfies the first two properties of the lemma. Finally, by the property (3) of Proposition 2.3 and the definition of the map $[\phi_{\mathcal{F}_\phi}]$, it is easy to see that $[\phi_{\mathcal{F}_\phi}] = [\phi]$. \square

Combining Lemmas 2.2 and 2.4, we get the following theorem:

THEOREM 2.5. *Let X be any irreducible projective variety. Then, the following two conditions are equivalent:*

- (a) *Any rank-2 vector bundle \mathcal{F} on $X \times D$ with trivial determinant, such that $\mathcal{F}|_{X \times D^*}$ is trivial, is itself trivial.*
- (b) *There exists no nonconstant morphism $X \rightarrow \mathcal{X}$.*

By virtue of the above theorem, an affirmative answer of Question 1.3 is equivalent to an affirmative answer of the following question. Observe that under the assumptions of Question 1.3, the family V_t , thought of as a rank-2 vector bundle \mathcal{V} on $\mathbb{P}^3 \times D$, has trivial determinant by virtue of [H, Exercise 12.6(b), Chap. III]. Also, $\mathcal{V}|_{\mathbb{P}^3 \times D^*}$ is trivial by the semicontinuity theorem (cf. [H, §12, Chap. III]).

QUESTION 2.6. *Does there exist no nonconstant morphism $\mathbb{P}^3 \rightarrow \mathcal{X}$?*

DEFINITION 2.7. Recall (cf. [K, Proposition 13.2.19 and its proof]) that the singular homology $H_2(\mathcal{X}, \mathbb{Z}) \simeq \mathbb{Z}$ and it has a canonical generator given by the Schubert cycle of complex dimension 1. For any morphism $\phi : \mathbb{P}^3 \rightarrow \mathcal{X}$, define its *degree* to be the integer $d = d_\phi$ such that the induced map in homology $\phi_* : H_2(\mathbb{P}^3, \mathbb{Z}) \rightarrow H_2(\mathcal{X}, \mathbb{Z})$ induced by ϕ is given via multiplication by d .

Since the pull-back of the ample generator of $\text{Pic } \mathcal{X} \simeq H^2(\mathcal{X}, \mathbb{Z})$ (which is globally generated) is a globally generated line bundle on \mathbb{P}^3 , $d \geq 0$ and $d = 0$ if and only if ϕ is a constant map.

For any rank-2 bundle \mathcal{F} on $\mathbb{P}^3 \times D$ with trivial determinant such that $\mathcal{F}|_{X \times D^*}$ is trivial, we define its *deformation index* $d(\mathcal{F}) = d_{[\phi_{\mathcal{F}}]}$.

PROPOSITION 2.8. *For any morphism $\phi : \mathbb{P}^3 \rightarrow \mathcal{X}$, d_ϕ is divisible by 6. Equivalently, for any \mathcal{F} as in the above definition, $d(\mathcal{F})$ is divisible by 6.*

PROOF. Consider the induced algebra homomorphism in cohomology:

$$\phi^* : H^*(\mathcal{X}, \mathbb{Z}) \rightarrow H^*(\mathbb{P}^3, \mathbb{Z}),$$

induced by ϕ . By the definition, the induced map at H^2 is multiplication by d_ϕ . Moreover, by [K, Exercise 11.3.E.4], for any $i \geq 0$, $H^{2i}(\mathcal{X}, \mathbb{Z})$ is a free \mathbb{Z} -module of rank 1 generated by the Schubert class ϵ_i . Moreover,

$$\epsilon_i \cdot \epsilon_j = \binom{i+j}{i} \epsilon_{i+j}.$$

In particular, $6\epsilon_3 = \epsilon_1^3$. From this the proposition follows. \square

3. Kollár-Peskine problem and the instanton moduli space

Take any morphism $\phi : \mathbb{P}^3 \rightarrow \mathcal{X}$, with degree $d = d_\phi$. Assume that $\phi([0, 0, 0, 1])$ is the base point $x_o := 1 \cdot \mathcal{P} \in \mathcal{X}$.

Define the map

$$\pi : \mathbb{C}^3 \setminus \{0\} \times \mathbb{P}^1 \longrightarrow \mathbb{P}^3, (x, [\lambda, \mu]) \longmapsto [\lambda x, \mu],$$

for $x \in \mathbb{C}^3 \setminus \{0\}$ and $[\lambda, \mu] \in \mathbb{P}^1$. There is an action of \mathbb{C}^* on $\mathbb{C}^3 \setminus \{0\} \times \mathbb{P}^1$ by

$$z \cdot (x, [\lambda, \mu]) = \left(zx, \left[\frac{1}{z} \lambda, \mu \right] \right), \text{ for } z \in \mathbb{C}^*.$$

Then, π factors through the \mathbb{C}^* -orbits. Consider the composite morphism

$$\bar{\phi} = \phi \circ \pi : \mathbb{C}^3 \setminus \{0\} \times \mathbb{P}^1 \rightarrow \mathcal{X}.$$

Observe that $\bar{\phi}(x, 0) = x_o$ for any $x \in \mathbb{C}^3 \setminus \{0\}$, where $0 \in \mathbb{P}^1$ is the base point $[0, 1]$.

Let $\text{Mor}_*^d(\mathbb{P}^1, \mathcal{X})$ denote the set of base point preserving morphisms from $\mathbb{P}^1 \rightarrow \mathcal{X}$ of degree d (taking 0 to x_o). Then, by Grothendieck's result on Hilbert schemes, $\text{Mor}_*^d(\mathbb{P}^1, \mathcal{X})$ acquires the structure of a complex algebraic variety. Moreover, the map

$$\hat{\phi} : \mathbb{C}^3 \setminus \{0\} \rightarrow \text{Mor}_*^d(\mathbb{P}^1, \mathcal{X}),$$

canonically induced from $\bar{\phi}$, is a morphism.

Let us consider the embedding $\mathbb{P}^1 \hookrightarrow \mathbb{P}^2$, $[\lambda, \mu] \mapsto [\lambda, \mu, 0]$. Fix $d \geq 0$ and let \mathcal{M}_d be the set of isomorphism classes of rank two vector bundles \mathcal{V} over \mathbb{P}^2 with trivial determinant and with second Chern class d together with a trivialization of $\mathcal{V}|_{\mathbb{P}^1}$. The isomorphism is required to preserve the trivialization of \mathcal{V} over \mathbb{P}^1 . Then, \mathcal{M}_d has a natural variety structure. Moreover, any bundle $\mathcal{V} \in \mathcal{M}_d$ is semistable. (By [OSS, Chapter I, Lemma 3.2.2], \mathcal{V} is trivial on generic lines $\ell \subset \mathbb{P}^2$. Thus, by [OSS, Chapter II, Lemma 2.2.1], \mathcal{V} is semistable.) We will refer to \mathcal{M}_d as the *Donaldson moduli space*. Donaldson [D] showed that there is a natural diffeomorphism between \mathcal{M}_d and the instanton moduli space \mathcal{I}_d of Yang-Mills d -instantons over the flat \mathbb{R}^4 with group $\text{SU}(2)$ modulo based gauge equivalence.

Define an action of \mathbb{C}^* on $\text{Mor}_*^d(\mathbb{P}^1, \mathcal{X})$ via:

$$(1) \quad (z \cdot \gamma)[\lambda, \mu] = \gamma[z\lambda, \mu],$$

for $z \in \mathbb{C}^*$, $\gamma \in \text{Mor}_*^d(\mathbb{P}^1, \mathcal{X})$ and $[\lambda, \mu] \in \mathbb{P}^1$.

Also, define an action of \mathbb{C}^* on \mathbb{P}^2 via:

$$(2) \quad z \cdot [\lambda, \mu, \nu] = [z^{-1}\lambda, \mu, \nu].$$

This gives rise to an action of \mathbb{C}^* on \mathcal{M}_d via the pull-back of bundles, i.e., for $\mathcal{V} \in \mathcal{M}_d$, $[X] \in \mathbb{P}^2$, the fiber of $z \cdot \mathcal{V}$ over $[X]$ is given by:

$$(3) \quad (z \cdot \mathcal{V})_{[X]} = \mathcal{V}_{z \cdot [X]}.$$

(Observe that $\mathbb{P}^1 \hookrightarrow \mathbb{P}^2$ is stable under \mathbb{C}^* and hence the trivialization of $\mathcal{V}|_{\mathbb{P}^1}$ pulls back to a trivialization.)

Recall the following result from [A, § 2].

PROPOSITION 3.1. *There is a natural embedding*

$$i : \text{Mor}_*^d(\mathbb{P}^1, \mathcal{X}) \hookrightarrow \mathcal{M}_d$$

as an open subset. Moreover, i is \mathbb{C}^ -equivariant with respect to the \mathbb{C}^* actions as in equations (1) and (3).*

The following result summarizes the above discussion.

THEOREM 3.2. *To any morphism $\phi : \mathbb{P}^3 \rightarrow \mathcal{X}$ of degree d preserving the base points, there is a canonically associated \mathbb{C}^* -equivariant morphism (defined above)*

$$\hat{\phi} : \mathbb{C}^3 \setminus \{0\} \rightarrow \mathcal{M}_d,$$

where \mathbb{C}^ acts on $\mathbb{C}^3 \setminus \{0\}$ via the multiplication.*

Moreover, ϕ is constant (i.e., $d = 0$) iff $\hat{\phi}$ is constant.

We would like to make the following conjecture.

CONJECTURE 3.3. *For any $d > 0$, there does not exist any \mathbb{C}^* -equivariant morphism $\hat{f} : \mathbb{C}^3 \setminus \{0\} \rightarrow \mathcal{M}_d$.*

Assuming the validity of the above conjecture, we get the following.

COROLLARY 3.4. *Assuming the validity of Conjecture 3.3, there does not exist any nonconstant morphism $\phi : \mathbb{P}^3 \rightarrow \mathcal{X}$.*

Thus, by Theorem 2.5, assuming the validity of Conjecture 3.3, any rank-2 bundle \mathcal{F} on $\mathbb{P}^3 \times D$ with trivial determinant, such that $\mathcal{F}|_{\mathbb{P}^3 \times D^*}$ is trivial, is itself trivial.

As a generalization of the above corollary, I would like to make the following conjecture.

CONJECTURE 3.5. *For any $n \geq 2$, let \mathcal{X}_n be the infinite Grassmannian associated to the group $G = \mathrm{SL}(n)$, i.e., $\mathcal{X}_n := \mathrm{SL}(n, K) / \mathrm{SL}(n, R)$. Then, there does not exist any nonconstant morphism $\phi : \mathbb{P}^{n+1} \rightarrow \mathcal{X}_n$.*

REMARK 3.6. An interesting aspect of this approach is that Question 1.3 involving an arbitrary family of (not necessarily semistable) vector bundles on \mathbb{P}^3 is reduced to a question about the Donaldson moduli space \mathcal{M}_d consisting of rank two semistable bundles on \mathbb{P}^2 .

4. Monad construction of \mathcal{M}_d

This section recalls an explicit construction of the moduli space \mathcal{M}_d via the monad construction. We refer to [OSS, §§3,4, Chap. II] for more details on the monad construction (see also [B] and [Hu]).

Fix an integer $d \geq 0$. Let H, K, L be complex vector spaces of dimensions $d, 2d+2, d$ respectively. By *monad* one means linear maps parameterized by $Z \in \mathbb{C}^3$, depending linearly on Z :

$$H \xrightarrow{A_Z} K \xrightarrow{B_Z} L,$$

such that the composite $B_Z \circ A_Z = 0$, for all $Z \in \mathbb{C}^3$. The monad is said to be *nondegenerate* if for all $Z \in \mathbb{C}^3 \setminus \{0\}$, B_Z is surjective and A_Z is injective. In this case, we get a vector bundle on \mathbb{P}^2 with fiber at the line $[Z]$ the vector space

$$\mathcal{E}(A, B) := \mathrm{Ker} B_Z / \mathrm{Im} A_Z.$$

Then, any rank-2 bundle on \mathbb{P}^2 with the second Chern class d , which is trivial on some line, is isomorphic with $\mathcal{E}(A, B)$, for some monad (A, B) . Moreover, such a monad (A, B) is unique up to the action of $\mathrm{GL}(H) \times \mathrm{GL}(K) \times \mathrm{GL}(L)$. Let $[\lambda, \mu, \nu]$ be the homogeneous coordinates on \mathbb{P}^2 . If we only consider bundles on \mathbb{P}^2 trivial on the fixed line $\nu = 0$, the condition on the corresponding monad is that the composite $B_\lambda A_\mu = -B_\mu A_\lambda$ is an isomorphism, where (for $Z = (\lambda, \mu, \nu)$)

$$A_Z := A_\lambda \lambda + A_\mu \mu + A_\nu \nu, \text{ and } B_Z := B_\lambda \lambda + B_\mu \mu + B_\nu \nu.$$

In the following, t denotes the transpose, $I_{d \times d}$ denotes the identity matrix of size $d \times d$, $0_{d \times d}$ denotes the zero matrix of size $d \times d$ and α, β, a and b are matrices of indicated sizes. For such bundles (i.e., rank-2 bundles on \mathbb{P}^2 with the second Chern

class d , which are trivial on the line $\nu = 0$), using the action of $\mathrm{GL}(H) \times \mathrm{GL}(K) \times \mathrm{GL}(L)$, one can choose bases for H, K, L so that the maps are given as follows:

$$A_\lambda = (I_{d \times d}, 0_{d \times d}, 0_{d \times 2})^t, A_\mu = (0_{d \times d}, I_{d \times d}, 0_{d \times 2})^t, A_\nu = (\alpha_{d \times d}^t, \beta_{d \times d}^t, a_{d \times 2})^t,$$

$$B_\lambda = (0_{d \times d}, I_{d \times d}, 0_{d \times 2}), B_\mu = (-I_{d \times d}, 0_{d \times d}, 0_{d \times 2}), B_\nu = (-\beta_{d \times d}, \alpha_{d \times d}, b_{d \times 2}),$$

and the following condition is satisfied:

$$B_\nu A_\nu = 0, \text{ which is equivalent to the condition } [\alpha, \beta] + ba^t = 0.$$

The restriction of the bundle $\mathcal{E}(A, B)$ to the line $\nu = 0$ has a standard frame given by the last 2 basis vectors of $K \simeq \mathbb{C}^{2d+2}$.

For any $d \geq 0$, let $\hat{\mathcal{S}}_d$ be the closed subvariety of matrices (α, β, a, b) such that α, β are $d \times d$ matrices and a, b are $d \times 2$ matrices and they satisfy:

$$(1) [\alpha, \beta] + ba^t = 0.$$

Let \mathcal{S}_d be the open subset of $\hat{\mathcal{S}}_d$ satisfying, in addition, the following condition:

$$(2) \text{ For all } \lambda, \mu \in \mathbb{C}, (\alpha^t + \lambda I_{d \times d}, \beta^t + \mu I_{d \times d}, a)^t \text{ is injective and } (-(\beta + \mu I_{d \times d}), \alpha + \lambda I_{d \times d}, b) \text{ is surjective.}$$

We recall the following result due to Barth from [D, Proposition 1].

THEOREM 4.1. *For any $d \geq 0$, the variety \mathcal{M}_d is isomorphic with the quotient of the variety \mathcal{S}_d by the action of $\mathrm{GL}(d)$ under:*

$$g \cdot (\alpha, \beta, a, b) = (g\alpha g^{-1}, g\beta g^{-1}, (g^{-1})^t a, gb),$$

for $g \in \mathrm{GL}(d)$, and $(\alpha, \beta, a, b) \in \mathcal{S}_d$.

REMARK 4.2. The affine variety $\hat{\mathcal{S}}_d$ is stable under the above action of $\mathrm{GL}(d)$. Moreover, the open subset of stable points of $\hat{\mathcal{S}}_d$ (under the $\mathrm{GL}(d)$ -action) is precisely equal to \mathcal{S}_d (cf. [D, Lemma on page 458 and its proof]).

LEMMA 4.3. *Under the above isomorphism of the variety \mathcal{M}_d with the quotient of \mathcal{S}_d by $\mathrm{GL}(d)$, the action of \mathbb{C}^* transports to the action:*

$$z \cdot (\alpha, \beta, a, b) = (z\alpha, \beta, za, b), \text{ for } z \in \mathbb{C}^*, (\alpha, \beta, a, b) \in \mathcal{S}_d.$$

PROOF. The \mathbb{C}^* -action on \mathcal{M}_d via the pull-back corresponds to the bundle:

$$\begin{aligned} \frac{\mathrm{Ker}(z^{-1}\lambda B_\lambda + \mu B_\mu + \nu B_\nu)}{\mathrm{Im}(z^{-1}\lambda A_\lambda + \mu A_\mu + \nu A_\nu)} &= \frac{\mathrm{Ker}(-\mu I_{d \times d} - \nu \beta, z^{-1}\lambda I_{d \times d} + \nu \alpha, \nu b)}{\mathrm{Im}((z^{-1}\lambda I_{d \times d} + \nu \alpha^t, \mu I_{d \times d} + \nu \beta^t, \nu a)^t)} \\ &= \frac{\mathrm{Ker}(-\mu I_{d \times d} - \nu \beta, z^{-1}\lambda I_{d \times d} + \nu \alpha, \nu b)}{\mathrm{Im}((\lambda I_{d \times d} + z\nu \alpha^t, z\mu I_{d \times d} + z\nu \beta^t, z\nu a)^t)}. \end{aligned}$$

Changing the basis in $\mathbb{C}^{2d+2} = \mathbb{C}^d \times \mathbb{C}^d \times \mathbb{C}^2$ in the second factor to $\{ze_j\}_{1 \leq j \leq d}$, where $\{e_j\}$ is the original basis, we get that the last term in the above equation is equal to

$$\frac{\mathrm{Ker}(-\mu I_{d \times d} - \nu \beta, \lambda I_{d \times d} + z\nu \alpha, \nu b)}{\mathrm{Im}((\lambda I_{d \times d} + z\nu \alpha^t, \mu I_{d \times d} + \nu \beta^t, z\nu a)^t)}.$$

This proves the lemma. \square

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