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Characters of simplylaced nonconnected groups versus characters of nonsimplylaced connected groups

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ABSTRACT. Let G be a connected, simply-connected, almost simple semisimple group over $\mathbf{C}$ of simplylaced type and let σ be a nontrivial diagram automorphism of G. Let $G\langle\sigma\rangle$ be the (disconnected) group generated by G and σ. As a consequence of a theorem of Jantzen the character of an irreducible representation of $G\langle\sigma\rangle$ (also irreducible on G) on $G\sigma$ can be expressed in terms of a character of an irreducible representation of a certain connected simply connected semisimple group G_σ of nonsimplylaced type. We show how Jantzen's theorem can be deduced from properties of the canonical bases.

Let G be a connected, simply-connected, almost simple algebraic group of simplylaced type over $\mathbf{C}$. Let T be a maximal torus of G. Let $x_i : \mathbf{C} \to G$, $y_i : \mathbf{C} \to G$ $(i \in I)$ be homomorphisms which together with T form a pinning (épinglage) of G. We fix a nontrivial automorphism σ of G such that $\sigma(T) = T$, and such that for some permutation $i \mapsto \tilde{i}$ of I we have $\sigma(x_i(a)) = x_{\tilde{i}}(a)$, $\sigma(y_i(a)) = y_{\tilde{i}}(a)$ for all $a \in \mathbf{C}$. For $i \in I$ we write $\sigma(i) = \tilde{i}$. Let $\langle\sigma\rangle$ be the finite subgroup of the automorphism group of G generated by σ and let $G\langle\sigma\rangle$ be the semidirect product of G with $\langle\sigma\rangle$.

Let X be the group of characters $T \to \mathbf{C}^*$; let Y be the group of one parameter subgroups $\mathbf{C}^* \to T$ and let $\langle,\rangle : Y \times X \to \mathbf{Z}$ be the standard pairing. For $i \in I$ we define $\alpha_i \in X$ by $x_i(\alpha_i(t)) = tx_i(1)t^{-1}$, $y_i(\alpha_i(t)^{-1}) = ty_i(1)t^{-1}$ for all $t \in T$. This is a root of G. Let $\check{\alpha}_i \in Y$ be the corresponding coroot. Note that

(a) $(Y, X, \langle,\rangle, \check{\alpha}_i, \alpha_i (i \in I))$

is the root datum of G. Now σ induces automorphisms of X, Y denoted again by σ; these are compatible with $\langle,\rangle$ and we have $\sigma(\alpha_i) = \alpha_{\sigma(i)}, \sigma(\check{\alpha}_i) = \check{\alpha}_{\sigma(i)}$ for $i \in I$. Let $X^+ = \{\lambda \in X; \langle\check{\alpha}_i, \lambda\rangle \in \mathbf{N} \forall i \in I\}$.

We set $Y_\sigma = Y/(\sigma-1)Y$, ${}^\sigma X = \{\lambda \in X; \sigma(\lambda) = \lambda\}$. Note that $\langle,\rangle : Y \times X \to \mathbf{Z}$ induces a perfect pairing $Y_\sigma \times {}^\sigma X \to \mathbf{Z}$ denoted again by $\langle,\rangle$. Let I_σ be the set of σ-orbits on I. For any $\mathcal{O} \in I_\sigma$ let $\check{\alpha}_\mathcal{O} \in Y_\sigma$ be the image of $\check{\alpha}_i$ under $Y \to Y_\sigma$ where i is any element of $\mathcal{O}$. Since $\{\check{\alpha}_i; i \in I\}$ is a $\mathbf{Z}$-basis of Y we see that $\{\check{\alpha}_\mathcal{O}; \mathcal{O} \in I_\sigma\}$ is a $\mathbf{Z}$-basis of Y_σ. For any $\mathcal{O} \in I_\sigma$ let $\alpha_\mathcal{O} = 2^h \sum_{i \in \mathcal{O}} \alpha_i \in {}^\sigma X$ where h is the number of unordered pairs (i,j) such that $i,j \in \mathcal{O}$, and $\alpha_i + \alpha_j$ is a root. Note that $h = 0$

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except when G is of type A_{2n} when $h = 0$ for all $\mathcal{O}$ but one and $h = 1$ for one $\mathcal{O}$. Note that

(b) $(Y_\sigma, {}^\sigma X, \langle , \rangle, \check{\alpha}_\mathcal{O}, \alpha_\mathcal{O} (\mathcal{O} \in I_\sigma))$

is a root datum, see [**Ja, p.29**]. Let ${}^\sigma X^+ = \{\lambda \in {}^\sigma X; \langle \check{\alpha}_\mathcal{O}, \lambda \rangle \in \mathbf{N} \forall \mathcal{O} \in I_\sigma\} = {}^\sigma X \cap X^+$. Let G_σ be the connected semisimple group over $\mathbf{C}$ with root datum (b). By definition, G_σ is provided with an épinglage $(T_\sigma, x_\mathcal{O}, y_\mathcal{O} \quad (\mathcal{O} \in I_\sigma)$ where $T_\sigma := \mathbf{C}^* \otimes Y_\sigma = T/\{\sigma(t)t^{-1}; t \in T\}$ is a maximal torus of G_σ and $x_\mathcal{O} : \mathbf{C} \longrightarrow G_\sigma$, $y_\mathcal{O} : \mathbf{C} \longrightarrow G_\sigma$ satisfy $x_\mathcal{O}(\alpha_\mathcal{O}(t_1)) = t_1 x_\mathcal{O}(1) t_1^{-1}$, $y_\mathcal{O}(\alpha_\mathcal{O}(t_1)^{-1}) = t_1 y_\mathcal{O}(1) t_1^{-1}$ for all $t_1 \in T_\sigma$. (We have ${}^\sigma X = \mathrm{Hom}(T_\sigma, \mathbf{C}^*)$ canonically.)

Note that G_σ is simply connected and that $G_\sigma \cong {}^L((({}^L G)^\sigma)^0)$ where ${}^L()$ denotes the Langlands dual group and $({}^L G)^\sigma$ denotes the fixed point set of the automorphism of ${}^L G$ induced by σ. Now G_σ is only isogenous to ${}^L(G^\sigma)$ where G^σ is the fixed point set of $\sigma : G \longrightarrow G$.

Let $\lambda \in {}^\sigma X^+$. We can view λ both as a character of T and as a character of T_σ. Let V (resp. V') be a finite dimensional complex irreducible representation of G (resp. G_σ) with a non-zero vector η (resp. η') such that $x_i(a)\eta = 0$ for all $i \in I, a \in \mathbf{C}$ (resp. $x_\mathcal{O}(a)\eta' = 0$ for all $\mathcal{O} \in I_\sigma, a \in \mathbf{C}$) and $t\eta = \lambda(t)\eta$ for all $t \in T$ (resp. $t'\eta' = \lambda(t')\eta'$ for all $t' \in T_\sigma$). Now V can be regarded as a representation of $G\langle\sigma\rangle$ whose restriction to G is as above and on which the action of σ satisfies $\sigma(\eta) = \eta$.

Let $\mu \in X$. Let $V_\mu = \{x \in V; tx = \mu(t)x \quad \forall t \in T\}$. Note that $\sigma : V \longrightarrow V$ permutes the weight spaces V_μ among themselves. A weight space V_μ is σ-stable if and only if $\mu \in {}^\sigma X$; in this case μ can be viewed as a character of T_σ and we set $V'_\mu = \{x' \in V'; t'x' = \mu(t')x' \forall t' \in T_\sigma\}$.

THEOREM (JANTZEN [**Ja, Satz 9**]). *For* $\mu \in {}^\sigma X$ *we have* $\mathrm{tr}(\sigma : V_\mu \longrightarrow V_\mu) = \dim V'_\mu$.

COROLLARY. *Let* $\varpi : T \longrightarrow T_\sigma$ *be the canonical homomorphism. For any* $t \in T$ *we have* $\mathrm{tr}(t\sigma : V \longrightarrow V) = \mathrm{tr}(\varpi(t), V')$.

The corollary describes completely the character of V on $G\sigma$ in terms of the character of V' since any semisimple element in $G\sigma$ is G-conjugate to an element of the form $t\sigma$ with $t \in T$. Note also that there is a well defined bijection between the set of semisimple G-conjugacy classes in $G\sigma$ and the set of semisimple G_σ-conjugacy classes in G_σ which for any $t \in T$ maps the G-conjugacy class of $t\sigma$ to the G_σ-conjugacy class of $\varpi(t)$; see [**L2, 6.26**], [**Mo**].

We now show (assuming that G is not of type A_{2n}) how Jantzen's theorem can be deduced from properties of canonical bases in [**L1**]. According to [**L1**], V has a canonical basis B_λ and V' has a canonical basis B'_λ. Also, B_λ (resp. B'_λ) can be naturally viewed as a subset of $\mathbf{B}$ (resp. $\mathbf{B}'$), the canonical basis of the $+$ part of the universal enveloping algebra attached to the root datum (a) (resp. (b)). Now σ acts naturally on $\mathbf{B}$ (preserving the subset B_λ) and [**L1, Theorem 14.4.9**] provides a canonical bijection between $\mathbf{B}'$ and the fixed point set of σ on $\mathbf{B}$. (This theorem is applicable since the Cartan datum of (b) is obtained from the Cartan datum of (a) by the general "folding" procedure [**L1, 14.1**] which appplies to any simplylaced Cartan datum of not necessarily finite type together with an admissible automorphism; here we use that G is not of type A_{2n}.) This restricts to a bijection between B'_λ and the fixed point set ${}^\sigma B_\lambda$ of σ on B_λ. Next we note that B_λ (resp. B'_λ) is compatible with the decomposition of V (resp. V') into weight spaces and from the definitions we see that the bijection above carries $B'_\lambda \cap V'_\mu$

bijectively onto ${}^{\sigma}B_{\lambda} \cap V_{\mu}$. Since $B_{\lambda} \cap V_{\mu}$ is a basis of V_{μ} which is σ-stable we have $\mathrm{tr}(\sigma : V_{\mu} \to V_{\mu}) = \sharp({}^{\sigma}B_{\lambda} \cap V_{\mu})$. Using the bijection above this equals $\sharp(B'_{\lambda} \cap V'_{\mu})$ and this is equal to $\dim V'_{\mu}$ since $B'_{\lambda} \cap V'_{\mu}$ is a basis of V'_{μ}. This gives the desired result.

We refer the reader to [**FSS, FRS, NS, N1, N2, We**] for other approaches to Jantzen's theorem. We thank S. Naito and the referee for pointing out these references to us.

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