

Eigenvalue problem and a new product in cohomology of flag varieties

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1. Introduction

Let G be a connected semisimple complex algebraic group and let P be a parabolic subgroup. In this paper we define a new (commutative and associative) product \odot_0 on the cohomology of the homogenous space G/P and use this to give a more efficient solution of the eigenvalue problem and also for the problem of determining the existence of G -invariants in the tensor product of irreducible representations of G . On the other hand, we show that this new product is intimately connected with the Lie algebra cohomology of the nil-radical of P via some works of Kostant and Kumar. We also initiate a uniform study of the geometric Horn problem for an arbitrary group G by obtaining two (a priori) different sets of necessary recursive conditions to determine when a cohomology product of Schubert classes in G/P is nonzero. Hitherto, this was studied largely only for the group $SL(n)$ and its maximal parabolics P .

This new cohomology product is a certain deformation of the classical product. For $w \in W^P$ (W^P being the set of minimal length representatives in the cosets of W/W_P), let $[\bar{\Lambda}_w^P] \in H^*(G/P)$ be the cohomology class of the subvariety $\bar{\Lambda}_w^P := \overline{w^{-1}BwP} \subseteq G/P$. If the structure coefficients for the classical product are written as

$$[\bar{\Lambda}_u^P] \cdot [\bar{\Lambda}_v^P] = \sum_w c_{u,v}^w [\bar{\Lambda}_w^P],$$

then the new product \odot_0 is a restricted sum

$$[\bar{\Lambda}_u^P] \odot_0 [\bar{\Lambda}_v^P] = \sum_w' c_{u,v}^w [\bar{\Lambda}_w^P],$$

where the summation is only over a smaller set of w which satisfy a certain numerical condition involving u , v and w (Definition 18). This numerical

condition is best understood in terms of a notion which we call L -movability (cf. Definition 4 and also Sect. 5). If P is a minuscule maximal parabolic (e.g., any maximal parabolic in $SL(n)$ is minuscule), \odot_0 coincides with the classical product. However, for any nonmaximal parabolic P even in $SL(n)$, \odot_0 differs from the classical product.

1.1. Eigenvalue problem

Choose a Borel subgroup B and a maximal torus $H \subset B$ of G . Let K be a maximal compact subgroup of G with Lie algebra \mathfrak{k} . Then, there is a natural homeomorphism $C : \mathfrak{k}/K \rightarrow \mathfrak{h}_+$, where K acts on \mathfrak{k} via the adjoint representation and \mathfrak{h}_+ is the positive Weyl chamber in the Lie algebra \mathfrak{h} of H .

The eigenvalue problem is concerned with the following question:

- (E) Determine all the s -tuples $(h_1, \dots, h_s) \in \mathfrak{h}_+^s$ for which there exist $(k_1, \dots, k_s) \in \mathfrak{k}^s$ such that $C(k_j) = h_j$ for $j = 1, \dots, s$, and

$$\sum_{j=1}^s k_j = 0.$$

Given a standard maximal parabolic subgroup P , let $\omega_P \in \mathfrak{h}^*$ denote the corresponding fundamental weight. This is invariant under the Weyl group of P .

For (E), Kapovich-Leeb-Millson [KaLM] (following the works of Klyachko [Kl], Belkale [Bel1] and Berenstein-Sjamaar [BeSj]) obtained the following:

Let $(h_1, \dots, h_s) \in \mathfrak{h}_+^s$. Then, the following are equivalent:

- (A) $\exists(k_1, \dots, k_s) \in \mathfrak{k}^s$ such that $k_1 + \dots + k_s = 0$, and $C(k_j) = h_j$, for $j = 1, \dots, s$.
- (B) For every standard maximal parabolic subgroup P in G and every choice of s -tuples $(w_1, \dots, w_s) \in (W^P)^s$ such that

$$[\bar{\Lambda}_{w_1}^P] \cdot \dots \cdot [\bar{\Lambda}_{w_s}^P] = [\bar{\Lambda}_e^P] \in H^*(G/P, \mathbb{Z}),$$

the following inequality holds:

$$\omega_P \left(\sum_{j=1}^s w_j^{-1} h_j \right) \leq 0. \tag{1}$$

In our Theorem 28, we show that we can replace the condition (B) by a smaller set of inequalities. This is our main theorem on solution of the eigenvalue problem stated below as well.

Theorem: (A) is equivalent to

(B') For every standard maximal parabolic subgroup P in G and every choice of s -tuples $(w_1, \dots, w_s) \in (W^P)^s$ such that

$$[\bar{\Lambda}_{w_1}^P] \odot_0 \cdots \odot_0 [\bar{\Lambda}_{w_s}^P] = [\bar{\Lambda}_e^P] \in (H^*(G/P, \mathbb{Z}), \odot_0),$$

the inequality (1) holds.

Knutson-Tao-Woodward [KTW] proved that for $G = \text{SL}(n)$ the inequalities (B) (which are, in this case, the same set of inequalities as (B') because of Lemma 19) are irredundant. However, in the case of the groups of type B_3 and C_3 , Kumar-Leeb-Millson [KLM] proved that of the total of 135 inequalities corresponding to the system (B), 33 are redundant. In contrast, making use of the tables in Sect. 10, it can be seen that the set of inequalities given by (B') is irredundant for both of B_3 and C_3 . It will be very interesting to know if the set of inequalities given by (B') forms an irredundant system for any G .

Let ν_1, \dots, ν_s be dominant integral weights. Then, consider the problem of finding conditions on ν_j 's such that the space of G -invariants

$$H^0((G/B)^s, \mathcal{L}(N\nu_1) \boxtimes \cdots \boxtimes \mathcal{L}(N\nu_s))^G$$

is nonzero for some $N > 0$. As is well known, by using symplectic geometry, this problem is equivalent to the eigenvalue problem. For a precise solution of this problem, see Theorem 22.

1.2. Relation to Lie algebra cohomology

Using a celebrated theorem of Kostant on the cohomology of some nilpotent Lie algebras (Theorem 42), we relate the new cohomology product \odot_0 to the Lie algebra cohomology. More specifically, in Theorem 43, we exhibit an explicit isomorphism of graded rings

$$\phi : (H^*(G/P, \mathbb{C}), \odot_0) \simeq [H^*(\mathfrak{u}_P) \otimes H^*(\mathfrak{u}_P^-)]^l,$$

where \mathfrak{u}_P (resp. \mathfrak{l}) is the nil-radical (resp. a Levi component) of the Lie algebra of P and we take the tensor product algebra structure on the right side. This isomorphism is used to determine the structure of $(H^*(G/B), \odot_0)$ completely (cf. Corollary 44).

1.3. Geometric Horn problem

One of the consequences of the work of Klyachko [Kl], and the saturation theorem of Knutson-Tao [KT] is that one can tell when a product of Schubert classes in a given Grassmannian $\text{Gr}(r, n)$ (which is a homogenous space for $\text{SL}(n)$ corresponding to a maximal parabolic) is nonzero by writing

down a series of inequalities coming from knowing the answer to the same question for smaller Grassmannians. These inequalities are the eigenvalue inequalities (1) for $SL(r)$ where the conjugacy classes are determined from the Schubert classes. We refer the reader to the survey article of Fulton [Fu2] and our Remark 38 for details (also see [Bel2] for a geometric proof).

In this paper we initiate a uniform study of solving the same problem for arbitrary semisimple groups.

We obtain two (a priori) different sets of necessary conditions (Theorems 29 and 36) for a product (with any number of factors) of Schubert classes to be nonzero in the cohomology of G/P . The inequalities given by Theorem 29, which we call *character inequalities*, are similar to the eigenvalue inequalities for the Levi subgroup L of P . These inequalities provide the first known numerical criterion for the vanishing of intersection multiplicities beyond the codimension condition for an arbitrary semisimple G .

The second set of inequalities, which we call the *dimension inequalities*, given by Theorem 36 are based on dimension counts. In the case of $G = SL(n)$ and P a maximal parabolic, the dimension and character inequalities coincide. In this case, we identify these inequalities with the classical Horn inequalities (cf. Remark 38). In general, even for minuscule maximal parabolics, these character and dimension inequalities are (a priori) different.

The character inequalities can be refined further if we consider the new product \odot_0 instead of the classical product (Theorem 32).

1.4. Examples

In Sect. 10 we give the multiplication tables determining the deformed product \odot for G/P for all the rank 3 complex simple groups G and maximal parabolic subgroups P .

Sections 7, 8 and 9 are independent of each other.

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2. Notation and preliminaries on the intersection theory in G/P

Let G be a connected reductive complex algebraic group. We choose a Borel subgroup B and a maximal torus $H \subset B$ and let $W = W_G := N_G(H)/H$ be the associated Weyl group, where $N_G(H)$ is the normalizer of H in G . Let $P \supseteq B$ be a (standard) parabolic subgroup of G and let $U = U_P$ be its unipotent radical. Consider the Levi subgroup $L = L_P$ of P containing H , so that P is the semi-direct product of U and L . Then, $B_L := B \cap L$ is a Borel subgroup of L . Let $X(H)$ denote the character group of H , i.e., the group of

all the algebraic group morphisms $H \rightarrow \mathbb{G}_m$. Then, B_L being the semidirect product of its commutator $[B_L, B_L]$ and H , any $\lambda \in X(H)$ extends uniquely to a character of B_L . Similarly, for any algebraic subgroup S of G , let $O(S)$ denote the set of all the one parameter subgroups in S , i.e., algebraic group morphisms $\mathbb{G}_m \rightarrow S$. We denote the Lie algebras of G, B, H, P, U, L, B_L by the corresponding Gothic characters: $\mathfrak{g}, \mathfrak{b}, \mathfrak{h}, \mathfrak{p}, \mathfrak{u}, \mathfrak{l}, \mathfrak{b}_L$ respectively. Let $R = R_{\mathfrak{g}}$ be the set of roots of \mathfrak{g} with respect to the Cartan subalgebra \mathfrak{h} and let R^+ be the set of positive roots (i.e., the set of roots of \mathfrak{b}). Similarly, let $R_{\mathfrak{l}}$ be the set of roots of \mathfrak{l} with respect to \mathfrak{h} and $R_{\mathfrak{l}}^+$ be the set of roots of \mathfrak{b}_L . Let $\Delta = \{\alpha_1, \dots, \alpha_{\ell}\} \subset R^+$ be the set of simple roots, where ℓ is the semisimple rank of G (i.e., the dimension of $\mathfrak{h}' := \mathfrak{h} \cap [\mathfrak{g}, \mathfrak{g}]$). We denote by $\Delta(P)$ the set of simple roots contained in $R_{\mathfrak{l}}$. For any $1 \leq j \leq \ell$, define the element $x_j \in \mathfrak{h}'$ by

$$\alpha_i(x_j) = \delta_{i,j}, \quad \forall 1 \leq i \leq \ell. \tag{2}$$

For an H -invariant subspace $V \subset \mathfrak{g}$ under the adjoint action, let $R(V)$ denote the set of roots of \mathfrak{g} appearing in V , i.e., $V = \bigoplus_{\alpha \in R(V)} \mathfrak{g}_{\alpha}$.

Recall that if W_P is the Weyl group of P (which is, by definition, the Weyl Group W_L of L ; thus $W_P := W_L$), then in each coset of W/W_P we have a unique member w of minimal length. This satisfies (cf. [Ku2, Exercise 1.3.E]):

$$wB_Lw^{-1} \subseteq B. \tag{3}$$

Let W^P be the set of the minimal length representatives in the cosets of W/W_P .

For any $w \in W^P$, define the (shifted) Schubert cell:

$$\Lambda_w^P := w^{-1}BwP \subset G/P.$$

Then, it is a locally closed subvariety of G/P isomorphic with the affine space $\mathbb{A}^{\ell(w)}$, $\ell(w)$ being the length of w (cf. [J, Part II, Chap. 13]). Its closure is denoted by $\bar{\Lambda}_w^P$, which is an irreducible (projective) subvariety of G/P of dimension $\ell(w)$. Considered as an element in the group of rational equivalence classes $A_{\ell(w)}(G/P)$ of algebraic degree $\ell(w)$ on G/P , it is denoted by $[\bar{\Lambda}_w^P]$. Since G/P is smooth, setting $A^*(G/P) := A_{\dim G/P - *}(G/P)$, $A^*(G/P)$ is a commutative associative graded ring under the intersection product \cdot (cf. [Fu1, §8.3]).

Let $\mu(\bar{\Lambda}_w^P)$ denote the fundamental class of $\bar{\Lambda}_w^P$ considered as an element of the singular homology with integral coefficients $H_{2\ell(w)}(G/P, \mathbb{Z})$ of G/P . Then, from the Bruhat decomposition, the elements $\{\mu(\bar{\Lambda}_w^P)\}_{w \in W^P}$ form a \mathbb{Z} -basis of $H_*(G/P, \mathbb{Z})$. Let $\{\epsilon_w^P\}_{w \in W^P}$ be the dual basis of the singular cohomology with integral coefficients $H^*(G/P, \mathbb{Z})$, i.e., for any $v, w \in W^P$ we have

$$\epsilon_v^P(\mu(\bar{\Lambda}_w^P)) = \delta_{v,w}.$$

By [Fu1, Example 19.1.11(b)] and [KLM, Proposition 2.6], there is a graded ring isomorphism, the cycle class map,

$$c : A^*(G/P) \rightarrow H^*(G/P, \mathbb{Z}), \quad [\bar{\Lambda}_w^P] \mapsto \epsilon_{w_o w w_o^P}^P,$$

where w_o (resp. w_o^P) is the longest element of W (resp. W_P). (For $w \in W^P$, we have $w_o w w_o^P \in W^P$ by [KLM, Proposition 2.6].) *From now on, we will identify $A^*(G/P)$ with $H^*(G/P, \mathbb{Z})$ under c . Thus, under the identification c , $H^*(G/P, \mathbb{Z})$ has two \mathbb{Z} -bases: $\{[\bar{\Lambda}_w^P] = \epsilon_{w_o w w_o^P}^P\}_{w \in W^P}$ and $\{\epsilon_w^P\}_{w \in W^P}$.*

Let $T^P = T(G/P)_e$ be the tangent space of G/P at $e \in G/P$. It carries a canonical action of P . For $w \in W^P$, define T_w^P to be the tangent space of Λ_w^P at e . We shall abbreviate T^P and T_w^P by T and T_w respectively when the reference to P is clear. By (3), B_L stabilizes Λ_w^P keeping e fixed. Thus,

$$B_L T_w \subset T_w. \tag{4}$$

Lemma 1. *For any $g \in G$, $g\Lambda_w^P$ passes through $e \Leftrightarrow g\Lambda_w^P = p\Lambda_w^P$ for some $p \in P$.*

Proof. Since $g\Lambda_w^P$ passes through e , $g^{-1} \in \Lambda_w^P$, i.e., $g \in Pw^{-1}Bw$. Write $g = pw^{-1}bw$, for some $p \in P$ and $b \in B$. Then, $g\Lambda_w^P = pw^{-1}bw\Lambda_w^P = p\Lambda_w^P$. \square

The following result is the starting point of our analysis.

Proposition 2. *Take any $s \geq 1$ and any $(w_1, \dots, w_s) \in (W^P)^s$ such that*

$$\sum_{j=1}^s \text{codim } \Lambda_{w_j}^P \leq \dim G/P. \tag{5}$$

(Clearly, (5) is equivalent to the following equation:

$$\sum_{j=1}^s \ell(w_j) \geq (s - 1) \dim G/P.) \tag{6}$$

Then, the following three conditions are equivalent:

(a) $[\bar{\Lambda}_{w_1}^P] \cdot \dots \cdot [\bar{\Lambda}_{w_s}^P] \neq 0 \in A^*(G/P)$.
(Observe that, by the above isomorphism c , this is equivalent to the condition: $\epsilon_{w_o w_1 w_o^P}^P \cdots \epsilon_{w_o w_s w_o^P}^P \neq 0$.)

(b) *For generic $(p_1, \dots, p_s) \in P^s$, the intersection*

$$p_1 \Lambda_{w_1}^P \cap \dots \cap p_s \Lambda_{w_s}^P$$

is transverse at e .

(c) *For generic $(p_1, \dots, p_s) \in P^s$,*

$$\dim(p_1 T_{w_1} \cap \dots \cap p_s T_{w_s}) = \dim G/P - \sum_{j=1}^s \text{codim } \Lambda_{w_j}^P.$$

As proved below, the set of s -tuples in (b) as well as (c) is an open subset of P^s .

Proof. For $p \in P$ and $w \in W^P$, the tangent space to $p\Lambda_w^P$ at $e \in G/P$ is pT_w . Therefore, by the definition of transversality (cf. [S, Chap. II, §2.1]), (c) is equivalent to (b). The subset in (c) is the set of points $(p_1, \dots, p_s) \in P^s$ for which the canonical morphism

$$T \rightarrow \bigoplus_{j=1}^s \frac{T}{p_j T_{w_j}}$$

is surjective. Therefore, this set is open in P^s .

If $(p_1, \dots, p_s) \in P^s$ satisfies the property in (b), the smooth varieties $p_1\Lambda_{w_1}^P, \dots, p_s\Lambda_{w_s}^P$ meet transversally and hence properly at $e \in G/P$. By [Fu1, Proposition 7.1 and Sect. 12.2] this implies that (a) holds.

To show that (a) implies (b), find $g_j \in G$ for $j = 1, \dots, s$ so that $g_1\Lambda_{w_1}^P, \dots, g_s\Lambda_{w_s}^P$ meet transversally at a nonempty set of points (cf. Proposition 3). By translation, assume that $e \in G/P$ is one of these points. By Lemma 1, for any $j = 1, \dots, s$, $g_j\Lambda_{w_j}^P = p_j\Lambda_{w_j}^P$, for some $p_j \in P$. Thus, we have found a point $(p_1, \dots, p_s) \in P^s$ satisfying the condition in (b). Since the condition is an open condition, we are assured of a nonempty open subset of P^s satisfying the property in (b). \square

Proposition 3 (Kleiman). *Let a connected algebraic group G act transitively on a smooth variety X and let X_1, \dots, X_s be irreducible locally closed subvarieties of X . Then, there exists a non empty open subset $U \subseteq G^s$ such that for $(g_1, \dots, g_s) \in U$, the intersection $\bigcap_{j=1}^s g_j X_j$ is proper (possibly empty) and dense in $\bigcap_{j=1}^s g_j \bar{X}_j$.*

Moreover, if $X_j, j = 1, \dots, s$, are smooth varieties, we can find such a U with the additional property that for $(g_1, \dots, g_s) \in U$, $\bigcap_{j=1}^s g_j X_j$ is transverse at each point of intersection.

Proof. We include a proof of the density statement (the rest of the conclusion is standard, see [Kle]).

Let $Y_j := \bar{X}_j \setminus X_j$. Let U be a nonempty open subset of G^s such that for $(g_1, \dots, g_s) \in U$, the following intersections are proper:

1. $\bigcap_{j=1}^s g_j \bar{X}_j$.
2. For $\ell \in \{1, \dots, s\}$, $\{\bigcap_{j \in \{1, \dots, s\} \setminus \{\ell\}} g_j \bar{X}_j\} \cap g_\ell Y_\ell$.

For $(g_1, \dots, g_s) \in U$ and $\ell \in \{1, \dots, s\}$, each irreducible component of the intersection $\{\bigcap_{j \in \{1, \dots, s\} \setminus \{\ell\}} g_j \bar{X}_j\} \cap g_\ell Y_\ell$ is therefore of dimension strictly less than that of each irreducible component of $\bigcap_{j=1}^s g_j \bar{X}_j$ (since $\dim(Y_\ell) < \dim(X_\ell)$). This proves the density statement. \square

3. Preliminary analysis of Levi-movability

We begin by introducing the central concept of this paper:

Definition 4. Let $w_1, \dots, w_s \in W^P$ be such that

$$\sum_{j=1}^s \text{codim } \Lambda_{w_j}^P = \dim G/P. \tag{7}$$

This of course is equivalent to the condition:

$$\sum_{j=1}^s \ell(w_j) = (s - 1) \dim G/P. \tag{8}$$

We then call the s -tuple (w_1, \dots, w_s) *Levi-movable* for short *L-movable* if, for generic $(l_1, \dots, l_s) \in L^s$, the intersection $l_1 \Lambda_{w_1} \cap \dots \cap l_s \Lambda_{w_s}$ is transverse at e .

Observe that, even though in the definition of *L-movability* we took minimal coset representatives in W/W_P , our definition is independent of the choice of coset representatives. Thus, we have the notion of *L-movability* for any s -tuple $(w_1, \dots, w_s) \in (W/W_P)^s$.

By Proposition 2, if (w_1, \dots, w_s) is *L-movable*, then $[\bar{\Lambda}_{w_1}^P] \cdot \dots \cdot [\bar{\Lambda}_{w_s}^P] = d[\bar{\Lambda}_e^P]$ in $H^*(G/P)$, for some nonzero d . The converse is not true in general (cf. Theorem 15).

Definition 5. Let $w \in W^P$. Since $T_w := T_e(\Lambda_w^P)$ is a B_L -module (by (4)), we have the P -equivariant vector bundle $\mathcal{T}_w := P \times T_w$ on P/B_L associated to the principal B_L -bundle $P \rightarrow P/B_L$ via the B_L -module T_w . In particular, we have the P -equivariant vector bundle $\mathcal{T} := P \times T$ (where, as in Sect. 2, $T := T_e(G/P)$) and \mathcal{T}_w is canonically a P -equivariant subbundle of \mathcal{T} . Take the top exterior powers $\det(\mathcal{T}/\mathcal{T}_w)$ and $\det(\mathcal{T}_w)$, which are P -equivariant line bundles on P/B_L . Observe that, since T is a P -module, the P -equivariant vector bundle \mathcal{T} is P -equivariantly isomorphic with the product bundle $P/B_L \times T$ under the map $\xi : P/B_L \times T \rightarrow \mathcal{T}$ taking $(pB_L, v) \mapsto (p, p^{-1}v)$ mod B_L , for $p \in P$ and $v \in T$; where P acts on $P/B_L \times T$ diagonally. We will often identify \mathcal{T} with the product bundle $P/B_L \times T$ under ξ .

Similarly, for any $\lambda \in X(H)$, we have a P -equivariant line bundle $\mathcal{L}(\lambda) = \mathcal{L}_P(\lambda)$ on P/B_L associated to the principal B_L -bundle $P \rightarrow P/B_L$ via the one dimensional B_L -module λ^{-1} . (As observed in Sect. 2, any $\lambda \in X(H)$ extends uniquely to a character of B_L .) The twist in the definition of $\mathcal{L}(\lambda)$ is introduced so that the dominant characters correspond to the dominant line bundles.

For $w \in W^P$, define the character $\chi_w \in \mathfrak{h}^*$ by

$$\chi_w = \sum_{\beta \in (R^+ \setminus R_1^+) \cap w^{-1}R^+} \beta.$$

Then, from [Ku2, 1.3.22.3] and (3),

$$\chi_w = \rho - 2\rho^L + w^{-1}\rho, \tag{9}$$

where ρ (resp. ρ^L) is half the sum of roots in R^+ (resp. in R_1^+).

The following lemma is easy to establish.

Lemma 6. *For $w \in W^P$, as P -equivariant line bundles on P/B_L , we have:*

$$\det(\mathcal{T}/\mathcal{T}_w) = \mathcal{L}(\chi_w).$$

Observe that, since P acts on $\wedge^{\text{top}}(T)$ with character $-\chi_1$, the line bundle $\mathcal{L}(\chi_1)$ is a trivial line bundle. However, as an H -equivariant line bundle, it is nontrivial in general as the character χ_1 restricted to the connected center of L is nontrivial.

Let \mathcal{T}_s be the P -equivariant product bundle $(P/B_L)^s \times T \rightarrow (P/B_L)^s$ under the diagonal action of P on $(P/B_L)^s \times T$. Then, \mathcal{T}_s is canonically P -equivariantly isomorphic with the pull-back bundle $\pi_j^*(\mathcal{T})$, for any $1 \leq j \leq s$, where $\pi_j : (P/B_L)^s \rightarrow P/B_L$ is the projection onto the j -th factor. For any $w_1, \dots, w_s \in W^P$, we have a P -equivariant map of vector bundles on $(P/B_L)^s$:

$$\Theta = \Theta_{(w_1, \dots, w_s)} : \mathcal{T}_s \rightarrow \bigoplus_{j=1}^s \pi_j^*(\mathcal{T}/\mathcal{T}_{w_j}) \tag{10}$$

obtained as the direct sum of the canonical projections $\mathcal{T}_s \rightarrow \pi_j^*(\mathcal{T}/\mathcal{T}_{w_j})$ under the identification $\mathcal{T}_s \simeq \pi_j^*(\mathcal{T})$. Now, assume that $w_1, \dots, w_s \in W^P$ satisfies the condition (7). In this case, we have the same rank on the two sides of the map (10). Let θ be the bundle map obtained from Θ by taking the top exterior power:

$$\theta = \det(\Theta) : \det(\mathcal{T}_s) \rightarrow \det(\mathcal{T}/\mathcal{T}_{w_1}) \boxtimes \dots \boxtimes \det(\mathcal{T}/\mathcal{T}_{w_s}), \tag{11}$$

where \boxtimes denotes the external tensor product. Clearly, θ is P -equivariant and hence one can view θ as a P -invariant element in

$$\begin{aligned} & H^0\left((P/B_L)^s, \det(\mathcal{T}_s)^* \otimes \left(\det(\mathcal{T}/\mathcal{T}_{w_1}) \boxtimes \dots \boxtimes \det(\mathcal{T}/\mathcal{T}_{w_s})\right)\right) \\ &= H^0\left((P/B_L)^s, \mathcal{L}(\chi_{w_1} - \chi_1) \boxtimes \dots \boxtimes \mathcal{L}(\chi_{w_s})\right), \end{aligned} \tag{12}$$

where the above equality follows from Lemma 6.

The following lemma is immediate:

Lemma 7. *Let $\bar{p} = (\bar{p}_1, \dots, \bar{p}_s) \in (P/B_L)^s$. Then, under the assumption (7), the following are equivalent:*

1. *The restriction of the map Θ to the fiber over \bar{p} is an isomorphism.*
2. *The section θ does not vanish at \bar{p} .*
3. *The locally closed subvarieties $p_1\Lambda_{w_1}^P, \dots, p_s\Lambda_{w_s}^P$ meet transversally at $e \in G/P$.*

The following corollary follows immediately from Proposition 2 and Lemma 7.

Corollary 8. *Let (w_1, \dots, w_s) be an s -tuple of elements of W^P satisfying the condition (7). Then, we have the following:*

1. *The section θ is nonzero if and only if*

$$[\bar{\Lambda}_{w_1}^P] \cdot \dots \cdot [\bar{\Lambda}_{w_s}^P] \neq 0 \in H^*(G/P).$$

2. *The s -tuple (w_1, \dots, w_s) is L -movable if and only if the section θ restricted to $(L/B_L)^s$ is not identically 0.*

4. Geometric invariant theory revisited

We need to consider the Geometric Invariant Theory (GIT) in a nontraditional setting, where a *nonreductive* group acts on a *nonprojective* variety. To handle such a situation, we need to introduce the notion of P -admissible one parameter subgroups (cf. Definition 11). But first we recall the following definition due to Mumford.

Definition 9. Let S be any (not necessarily reductive) algebraic group acting on a (not necessarily projective) variety \mathbb{X} and let \mathbb{L} be an S -equivariant line bundle on \mathbb{X} . Take any $x \in \mathbb{X}$ and a one parameter subgroup (for short OPS) $\lambda \in O(S)$ such that the limit

$$\lim_{t \rightarrow 0} \lambda(t)x$$

exists in \mathbb{X} (i.e., the morphism $\lambda_x : \mathbb{G}_m \rightarrow X$ given by $t \mapsto \lambda(t)x$ extends to a morphism $\tilde{\lambda}_x : \mathbb{A}^1 \rightarrow X$). This condition is of course satisfied for any $x \in \mathbb{X}$, if \mathbb{X} is projective. Then, following Mumford, define a number $\mu^{\mathbb{L}}(x, \lambda)$ as follows: Let $x_o \in X$ be the point $\tilde{\lambda}_x(0)$. Since x_o is \mathbb{G}_m -invariant via λ , the fiber of \mathbb{L} over x_o is a \mathbb{G}_m -module; in particular, is given by a character of \mathbb{G}_m . This integer is defined as $\mu^{\mathbb{L}}(x, \lambda)$.

Alternatively, $\mu^{\mathbb{L}}(x, \lambda)$ is given as follows. Let $\mathbb{L}' := \tilde{\lambda}_x^*(\mathbb{L})$ be the pull-back line bundle on \mathbb{A}^1 . Let σ_1 be a nonzero vector in the fiber of \mathbb{L} over x . Then, using the \mathbb{G}_m -action, σ_1 extends to a \mathbb{G}_m -invariant section σ of \mathbb{L}' over \mathbb{G}_m . Since \mathbb{L}' is a line bundle on \mathbb{A}^1 we can speak of the order of vanishing of σ at 0 (this number is negative if σ has a pole at $0 \in \mathbb{A}^1$). This number can be seen to be equal to $\mu^{\mathbb{L}}(x, \lambda)$.

Let V be a finite dimensional representation of S and let $i : \mathbb{X} \hookrightarrow \mathbb{P}(V)$ be an S -equivariant embedding. Take $\mathbb{L} := i^*(\mathcal{O}(1))$. Let $\lambda \in O(S)$ and let $\{e_1, \dots, e_n\}$ be a basis of V consisting of eigenvectors, i.e., $\lambda(t) \cdot e_l = t^{\lambda_l} e_l$, for $l = 1, \dots, n$. For any $x \in \mathbb{X}$ such that $\lim_{t \rightarrow 0} \lambda(t)x$ exists in \mathbb{X} , write $i(x) = [\sum_{l=1}^n x_l e_l]$. Then, it is easy to see that, we have ([MFK, Proposition 2.3, p. 51])

$$\mu^{\mathbb{L}}(x, \lambda) = \max_{l: x_l \neq 0} (-\lambda_l). \tag{13}$$

We record the following simple properties of $\mu^{\mathbb{L}}(x, \lambda)$:

Proposition 10. *For any $x \in \mathbb{X}$ and $\lambda \in O(S)$ such that $\lim_{t \rightarrow 0} \lambda(t)x$ exists in \mathbb{X} , we have the following (for any S -equivariant line bundles $\mathbb{L}, \mathbb{L}_1, \mathbb{L}_2$):*

- (a) $\mu^{\mathbb{L}_1 \otimes \mathbb{L}_2}(x, \lambda) = \mu^{\mathbb{L}_1}(x, \lambda) + \mu^{\mathbb{L}_2}(x, \lambda)$.
- (b) *If there exists $\sigma \in H^0(\mathbb{X}, \mathbb{L})^S$ such that $\sigma(x) \neq 0$, then $\mu^{\mathbb{L}}(x, \lambda) \geq 0$.*
- (c) *If $\mu^{\mathbb{L}}(x, \lambda) = 0$, then any element of $H^0(\mathbb{X}, \mathbb{L})^S$ which does not vanish at x does not vanish at $\lim_{t \rightarrow 0} \lambda(t)x$ as well.*
- (d) *For any S -variety \mathbb{X}' together with an S -equivariant morphism $f : \mathbb{X}' \rightarrow \mathbb{X}$ and any $x' \in \mathbb{X}'$ such that $\lim_{t \rightarrow 0} \lambda(t)x'$ exists in \mathbb{X}' , we have*

$$\mu^{f^*\mathbb{L}}(x', \lambda) = \mu^{\mathbb{L}}(f(x'), \lambda).$$

Definition 11. We call an OPS $\lambda \in O(P)$, P -admissible if the limit $\lim_{t \rightarrow 0} \lambda(t)x$ exists in P/B_L for every $x \in P/B_L$. When the reference to P is clear from the context, we will abbreviate P -admissible by *admissible*.

From the conjugacy of L and B_L , it is easy to see that the notion of the admissibility of λ does not depend upon the choice of the Levi subgroup L or the Borel subgroup B_L of L .

For an OPS $\lambda \in O(G)$, set

$$\dot{\lambda} := \left. \frac{d\lambda(t)}{dt} \right|_{t=1} \in \mathfrak{g},$$

its tangent vector.

Also define the *associated parabolic subgroup* $P(\lambda)$ of G by

$$P(\lambda) := \left\{ g \in G : \lim_{t \rightarrow 0} \lambda(t)g\lambda(t)^{-1} \text{ exists in } G \right\}.$$

We also denote $P(\lambda)$ sometimes by $P(\dot{\lambda})$. This should not create any confusion since $\dot{\lambda}$ uniquely determines λ .

The following lemma gives a characterization of admissible one parameter subgroups in P .

Lemma 12. *Let $\lambda \in O(P)$. Then, λ is admissible iff*

$$P(\lambda) \supset U.$$

Equivalently, λ is admissible iff there exists $p \in P$ such that $\lambda_o := p\lambda p^{-1}$ lies in H and $\beta(\dot{\lambda}_o) \geq 0$ for all $\beta \in R^+ \setminus R_1^+$.

Proof. Clearly, λ is admissible iff any conjugate $p\lambda p^{-1}$ is admissible, for $p \in P$. Moreover, $P(p\lambda p^{-1}) = pP(\lambda)p^{-1}$. Thus, U being normal in P , the condition that $P(\lambda) \supset U$ is invariant under the conjugation of λ via p . Further, since any OPS in P can be conjugated via an element $p \in P$ inside the maximal torus H , we can assume that $\text{Im } \lambda \subset H$. Decompose

$$P/B_L \simeq U \times L/B_L \text{ as varieties.}$$

For any OPS λ in H (in particular in L), and any $u \in U, l \in L$,

$$\lim_{t \rightarrow 0} \lambda(t)(ulB_L) = \lim_{t \rightarrow 0} (\lambda(t)u\lambda(t)^{-1}(\lambda(t)lB_L)).$$

But, since L/B_L is projective, the limit on the left side exists iff $\lim_{t \rightarrow 0} \lambda(t)u\lambda(t)^{-1}$ exists in U , which is equivalent to the condition that $P(\lambda) \supset U$. This proves the first part of the lemma.

The second part follows readily from the first part. □

Definition 13. Let $x = ulB_L \in P/B_L$, for $u \in U$ and $l \in L$. Let λ be an admissible OPS and let $P_L(\lambda) := P(\lambda) \cap L$. Then, $P_L(\lambda)$ is a parabolic subgroup of L . To see this, write $\lambda = u'\lambda_o u'^{-1}$, for $u' \in U$ and with λ_o an OPS in L . This gives $P(\lambda) = u'P(\lambda_o)u'^{-1}$. But since λ is admissible by assumption, and hence so is λ_o ; and thus by Lemma 12, $P(\lambda_o) \supset U$. This gives $P(\lambda) = P(\lambda_o)$. But, as is well known, $P_L(\lambda_o)$ is a parabolic subgroup of L and hence so is $P_L(\lambda)$.

Write $P_L(\lambda) = l_1 Q l_1^{-1}$, for some $l_1 \in L$, where Q is a standard parabolic subgroup of L , i.e., $Q \supset B_L$. Let $l^{-1}l_1 \in B_L w Q$, where $w \in W_L/W_Q$. Then, clearly $w \in W_L/W_Q$ does not depend upon the choices of the representatives l (in lB_L) and l_1 . We define the *relative position* $[x, \lambda]$ to be $w \in W_L/W_Q$. This satisfies:

$$[px, p\lambda p^{-1}] = [x, \lambda], \text{ for any } p \in P. \tag{14}$$

Observe that, if λ is an OPS lying in the center of L , then for any $x \in P/B_L$,

$$[x, \lambda] = 1. \tag{15}$$

For an OPS λ in P , we can choose $p \in P$ such that $\lambda_o := p\lambda p^{-1}$ is an OPS in H and, moreover, $\dot{\lambda}_o \in \mathfrak{h}$ is L -dominant (i.e., $\alpha_i(\dot{\lambda}_o) \geq 0$ for all $\alpha_i \in \Delta(P)$). Set,

$$X_\lambda^P = \dot{\lambda}_o. \tag{16}$$

Then, X_λ^P is well defined. Moreover, W_Q fixes X_λ^P since $Q = P_L(\lambda_o)$. We shall abbreviate X_λ^P by X_λ when the reference to P is clear.

The following lemma is a generalization of the corresponding result in [BeSj, Sect. 4.2].

Lemma 14. *Let λ be an admissible OPS in P , $x \in P/B_L$ and $\chi \in X(H)$. Then, we have the following formula:*

$$\mu^{\mathcal{L}(\chi)}(x, \lambda) = -\chi([x, \lambda]X_\lambda).$$

Proof. Let $p \in P$ be such that $\lambda = p\lambda_o p^{-1}$ (where λ_o is an OPS in H such that $\lambda_o = X_\lambda$). Let $\bar{x} \in P$ be a lift of x . We seek an OPS $b(t)$ in B_L so that $\lambda_o(t)p^{-1}\bar{x}b(t)$ has a limit in P as $t \rightarrow 0$. Let $p = u_1 l_1$, $\bar{x} = ul$, for $u, u_1 \in U$ and $l, l_1 \in L$, and let Q be the standard parabolic subgroup of L as in Definition 13. Choose $w \in W_L/W_Q$, $b_l \in [B_L, B_L]$ and $q \in Q$ so that $l_1 = lb_l wq$. Then, by the definition,

$$w = [x, \lambda]. \tag{17}$$

Thus, for any $b(t) \in B_L$,

$$\begin{aligned} \lambda_o(t)p^{-1}\bar{x}b(t) &= \lambda_o(t)l_1^{-1}u_1^{-1}ul_1^{-1}lb(t) \\ &= \lambda_o(t)\hat{u}\lambda_o(t)^{-1}\lambda_o(t)q^{-1}\lambda_o(t)^{-1}\lambda_o(t)w^{-1}b_l^{-1}b(t), \end{aligned}$$

where $\hat{u} := l_1^{-1}u_1^{-1}ul_1 \in U$. In view of Lemma 12 and since $P_L(\lambda_o) = Q$, we find that the OPS

$$b(t) := b_l w \lambda_o(t)^{-1} w^{-1} b_l^{-1}$$

“works”, i.e., $\lambda_o(t)p^{-1}\bar{x}b(t)$ has a limit in P as $t \rightarrow 0$.

Consider the \mathbb{G}_m -invariant section $\sigma(t) := (\lambda(t)\bar{x}, 1) \bmod B_L$ of $\lambda_x^*(\mathcal{L}_\chi)$ over \mathbb{G}_m , where, as in Definition 9, $\lambda_x : \mathbb{G}_m \rightarrow P/B_L$ is the map $t \mapsto \lambda(t) \cdot x$. Then, the section $\sigma(t)$ corresponds to the function $\mathbb{G}_m \rightarrow \mathbb{A}^1$, $t \mapsto \chi^{-1}(b(t)^{-1})$. From this we get the lemma by using (14). \square

5. Criterion for L -movability

The aim of this section is to prove the following characterization of L -movability.

Theorem 15. *Assume that $(w_1, \dots, w_s) \in (W^P)^s$ satisfies equation (7). Then, the following are equivalent.*

- (a) (w_1, \dots, w_s) is L -movable.
- (b) $[\bar{\Lambda}_{w_1}^P] \cdot \dots \cdot [\bar{\Lambda}_{w_s}^P] = d[\bar{\Lambda}_e^P]$ in $H^*(G/P)$, for some nonzero d ; and for each $\alpha_i \in \Delta \setminus \Delta(P)$, we have

$$\left(\left(\sum_{j=1}^s \chi_{w_j} \right) - \chi_1 \right) (x_i) = 0,$$

where χ_w is as defined in Definition 5 and $x_i \in \mathfrak{h}$ is defined by (2).

Proof. (a) \Rightarrow (b): Let $(w_1, \dots, w_s) \in (W^P)^s$ be L -movable. Consider the restriction $\hat{\theta}$ of the P -invariant section

$$\theta \in H^0 \left((P/B_L)^s, \mathcal{L}(\chi_{w_1} - \chi_1) \boxtimes \dots \boxtimes \mathcal{L}(\chi_{w_s}) \right)$$

to $(L/B_L)^s$, where θ is as defined in equations (11)–(12). Then, $\hat{\theta}$ is non-vanishing by Corollary 8. But, for

$$H^0\left((L/B_L)^s, \mathcal{L}(\chi_{w_1} - \chi_1) \boxtimes \cdots \boxtimes \mathcal{L}(\chi_{w_s})\right)^L$$

to be nonzero, the center of L should act trivially (under the diagonal action) on $\mathcal{L}(\chi_{w_1} - \chi_1) \boxtimes \cdots \boxtimes \mathcal{L}(\chi_{w_s})$ restricted to $(L/B_L)^s$. This gives

$$\sum_{j=1}^s \chi_{w_j}(h) = \chi_1(h),$$

for all h in the Lie algebra \mathfrak{z}_L of the center of L . For any $\alpha_i \in \Delta \setminus \Delta(P)$, $h = x_i$ clearly lies in \mathfrak{z}_L . Taking $h = x_i$ in the above equation, we obtain the implication (a) \Rightarrow (b) by using Corollary 8.

We now prove the implication (b) \Rightarrow (a). Thus, we are given that for each $\alpha_i \in \Delta \setminus \Delta(P)$,

$$\left(\sum_{j=1}^s \chi_{w_j}\right) - \chi_1(x_i) = 0.$$

Moreover, by Corollary 8, $\theta(\bar{p}_1, \dots, \bar{p}_s) \neq 0$, for some $\bar{p}_j \in P/B_L$. Consider the central OPS of $L: \lambda(t) := \prod_{\alpha_i \in \Delta \setminus \Delta(P)} t^{x_i}$. Clearly,

$$\dot{\lambda} = \sum_{\alpha_i \in \Delta \setminus \Delta(P)} x_i. \tag{18}$$

Thus, by Lemma 12, λ is admissible. For any $x = ulB_L \in P/B_L$, with $u \in U$ and $l \in L$,

$$\lim_{t \rightarrow 0} \lambda(t)x = \lim_{t \rightarrow 0} \lambda(t)u\lambda(t)^{-1}(\lambda(t)l)B_L.$$

But, since $\beta(\dot{\lambda}) > 0$, for all $\beta \in R^+ \setminus R_1^+$, we get

$$\lim_{t \rightarrow 0} \lambda(t)u\lambda(t)^{-1} = 1.$$

Moreover, since $\lambda(t)$ is central in L , the limit $\lim_{t \rightarrow 0} \lambda(t)lB_L$ exists and equals lB_L . Thus, $\lim_{t \rightarrow 0} \lambda(t)x$ exists and lies in L/B_L .

Now, let \mathbb{L} be the P -equivariant line bundle $\mathcal{L}(\chi_{w_1} - \chi_1) \boxtimes \cdots \boxtimes \mathcal{L}(\chi_{w_s})$ on $\mathbb{X} := (P/B_L)^s$, and $\bar{p} := (\bar{p}_1, \dots, \bar{p}_s) \in \mathbb{X}$. Then, by Lemma 14 and equations (15) and (18), we get

$$\begin{aligned} \mu^{\mathbb{L}}(\bar{p}, \lambda) &= (\chi_1 - \chi_{w_1})([\bar{p}_1, \lambda]\dot{\lambda}) - \sum_{j=2}^s \chi_{w_j}([\bar{p}_j, \lambda]\dot{\lambda}) \\ &= - \sum_{\alpha_i \in \Delta \setminus \Delta(P)} \left(\left(\sum_{j=1}^s \chi_{w_j} \right) - \chi_1(x_i) \right) \\ &= 0, \text{ by assumption.} \end{aligned}$$

Therefore, using Proposition 10(c) for $S = P$, θ does not vanish at $\lim_{t \rightarrow 0} \lambda(t) \bar{p}$. But, from the above, this limit exists as an element of $(L/B_L)^s$. Hence, (w_1, \dots, w_s) is L -movable by Corollary 8. \square

6. Deformation of cup product in $H^*(G/P)$

Define the structure coefficients $c_{u,v}^w$ under the intersection product in $H^*(G/P, \mathbb{Z})$ via the formula

$$[\bar{\Lambda}_u^P] \cdot [\bar{\Lambda}_v^P] = \sum_{w \in W^P} c_{u,v}^w [\bar{\Lambda}_w^P]. \tag{19}$$

The number $c_{u,v}^w$ is the number of points (counted with multiplicity) in the intersection $g_1[\bar{\Lambda}_u^P] \cap g_2[\bar{\Lambda}_v^P] \cap g_3[\bar{\Lambda}_{w_o w w_o^P}^P]$ for generic $(g_1, g_2, g_3) \in G^3$. If the generic intersection is infinite, we set $c_{u,v}^w = 0$.

Introduce the indeterminates τ_i for each $\alpha_i \in \Delta \setminus \Delta(P)$ and write a deformed cup product

$$[\bar{\Lambda}_u^P] \odot [\bar{\Lambda}_v^P] = \sum_{w \in W^P} \left(\prod_{\alpha_i \in \Delta \setminus \Delta(P)} \tau_i^{(\chi_w - (\chi_u + \chi_v))(x_i)} \right) c_{u,v}^w [\bar{\Lambda}_w^P], \tag{20}$$

where χ_w is defined in Definition 5. Extend this to a $\mathbb{Z}[\tau_i]_{\alpha_i \in \Delta \setminus \Delta(P)}$ -linear product structure on $H^*(G/P, \mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{Z}[\tau_i]$, where $\mathbb{Z}[\tau_i]$ is the polynomial ring in the variables $\{\tau_i\}_{\alpha_i \in \Delta \setminus \Delta(P)}$. By the next Proposition 17, if $c_{u,v}^w \neq 0$,

$$(\chi_w - (\chi_u + \chi_v))(x_i) \geq 0, \quad \text{for any } \alpha_i \in \Delta \setminus \Delta(P),$$

and hence the above product indeed lies in $H^*(G/P, \mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{Z}[\tau_i]$. Clearly, \odot is commutative. *This product should not be confused with the product in the quantum cohomology of G/P .*

Recall the definition of T, T_w from Sect. 2.

Lemma 16. *For any $w \in W^P$,*

- (a) $T_w \oplus w_o^P(T_{w_o w w_o^P}) = T$.
- (b) $\chi_w + w_o^P \chi_{w_o w w_o^P} = \chi_1$.
- (c) $(\chi_w + \chi_{w_o w w_o^P} - \chi_1)(x_i) = 0$, for all $\alpha_i \in \Delta \setminus \Delta(P)$.
- (d) For any $v, w \in W^P$ such that $\ell(v) = \ell(w)$, $[\bar{\Lambda}_v^P] \odot [\bar{\Lambda}_{w_o w w_o^P}^P] = \delta_{v,w} [\bar{\Lambda}_1^P]$.

Proof. As in Sect. 2, let $R(T_w)$ denote the set of roots α such that the root space $\mathfrak{g}_\alpha \subset T_w$, i.e., $T_w = \bigoplus_{\alpha \in R(T_w)} \mathfrak{g}_\alpha$. Then,

$$R(T_w) = w^{-1} R^+ \cap (R^- \setminus R_1^-), \tag{21}$$

where $R^- := R \setminus R^+$ and $R_{\tau}^- := R_{\tau} \setminus R_{\tau}^+$. To prove (a), it suffices to show that we have the disjoint union:

$$R^- \setminus R_{\tau}^- = R(T_w) \sqcup w_o^P \cdot R(T_{w_o w w_o^P}). \tag{22}$$

Since w_o^P keeps $R^- \setminus R_{\tau}^-$ stable, by (21),

$$\begin{aligned} w_o^P \cdot R(T_{w_o w w_o^P}) &= w^{-1} w_o R^+ \cap (R^- \setminus R_{\tau}^-) \\ &= w^{-1} R^- \cap (R^- \setminus R_{\tau}^-). \end{aligned}$$

From this and equation (21), equation (22) follows.

To prove (b), by equation (9),

$$\begin{aligned} \chi_w + w_o^P \chi_{w_o w w_o^P} &= \rho - 2\rho^L + w^{-1} \rho + w_o^P \rho - 2w_o^P \rho^L + w^{-1} w_o \rho \\ &= \rho - 2\rho^L + w^{-1} \rho + w_o^P (\rho - \rho^L) - w_o^P \rho^L - w^{-1} \rho \\ &= \rho - 2\rho^L + \rho - \rho^L + \rho^L, \text{ since } w_o^P \text{ permutes } R^+ \setminus R_{\tau}^+ \\ &= 2\rho - 2\rho^L \\ &= \chi_1. \end{aligned}$$

This proves (b).

For any $\alpha_i \in \Delta \setminus \Delta(P)$, x_i is central in \mathfrak{g} ; in particular, w_o^P acts trivially on x_i . Thus (c) follows from (b).

By [KLM, Lemma 2.9],

$$[\bar{\Lambda}_v^P] \cdot [\bar{\Lambda}_{w_o w w_o^P}^P] = \delta_{v,w} [\bar{\Lambda}_1^P]. \tag{23}$$

Thus, (d) follows from the defining equation (20) and the (c)-part. □

Proposition 17. (a) For any $u, v, w \in W^P$ such that $c_{u,v}^w \neq 0$, we have

$$(\chi_w - \chi_u - \chi_v)(x_i) \geq 0, \text{ for each } \alpha_i \in \Delta \setminus \Delta(P). \tag{24}$$

(b) The product \odot in $H^*(G/P) \otimes \mathbb{Z}[\tau_i]$ is associative.

(c) For $(w_1, \dots, w_s) \in (W^P)^s$ and $w \in W^P$, the coefficient of $[\bar{\Lambda}_w^P]$ in

$$[\bar{\Lambda}_{w_1}^P] \odot \dots \odot [\bar{\Lambda}_{w_s}^P]$$

is

$$\prod_{\alpha_i \in \Delta \setminus \Delta(P)} \tau_i^{(\chi_w - \sum_{j=1}^s \chi_{w_j})(x_i)}$$

times the coefficient of $[\bar{\Lambda}_w^P]$ in the usual cohomology product

$$[\bar{\Lambda}_{w_1}^P] \cdot \dots \cdot [\bar{\Lambda}_{w_s}^P].$$

Proof. By equation (23), $c_{u,v}^w \neq 0$ iff

$$[\bar{\Lambda}_u^P] \cdot [\bar{\Lambda}_v^P] \cdot [\bar{\Lambda}_{w_o w w_o^P}^P] = d[\bar{\Lambda}_1^P],$$

for some nonzero d . Thus, by (subsequent) Theorem 29(a), for any $\alpha_i \in \Delta \setminus \Delta(P)$,

$$(\chi_u + \chi_v + \chi_{w_o w w_o^P} - \chi_1)(x_i) \leq 0.$$

By Lemma 16(c), this gives

$$(\chi_u + \chi_v - \chi_w)(x_i) \leq 0,$$

proving (a).

To prove (b), write

$$\begin{aligned} ([\bar{\Lambda}_u^P] \odot [\bar{\Lambda}_v^P]) \odot [\bar{\Lambda}_w^P] &= \sum_{\theta \in W^P} \left(\prod \tau_i^{(\chi_\theta - (\chi_u + \chi_v))(x_i)} \right) c_{u,v}^\theta [\bar{\Lambda}_\theta^P] \odot [\bar{\Lambda}_w^P] \\ &= \sum_{\theta, \eta} \left(\prod \tau_i^{(\chi_\eta - \chi_w - \chi_u - \chi_v)(x_i)} \right) c_{u,v}^\theta c_{\theta,w}^\eta [\bar{\Lambda}_\eta^P]. \end{aligned}$$

Similarly,

$$[\bar{\Lambda}_u^P] \odot ([\bar{\Lambda}_v^P] \odot [\bar{\Lambda}_w^P]) = \sum_{\theta, \eta} \left(\prod \tau_i^{(\chi_\eta - \chi_u - \chi_v - \chi_w)(x_i)} \right) c_{u,\theta}^\eta c_{v,w}^\theta [\bar{\Lambda}_\eta^P].$$

So, the associativity of \odot follows from the corresponding associativity in $H^*(G/P)$ under the standard cup product. The property (c) is immediate from the definitions and property (b). \square

Definition 18. The cohomology of G/P obtained by setting each $\tau_i = 0$ in $(H^*(G/P, \mathbb{Z}) \otimes \mathbb{Z}[\tau_i], \odot)$ is denoted by $(H^*(G/P, \mathbb{Z}), \odot_0)$. Thus, as a \mathbb{Z} -module, this is the same as the singular cohomology $H^*(G/P, \mathbb{Z})$. This has essentially the effect of ignoring all the non L -movable intersections. By the above proposition, $(H^*(G/P, \mathbb{Z}), \odot_0)$ is associative (and commutative). Moreover, by Lemma 16(d), it continues to satisfy the Poincaré duality.

Lemma 19. Let P be a minuscule maximal standard parabolic subgroup of G (i.e., the simple root $\alpha_p \in \Delta \setminus \Delta(P)$ appears with coefficient 1 in the highest root of R^+ and hence in all the roots in $R(\mathfrak{u}_P)$). Then, for any $u, v \in W^P$,

$$[\bar{\Lambda}_u^P] \odot [\bar{\Lambda}_v^P] = [\bar{\Lambda}_u^P] \cdot [\bar{\Lambda}_v^P].$$

Proof. By the definition of \odot , it suffices to show that for any $w \in W^P$ such that $c_{u,v}^w \neq 0$,

$$(\chi_w - (\chi_u + \chi_v))(x_p) = 0. \tag{25}$$

By the definition of χ_w (cf. Definition 5), since P is minuscule,

$$\chi_w(x_p) = |w^{-1}R^+ \cap (R^+ \setminus R_\Gamma^+)| = \text{codim}(\Lambda_w^P; G/P), \tag{26}$$

where the last equality follows from equation (21). Moreover, since $c_{u,v}^w \neq 0$,

$$\text{codim}(\Lambda_u^P; G/P) + \text{codim}(\Lambda_v^P; G/P) = \text{codim}(\Lambda_w^P; G/P). \tag{27}$$

Combining equations (26) and (27), we get equation (25). □

Remark 20. Write the cup product in $H^*(G/P, \mathbb{Z})$ in the $\{\epsilon_w^P\}$ basis as follows:

$$\epsilon_u^P \cdot \epsilon_v^P = \sum_{w \in W^P} d_{u,v}^w \epsilon_w^P. \tag{28}$$

Then, the deformed product in the $\{\epsilon_w^P\}$ basis is given as follows:

$$\epsilon_u^P \odot \epsilon_v^P = \sum_{w \in W^P} \left(\prod_{\alpha_i \in \Delta \setminus \Delta(P)} \tau_i^{(u^{-1}\rho + v^{-1}\rho - w^{-1}\rho - \rho)(x_i)} \right) d_{u,v}^w \epsilon_w^P, \tag{29}$$

7. Solution of the eigenvalue problem

Our aim in this section is to prove that, for the solution of the eigenvalue problem, one may restrict to those inequalities coming from L -movable intersections with intersection number one.

7.1. Principal criterion for the nontriviality of the space of invariants

Let $\nu_1, \dots, \nu_s \in X(H)$ be dominant weights and let \mathbb{L} be the G -linearized line bundle $\mathcal{L}(\nu_1) \boxtimes \dots \boxtimes \mathcal{L}(\nu_s)$ on $(G/B)^s$.

In this subsection, we give our principal criterion to decide whether there exists an integer $N > 0$ such that $H^0((G/B)^s, \mathbb{L}^N)^G \neq 0$.

Let P_1, \dots, P_s be the standard parabolic subgroups such that the line bundle $\mathcal{L}(\nu_j)$ on G/B descends as an ample line bundle on G/P_j , still denoted by $\mathcal{L}(\nu_j)$. Consider $\mathbb{X} := G/P_1 \times \dots \times G/P_s$ with the diagonal action of G and \mathbb{L} the G -linearized ample line bundle $\mathcal{L}(\nu_1) \boxtimes \dots \boxtimes \mathcal{L}(\nu_s)$ on \mathbb{X} . Let $\pi : (G/B)^s \rightarrow \mathbb{X}$ be the canonical projection map.

Lemma 21. *Let G be a connected reductive group. Let $x \in (G/B)^s$ and let $x' = \pi(x)$. The following are equivalent.*

- (a) *For some integer $N > 0$, there exists $\sigma \in H^0((G/B)^s, \mathbb{L}^N)^G$ such that $\sigma(x) \neq 0$, where G acts diagonally.*
- (b) *For every OPS $\lambda \in O(G)$, $\mu^\mathbb{L}(x, \lambda) \geq 0$.*
- (c) *For every OPS $\lambda \in O(G)$, $\mu^\mathbb{L}(x', \lambda) \geq 0$.*
- (d) *For some integer $N > 0$, there exists $\sigma' \in H^0(\mathbb{X}, \mathbb{L}^N)^G$ such that $\sigma'(x') \neq 0$, i.e., x' is a semistable point of \mathbb{X} (with respect to the G -linearized ample line bundle \mathbb{L}).*

Proof. We first note that for any integer $N > 0$, the map π induces an isomorphism of G -modules:

$$H^0(\mathbb{X}, \mathbb{L}^N) \simeq H^0((G/B)^s, \mathbb{L}^N). \tag{30}$$

It follows from Proposition 10(d) that (b) is equivalent to (c).

By Hilbert-Mumford theorem [MFK, Theorem 2.1], since \mathbb{L} is ample on \mathbb{X} , (c) is equivalent to (d).

By equation (30), (d) is equivalent to (a) and we are done. □

We now state one of the main theorems of this paper.

Theorem 22. *Let G be a connected semisimple group. With the notation as above, the following are equivalent:*

(i) *For some integer $N > 0$,*

$$H^0((G/B)^s, \mathbb{L}^N)^G \neq 0.$$

(ii) *For every standard maximal parabolic subgroup P in G and every choice of L -movable s -tuples $(w_1, \dots, w_s) \in (W^P)^s$ such that*

$$[\bar{\Lambda}_{w_1}^P] \cdot \dots \cdot [\bar{\Lambda}_{w_s}^P] = [\bar{\Lambda}_e^P] \in H^*(G/P, \mathbb{Z}),$$

the following inequality holds:

$$\sum_{j=1}^s \nu_j(w_j x_{i_P}) \leq 0, \tag{31}$$

where α_{i_P} is the simple root in $\Delta \setminus \Delta(P)$.

(iii) *For every standard maximal parabolic subgroup P in G and every choice of s -tuples $(w_1, \dots, w_s) \in (W^P)^s$ such that*

$$[\bar{\Lambda}_{w_1}^P] \odot_0 \dots \odot_0 [\bar{\Lambda}_{w_s}^P] = [\bar{\Lambda}_e^P] \in (H^*(G/P, \mathbb{Z}), \odot_0),$$

the above inequality (31) holds.

The proof of this theorem will be given in Subsect. 7.3.

Remark 23. (a) The above theorem remains true for any connected reductive G provided we assume that $\sum_j \nu_j|_{\mathfrak{z}(\mathfrak{g})} = 0$, where $\mathfrak{z}(\mathfrak{g})$ is the center of \mathfrak{g} .

(b) The implication (i) \Rightarrow (ii) (and hence also the implication (i) \Rightarrow (iii)) in the above theorem remains true for any (not necessarily maximal) standard parabolic P of G .

7.2. Maximally destabilizing one parameter subgroups

We recall the definition of the Kempf’s OPS attached to an unstable point, which is in some sense ‘most destabilizing’ OPS. The exposition below follows the paper of Hesselink [H].

Let X be a projective variety with the action of a reductive group G and let \mathcal{L} be a G -linearized ample line bundle on X .

We introduce the set $M(G)$ of fractional OPS in G . This is the set consisting of the ordered pairs (δ, a) , where $\delta : \mathbb{G}_m \rightarrow G$ is an OPS of G and $a \in \mathbb{Z}_{>0}$, modulo the equivalence relation $(\delta, a) \simeq (v, b)$ if $\delta^b = v^a$. An OPS δ of G gives the element $(\delta, 1) \in M(G)$. The group G acts on $M(G)$ via conjugation: $g \cdot (\delta, a) = (g\delta g^{-1}, a)$. Choose a G -invariant norm $q : M(G) \rightarrow \mathbb{R}_+$ in the sense of [H, §1.3], \mathbb{R}_+ being the set of nonnegative real numbers. (Actually our norm q is the square root of Hesselink’s norm.)

We can extend the definition of $\mu^{\mathcal{L}}(x, \lambda)$ for any element $\lambda = (\delta, a) \in M(G)$ and $x \in X$ by setting $\mu^{\mathcal{L}}(x, \lambda) = \frac{\mu^{\mathcal{L}}(x, \delta)}{a}$.

For an OPS λ of G , recall the definition of the associated parabolic subgroup $P(\lambda)$ of G from Definition 11. We extend the definition of $P(\lambda)$ for any $\lambda = (\delta, a) \in M(G)$ by setting

$$P(\lambda) = P(\delta).$$

Then, the group

$$L(\lambda) := \{p \in G \mid \lim_{t \rightarrow 0} \lambda(t)p\lambda(t)^{-1} = p\}$$

is a Levi subgroup of $P(\lambda)$ and, moreover,

$$U(\lambda) := \{p \in G \mid \lim_{t \rightarrow 0} \lambda(t)p\lambda(t)^{-1} = e \in G\}$$

is the unipotent radical of $P(\lambda)$.

We note the following elementary property: If $\lambda \in M(G)$ and $p \in P(\lambda)$ then

$$\mu^{\mathcal{L}}(x, \lambda) = \mu^{\mathcal{L}}(x, p\lambda p^{-1}). \tag{32}$$

For any unstable point $x \in X$, define

$$q^*(x) = \inf_{\lambda \in M(G)} \{q(\lambda) \mid \mu^{\mathcal{L}}(x, \lambda) \leq -1\},$$

and the *optimal class*

$$\Lambda(x) = \{\lambda \in M(G) \mid \mu^{\mathcal{L}}(x, \lambda) \leq -1, q(\lambda) = q^*(x)\}.$$

Any $\lambda \in \Lambda(x)$ is called *Kempf’s OPS associated to x* .

The following theorem is due to Kempf (cf. [Ke] or [K, Lemma 12.13]).

Theorem 24. *For any unstable point $x \in X$, $\Lambda(x)$ is nonempty and for $\lambda_1, \lambda_2 \in \Lambda(x)$, $P(\lambda_1) = P(\lambda_2)$. Moreover, there exist $p_1, p_2 \in P(\lambda_1)$ so that $p_1\lambda_1p_1^{-1} = p_2\lambda_2p_2^{-1}$.*

Conversely, for $\lambda \in \Lambda(x)$ and $p \in P(\lambda)$, we have $p\lambda p^{-1} \in \Lambda(x)$ by equation (32).

The parabolic $P(\lambda)$ for $\lambda \in \Lambda(x)$ will be denoted by $P(x)$ and called the *Kempf's parabolic associated to the unstable point x* . We recall the following theorem due to Ramanan-Ramanathan [RR, Proposition 1.9].

Theorem 25. *For any unstable point $\bar{x} \in X$ and $\lambda = (\delta, a) \in \Lambda(x)$, let*

$$x_o = \lim_{t \rightarrow 0} \delta(t) \cdot x \in X.$$

Then, x_o is unstable and $\lambda \in \Lambda(x_o)$.

7.3. Proof of Theorem 22

We return to the notation and assumptions of Subsect. 7.1. In particular, $\nu_1, \dots, \nu_s \in X(H)$ are dominant weights and \mathbb{L} denotes the G -linearized line bundle $\mathcal{L}(\nu_1) \boxtimes \dots \boxtimes \mathcal{L}(\nu_s)$ on $(G/B)^s$. Also, P_1, \dots, P_s are the standard parabolic subgroups such that \mathbb{L} descends as an ample line bundle (still denoted by) \mathbb{L} on $\mathbb{X} := G/P_1 \times \dots \times G/P_s$. We call a point $x \in (G/B)^s$ *semistable* (with respect to, not necessarily ample, \mathbb{L}) if its image in \mathbb{X} under the canonical map $\pi : (G/B)^s \rightarrow \mathbb{X}$ is semistable.

By Lemma 21 and equation (30), condition (i) of Theorem 22 is equivalent to the following condition:

(iv) *The set of semistable points of $(G/B)^s$ with respect to \mathbb{L} is non empty.*

Moreover, by the definition of \odot_0 and Theorem 15, the conditions (ii) and (iii) of Theorem 22 are equivalent.

Proof of the implication (iv) \Rightarrow (ii) of Theorem 22. Let $x = (\bar{g}_1, \dots, \bar{g}_s) \in (G/B)^s$ be a semistable point, where $\bar{g}_j = g_j B$. Since the set of semistable points is clearly open, by Proposition 3 we can choose a generic enough x such that the intersection $\cap g_j B w_j P$ itself is nonempty. (By assumption, $\cap \overline{g_j B w_j P}$ is nonempty for any g_j .) Pick $f \in \cap g_j B w_j P$. Consider the OPS $\lambda = t^{x_{i_p}}$, which is central in L . Since $P(\lambda) \supset L$ and, clearly, $P(\lambda) \supset B$, we have $P(\lambda) \supset P$. But, by assumption, P is a maximal parabolic subgroup and hence $P(\lambda) = P$ (it is easy to see that $P(\lambda) \neq G$). Moreover, from Definition 13 applied to the case $P = G$, it is easy to see that $[\bar{g}_j, f^\lambda f^{-1}] = w_j$ and, $X_{f^\lambda f^{-1}} = x_{i_p}$. Thus, applying Lemma 14 for $P = G$, the required inequality (31) is the same as $\mu^{\mathbb{L}}(x, f^\lambda f^{-1}) \geq 0$; but this follows from Lemma 21, since x is semistable by assumption. \square

Before we come to the proof of the implication $(ii) \Rightarrow (i)$ in Theorem 22, we need to recall the following result due to Kapovich-Leeb-Millson.

Suppose that $x = (\bar{g}_1, \dots, \bar{g}_s) \in (G/B)^s$ is an unstable point and $P(x)$ the Kempf's parabolic associated to $\pi(x)$. Let $\lambda = (\delta, a)$ be a Kempf's OPS associated to $\pi(x)$. Express $\delta(t) = f\gamma(t)f^{-1}$, where $\dot{\gamma} \in \mathfrak{h}_+ := \{h \in \mathfrak{h} : \alpha_i(h) \geq 0 \forall \text{ simple roots } \alpha_i\}$. Then, $P(\gamma)$ is a standard parabolic. Let P be a maximal parabolic containing $P(\gamma)$. Define $w_j \in W/W_{P(\gamma)}$ by $fP(\gamma) \in g_j Bw_j P(\gamma)$ for $j = 1, \dots, s$. We recall the following theorem due to Kapovich-Leeb-Millson applicable to any unstable point $x \in (G/B)^s$. We postpone the proof of this theorem to the next subsection.

Theorem 26. (a) *The intersection $\bigcap_{j=1}^s g_j Bw_j P \subset G/P$ is the singleton $\{fP\}$.*

(b) *For the simple root $\alpha_{i_p} \in \Delta \setminus \Delta(P)$, $\sum_{j=1}^s v_j(w_j x_{i_p}) > 0$.*

Now, we come to the proof of the implication $(ii) \Rightarrow (i)$ in Theorem 22. Assume, if possible, that (i) equivalently (iv) as above is false, i.e., the set of semistable points of $(G/B)^s$ is empty (and (ii) is true). Thus, any point $x = (\bar{g}_1, \dots, \bar{g}_s) \in (G/B)^s$ is unstable. Choose a generic x so that for each standard parabolic \tilde{P} in G and any $(z_1, \dots, z_s) \in W^s$, the intersection

$$g_1 Bz_1 \tilde{P} \cap \dots \cap g_s Bz_s \tilde{P}$$

is transverse (possibly empty) and dense in $\overline{g_1 Bz_1 \tilde{P}} \cap \dots \cap \overline{g_s Bz_s \tilde{P}}$. Let $\lambda = (\delta, a)$, P, γ, f, w_j be as above. It follows from Theorem 26 that $\bigcap_{j=1}^s g_j Bw_j P \subset G/P$ is the single point f and, since x is generic, we get

$$[\bar{\Lambda}_{w_1}^P] \cdot \dots \cdot [\bar{\Lambda}_{w_s}^P] = [\bar{\Lambda}_e^P] \in H^*(G/P, \mathbb{Z}). \tag{33}$$

We now claim that the s -tuple $(w_1, \dots, w_s) \in (W/W_P)^s$ is L -movable.

Write $g_j = fp_j w_j^{-1} b_j$, for some $p_j \in P(\gamma)$ and $b_j \in B$. Hence,

$$\delta(t)\bar{g}_j = f\gamma(t)p_j w_j^{-1} B = f\gamma(t)p_j \gamma^{-1}(t)w_j^{-1} B \in G/B.$$

Define,

$$l_j = \lim_{t \rightarrow 0} \gamma(t)p_j \gamma^{-1}(t).$$

Then, $l_j \in L(\gamma)$. Therefore,

$$\lim_{t \rightarrow 0} \delta(t)x = (fl_1 w_1^{-1} B, \dots, fl_s w_s^{-1} B).$$

By Theorem 25, $\lambda \in \Lambda(\lim_{t \rightarrow 0} \delta(t)x)$. We note that, for $j = 1, \dots, s$,

$$fP(\gamma) \in (fl_j w_j^{-1})Bw_j P(\gamma).$$

Applying Theorem 26 to the unstable point $x_o = \lim_{t \rightarrow 0} \delta(t)x$ yields

(†) fP is the only point in the intersection $\bigcap_{j=1}^s fl_j w_j^{-1} Bw_j P$.

Translating by f , we get:

(\ddagger) eP is the only point in the intersection

$$\Omega := \bigcap l_j w_j^{-1} B w_j P.$$

Since the sequence (w_1, \dots, w_s) satisfies equation (33); in particular, it satisfies equation (7). Therefore, the expected dimension of Ω is 0 and so is its actual dimension by (\ddagger). If this intersection Ω is not transverse at e , then by intersection theory ([Fu1, Remark 8.2]), the local multiplicity at e is > 1 , each $w_j^{-1} B w_j P$ being smooth.

Further, G/P being a homogenous space, any other component of the intersection

$$\bigcap \overline{l_j w_j^{-1} B w_j P}.$$

contributes nonnegatively to the intersection product $[\bar{\Lambda}_{w_1}^P] \cdot \dots \cdot [\bar{\Lambda}_{w_s}^P]$ (cf. [Fu1, §12.2]). Thus, from equation (33), we get that the intersection $\bigcap l_j w_j^{-1} B w_j P$ is transverse at $e \in G/P$, proving that (w_1, \dots, w_s) is L -movable. Clearly, by condition (ii), we see that the s -tuple (w_1, \dots, w_s) satisfies

$$\sum_{j=1}^s v_j(w_j x_{i_P}) \leq 0.$$

This contradicts the (b)-part of Theorem 26. Thus, the set of semistable points of $(G/B)^s$ is nonempty, proving condition (i) of Theorem 22. \square

7.4. Proof of Theorem 26

Even though proof of Theorem 26 can be extracted from [KaLM], we did not find it explicitly stated there. So, for completeness, we give its proof couched entirely in algebro-geometric language.

Let $x = (g_1 B, \dots, g_s B) \in (G/B)^s$ be any unstable point and let $\lambda = (\delta, a)$ be a Kempf OPS associated to x . Write

$$Y = \frac{X_\delta}{a} = \sum_{\alpha_i \in S_\delta} c_i x_{i_P},$$

where S_δ is a (nonempty) subset of the set of simple roots such that each c_i is nonzero (and hence strictly positive). Let $\gamma \in O(G)$ and $f \in G$ be such that $f\gamma f^{-1} = \delta$ and $\dot{\gamma} = X_\delta \in \mathfrak{h}_+$. Define $w_j \in W/W_{P(\gamma)}$ by $fP(\gamma) \in g_j B w_j P(\gamma)$ for $j = 1, \dots, s$.

With this notation, Theorem 26 can equivalently be formulated as the following:

(a)' For $\alpha_i \in S_\delta$, $fP(\lambda_i)$ is the only point in $\bigcap_j g_j B w_j P(\lambda_i)$, where λ_i is the OPS with $\hat{\lambda}_i = x_i$. Moreover,

$$\sum_{j=1}^s v_j(w_j x_i) > 0. \tag{34}$$

Note that

$$P(\lambda_i) \supset P(\gamma). \tag{35}$$

We first prove the following additional property:

(a₁) $fP(\gamma)$ is the only point in $\bigcap_j g_j B w_j P(\gamma)$.

For (a₁), let $hP(\gamma)$ be some other point in the intersection. Then, from Lemma 14, we have $\mu^\mathbb{L}(x, h\gamma h^{-1}) = \mu^\mathbb{L}(x, f\gamma f^{-1})$. Further, by the definition of norm, $f\gamma f^{-1}$ and $h\gamma h^{-1}$ have the same norm. Therefore, by Theorem 24, $fP(\gamma) f^{-1} = hP(\gamma) h^{-1}$, i.e., $hP(\gamma) = fP(\gamma)$ as elements of $G/P(\gamma)$.

To prove (a)', it is convenient to prove the following auxiliary lemma.

Lemma 27. *The inequality (34) holds. Moreover, let θ be an OPS such that $X_\theta = x_i$ for $\alpha_i \in S_\delta$ and*

$$\frac{-\mu^\mathbb{L}(x, \theta)}{q(\theta)} \geq \frac{-\mu^\mathbb{L}(x, f\lambda_i f^{-1})}{q(\lambda_i)}. \tag{36}$$

Then, equality holds in the above equation (36) and $P(\theta) = P(f\lambda_i f^{-1})$.

This lemma will yield (a)', because if $hP(\lambda_i)$ were another point of the intersection $\bigcap_j g_j B w_j P(\lambda_i)$, then (by using Lemma 14 again)

$$\mu^\mathbb{L}(x, h\lambda_i h^{-1}) = \mu^\mathbb{L}(x, f\lambda_i f^{-1}),$$

and of course $q(h\lambda_i h^{-1}) = q(f\lambda_i f^{-1})$. Thus, from the above lemma, we get that $hP(\lambda_i) = fP(\lambda_i)$. This completes the proof of (a)'.

Proof of Lemma 27. Let \tilde{H} be a maximal torus of G contained in $P(\theta) \cap P(\delta)$, where $\lambda = (\delta, a)$ is a Kempf's OPS associated to x . Using equation (32) and the conjugacy of maximal tori in the algebraic group $P(\theta)$, replace θ in Lemma 27 with $p\theta p^{-1}$ for some $p \in P(\theta)$ chosen so that $p\theta p^{-1} \in O(\tilde{H})$ (and the inequality (36) is satisfied for θ replaced by $p\theta p^{-1}$).

Let $(\tilde{\delta}, a)$ be a Kempf's OPS (corresponding to the point $x \in \mathbb{X}$) such that $\tilde{\delta} \in O(\tilde{H})$. Find $b \in G$ so that $b^{-1}\tilde{H}b = H$ and $b^{-1}\tilde{\delta}b = \gamma$.

From the uniqueness of the parabolics associated to Kempf's OPS (cf. Theorem 24), we get:

$$bP(\gamma) = fP(\gamma) \in G/P(\gamma). \tag{37}$$

Now, $b^{-1}\theta b \in O(H)$ and $X_\theta = x_i$. Therefore,

$$\text{Ad}(b^{-1})\dot{\theta} = wx_i, \quad \text{for some } w \in W. \tag{38}$$

Let $\mathfrak{h}^{\mathbb{Q}}$ be the \mathbb{Q} -vector subspace of \mathfrak{h} spanned by $\dot{\nu}$, where ν runs over $O(H)$, and let $\mathfrak{h}_+^{\mathbb{Q}} := \mathfrak{h}^{\mathbb{Q}} \cap \mathfrak{h}_+$. Define the function $\mathcal{L} = \mathcal{L}_{\mathbb{L},x,b} : \mathfrak{h}^{\mathbb{Q}} \rightarrow \mathbb{Q}$ as follows. For any $\beta \in O(H)$ and $r \geq 0 \in \mathbb{Q}$,

$$\mathcal{L}(r\dot{\beta}) = -r\mu^{\mathbb{L}}(x, b\beta b^{-1}).$$

Since $\mu^{\mathbb{L}}(x, b\beta^p b^{-1}) = p\mu^{\mathbb{L}}(x, b\beta b^{-1})$ for any positive integer p , \mathcal{L} is well defined. Let V be a finite dimensional representation of G together with a G -equivariant embedding $\iota : \mathbb{X} \rightarrow \mathbb{P}(V)$ such that $\iota^*(\mathcal{O}(1))$ is G -equivariantly isomorphic with \mathbb{L}^M for some $M > 0$. We can take, e.g., $V = H^0((G/B)^s, \mathbb{L}^M)^*$ for any $M > 0$. Define a twisted action of G on V via

$$g \odot v = (bgb^{-1}) \cdot v, \quad \text{for } g \in G, v \in V.$$

Find a basis $\{e_1, \dots, e_n\}$ of V so that, under the twisted action, H acts by the character η_l on e_l , i.e., $t \odot e_l = \eta_l(t)e_l$, for $t \in H$. Write $\iota(\pi(x)) = [\sum x_l e_l]$. Then, by equation (13), for any $h \in \mathfrak{h}^{\mathbb{Q}}$,

$$\mathcal{L}(h) = -\frac{1}{M} \max_{l: x_l \neq 0} (-\dot{\eta}_l(h)) = \frac{1}{M} \min_{l: x_l \neq 0} (\dot{\eta}_l(h)).$$

The function \mathcal{L} satisfies the following properties:

- (P₁) \mathcal{L} is convex: $\mathcal{L}(ah_1 + bh_2) \geq a\mathcal{L}(h_1) + b\mathcal{L}(h_2)$ for $h_1, h_2 \in \mathfrak{h}^{\mathbb{Q}}$ and positive rational numbers a, b .
- (P₂) $\mathcal{L}(h) = \sum_j \nu_j \langle \nu_j h, h \rangle$, for $h \in \mathfrak{h}_+^\delta := \sum_{\alpha_i \in S_\delta} \mathbb{Q}_+ x_i$; in particular, \mathcal{L} is linear restricted to \mathfrak{h}_+^δ .
- (P₃) From the Kempf's theory, the function $J(h) := \frac{\mathcal{L}(h)}{q(h)}$ on $\mathfrak{h}^{\mathbb{Q}} \setminus \{0\}$ (constant on positive rays) achieves its maximum uniquely at the positive ray through $Y \in \mathfrak{h}_+^\delta$, where Y is defined in the beginning of this Subsect. 7.4, and the norm q on $M(G)$ clearly gives rise to a W -invariant norm (again denoted by) q on $\mathfrak{h}^{\mathbb{Q}}$.

Fix a nonzero $h \in \mathfrak{h}^{\mathbb{Q}}$ and consider the function $v \mapsto J(Y + vh)$, for rational $v \geq 0$. Then, by the convexity of \mathcal{L} as in (P₁), $J(Y + vh) \geq \frac{\mathcal{L}(Y) + v\mathcal{L}(h)}{q(Y + vh)}$. View the right hand side as a function of $v \geq 0$. It clearly takes the maximum value at $v = 0$. Thus, taking its derivative at $v = 0$, we get:

$$\mathcal{L}(h)q(Y)^2 - \mathcal{L}(Y)\langle Y, h \rangle \leq 0,$$

or, that

$$J(h) \leq J(Y) \frac{\langle Y, h \rangle}{q(Y)q(h)}, \tag{39}$$

where $2\langle Y, h \rangle := q(Y + h)^2 - q(Y)^2 - q(h)^2$. From now on till the end of this proof, we take i such that $\alpha_i \in S_\delta$. We note that $Y + vx_i \in \mathfrak{h}_+^\delta$ for small (positive or negative) values of v . Moreover, by (P_2) , $\mathfrak{L}(Y + vx_i) = \mathfrak{L}(Y) + v\mathfrak{L}(x_i)$ for small values of v . Thus, the function $v \mapsto \frac{\mathfrak{L}(Y) + v\mathfrak{L}(x_i)}{q(Y + vx_i)}$ has a local maximum at 0; in particular, its derivative at $v = 0$ is zero. This gives:

$$J(x_i) = J(Y) \frac{\langle Y, x_i \rangle}{q(Y)q(x_i)}. \tag{40}$$

Moreover, since $\langle x_i, x_j \rangle \geq 0$ for simple roots α_i and α_j and $J(Y) > 0$ (since $\mathfrak{L}(\text{Ad}(b^{-1}) \cdot \tilde{\delta}) > 0$), we get that $J(x_i) > 0$. That is,

$$\mu^\mathbb{L}(x, b\lambda_i b^{-1}) < 0,$$

where $\lambda_i \in O(H)$ is defined by $\dot{\lambda}_i = x_i$. But, it follows from equation (37) that

$$\mu^\mathbb{L}(x, b\lambda_i b^{-1}) = \mu^\mathbb{L}(x, f\lambda_i f^{-1}). \tag{41}$$

We conclude using Lemma 14 that the inequality (34) holds. Our assumption (36) now reads as (in view of equations (38) and (41))

$$J(wx_i) \geq J(x_i). \tag{42}$$

But, according to the inequality (39),

$$J(wx_i) \leq J(Y) \frac{\langle Y, wx_i \rangle}{q(Y)q(wx_i)}, \tag{43}$$

and, by equation (40),

$$J(x_i) = J(Y) \frac{\langle Y, x_i \rangle}{q(Y)q(x_i)}. \tag{44}$$

It is also easy to see that $\langle Y, x_i \rangle > \langle Y, wx_i \rangle$, if $wx_i \neq x_i$. Combining equations (42)–(44), we therefore conclude that $wx_i = x_i$. Thus, by equation (38), $\theta = b\lambda_i b^{-1}$ and hence

$$P(\theta) = P(b\lambda_i b^{-1}) = P(f\lambda_i f^{-1}),$$

by equations (35) and (37). Finally, by equations (38) and (41), the inequality (36) is in fact an equality. This proves Lemma 27 and hence (a)' and hence Theorem 26 is proved. \square

7.5. *L*-movability and the eigenvalue problem

Let G be a connected (complex) semisimple group. Choose a maximal compact subgroup K of G with Lie algebra \mathfrak{k} . Then, there is a natural homeomorphism $C : \mathfrak{k}/K \rightarrow \mathfrak{h}_+$, where K acts on \mathfrak{k} by the adjoint representation

and $\mathfrak{h}_+ := \{h \in \mathfrak{h} : \alpha_j(h) \geq 0 \ \forall \text{ the simple roots } \alpha_j\}$ is the positive Weyl chamber in \mathfrak{h} . The inverse map C^{-1} takes any $h \in \mathfrak{h}_+$ to the K -conjugacy class of ih .

Now, one of the main aims of the *eigenvalue problem* is to describe the set $\Gamma(s, K) := \{(h_1, \dots, h_s) \in \mathfrak{h}_+^s \mid \exists(k_1, \dots, k_s) \in \mathfrak{k}^s : \sum_{j=1}^s k_j = 0 \text{ and } C(k_j) = h_j \ \forall j = 1, \dots, s\}$.

Given a standard maximal parabolic subgroup P , let ω_P denote the corresponding fundamental weight, i.e., $\omega_P(\alpha_i^\vee) = 1$, if $\alpha_i \in \Delta \setminus \Delta(P)$ and 0 otherwise, where α_i^\vee is the fundamental coroot corresponding to the simple root α_i . This is invariant under the Weyl group W_P of P .

The following theorem is our main result on solution of the eigenvalue problem.

Theorem 28. *Let $(h_1, \dots, h_s) \in \mathfrak{h}_+^s$. Then, the following are equivalent:*

- (a) $(h_1, \dots, h_s) \in \Gamma(s, K)$.
- (b) For every standard maximal parabolic subgroup P in G and every choice of s -tuples $(w_1, \dots, w_s) \in (W^P)^s$ such that

$$[\bar{\Lambda}_{w_1}^P] \odot_0 \cdots \odot_0 [\bar{\Lambda}_{w_s}^P] = [\bar{\Lambda}_e^P] \in (H^*(G/P, \mathbb{Z}), \odot_0),$$

the following inequality holds:

$$\omega_P\left(\sum_{j=1}^s w_j^{-1}h_j\right) \leq 0.$$

Proof. Observe first that, under the identification of \mathfrak{h} with \mathfrak{h}^* induced from the Killing form, \mathfrak{h}_+ is isomorphic with the set D of dominant weights in \mathfrak{h}^* . In fact, under this identification, x_i corresponds with $2\omega_i/\langle \alpha_i, \alpha_i \rangle$, where ω_i denotes the i -th fundamental weight. Let $D_{\mathbb{Z}}$ be the set of dominant integral weights. Define

$$\hat{\Gamma}(s) := \{(v_1, \dots, v_s) \in D^s : Nv_j \in D_{\mathbb{Z}} \text{ for all } j \text{ and } H^0((G/B)^s, \mathcal{L}(Nv_1) \boxtimes \cdots \boxtimes \mathcal{L}(Nv_s))^G \neq 0 \text{ for some } N > 0\}.$$

Then, under the identification of \mathfrak{h}_+ with D (and hence of \mathfrak{h}_+^s with D^s), $\Gamma(s, K)$ corresponds to the closure of $\hat{\Gamma}(s)$. In fact, $\hat{\Gamma}(s)$ consists of the rational points of the image of $\Gamma(s, K)$ (cf., e.g., [Sj, Theorem 7.6]). Since x_i corresponds with $2\omega_i/\langle \alpha_i, \alpha_i \rangle$, the theorem follows from Theorem 22. \square

8. Nonvanishing of products in the cohomology of flag varieties (Horn inequalities)

We give two inductive criteria (actually, only necessary conditions) to determine when the product of a number of Schubert cohomology classes of G/P is nonzero. The first criterion (Theorem 29) is in terms of the characters, whereas the second one (Theorem 36) is in terms of dimension counts. In this section G denotes a connected reductive group.

Theorem 29. Assume that $(w_1, \dots, w_s) \in (W^P)^s$ satisfies equation (7) and that $[\bar{\Lambda}_{w_1}^P] \cdot \dots \cdot [\bar{\Lambda}_{w_s}^P] = d[\bar{\Lambda}_1^P]$ in $H^*(G/P)$, for some nonzero d . Then,

(a) For each $\alpha_i \in \Delta \setminus \Delta(P)$, the following inequality holds:

$$\left(\left(\sum_{j=1}^s \chi_{w_j} \right) - \chi_1 \right) (x_i) \leq 0, \tag{45}$$

where χ_w is defined in Definition 5.

(b) For any standard parabolic Q_L of L (i.e., $Q_L \supset B_L$), and $u_1, \dots, u_s \in W_L/W_{Q_L}$ such that

$$[\bar{\Lambda}_{u_1}^{Q_L}] \cdot \dots \cdot [\bar{\Lambda}_{u_s}^{Q_L}] \neq 0 \in H^*(L/Q_L),$$

the inequality

$$\sum_{j=1}^s \chi_{w_j}(u_j x_p) \leq \chi_1(x_p) \tag{46}$$

holds for any p such that $\alpha_p \in \Delta(P) \setminus \Delta(Q_L)$.

(Observe that W_{Q_L} acts trivially on x_p for any $\alpha_p \in \Delta(P) \setminus \Delta(Q_L)$.)

Proof. Let $v_1, \dots, v_s \in X(H)$ and let

$$0 \neq \sigma \in H^0 \left((P/B_L)^s, \mathcal{L}(v_1) \boxtimes \dots \boxtimes \mathcal{L}(v_s) \right)^P.$$

Assume that σ does not vanish at $(\bar{p}_1, \dots, \bar{p}_s) \in (P/B_L)^s$. Then, for every admissible OPS $\lambda \in O(P)$, we have the following inequality obtained from Proposition 10(b) and Lemma 14:

$$v_1([\bar{p}_1, \lambda]X_\lambda) + \dots + v_s([\bar{p}_s, \lambda]X_\lambda) \leq 0, \tag{47}$$

where X_λ is defined by equation (16).

Now, Theorem 29 follows immediately from the following proposition together with Proposition 2 and Lemma 7. (Observe that χ_1 is fixed by W_L .)

□

Proposition 30. Let $v_1, \dots, v_s \in X(H)$ and let

$$H^0 \left((P/B_L)^s, \mathcal{L}(v_1) \boxtimes \dots \boxtimes \mathcal{L}(v_s) \right)^P \neq 0. \tag{48}$$

Then,

(a) For each $\alpha_i \in \Delta \setminus \Delta(P)$, we have $\sum_{j=1}^s v_j(x_i) \leq 0$.

(b) For any standard parabolic subgroup Q_L of L and $u_1, \dots, u_s \in W_L/W_{Q_L}$ such that $[\bar{\Lambda}_{u_1}^{Q_L}] \cdot \dots \cdot [\bar{\Lambda}_{u_s}^{Q_L}] \neq 0 \in H^*(L/Q_L)$, we have:

$$\sum_{j=1}^s v_j(u_j x_p) \leq 0, \text{ for all } \alpha_p \in \Delta(P) \setminus \Delta(Q_L).$$

Proof. For (a), apply equation (47) to the admissible OPS $\lambda = t^{x_i}$, which is central in L . In this case, by equation (15), $[\bar{p}, \lambda] = 1$ for any $\bar{p} \in P/B_L$ and $X_\lambda = \dot{\lambda} = x_i$.

To prove (b), pick a nonzero σ in the vector space (48). Let $Z \subset L^s$ be the set of points (l_1, \dots, l_s) such that

$$l_1 B_L u_1 Q_L \cap \dots \cap l_s B_L u_s Q_L \neq \emptyset.$$

By the assumption in (b), Z contains a nonempty open set Z^0 of L^s (cf. Proposition 3). Let Z_σ be the subset of P^s consisting of (p_1, \dots, p_s) such that σ does not vanish at $(\bar{p}_1, \dots, \bar{p}_s) \in (P/B_L)^s$, where $\bar{p}_j := p_j B_L$. Since $\sigma \neq 0$, Z_σ is a nonempty open subset of P^s . Consider the projection $\pi : P^s \rightarrow L^s$ under the decomposition $P = U \cdot L$ and pick

$$(p_1, \dots, p_s) \in Z_\sigma \cap \pi^{-1}(Z^0).$$

Since Z^0 and Z_σ are nonempty Zariski open subsets, the intersection $Z_\sigma \cap \pi^{-1}(Z^0)$ is nonempty and open.

Let $p_j = v_j l_j$, for $j = 1, \dots, s$, where $v_j \in U$ and $l_j \in L$. Pick

$$l \in l_1 B_L u_1 Q_L \cap \dots \cap l_s B_L u_s Q_L.$$

Consider the admissible OPS $\lambda(t) = lt^{x_p}l^{-1}$, for $\alpha_p \in \Delta(P) \setminus \Delta(Q_L)$. (To prove that this is admissible, use Lemma 12.) Since λ is conjugate to the OPS $\lambda_o := t^{x_p}$ lying in H and, moreover, $x_p \in \mathfrak{h}$ is L -dominant, we get $X_\lambda = x_p$. Clearly, $[\bar{p}_j, \lambda] = u_j$, for any $j = 1, \dots, s$. We can therefore use equation (47) to conclude that

$$\sum_{j=1}^s v_j (u_j x_p) \leq 0.$$

This proves (b). □

Remark 31. (a) Let $v_1, \dots, v_s \in X(H)$ and let \mathbb{L} be the line bundle $\mathcal{L}(v_1) \boxtimes \dots \boxtimes \mathcal{L}(v_s)$ on $(P/B_L)^s$. Assume that for every $\alpha_i \in \Delta \setminus \Delta(P)$, we have $\sum_{j=1}^s v_j(x_i) = 0$. Then, the restriction map

$$H^0((P/B_L)^s, \mathbb{L})^P \rightarrow H^0((L/B_L)^s, \mathbb{L})^L$$

is an isomorphism. If the above equality is violated for some $\alpha_i \in \Delta \setminus \Delta(P)$, then $H^0((L/B_L)^s, \mathbb{L})^L = 0$.

This is proved by the same technique that was used in the proof of Theorem 15 (cf. Sect. 5).

(b) In Theorem 29, the validity of property (b) for every standard parabolic subgroup Q_L of L is equivalent to the corresponding property only for the standard maximal parabolic subgroups Q_L of L . This can be proved by the same technique as developed in Sect. 7.

In the L -movable case, we have the following refinement of Theorem 29. To state this refinement, we need the following notation.

For any $w \in W^P$ and a central character c of L (i.e., an algebraic group homomorphism $Z(L) \rightarrow \mathbb{G}_m$, $Z(L) \subset H$ being the center of the Levi subgroup L of P), define

$$\chi_w^c = \sum_{\beta \in R(w,c)} \beta, \tag{49}$$

where

$$R(w, c) := \{ \beta \in (R^+ \setminus R_1^+) \cap w^{-1}R^+ : e_{|Z(L)}^\beta = c \}. \tag{50}$$

Observe that

$$\chi_w = \sum_c \chi_w^c, \tag{51}$$

where the sum runs over all the central characters of L such that $\chi_1^c \neq 0$.

Theorem 32. *Assume that the s -tuple $(w_1, \dots, w_s) \in (W^P)^s$ is L -movable. Then,*

1. *For any central character c of L such that $\chi_1^c \neq 0$, we have*

$$\sum_{j=1}^s |R(w_j, c)| = |R(1, c)|, \tag{52}$$

where $|\cdot|$ denotes the cardinality of the enclosed set.

2. *For any standard parabolic Q_L of L and $u_1, \dots, u_s \in W_L/W_{Q_L}$ such that*

$$[\bar{\Lambda}_{u_1}^{Q_L}] \cdot \dots \cdot [\bar{\Lambda}_{u_s}^{Q_L}] \neq 0 \in H^*(L/Q_L),$$

and any central character c of L such that $\chi_1^c \neq 0$, the following inequality is satisfied for any $\alpha_p \in \Delta(P) \setminus \Delta(Q_L)$:

$$\sum_{j=1}^s \chi_{w_j}^c(u_j x_p) \leq \chi_1^c(x_p). \tag{53}$$

Remark 33. (a) Observe that in the L -movable case, by virtue of Theorem 15, the inequality (45) is, in fact, an equality. Moreover, for any $\alpha_i \in \Delta \setminus \Delta(P)$, since $\exp x_i \in Z(L)$, $\chi_w^c(x_i) = |R(w, c)|c(\exp x_i)$.

(b) The inequalities (53) summed over all the central characters c of L such that $\chi_1^c \neq 0$ is nothing but the inequality (46) (use the identity (51)). Thus, Theorem 32 is a refinement of Theorem 29 in the L -movable case.

Proof of Theorem 32. For any central character c of L and $w \in W^P$, let T_w^c be the B_L -submodule of T_w defined by

$$T_w^c := \{v \in T_w : t \cdot v = c(t)v, \forall t \in Z(L) \subset B_L\}.$$

This gives rise to the L -equivariant vector bundle on L/B_L :

$$\mathcal{T}_w^c := L \times_{B_L} T_w^c.$$

In particular, we have the B_L -submodule T^c of T and the associated L -equivariant vector bundle \mathcal{T}^c on L/B_L and \mathcal{T}_s^c on $(L/B_L)^s$. Analogous to Lemma 6, we have:

$$\det(\mathcal{T}^c / \mathcal{T}_w^c) = \mathcal{L}(\chi_w^c), \tag{54}$$

as L -equivariant line bundles on L/B_L . Let Θ_o denote the restriction of the bundle map Θ (defined by (10)) to the subvariety $(L/B_L)^s \subset (P/B_L)^s$. Then, by Corollary 8, Θ_o is an isomorphism over a dense open subset of $(L/B_L)^s$. From this we see that, for any central character c of L such that $\chi_1^c \neq 0$,

$$\Theta_{o|\mathcal{T}_s^c} : \mathcal{T}_s^c \rightarrow \bigoplus_{j=1}^s \pi_j^*(\mathcal{T}^c / \mathcal{T}_{w_j}^c) \tag{55}$$

is an isomorphism over a dense open subset of $(L/B_L)^s$, where $\pi_j : (L/B_L)^s \rightarrow L/B_L$ is the projection onto the j -th factor. Therefore, the ranks of the two sides of equation (55) coincide. This gives the equality (52).

Taking the determinant of $\Theta_{o|\mathcal{T}_s^c}$ and using the equation (54), we get a nonzero bundle map

$$\theta_o^c : \det(\mathcal{T}_s^c) \rightarrow \mathcal{L}(\chi_{w_1}^c) \boxtimes \cdots \boxtimes \mathcal{L}(\chi_{w_s}^c),$$

i.e., a nonzero section of the line bundle

$$\mathcal{L}(\chi_{w_1}^c - \chi_1^c) \boxtimes \cdots \boxtimes \mathcal{L}(\chi_{w_s}^c)$$

on $(L/B_L)^s$. Now, applying Proposition 30 for the case $G = L, P = L$, and observing that χ_1^c is fixed by W_L , we get the theorem. \square

Question 34. We would like to ask if the converse of Theorem 29 is true. Specifically, take $(w_1, \dots, w_s) \in (W^P)^s$ satisfying the equation (7) and assume that the conditions (a) and (b) of Theorem 29 are satisfied. Then, is it true that

$$[\bar{\Lambda}_{w_1}^P] \cdot \dots \cdot [\bar{\Lambda}_{w_s}^P] = d[\bar{\Lambda}_1^P]$$

in $H^*(G/P)$, for some nonzero d . For $G = SL_n$ and any maximal parabolic subgroup P , this question has an affirmative answer (cf. Remark 38).

One could ask the corresponding (weaker) question in the L -movable case, i.e., is the converse of Theorem 32 true? It may be remarked here that this question for G/B has an affirmative answer trivially.

We now come to our second criterion (Theorem 36) to determine when the products of cohomology classes in G/P are nonzero. This criterion is in terms of inequalities involving certain dimension counts.

Let Q be a standard parabolic subgroup of G contained in P and let $Q_L := Q \cap L$ be the associated parabolic in L . Let \hat{Q} be a standard parabolic in G containing Q . Then, we have the canonical identification $L/Q_L \simeq P/Q$ and the standard projection $\tau : P/Q \rightarrow G/\hat{Q}$.

Lemma 35. *For any $w \in W^P$, $v \in W_L$, $p \in P$ and $b \in B$, we have (setting $g^{-1} = bwp$):*

$$\tau(p^{-1}BvQ) \subset gB(wv)\hat{Q}.$$

Proof. Since $w \in W^P$, by equation (3), we have $wB_Lw^{-1} \subset B$. Further,

$$\begin{aligned} \tau(p^{-1}BvQ) &= \tau(p^{-1}B_LUvQ) = \tau(p^{-1}B_LvQ) \\ &= gbwB_Lv\hat{Q} = gbwB_Lw^{-1}(wv)\hat{Q}, \end{aligned}$$

where U is the unipotent radical of P . We therefore have

$$\tau(p^{-1}BvQ) \subset gbBwv\hat{Q} = gB(wv)\hat{Q}.$$

□

Theorem 36. *Let $w_1, \dots, w_s \in W^P$ be such that*

$$[\bar{\Lambda}_{w_1}^P] \cdot \dots \cdot [\bar{\Lambda}_{w_s}^P] \neq 0 \tag{56}$$

in $H^(G/P, \mathbb{Z})$. Then, given the data: $Q \subset P$ a standard parabolic, $\hat{Q} \subset G$ a parabolic containing Q , elements $u_1, \dots, u_s \in W_L/W_{Q_L}$ such that*

$$[\bar{\Lambda}_{u_1}^{Q_L}] \cdot \dots \cdot [\bar{\Lambda}_{u_s}^{Q_L}] \neq 0 \tag{57}$$

in $H^(L/Q_L, \mathbb{Z})$, the following hold (with $\hat{w}_j := w_ju_j$):*

(a) $[\bar{\Lambda}_{\hat{w}_1}^{\hat{Q}}] \cdot \dots \cdot [\bar{\Lambda}_{\hat{w}_s}^{\hat{Q}}] \neq 0$ in $H^*(G/\hat{Q}, \mathbb{Z})$. In particular,

$$\sum_j \text{codim}(\Lambda_{\hat{w}_j}^{\hat{Q}}; G/\hat{Q}) \leq \dim(G/\hat{Q}). \tag{58}$$

(b) If $\hat{Q} \cap P = Q$, then

$$\begin{aligned} \dim(G/\hat{Q}) - \sum_j \text{codim}(\Lambda_{\hat{w}_j}^{\hat{Q}}; G/\hat{Q}) \\ \geq \dim(L/Q_L) - \sum_j \text{codim}(\Lambda_{u_j}^{Q_L}; L/Q_L). \end{aligned} \tag{59}$$

The above inequality (59) is equivalent to the following inequality (in the case $\hat{Q} \cap P = Q$):

$$|R(u_{\hat{Q}}) \cap R(u_P)| \geq \sum_{j=1}^s |R(u_{\hat{Q}}) \cap R(u_P) \cap \hat{w}_j^{-1} R^+|, \tag{60}$$

where, as earlier, $|\cdot|$ denotes the cardinality of the enclosed set.

Proof. Take generic g_j for $j = 1, \dots, s$ so that for each standard parabolic \tilde{P} in G and any $(z_1, \dots, z_s) \in W^s$, the intersection

$$g_1 B z_1 \tilde{P} \cap \dots \cap g_s B z_s \tilde{P}$$

is transverse (possibly empty) and dense in $\overline{g_1 B z_1 \tilde{P}} \cap \dots \cap \overline{g_s B z_s \tilde{P}}$. Moreover, the latter intersection is proper.

We may further assume, by left multiplying each g_j by the same element, that for the given s -tuple $(w_1, \dots, w_s) \in (W^P)^s$,

$$e \in g_1 B w_1 P \cap \dots \cap g_s B w_s P.$$

Choose $p_j \in P, b_j \in B$ such that $e = g_j b_j w_j p_j$, for $j = 1, \dots, s$. Now, consider the intersection in $L/Q_L \simeq P/Q$:

$$\Omega = \overline{p_1^{-1} B u_1 Q} \cap \dots \cap \overline{p_s^{-1} B u_s Q}.$$

Then, Ω is nonempty because of the assumption (57). Each irreducible component of Ω is of dimension at least

$$\dim(L/Q_L) - \sum_{j=1}^s \text{codim}(\Lambda_{u_j}^{Q_L}; L/Q_L). \tag{61}$$

By Lemma 35, $\tau(p_j^{-1} B u_j Q) \subset g_j B w_j u_j \hat{Q} \subset \overline{g_j B \hat{w}_j \hat{Q}}$, for $j = 1, \dots, s$, under the projection $\tau : P/Q \rightarrow G/\hat{Q}$. So,

$$\tau(\Omega) \subset \bigcap_{j=1}^s \overline{g_j B \hat{w}_j \hat{Q}}. \tag{62}$$

The intersection $\Omega' := \bigcap_{j=1}^s \overline{g_j B \hat{w}_j \hat{Q}}$ is therefore nonempty (and proper by the choice of g_j). This gives the part (a) of the theorem.

In the case $\hat{Q} \cap P = Q$, τ is an embedding. Comparison of equations (61) and (62) gives the inequality (59) of (b). The inequality (60) follows from (59) and the following Lemma 40. \square

Remark 37. This theorem is of interest even in the case $P = B$. In this case, $Q = B$ and any standard parabolic is allowed for \hat{Q} .

Remark 38. Fix positive integers $r < n$. Let $G = SL(n)$ and $G/P = \text{Gr}(r, n)$ (the Grassmannian of r -dimensional subspaces of \mathbb{C}^n). The Schubert cells in G/P are indexed by the subsets $I = \{i_1 < \dots < i_r\} \subset [n] := \{1, \dots, n\}$ of cardinality r . Such a set I gives a canonical element w_I in the symmetric group on n letters (the Weyl group of $SL(n)$) by letting $w_I(a) = i_a$ for $a = 1, \dots, r$ and $(w_I(r + 1), \dots, w_I(n))$ the sequence obtained by arranging the elements of $\{1, \dots, n\} \setminus I$ in ascending order. Then, $w_I \in W^P$. Let $\omega_I = [\bar{\Lambda}_{w_I}^P] \in H^*(\text{Gr}(r, n))$. Let I' be the set (arranged in the ascending order) $I' = \{n + 1 - i, i \in I\}$. Then, $\omega_{I'}$ is dual to ω_I under the intersection pairing.

Define the set S_n^r as the set of triples (I, J, K) of subsets of $[n]$ of cardinality r such that $\omega_{I'} \cdot \omega_{J'} \cdot \omega_K$ is a nonzero multiple of the class of a point in $H^*(\text{Gr}(r, n))$. Then, as in [Fu2, Inequality 9 and Theorem 12, also see Theorem 18], S_n^r admits the following inductive characterization: The triple (I, J, K) belongs to S_n^r if and only if

$$\sum_{i \in I} i + \sum_{j \in J} j = r(r + 1)/2 + \sum_{k \in K} k \tag{63}$$

and for all $0 < p < r$ and $(F, G, H) \in S_p^r$ the following inequality holds:

$$\sum_{f \in F} i_f + \sum_{g \in G} j_g \leq p(p + 1)/2 + \sum_{h \in H} k_h. \tag{64}$$

Let $w_1 = w_{I'}$, $w_2 = w_{J'}$ and $w_3 = w_K$. The condition in Equation (63) is precisely the equality in Equation (7) of our paper. Let Q be the maximal parabolic of P so that $P/Q = \text{Gr}(p, r)$. Set $u_1 = w_{F'}$, $u_2 = w_{G'}$ and $u_3 = w_H$. By a simple computation we see that inequality (60) is the same as the inequality (64) if we take \hat{Q} to be the maximal parabolic subgroup of $SL(n)$ such that $\hat{Q} \cap P = Q$.

Thus, for $H^*(\text{Gr}(r, n))$, the converse of our Theorem 36 is true. In fact, in this case, the set of inequalities (60) coming from only maximal parabolic subgroups Q of P and maximal parabolic subgroups \hat{Q} of $SL(n)$ such that $\hat{Q} \cap P = Q$ suffice for the converse of the theorem.

We recall the following simple lemma.

Lemma 39. For $w \in W$ (not necessarily in W^P), we have:

$$\text{codim}(\Lambda_w^P; G/P) = |R(u_P) \cap (w^{-1}R^+)|. \tag{65}$$

Lemma 40. Let $w \in W^P$, $u \in W_P$ and $\hat{w} := wu$. Let $Q \subset P$ be a standard parabolic and let \hat{Q} be a parabolic in G such that $\hat{Q} \cap P = Q$. Then,

$$\text{codim}(\Lambda_{\hat{w}}^{\hat{Q}}; G/\hat{Q}) - \text{codim}(\Lambda_u^{Q_L}; L/Q_L) = |R(u_{\hat{Q}}) \cap R(u_P) \cap \hat{w}^{-1}R^+|. \tag{66}$$

Proof. Write R^+ as a disjoint union of three parts (note that since $w \in W^P$, $wR_1^+ \subset R^+$ by (3))

$$R^+ = wR_1^+ \sqcup (wR(u_{\bar{p}}) \cap R^+) \sqcup (wR(u_p) \cap R^+),$$

where $u_{\bar{p}}$ is the opposite nil-radical of \mathfrak{p} . Take $\hat{w}^{-1} = u^{-1}w^{-1}$ of this decomposition and intersect with $R(u_{\hat{Q}})$. The first piece is

$$R(u_{\hat{Q}}) \cap u^{-1}R_1^+ = R(u_Q) \cap u^{-1}R_1^+, \text{ since } \hat{Q} \cap P = Q.$$

Thus, by Lemma 39 applied to W_L ,

$$|R(u_{\hat{Q}}) \cap u^{-1}R_1^+| = \text{codim}(\Lambda_u^{Q_L}; L/Q_L). \tag{67}$$

The second piece is

$$R(u_{\hat{Q}}) \cap u^{-1}R(u_{\bar{p}}) \cap u^{-1}w^{-1}R^+.$$

But $u^{-1}R(u_{\bar{p}}) = R(u_{\bar{p}})$ and clearly

$$R(u_{\hat{Q}}) \cap R(u_{\bar{p}}) = \emptyset. \tag{68}$$

So the second piece gives us the empty set.

The third piece is

$$R(u_{\hat{Q}}) \cap u^{-1}R(u_p) \cap u^{-1}w^{-1}R^+ = R(u_{\hat{Q}}) \cap R(u_p) \cap u^{-1}w^{-1}R^+. \tag{69}$$

Finally, by Lemma 39,

$$|R(u_{\hat{Q}}) \cap u^{-1}w^{-1}R^+| = \text{codim}(\Lambda_{\hat{w}}^{\hat{Q}}; G/\hat{Q}). \tag{70}$$

Combining the equations (67)–(70), we get equation (66), proving the lemma. \square

Remark 41. (a) It is easy to see that in the case when \hat{Q} is a minuscule maximal parabolic subgroup and $Q \neq P$ (and $\hat{Q} \cap P = Q$), the inequality (60) is the same as the inequality (46).

Also, in the case $\hat{Q} = Q$ and P is a minuscule maximal parabolic subgroup, the inequality (60) is the same as the inequality (45).

It may be remarked that, in general, the inequality (46) is a certain ‘weighted’ version of the inequality (60).

- (b) In the general case, we do not know if the system of inequalities (45) and (46) is equivalent to the system of inequalities (60).
- (c) K. Purbhoo [P] has given some criteria for determining which of the Schubert intersection numbers vanish in terms of a combinatorial recipe which he calls ‘root game.’ We do not know a connection between his recipe and our Theorems 29, 32, and 36.

9. A further study of $(H^*(G/P, \mathbb{C}), \odot_0)$

For any Lie algebra \mathfrak{s} and a subalgebra \mathfrak{t} , let $H^*(\mathfrak{s}, \mathfrak{t})$ be the Lie algebra cohomology of the pair $(\mathfrak{s}, \mathfrak{t})$ with trivial coefficients. Recall (cf. [Ku2, Sect. 3.1]) that this is the cohomology of the cochain complex

$$C^\bullet(\mathfrak{s}, \mathfrak{t}) = \{C^p(\mathfrak{s}, \mathfrak{t})\}_{p \geq 0}, \quad \text{where}$$

$$C^p(\mathfrak{s}, \mathfrak{t}) := \text{Hom}_{\mathfrak{t}}(\wedge^p(\mathfrak{s}/\mathfrak{t}), \mathbb{C}).$$

We now return to the notation of Sect. 2. For any (positive) root $\beta \in R^+$, let $y_\beta \in \mathfrak{g}_\beta$ be a nonzero root vector and let $y_{-\beta} \in \mathfrak{g}_{-\beta}$ be the vector such that $\langle y_\beta, y_{-\beta} \rangle = 1$ under the Killing form. For any $w \in W^P$, let $\Phi_w := w^{-1}R^- \cap R^+ \subset R(\mathfrak{u}_P)$. Then,

$$\sum_{\beta \in \Phi_w} \beta = \rho - w^{-1}\rho. \tag{71}$$

In particular, $\Phi_v = \Phi_w$ iff $v = w$. Let $\Phi_w = \{\beta_1, \dots, \beta_p\} \subset R(\mathfrak{u}_P)$. Set $y_w := y_{\beta_1} \wedge \dots \wedge y_{\beta_p} \in \wedge^p(\mathfrak{u}_P)$, determined up to a nonzero scalar multiple. Then, up to scalar multiples, y_w is the unique weight vector of $\wedge(\mathfrak{u}_P)$ with weight $\rho - w^{-1}\rho$ (cf. [Ko1, Lemma 5.12]). Similarly, we can define $y_w^- := y_{-\beta_1} \wedge \dots \wedge y_{-\beta_p} \in \wedge^p(\mathfrak{u}_P^-)$ of weight $w^{-1}\rho - \rho$.

We recall the following fundamental result due to Kostant [Ko1].

Theorem 42. *For any standard parabolic subgroup P of G ,*

$$H^P(\mathfrak{u}_P) = \bigoplus_{\substack{w \in W^P: \\ \ell(w)=p}} M_w,$$

as \mathfrak{l} -modules, where M_w is the unique irreducible \mathfrak{l} -submodule of $H^P(\mathfrak{u}_P)$ with highest weight $w^{-1}\rho - \rho$ (which is \mathfrak{l} -dominant for any $w \in W^P$). This has a highest weight vector $\phi_w \in \wedge^p(\mathfrak{u}_P)^*$ defined by $\phi_w(y_w) = 1$ and $\phi_w(y) = 0$ for any weight vector of $\wedge^p(\mathfrak{u}_P)$ of weight $\neq \rho - w^{-1}\rho$.

Similarly, for the opposite nil-radical \mathfrak{u}_P^- ,

$$H^P(\mathfrak{u}_P^-) = \bigoplus_{\substack{w \in W^P: \\ \ell(w)=p}} N_w,$$

as \mathfrak{l} -modules, where N_w is the unique irreducible \mathfrak{l} -submodule of $H^P(\mathfrak{u}_P^-)$ isomorphic with the dual M_w^* and it has a lowest weight vector $\phi_w^- \in \wedge^p(\mathfrak{u}_P^-)^*$ defined by $\phi_w^-(y_w^-) = 1$ and $\phi_w^-(y) = 0$ for any weight vector of $\wedge^p(\mathfrak{u}_P^-)$ of weight $\neq w^{-1}\rho - \rho$.

Thus,

$$\begin{aligned} [H^p(\mathfrak{u}_P) \otimes H^q(\mathfrak{u}_{\bar{P}})]^{\mathbb{1}} &= 0, \quad \text{unless } p = q, \text{ and} \\ [H^p(\mathfrak{u}_P) \otimes H^p(\mathfrak{u}_{\bar{P}})]^{\mathbb{1}} &\simeq \bigoplus_{\substack{w \in W^P, \\ \ell(w) = p}} \mathbb{C} \xi^w, \end{aligned}$$

where $\xi^w \in [M_w \otimes N_w]^{\mathbb{1}}$ is the unique element whose H -equivariant projection to $(M_w)_{w^{-1}\rho - \rho} \otimes N_w$ is the element $\phi_w \otimes \phi_w^-$, $(M_w)_{w^{-1}\rho - \rho}$ being the weight space of M_w corresponding to the weight $w^{-1}\rho - \rho$. (Observe that the ambiguity in the choice of y_w disappears in the definition of ξ^w giving rise to a completely unique element.)

Theorem 43. *For any standard parabolic subgroup P of G , there is a graded algebra isomorphism*

$$\phi : (H^*(G/P, \mathbb{C}), \odot_0) \simeq [H^*(\mathfrak{u}_P) \otimes H^*(\mathfrak{u}_{\bar{P}})]^{\mathbb{1}}$$

such that, for any $w \in W^P$,

$$\phi(\epsilon_w^P) = (-1)^{\frac{p(p-1)}{2}} \left(\frac{i}{2\pi}\right)^p \langle \rho, \Phi_{w^{-1}} \rangle \xi^w, \tag{72}$$

where $p := \ell(w)$, $\langle \rho, \Phi_{w^{-1}} \rangle := \prod_{\alpha \in wR^- \cap R^+} \langle \rho, \alpha \rangle$ (for any $w \in W$), $H^*(G/P, \mathbb{C})$ is equipped with the product \odot_0 as in Definition 18, and we take the tensor product algebra structure on the right side.

Proof. Let $x_P := \sum_{\alpha_i \in \Delta \setminus \Delta(P)} x_i$. Consider the graded, multiplicative decreasing filtration $\mathcal{A} = \{\mathcal{A}_m\}_{m \geq 0}$ of $H^*(G/P, \mathbb{C})$ under the intersection product defined as follows:

$$\mathcal{A}_m := \bigoplus_{w \in W_m^P} \mathbb{C} [\bar{\Lambda}_w^P],$$

where $W_m^P := \{w \in W^P : \chi_w(x_P) \geq m\}$. Observe that, by equation (9), $\chi_w(x_P) = \rho + w^{-1}\rho(x_P) \in \mathbb{Z}_+$. Thus, by Proposition 17(a), $\mathcal{A}_m \cdot \mathcal{A}_n \subset \mathcal{A}_{m+n}$, i.e., \mathcal{A} respects the algebra structure.

Let $\text{gr}(\mathcal{A}) := \bigoplus_{m \geq 0} \frac{\mathcal{A}_m}{\mathcal{A}_{m+1}}$ be the associated ‘gr’ algebra. Then $\text{gr}(\mathcal{A})$ acquires two gradings: the first one the ‘homogeneous grading’ m assigned to the elements of $\frac{\mathcal{A}_m}{\mathcal{A}_{m+1}}$ and the second one the ‘cohomological grading’ coming from the cohomological degree in $H^*(G/P, \mathbb{C})$. For example, $[\bar{\Lambda}_w^P]$ has bidegree $(\chi_w(x_P), \dim G/P - \ell(w))$.

By the definition of the product \odot_0 and Proposition 17(a), the linear map

$$\phi : (H^*(G/P, \mathbb{C}), \odot_0) \rightarrow \text{gr}(\mathcal{A}), [\bar{\Lambda}_w^P] \mapsto [\bar{\Lambda}_w^P] \pmod{\mathcal{A}_{\chi_w(x_P)+1}},$$

is, in fact, a graded algebra isomorphism with respect to the cohomological grading on $\text{gr}(\mathcal{A})$.

We next introduce another filtration $\{\tilde{\mathcal{F}}_m\}_{m \geq 0}$ of $H^*(G/P, \mathbb{C})$ in terms of the Lie algebra cohomology. Recall that choosing a maximal compact subgroup K of G , we can identify $H^*(G/P, \mathbb{C})$ with the cohomology of the K -invariant forms on G/P under the de Rham differential, i.e., with the Lie algebra cohomology $H^*(\mathfrak{g}, \mathfrak{l})$. The underlying cochain complex $C^\bullet = C^\bullet(\mathfrak{g}, \mathfrak{l})$ for $H^*(\mathfrak{g}, \mathfrak{l})$ can be rewritten as

$$C^\bullet := [\wedge^\bullet(\mathfrak{g}/\mathfrak{l})^*]^{\mathfrak{l}} = \text{Hom}_{\mathfrak{l}}(\wedge^\bullet(\mathfrak{u}_P) \otimes \wedge^\bullet(\mathfrak{u}_{\bar{P}}), \mathbb{C}).$$

Define a decreasing multiplicative filtration $\mathcal{F} = \{\mathcal{F}_m\}_{m \geq 0}$ of the cochain complex C^\bullet by subcomplexes:

$$\mathcal{F}_m := \text{Hom}_{\mathfrak{l}}\left(\frac{\wedge^\bullet(\mathfrak{u}_P) \otimes \wedge^\bullet(\mathfrak{u}_{\bar{P}})}{\bigoplus_{s+t \leq m-1} \wedge_{(s)}^\bullet(\mathfrak{u}_P) \otimes \wedge_{(t)}^\bullet(\mathfrak{u}_{\bar{P}})}, \mathbb{C}\right),$$

where $\wedge_{(s)}^\bullet(\mathfrak{u}_P)$ (resp. $\wedge_{(s)}^\bullet(\mathfrak{u}_{\bar{P}})$) denotes the subspace of $\wedge^\bullet(\mathfrak{u}_P)$ (resp. $\wedge^\bullet(\mathfrak{u}_{\bar{P}})$) spanned by the \mathfrak{h} -weight vectors of weight β with P -relative height

$$\text{ht}_P(\beta) := |\beta(x_P)| = s.$$

Now, define the filtration $\tilde{\mathcal{F}} = \{\tilde{\mathcal{F}}_m\}_{m \geq 0}$ of $H^*(\mathfrak{g}, \mathfrak{l}) \simeq H^*(G/P)$ by

$$\tilde{\mathcal{F}}_m := \text{Image of } H^*(\mathcal{F}_m) \rightarrow H^*(C^\bullet).$$

The filtration \mathcal{F} of C^\bullet gives rise to the cohomology spectral sequence $\{E_r\}_{r \geq 1}$ converging to $H^*(C^\bullet) = H^*(G/P, \mathbb{C})$. By [Ku2, Proof of Proposition 3.2.11], for any $m \geq 0$,

$$E_1^m = \bigoplus_{s+t=m} [H_{(s)}^\bullet(\mathfrak{u}_P) \otimes H_{(t)}^\bullet(\mathfrak{u}_{\bar{P}})]^{\mathfrak{l}},$$

where $H_{(s)}^\bullet(\mathfrak{u}_P)$ denotes the cohomology of the subcomplex $(\wedge_{(s)}^\bullet(\mathfrak{u}_P))^*$ of the standard cochain complex $\wedge^\bullet(\mathfrak{u}_P)^*$ associated to the Lie algebra \mathfrak{u}_P and similarly for $H_{(t)}^\bullet(\mathfrak{u}_{\bar{P}})$. Moreover, by loc. cit., the spectral sequence degenerates at the E_1 term, i.e.,

$$E_1^m = E_\infty^m. \tag{73}$$

Further, by the definition of P -relative height,

$$[H_{(s)}^\bullet(\mathfrak{u}_P) \otimes H_{(t)}^\bullet(\mathfrak{u}_{\bar{P}})]^{\mathfrak{l}} \neq 0 \Rightarrow s = t.$$

Thus,

$$\begin{aligned} E_1^m &= 0, && \text{unless } m \text{ is even and} \\ E_1^{2m} &= [H_{(m)}^\bullet(\mathfrak{u}_P) \otimes H_{(m)}^\bullet(\mathfrak{u}_{\bar{P}})]^{\mathfrak{l}}. \end{aligned}$$

In particular, from (73) and the general properties of spectral sequences (cf. [Ku2, Theorem E.9]), we have a canonical algebra isomorphism:

$$\text{gr}(\bar{\mathcal{F}}) \simeq \bigoplus_{m \geq 0} [H_{(m)}^\bullet(u_P) \otimes H_{(m)}^\bullet(u_{\bar{P}})]^1, \tag{74}$$

where $[H_{(m)}^\bullet(u_P) \otimes H_{(m)}^\bullet(u_{\bar{P}})]^1$ sits inside $\text{gr}(\bar{\mathcal{F}})$ precisely as the homogeneous part of degree $2m$; homogeneous parts of $\text{gr}(\bar{\mathcal{F}})$ of odd degree being zero.

Finally, we claim that, for any $m \geq 0$,

$$\mathcal{A}_m = \bar{\mathcal{F}}_{2m} : \tag{75}$$

Following Kostant [Ko2], take the d - ∂ harmonic representative \hat{s}^w in C^\bullet for the cohomology class $[\bar{\Lambda}_w^P]$. An explicit expression is given by [Ko2, Theorem 4.6] together with [KK, Theorem 3.1]. From this explicit expression and [Ko2, Proposition 3.2], we easily see that

$$\mathcal{A}_m \subset \bar{\mathcal{F}}_{2m}. \tag{76}$$

Moreover, from the definition of \mathcal{A} , for any $m \geq 0$,

$$\dim \frac{\mathcal{A}_m}{\mathcal{A}_{m+1}} = \#\{w \in W^P : \chi_w(x_P) = \rho + w^{-1}\rho(x_P) = m\}.$$

Also, by the isomorphism (74) and Theorem 42,

$$\begin{aligned} \dim \frac{\bar{\mathcal{F}}_{2m}}{\bar{\mathcal{F}}_{2m+1}} &= \#\{w \in W^P : \rho - w^{-1}\rho(x_P) = m\} \\ &= \#\{w \in W^P : (\rho - (w_o w w_o^P)^{-1}\rho)(x_P) = m\}, \\ &\quad \text{using the involution } w \mapsto w_o w w_o^P \text{ of } W^P \\ &= \#\{w \in W^P : \rho + w^{-1}\rho(x_P) = m\}, \text{ since } w_o^P \text{ keeps } x_P \text{ fixed.} \end{aligned}$$

Thus,

$$\dim \frac{\mathcal{A}_m}{\mathcal{A}_{m+1}} = \dim \frac{\bar{\mathcal{F}}_{2m}}{\bar{\mathcal{F}}_{2m+1}}.$$

Further, for large enough m_o , $\mathcal{A}_{m_o} = \bar{\mathcal{F}}_{2m_o} = 0$. Thus, by decreasing induction on m , we get (75) from (76).

Thus, combining the isomorphisms φ and (74) and using (75), we get the isomorphism ϕ as in the theorem. The assertion (72) follows from the description of the map φ and the identification (74) together with the explicit expression of the d - ∂ harmonic representative \hat{s}^w in C^\bullet . This proves the theorem. □

Corollary 44. *The product in $(H^*(G/B), \odot_0)$ is given by*

$$\begin{aligned} \epsilon_u^B \odot_0 \epsilon_v^B &= 0, \text{ if } \Phi_u \cap \Phi_v \neq \emptyset \\ &= 0, \text{ if } \Phi_u \cap \Phi_v = \emptyset \text{ and } \nexists w \in W \text{ with } \Phi_w = \Phi_u \sqcup \Phi_v \\ &= \frac{\langle \rho, \Phi_{u^{-1}} \rangle \langle \rho, \Phi_{v^{-1}} \rangle}{\langle \rho, \Phi_{w^{-1}} \rangle} \epsilon_w^B, \text{ if } \Phi_u \cap \Phi_v = \emptyset \\ &\text{ and } \exists w \in W \text{ with } \Phi_w = \Phi_u \sqcup \Phi_v. \end{aligned}$$

Proof. By Theorem 43,

$$\phi(\epsilon_u^B \odot_0 \epsilon_v^B) = (-1)^{\binom{p(p-1)}{2} + \binom{q(q-1)}{2}} \left(\frac{i}{2\pi}\right)^{p+q} \langle \rho, \Phi_{u^{-1}} \rangle \langle \rho, \Phi_{v^{-1}} \rangle \xi^u \xi^v, \tag{77}$$

where $p = \ell(u)$ and $q = \ell(v)$. The right side is clearly 0 if $\Phi_u \cap \Phi_v \neq \emptyset$. So, let us consider the case when $\Phi_u \cap \Phi_v = \emptyset$. In this case, two subcases occur:

1. There exists $w \in W$ such that $\Phi_w = \Phi_u \sqcup \Phi_v$. In particular, $\ell(w) = \ell(u) + \ell(v)$. (Such a w is necessarily unique.)
2. There does not exist any such $w \in W$.

In the first case, $\xi^u \xi^v = (-1)^{pq} \xi^w$ and thus the right side of equation (77) is equal to

$$(-1)^{\binom{\ell(w)(\ell(w)-1)}{2}} \left(\frac{i}{2\pi}\right)^{\ell(w)} \langle \rho, \Phi_{u^{-1}} \rangle \langle \rho, \Phi_{v^{-1}} \rangle \xi^w = \frac{\phi(\epsilon_w^B) \langle \rho, \Phi_{u^{-1}} \rangle \langle \rho, \Phi_{v^{-1}} \rangle}{\langle \rho, \Phi_{w^{-1}} \rangle}.$$

Hence, in this case, $\epsilon_u^B \odot_0 \epsilon_v^B = \frac{\langle \rho, \Phi_{u^{-1}} \rangle \langle \rho, \Phi_{v^{-1}} \rangle}{\langle \rho, \Phi_{w^{-1}} \rangle} \epsilon_w^B$.

In the second subcase, by Theorem 42,

$$\xi^u \xi^v = 0, \quad \text{as an element of } [H^*(u_B) \otimes H^*(u_B^-)]^H.$$

Thus, $\epsilon_u \odot_0 \epsilon_v = 0$. This proves the corollary. □

Remark 45. a) For any $u, v \in W$, $\Phi_u \cap \Phi_v = \emptyset$ iff $\ell(uv^{-1}) = \ell(u) + \ell(v)$.

b) A subset $S \subset R^+$ is called *closed under addition* if for $\alpha, \beta \in S$ such that $\alpha + \beta \in R^+$, we have $\alpha + \beta \in S$. A subset $S \subset R^+$ is called *coclosed under addition* if $R^+ \setminus S$ is closed under addition.

Now, by a result of Kostant [Ko1, Proposition 5.10], a subset S of R^+ is closed and coclosed under addition iff there exists $w \in W$ such that $S = \Phi_w$.

c) For any $u, v, w \in W$, the following two conditions are equivalent:

- c₁) $\Phi_u \cap \Phi_v = \emptyset$ and $\Phi_w = \Phi_u \sqcup \Phi_v$.
- c₂) $\ell(w) = \ell(u) + \ell(v)$, $\ell(uv^{-1}) = \ell(u) + \ell(v)$, $\ell(wu^{-1}) = \ell(w) - \ell(u)$ and $\ell(wv^{-1}) = \ell(w) - \ell(v)$.

d) As a consequence of Corollary 44, it is easy to get the following analogue of Chevalley formula:

For a simple reflection s_i ,

$$\begin{aligned} \epsilon_{s_i}^B \odot_0 \epsilon_v^B &= \epsilon_{vs_i}^B, & \text{if } v\alpha_i \text{ is a (positive) simple root} \\ &= 0, & \text{otherwise.} \end{aligned}$$

Thus, the subalgebra of $(H^*(G/B, \mathbb{Z}), \odot_0)$ generated by $H^2(G/B, \mathbb{Z})$ is precisely equal to $\bigoplus \mathbb{Z}\epsilon^v$, where the summation runs over those $v \in W$ which can be written as a product of commuting simple reflections with no simple reflection appearing more than once.

10. Tables of the deformed product \odot for rank 3 groups

We give below the multiplication tables under the deformed product \odot (cf. equation (29)) for G/P for all the rank 3 complex simple groups G and maximal parabolic subgroups P . We will freely follow the convention as in [KLM] without explanation. Since we are only considering maximal parabolics, we have only one indeterminate, which we denote by τ . In the case of $G = SL_n$, all the maximal parabolic subgroups are minuscule. Similarly, for $G = B_3$ the maximal parabolic P_1 is minuscule and for $G = C_3$, P_3 is minuscule. So, by Lemma 19, the deformed product in the cohomology of the corresponding flag varieties coincides with the usual cup product, so we do not write them here. (The interested reader can find it, e.g., in [KLM].)

Example 1. $G = B_3, P = P_2$:

$H^*(G/P_2)$	b_1	b'_2	b''_2	b'_3	b''_3
b_1	$b'_2 + 2b''_2$	$2b'_3$	$b'_3 + b''_3$	$2\tau b'_4 + \tau b''_4$	$\tau b'_4 + 2\tau b''_4$
b'_2		$2\tau b'_4$	$\tau b'_4 + \tau b''_4$	$2\tau b'_5 + \tau b''_5$	$\tau b''_5$
b''_2			$\tau b'_4 + \tau b''_4$	$\tau b'_5 + \tau b''_5$	$\tau b'_5 + \tau b''_5$
b'_3				$2\tau b_6$	τb_6
b''_3					$2\tau b_6$

$H^*(G/P_2)$	b'_4	b''_4	b'_5	b''_5	b_6	b_7
b_1	$2b'_5 + b''_5$	b''_5	b_6	$2b_6$	b_7	0
b'_2	$2b_6$	0	b_7	0	0	0
b''_2	b_6	b_6	0	b_7	0	0
b'_3	b_7	0	0	0	0	0
b''_3	0	b_7	0	0	0	0

Example 2. $G = B_3, P = P_3$:

$H^*(G/P_3)$	b_1	b_2	b'_3	b''_3	b_4	b_5	b_6
b_1	τb_2	$\tau b'_3 + b''_3$	b_4	τb_4	τb_5	b_6	0
b_2		$2b_4$	b_5	τb_5	b_6	0	0
b'_3			0	b_6	0	0	0
b''_3				0	0	0	0

Example 3. $G = C_3, P = P_1$:

$H^*(G/P_1)$	a_1	a_2	a_3	a_4	a_5
a_1	a_2	τa_3	a_4	a_5	0
a_2		τa_4	a_5	0	0

Example 4. $G = C_3, P = P_2$:

$H^*(G/P_2)$	a_1	a'_2	a''_2	a'_3	a''_3
a_1	$a'_2 + \tau a''_2$	$\tau a'_3$	$a'_3 + a''_3$	$2\tau a'_4 + \tau a''_4$	$\tau a'_4 + 2\tau a''_4$
a'_2		$\tau^2 a'_4$	$\tau a'_4 + \tau a''_4$	$\tau^2 a'_5 + \tau a''_5$	$\tau a''_5$
a''_2			$2a'_4 + 2a''_4$	$\tau a'_5 + 2a''_5$	$\tau a'_5 + 2a''_5$
a'_3				$2\tau a_6$	τa_6
a''_3					$2\tau a_6$

$H^*(G/P_2)$	a'_4	a''_4	a'_5	a''_5	a_6	a_7
a_1	$\tau a'_5 + a''_5$	a''_5	a_6	τa_6	a_7	0
a'_2	τa_6	0	a_7	0	0	0
a''_2	a_6	a_6	0	a_7	0	0
a'_3	a_7	0	0	0	0	0
a''_3	0	a_7	0	0	0	0

Remark 46. We believe that our results can be generalized to the small quantum cohomology of homogenous spaces and the multiplicative eigenvalue problem. In particular, there should be an analogous deformation \odot_0^q of the quantum cohomology of homogenous spaces G/P with an analogous relation to the multiplicative eigenvalue problem ([AW,Bel1,Bel4,Bel5,TW]).

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