
Extensions of algebraic groups

Shrawan Kumar¹ and Karl-Hermann Neeb²

¹ Department of Mathematics
University of North Carolina
Chapel Hill
NC 27599-3250
USA
shrawan@email.unc.edu

² Fachbereich Mathematik
Darmstadt University of Technology
Schloßgartenstr. 7
D-64289, Darmstadt
Germany
neeb@mathematik.tu-darmstadt.de

Dedicated to Professor Anthony Joseph on his sixtieth birthday

Summary. Let G be a connected complex algebraic group and A an abelian connected algebraic group, together with an algebraic action of G on A via group automorphisms. The aim of this article is to study the group of isomorphism classes of extensions of G by A in the algebraic group category. We describe this group as a direct sum of the group $\text{Hom}(\pi_1([G, G]), A)$ and a relative Lie algebra cohomology space. We also prove a version of Van Est's theorem for algebraic groups, identifying the cohomology of G with values in a G -module \mathfrak{a} in terms of relative Lie algebra cohomology.

Subject Classification: 20G10

Introduction

Let G be a connected complex algebraic group and A an abelian connected algebraic group, together with an algebraic action of G on A via group automorphisms. The aim of this article is to study the set of isomorphism classes $\text{Ext}_{\text{alg}}(G, A)$ of extensions of G by A in the algebraic group category. The following is our main result (cf. Theorem 1.8).

0.1. Theorem. *For G and A as above, there exists a split exact sequence of abelian groups:*

$$0 \rightarrow \text{Hom}(\pi_1([G, G]), A) \rightarrow \text{Ext}_{\text{alg}}(G, A) \xrightarrow{\pi} H^2(\mathfrak{g}, \mathfrak{g}_{\text{red}}, \mathfrak{a}_u) \rightarrow 0,$$

where A_u is the unipotent radical of A , G_{red} is a Levi subgroup of G , $\mathfrak{g}_{\text{red}}, \mathfrak{g}, \mathfrak{a}_u$ are the Lie algebras of G_{red}, G, A_u , respectively, and $H^*(\mathfrak{g}, \mathfrak{g}_{\text{red}}, \mathfrak{a}_u)$ is the Lie algebra cohomology of the pair $(\mathfrak{g}, \mathfrak{g}_{\text{red}})$ with coefficients in the \mathfrak{g} -module \mathfrak{a}_u .

Our next main result is the following analogue of the Van-Est Theorem for the algebraic group cohomology (cf. Theorem 2.2).

0.2. Theorem. *Let G be a connected algebraic group and let \mathfrak{a} be a finite-dimensional algebraic G -module. Then, for any $p \geq 0$,*

$$H_{\text{alg}}^p(G, \mathfrak{a}) \simeq H^p(\mathfrak{g}, \mathfrak{g}_{\text{red}}, \mathfrak{a}).$$

By an algebraic group G we mean an affine algebraic group over the field of complex numbers \mathbb{C} and the varieties are considered over \mathbb{C} . The Lie algebra of G is denoted by $L(G)$.

1. Extensions of Algebraic Groups

1.1. Definition. Let G be an algebraic group and A an abelian algebraic group, together with an algebraic action of G on A via group automorphisms, i.e., a morphism of varieties $\rho : G \times A \rightarrow A$ such that the induced map $G \rightarrow \text{Aut } A$ is a group homomorphism. Such an A is called an *algebraic group with G -action*.

By $\text{Ext}_{\text{alg}}(G, A)$ we mean the set of isomorphism classes of extensions of G by A in the algebraic group category, i.e., surjective morphisms $q : \widehat{G} \rightarrow G$ with kernel isomorphic to A as an algebraic group with G -action. We obtain on $\text{Ext}_{\text{alg}}(G, A)$ the structure of an abelian group by assigning to two extensions $q_i : \widehat{G}_i \rightarrow G$ of G by A the fiber product extension $\widehat{G}_1 \times_G \widehat{G}_2$ of G by $A \times A$ and then applying the group morphism $m_A : A \times A \rightarrow A$ fiberwise to obtain an A -extension of G (this is the Baer sum of two extensions). Then, Ext_{alg} assigns to a pair of an algebraic group G and an abelian algebraic group A with G -action, an abelian group, and this assignment is contravariant in G (via pulling back the action of G and the extension) and if G is fixed, $\text{Ext}_{\text{alg}}(G, \cdot)$ is a covariant functor from the category of abelian algebraic groups with G -actions to the category of abelian groups. Here we assign to a G -equivariant morphism $\gamma : A_1 \rightarrow A_2$ of abelian algebraic groups and an extension $q : \widehat{G} \rightarrow G$ of G by A_1 the extension

$$\gamma_* \widehat{G} := (A_2 \rtimes \widehat{G}) / \Gamma(\gamma) \rightarrow G, \quad [(a, g)] \mapsto q(g),$$

where $\Gamma(\gamma)$ is the graph of γ in $A_2 \times A_1$ and the semidirect product refers to the action of \widehat{G} on A_2 obtained by pulling back the action of G on A_2 to \widehat{G} . In view of the equivariance of γ , its graph is a normal algebraic subgroup of $A_2 \rtimes \widehat{G}$ so that we can form the quotient $\gamma_* \widehat{G}$.

We define a map

$$D : \text{Ext}_{\text{alg}}(G, A) \rightarrow \text{Ext}(L(G), L(A))$$

by assigning to an extension

$$1 \rightarrow A \xrightarrow{i} \widehat{G} \xrightarrow{q} G \rightarrow 1$$

of algebraic groups the corresponding extension

$$0 \rightarrow L(A) \xrightarrow{di} L(\widehat{G}) \xrightarrow{dq} L(G) \rightarrow 0$$

of Lie algebras. Since i is injective, di is injective. Similarly, dq is surjective. Moreover, $\dim G = \dim L(G)$ and hence the above sequence of Lie algebras is indeed exact.

It is clear from the definition of D that it is a homomorphism of abelian groups. If \mathfrak{g} is the Lie algebra of G and \mathfrak{a} the Lie algebra of A , then the group $\text{Ext}(\mathfrak{g}, \mathfrak{a})$ is isomorphic to the second Lie algebra cohomology space $H^2(\mathfrak{g}, \mathfrak{a})$ of \mathfrak{g} with coefficients in the \mathfrak{g} -module \mathfrak{a} (with respect to the derived action) ([CE]). Therefore, the description of the group $\text{Ext}_{\text{alg}}(G, A)$ depends on a “good” description of kernel and cokernel of D which will be obtained below in terms of an exact sequence involving D .

In the following G is always assumed to be connected. The following lemma reduces the extension theory for connected algebraic groups A with G -actions to the two cases of a torus A_s and the case of a unipotent group A_u .

1.2. Lemma. *Let G be connected and let A be a connected algebraic group with G -action. Further, let $A = A_u A_s$ denote the decomposition of A into its unipotent and reductive factors. Then, $A \cong A_u \times A_s$ as a G -module, where G acts trivially on A_s and G acts on A_u as a G -stable subgroup of A . Thus, we have*

$$\text{Ext}_{\text{alg}}(G, A) \cong \text{Ext}_{\text{alg}}(G, A_u) \oplus \text{Ext}_{\text{alg}}(G, A_s). \tag{1}$$

Proof. Decompose

$$A = A_u A_s, \tag{2}$$

where A_s is the set of semisimple elements of A and A_u is the set of unipotent elements of A . Then, A_s and A_u are closed subgroups of A and (2) is a direct product decomposition (see [H, Theorem 15.5]). The action of G on A clearly keeps A_s and A_u stable separately. Also, G acts trivially on A_s since $\text{Aut}(A_s)$ is discrete and G is connected (by assumption). Thus, the action of G on A decomposes as the product of actions on A_s and A_u with the trivial action on A_s . Hence, the isomorphism (1) follows from the functoriality of $\text{Ext}_{\text{alg}}(G, \cdot)$. \square

If $G = G_u \rtimes G_{\text{red}}$ is a Levi decomposition of G , then G_u being simply-connected,

$$\pi_1(G) \cong \pi_1(G_{\text{red}}),$$

where G_u is the unipotent radical of G , G_{red} is a Levi subgroup of G and π_1 denotes the fundamental group. The connected reductive group G_{red} is a product of its connected center $Z := Z(G_{\text{red}})_0$ and its commutator group $G'_{\text{red}} := [G_{\text{red}}, G_{\text{red}}]$ which is

a connected semisimple group. Thus, G'_{red} has an algebraic universal covering group \tilde{G}'_{red} , with the finite abelian group $\pi_1(G'_{\text{red}})$ as its fiber. We write $\tilde{G}_{\text{red}} := Z \times \tilde{G}'_{\text{red}}$, which is an algebraic covering group of G_{red} ; denote its kernel by Π_G and observe that

$$\tilde{G} := G_u \rtimes \tilde{G}_{\text{red}}$$

is a covering of G with Π_G as its fiber. We write $q_G : \tilde{G} \rightarrow G$ for the corresponding covering map.

1.3. Lemma. *If G and A are tori, then $\text{Ext}_{\text{alg}}(G, A) = 0$.*

Proof. Let $q : \hat{G} \rightarrow G$ be an extension of the torus G by A . Then, as is well known, \hat{G} is again a torus (cf. [B, §11.5]). Since any character of a subtorus of a torus extends to a character of the whole torus ([B, §8.2]), the identity $I_A : A \rightarrow A$ extends to a morphism $f : \hat{G} \rightarrow A$. Now, $\ker f$ yields a splitting of the above extension. \square

The following proposition deals with the case $A = A_s$.

1.4. Proposition. *If $A = A_s$, then $D = 0$ and we obtain an exact sequence*

$$\text{Hom}(\tilde{G}, A_s) \xrightarrow{\text{res}} \text{Hom}(\Pi_G, A_s) \xrightarrow{\Phi} \text{Ext}_{\text{alg}}(G, A_s),$$

where Φ assigns to any $\gamma \in \text{Hom}(\Pi_G, A_s)$ the extension $\gamma_* \tilde{G}$. The kernel of Φ consists of those homomorphisms vanishing on the fundamental group $\pi_1(G'_{\text{red}})$ of G'_{red} and Φ factors through an isomorphism

$$\Phi' : \text{Hom}(\pi_1(G'_{\text{red}}), A_s) \simeq \text{Ext}_{\text{alg}}(G, A_s).$$

Proof. Consider an extension

$$1 \rightarrow A_s \rightarrow \hat{G} \rightarrow G \rightarrow 1.$$

Since A_s is a central torus in \hat{G} , the unipotent radical \hat{G}_u of \hat{G} maps isomorphically on G_u . Also

$$1 \rightarrow A_s \rightarrow \hat{G}_{\text{red}} \rightarrow G_{\text{red}} \rightarrow 1$$

is an extension whose restriction to Z splits by the preceding lemma. On the other hand the commutator group of \hat{G}_{red} has the same Lie algebra as G'_{red} , hence is a quotient of \tilde{G}'_{red} . Thus, \hat{G}_{red} is a quotient of $A_s \times Z \times \tilde{G}'_{\text{red}}$, which implies that \hat{G} is a quotient of $A_s \times \tilde{G}$. Hence, \hat{G} is obtained from $A_s \times \tilde{G}$ by taking its quotient by the graph of a homomorphism $\Pi_G \rightarrow A_s$. This proves that Φ is surjective. In particular, the pullback $q_G^* \hat{G}$ of \hat{G} to \tilde{G} always splits.

We next show that $\ker \Phi$ coincides with the image of the restriction map from $\text{Hom}(\tilde{G}, A_s)$ to $\text{Hom}(\Pi_G, A_s)$. Assume that the extension $\hat{G}_\gamma = \gamma_* \tilde{G}$ defined by $\gamma \in \text{Hom}(\Pi_G, A_s)$ splits. Let $\sigma : G \rightarrow \hat{G}_\gamma$ be a splitting morphism. Pulling σ back via q_G , we obtain a splitting morphism

$$\tilde{\sigma} : \tilde{G} \rightarrow q_G^* \hat{G}_\gamma \cong A_s \times \tilde{G}.$$

Thus, there exists a morphism $\delta : \tilde{G} \rightarrow A_s$ of algebraic groups such that σ satisfies $\sigma(q_G(g)) = \beta(\delta(g), g)$ for all $g \in \tilde{G}$, where $\beta : A_s \times \tilde{G} \rightarrow \tilde{G}_\gamma = (A_s \times \tilde{G})/\Gamma(\gamma)$ is the standard quotient map. For $g \in \Pi_G = \ker q_G$ we have $\beta(\delta(g), g) = 1$, and therefore $\delta(g) = \gamma(g)$ for all $g \in \Pi_G$. This shows that δ is an extension of γ to \tilde{G} . Conversely, if γ extends to \tilde{G} , \tilde{G}_γ is a trivial extension of G .

That $D = 0$ follows from the fact that \tilde{G} and $q_G^*\tilde{G}$ have the same Lie algebras, which is a split extension of \mathfrak{g} by \mathfrak{a}_s .

We recall that $\tilde{G} = G_u \rtimes (Z \times \tilde{G}'_{\text{red}})$. If a homomorphism $\gamma : \Pi_G \rightarrow A_s$ extends to \tilde{G} , then it must vanish on the subgroup $\pi_1(G'_{\text{red}})$ of Π_G since, \tilde{G}'_{red} being a semisimple group, there are no nonconstant homomorphisms from $\tilde{G}'_{\text{red}} \rightarrow A_s$. Conversely, if a homomorphism $\gamma : \Pi_G \rightarrow A_s$ vanishes on $\pi_1(G'_{\text{red}})$, then γ defines a homomorphism

$$Z \cap G'_{\text{red}} \cong \Pi_G/\pi_1(G'_{\text{red}}) \rightarrow A_s.$$

But since A_s is a torus, this extends to a morphism $f : Z \rightarrow A_s$ ([B, §8.2]) which in turn can be pulled back via $Z \cong \tilde{G}/(G_u \rtimes \tilde{G}'_{\text{red}})$ to a morphism $\tilde{f} : \tilde{G} \rightarrow A_s$ extending γ . This proves that the image of $\text{Hom}(\tilde{G}, A_s)$ under the restriction map in $\text{Hom}(\Pi_G, A_s)$ is the annihilator of $\pi_1(G'_{\text{red}})$ so that

$$\Phi : \text{Hom}(\Pi_G, A_s) \rightarrow \text{Ext}_{\text{alg}}(G, A_s)$$

factors through an isomorphism

$$\Phi' : \text{Hom}(\pi_1(G'_{\text{red}}), A_s) \simeq \text{Ext}_{\text{alg}}(G, A_s). \quad \square$$

1.5. Remark. A unipotent group A_u over \mathbb{C} has no nontrivial finite subgroups, so that

$$\text{Hom}(\pi_1(G'_{\text{red}}), A_s) \cong \text{Hom}(\pi_1(G'_{\text{red}}), A).$$

Now, we turn to the study of extensions by unipotent groups. In contrast to the situation for tori, we shall see that these extensions are faithfully represented by the corresponding Lie algebra extensions.

1.6. Lemma. *The canonical restriction map*

$$H^2(\mathfrak{g}, \mathfrak{g}_{\text{red}}, \mathfrak{a}_u) \longrightarrow H^2(\mathfrak{g}, \mathfrak{a}_u)$$

is injective.

Proof. Let $\omega \in Z^2(\mathfrak{g}, \mathfrak{a}_u)$ be a Lie algebra cocycle representing an element of $H^2(\mathfrak{g}, \mathfrak{g}_{\text{red}}, \mathfrak{a}_u)$ and suppose that the class $[\omega] \in H^2(\mathfrak{g}, \mathfrak{a}_u)$ vanishes so that the extension

$$\hat{\mathfrak{g}} := \mathfrak{a}_u \oplus_\omega \mathfrak{g} \rightarrow \mathfrak{g}, \quad (a, x) \mapsto x$$

with the bracket $[(a, x), (a', x')] = (x.a' - x'.a + \omega(x, x'), [x, x'])$ splits. We have to find a $\mathfrak{g}_{\text{red}}$ -module map $f : \mathfrak{g} \rightarrow \mathfrak{a}_u$ vanishing on $\mathfrak{g}_{\text{red}}$ with

$$\omega(x, x') = (d_{\mathfrak{g}}f)(x, x') := x.f(x') - x'.f(x) - f([x, x']), \quad x, x' \in \mathfrak{g}.$$

Since the space $C^1(\mathfrak{g}, \mathfrak{a}_u)$ of linear maps $\mathfrak{g} \rightarrow \mathfrak{a}_u$ is a semisimple $\mathfrak{g}_{\text{red}}$ -module (\mathfrak{a}_u being a G -module; in particular, a G_{red} -module), we have

$$C^1(\mathfrak{g}, \mathfrak{a}_u) = C^1(\mathfrak{g}, \mathfrak{a}_u)^{\mathfrak{g}_{\text{red}}} \oplus \mathfrak{g}_{\text{red}} \cdot C^1(\mathfrak{g}, \mathfrak{a}_u);$$

and similarly for the space $Z^2(\mathfrak{g}, \mathfrak{a}_u)$ of 2-cocycles. As the Lie algebra differential $d_{\mathfrak{g}} : C^1(\mathfrak{g}, \mathfrak{a}_u) \rightarrow Z^2(\mathfrak{g}, \mathfrak{a}_u)$ is a $\mathfrak{g}_{\text{red}}$ -module map, each $\mathfrak{g}_{\text{red}}$ -invariant coboundary is the image of a $\mathfrak{g}_{\text{red}}$ -invariant cochain in $C^1(\mathfrak{g}, \mathfrak{a}_u)$. We conclude, in particular, that $\omega = d_{\mathfrak{g}}h$ for some $\mathfrak{g}_{\text{red}}$ -module map $h : \mathfrak{g} \rightarrow \mathfrak{a}_u$. For $x \in \mathfrak{g}_{\text{red}}$ and $x' \in \mathfrak{g}$, it follows that

$$\begin{aligned} 0 &= \omega(x, x') = x.h(x') - x'.h(x) - h([x, x']) \\ &= h([x, x']) - x'.h(x) - h([x, x']) = -x'.h(x), \end{aligned}$$

showing that $h(\mathfrak{g}_{\text{red}}) \subseteq \mathfrak{a}_u^{\mathfrak{g}}$, which in turn leads to $[\mathfrak{g}_{\text{red}}, \mathfrak{g}_{\text{red}}] \subseteq \ker h$. As $\mathfrak{z}(\mathfrak{g}_{\text{red}}) \cap [\mathfrak{g}, \mathfrak{g}] = \{0\}$, the map $h|_{\mathfrak{z}(\mathfrak{g}_{\text{red}})}$ extends to a linear map $f : \mathfrak{g} \rightarrow \mathfrak{a}_u^{\mathfrak{g}}$ vanishing on $[\mathfrak{g}, \mathfrak{g}]$. Moreover, since f vanishes on $[\mathfrak{g}, \mathfrak{g}]$, f is clearly a \mathfrak{g} -module map; in particular, a $\mathfrak{g}_{\text{red}}$ -module map. Then, $d_{\mathfrak{g}}f = 0$, so that $d_{\mathfrak{g}}(h - f) = \omega$, and $h - f$ vanishes on $\mathfrak{g}_{\text{red}}$. \square

1.7. Proposition. *For $A = A_u$ the map $D : \text{Ext}_{\text{alg}}(G, A_u) \rightarrow H^2(\mathfrak{g}, \mathfrak{a}_u)$ induces a bijection*

$$D : \text{Ext}_{\text{alg}}(G, A_u) \rightarrow H^2(\mathfrak{g}, \mathfrak{g}_{\text{red}}, \mathfrak{a}_u).$$

Proof. In view of the preceding lemma, we may identify $H^2(\mathfrak{g}, \mathfrak{g}_{\text{red}}, \mathfrak{a}_u)$ with a subspace of $H^2(\mathfrak{g}, \mathfrak{a}_u)$. First we claim that $\text{im}(D)$ is contained in this subspace. For any extension

$$1 \rightarrow A_u \rightarrow \widehat{G} \rightarrow G \rightarrow 1, \tag{3}$$

we choose a Levi subgroup $\widehat{G}_{\text{red}} \subset \widehat{G}$ mapping to G_{red} under the above map $\widehat{G} \rightarrow G$. Then,

$$\widehat{G}_{\text{red}} \cap A_u = \{1\}.$$

Moreover, $\widehat{G}_{\text{red}} \rightarrow G_{\text{red}}$ is surjective and hence an isomorphism. This shows that the extension (3) restricted to G_{red} is trivial and that $\widehat{\mathfrak{g}}_u$ contains a $\widehat{\mathfrak{g}}_{\text{red}}$ -invariant complement to \mathfrak{a}_u . Therefore, $\widehat{\mathfrak{g}}$ can be described by a cocycle $\omega \in Z^2(\mathfrak{g}, \mathfrak{g}_{\text{red}}, \mathfrak{a}_u)$; in particular, ω vanishes on $\mathfrak{g} \times \mathfrak{g}_{\text{red}}$. This shows that $\text{Im } D \subset H^2(\mathfrak{g}, \mathfrak{g}_{\text{red}}, \mathfrak{a}_u)$.

If the image of the extension (3) under D vanishes, then the extension $\mathfrak{a}_u \hookrightarrow \widehat{\mathfrak{g}}_u \twoheadrightarrow \mathfrak{g}_u$ splits, which implies that the corresponding extension of unipotent groups $A_u \hookrightarrow \widehat{G}_u \twoheadrightarrow G_u$ splits. Moreover, the splitting map can be chosen to be G_{red} -equivariant since ω is G_{red} -invariant. This means that we have a morphism $G_u \times G_{\text{red}} \rightarrow \widehat{G} \cong \widehat{G}_u \times G_{\text{red}}$ splitting the extension (3). This proves that D is injective.

To see that D is surjective, let $\omega \in Z^2(\mathfrak{g}, \mathfrak{g}_{\text{red}}, \mathfrak{a}_u)$. Let $q : \widehat{\mathfrak{g}} := \mathfrak{a}_u \oplus_{\omega} \mathfrak{g} \rightarrow \mathfrak{g}$ denote the corresponding Lie algebra extension. Since \mathfrak{a}_u is a nilpotent module of \mathfrak{g}_u , the subalgebra $\widehat{\mathfrak{g}}_u := \mathfrak{a}_u \oplus_{\omega} \mathfrak{g}_u$ of $\widehat{\mathfrak{g}}$ is nilpotent, hence corresponds to a unipotent algebraic group \widehat{G}_u which is an extension of G_u by A_u . Further, the G_{red} -invariance of the decomposition $\widehat{\mathfrak{g}} = \mathfrak{a}_u \oplus \mathfrak{g}$ implies that G_{red} acts algebraically on $\widehat{\mathfrak{g}}_u$ and hence on \widehat{G}_u , so that we can form the semidirect product $\widehat{G} := \widehat{G}_u \rtimes G_{\text{red}}$ which is an extension of G by A_u mapped by D onto $\widehat{\mathfrak{g}}$. \square

Combining the previous results, we get the following.

1.8. Theorem. *For a connected algebraic group G and a connected abelian algebraic group A with G -action, we have the following isomorphisms of abelian groups induced respectively by the decomposition $A = A_u A_s$, the map Φ of Proposition 1.4 and the map D .*

- (a) $\text{Ext}_{\text{alg}}(G, A) \simeq \text{Ext}_{\text{alg}}(G, A_u) \oplus \text{Ext}_{\text{alg}}(G, A_s)$,
- (b) $\text{Ext}_{\text{alg}}(G, A_s) \simeq \text{Hom}(\pi_1([G, G]), A_s)$, and
- (c) $\text{Ext}_{\text{alg}}(G, A_u) \simeq H^2(\mathfrak{g}, \mathfrak{g}_{\text{red}}, \mathfrak{a}_u) \simeq H^2(\mathfrak{g}_u, \mathfrak{a}_u)^{\mathfrak{g}}$,

where $\mathfrak{a} = L(A)$, G_{red} is a Levi subgroup of G , $\mathfrak{g}_{\text{red}} = L(G_{\text{red}})$, $\mathfrak{g}_u = L(G_u)$, $\mathfrak{g} = L(G)$ and $\mathfrak{a}_u = L(A_u)$.

(Observe that, by the following proof, the fundamental group $\pi_1([G, G])$ is a finite group.)

Moreover, the isomorphisms in (a) and (b) are functorial in both A and G . Also, the first isomorphism in (c) is functorial in A as well as in the category of pairs (G, G_{red}) , where G is a connected algebraic group and G_{red} is a Levi subgroup. (Recall that given any algebraic group morphism $f : G \rightarrow H$, we can always choose Levi subgroups $G_{\text{red}} \subset G$ and $H_{\text{red}} \subset H$ such that $f(G_{\text{red}}) \subset H_{\text{red}}$.)

Proof. In view of the Levi decomposition of the commutator $[G, G] = [G, G]_u \rtimes G'_{\text{red}}$, we have $\pi_1([G, G]) = \pi_1(G'_{\text{red}})$. Now, we only have to use Lemma 1.2 and the preceding results Propositions 1.4 and 1.7 on extensions by A_s and A_u to complete the proof of (a), (b) and the first isomorphism of (c). The restriction map $\gamma : C^*(\mathfrak{g}, \mathfrak{g}_{\text{red}}, \mathfrak{a}) \rightarrow C^*(\mathfrak{g}_u, \mathfrak{a})^{\mathfrak{g}_{\text{red}}}$ between the standard cochain complexes is clearly an isomorphism. Thus, the last isomorphism in (c) follows since the action of $\mathfrak{g}_{\text{red}}$ on \mathfrak{g} and \mathfrak{a}_u is completely reducible and, moreover, the standard action of any Lie algebra \mathfrak{s} on $H^*(\mathfrak{s}, M)$ is trivial (for any \mathfrak{s} -module M).

The functoriality of the isomorphisms follows from their proofs. □

2. Analogue of Van-Est Theorem for algebraic group cohomology

2.1. Definition. Let G be an algebraic group and A an abelian algebraic group with G -action. For any $n \geq 0$, let $C_{\text{alg}}^n(G, A)$ be the abelian group consisting of all the variety morphisms $f : G^n \rightarrow A$ under the pointwise addition. Define the differential

$$\begin{aligned} \delta : C_{\text{alg}}^n(G, A) &\rightarrow C_{\text{alg}}^{n+1}(G, A) && \text{by} \\ (\delta f)(g_0, \dots, g_n) &= g_0 \cdot f(g_1, \dots, g_n) + (-1)^{n+1} f(g_0, \dots, g_{n-1}) \\ &+ \sum_{i=0}^{n-1} (-1)^{i+1} f(g_0, g_1, \dots, g_i g_{i+1}, \dots, g_n). \end{aligned}$$

Then, as is well known (and easy to see),

$$\delta^2 = 0. \tag{4}$$

The algebraic group cohomology $H_{\text{alg}}^*(G, A)$ of G with coefficients in A is defined as the cohomology of the complex

$$0 \rightarrow C_{\text{alg}}^0(G, A) \xrightarrow{\delta} C_{\text{alg}}^1(G, A) \xrightarrow{\delta} \dots$$

We have the following analogue of the Van-Est Theorem [V] for the algebraic group cohomology.

2.2. Theorem. *Let G be a connected algebraic group and let \mathfrak{a} be a finite-dimensional algebraic G -module. Then, for any $p \geq 0$,*

$$H_{\text{alg}}^p(G, \mathfrak{a}) \simeq H^p(\mathfrak{g}, \mathfrak{g}_{\text{red}}, \mathfrak{a}) \simeq H^p(\mathfrak{g}_u, \mathfrak{a})^{\mathfrak{g}},$$

where \mathfrak{g} is the Lie algebra of G , \mathfrak{g}_u is the Lie algebra of the unipotent radical G_u of G , and $\mathfrak{g}_{\text{red}}$ is the Lie algebra of a Levi subgroup G_{red} of G as in Section 1.

Proof. Consider the homogeneous affine variety $X := G/G_{\text{red}}$ and let $\{\Omega^q = \Omega^q(X, \mathfrak{a})\}_q$ denote the complex vector space of algebraic de Rham forms on X with values in the vector space \mathfrak{a} . Since X is a G -variety under the left multiplication of G and \mathfrak{a} is a G -module, Ω^q has a natural locally-finite algebraic G -module structure. Define a double cochain complex $A = \bigoplus_{p,q \geq 0} A^{p,q}$, where

$$A^{p,q} := C_{\text{alg}}^p(G, \Omega^q)$$

and $C_{\text{alg}}^p(G, \Omega^q)$ consists of all the maps $f : G^p \rightarrow \Omega^q$ such that $\text{im } f \subset M_f$, for some finite-dimensional G -stable subspace $M_f \subset \Omega^q$ and, moreover, the map $f : G^p \rightarrow M_f$ is algebraic. Let $\delta : A^{p,q} \rightarrow A^{p+1,q}$ be the group cohomology differential as in Section 2.1 and let $d : A^{p,q} \rightarrow A^{p,q+1}$ be induced from the standard de Rham differential $\Omega^q \rightarrow \Omega^{q+1}$, which is a G -module map. It is easy to see that $d\delta - \delta d = 0$ and, of course, $d^2 = \delta^2 = 0$. Thus, (A, δ, d) is a double cochain complex. This gives rise to two spectral sequences both converging to the cohomology of the associated single complex $(C, \delta + d)$ with their E_1 -terms given as follows:

$$\backslash E_1^{p,q} = H_d^q(A^{p,*}), \quad \text{and} \tag{5}$$

$$\backslash\backslash E_1^{p,q} = H_\delta^q(A^{*,p}). \tag{6}$$

We now determine $\backslash E_1$ and $\backslash\backslash E_1$ more explicitly in our case.

Since X is a contractible variety, by the algebraic de Rham theorem [GH, Chap. 3, §5], the algebraic de Rham cohomology

$$H_{\text{dR}}^q(X, \mathfrak{a}) \begin{cases} \simeq \mathfrak{a}, & \text{if } q = 0 \\ = 0, & \text{otherwise.} \end{cases}$$

Thus,

$$\backslash E_1^{p,q} \begin{cases} \simeq C_{\text{alg}}^p(G, \mathfrak{a}), & \text{if } q = 0 \\ = 0, & \text{otherwise.} \end{cases}$$

Therefore,

$$\backslash E_2^{p,q} = H_{\delta}^p(H_d^q(A)) = \begin{cases} H_{\text{alg}}^p(G, \mathfrak{a}), & \text{if } q = 0 \\ 0, & \text{otherwise.} \end{cases} \tag{7}$$

In particular, the spectral sequence $\backslash E_*$ collapses at $\backslash E_2$. From this we see that there is a canonical isomorphism

$$H_{\text{alg}}^p(G, \mathfrak{a}) \simeq H^p(C, \delta + d). \tag{8}$$

We next determine $\backslash\backslash E_1$ and $\backslash\backslash E_2$. But first we need the following two lemmas.

2.3. Lemma. *For any $p \geq 0$,*

$$H_{\text{alg}}^q(G, \Omega^p) = \begin{cases} \Omega^{pG}, & \text{if } q = 0 \\ 0, & \text{otherwise,} \end{cases}$$

where Ω^{pG} denotes the subspace of G -invariants in $\Omega^p = \Omega^p(X, \mathfrak{a})$.

Proof. The assertion for $q = 0$ follows from the general properties of group cohomology. So we need to consider the case $q > 0$ now.

Since $L := G_{\text{red}}$ is reductive, any algebraic L -module M is completely reducible. Let

$$\pi^M : M \rightarrow M^L$$

be the unique L -module projection onto the space of L -module invariants M^L of M . Taking M to be the ring of regular functions $\mathbb{C}[L]$ on L under the left regular representation, i.e., under the action

$$(k \cdot f)(k') = f(k^{-1}k'), \quad \text{for } f \in \mathbb{C}[L], k, k' \in L,$$

we get the L -module projection $\pi = \pi^{\mathbb{C}[L]} : \mathbb{C}[L] \rightarrow \mathbb{C}$. Thus, for any complex vector space V , we get the projection $\pi \otimes I_V : \mathbb{C}[L] \otimes V \rightarrow V$, which we abbreviate simply by π , where I_V is the identity map of V . We define a ‘‘homotopy operator’’ H , for any $q \geq 0$,

$$H : C_{\text{alg}}^{q+1}(G, \Omega^p) \rightarrow C_{\text{alg}}^q(G, \Omega^p)$$

by

$$\left((Hf)(g_1, \dots, g_q) \right)_{g_0L} = \pi \left(\Theta_{(g_0, \dots, g_q)}^f \right),$$

for $f \in C_{\text{alg}}^{q+1}(G, \Omega^p)$ and $g_0, \dots, g_q \in G$, where $\Theta_{(g_0, \dots, g_q)}^f : L \rightarrow \Omega_{g_0L}^p$ is defined by

$$\Theta_{(g_0, \dots, g_q)}^f(k) = \left((g_0k) \cdot f(k^{-1}g_0^{-1}, g_1, g_2, \dots, g_q) \right)_{g_0L},$$

for $k \in L$. (Here $\Omega_{g_0L}^p$ denotes the fiber at g_0L of the vector bundle of p -forms in X with values in \mathfrak{a} and, for a form ω , ω_{g_0L} denotes the value of the form ω at g_0L .) It is easy to see that on $C_{\text{alg}}^q(G, \Omega^p)$, for any $q \geq 1$,

$$H\delta + \delta H = I. \tag{9}$$

To prove this, take any $f \in C_{\text{alg}}^q(G, \Omega^p)$ and $g_0, \dots, g_q \in G$. Then,

$$\begin{aligned} & \left((H\delta f)(g_1, \dots, g_q) \right)_{g_0L} \\ &= \pi \left(\Theta_{(g_0, \dots, g_q)}^{\delta f} \right) \\ &= \left(f(g_1, \dots, g_q) \right)_{g_0L} \\ &+ (-1)^{q+1} \pi \left(((g_0k) \cdot f(k^{-1}g_0^{-1}, g_1, \dots, g_{q-1}))_{g_0L} \right) \\ &+ \sum_{i=1}^{q-1} (-1)^{i+1} \pi \left(((g_0k) \cdot f(k^{-1}g_0^{-1}, g_1, \dots, g_i g_{i+1}, \dots, g_q))_{g_0L} \right) \\ &- \pi \left(((g_0k) \cdot f(k^{-1}g_0^{-1}g_1, g_2, \dots, g_q))_{g_0L} \right), \end{aligned} \tag{10}$$

where $((g_0k) \cdot f(k^{-1}g_0^{-1}, g_1, \dots, g_{q-1}))_{g_0L}$ means the function from L to $\Omega_{g_0L}^p$ defined as $k \mapsto ((g_0k) \cdot f(k^{-1}g_0^{-1}, g_1, \dots, g_{q-1}))_{g_0L}$. Similarly,

$$\begin{aligned} & \left((\delta Hf)(g_1, \dots, g_q) \right)_{g_0L} \\ &= \left(g_1 \cdot ((Hf)(g_2, \dots, g_q)) \right)_{g_0L} \\ &+ (-1)^q \left((Hf)(g_1, \dots, g_{q-1}) \right)_{g_0L} \\ &+ \sum_{i=1}^{q-1} (-1)^i \left((Hf)(g_1, \dots, g_i g_{i+1}, \dots, g_q) \right)_{g_0L} \\ &= \left(g_1 \cdot ((Hf)(g_2, \dots, g_q)) \right)_{g_0L} \\ &+ (-1)^q \pi \left(((g_0k) \cdot f(k^{-1}g_0^{-1}, g_1, \dots, g_{q-1}))_{g_0L} \right) \\ &+ \sum_{i=1}^{q-1} (-1)^i \pi \left(((g_0k) \cdot f(k^{-1}g_0^{-1}, g_1, \dots, g_i g_{i+1}, \dots, g_q))_{g_0L} \right). \end{aligned} \tag{11}$$

From the definition of the G -action on Ω^p , it is easy to see that

$$\pi \left(((g_0k) \cdot f(k^{-1}g_0^{-1}g_1, g_2, \dots, g_q))_{g_0L} \right) = \left(g_1 \cdot ((Hf)(g_2, \dots, g_q)) \right)_{g_0L}. \tag{12}$$

Combining (10)–(12), we clearly get (9).

From the above identity (9), we see, of course, that any cocycle in $C_{\text{alg}}^q(G, \Omega^p)$ (for any $q \geq 1$) is a coboundary, proving the lemma. \square

2.4. Lemma. *The restriction map $\gamma : \Omega^{pG} \rightarrow C^p(\mathfrak{g}, \mathfrak{g}_{\text{red}}, \mathfrak{a})$ (defined below in the proof) is an isomorphism for all $p \geq 0$, where $C^*(\mathfrak{g}, \mathfrak{g}_{\text{red}}, \mathfrak{a})$ is the standard cochain complex for the Lie algebra pair $(\mathfrak{g}, \mathfrak{g}_{\text{red}})$ with coefficient in the \mathfrak{g} -module \mathfrak{a} . Moreover, γ commutes with differentials. Thus, γ induces an isomorphism in cohomology*

$$H^*(\Omega^G) \xrightarrow{\sim} H^*(\mathfrak{g}, \mathfrak{g}_{\text{red}}, \mathfrak{a}).$$

Proof. For any $\omega \in \Omega^{pG}$, define $\gamma(\omega)$ as the value of ω at eL . Since G acts transitively on X , and ω is G -invariant, γ is injective.

Since any $\omega_o \in C^p(\mathfrak{g}, \mathfrak{g}_{\text{red}}, \mathfrak{a})$ can be extended (uniquely) to a G -invariant form on X with values in \mathfrak{a} , γ is surjective. Further, from the definition of differentials on the two sides, it is easy to see that γ commutes with differentials. \square

2.5. Continuation of the proof of Theorem 2.2

We now determine ${}^{\parallel}E$. First, by (6) of (2.2),

$${}^{\parallel}E_1^{p,q} = H_{\delta}^q(A^{*,p}) = H_{\text{alg}}^q(G, \Omega^p).$$

Thus, by Lemma 2.3,

$${}^{\parallel}E_1^{p,0} = H_{\text{alg}}^0(G, \Omega^p) = \Omega^{pG}$$

and

$${}^{\parallel}E_1^{p,q} = 0, \quad \text{if } q > 0.$$

Moreover, under the above equality, the differential of the spectral sequence $d_1 : {}^{\parallel}E_1^{p,0} \rightarrow {}^{\parallel}E_1^{p+1,0}$ can be identified with the restriction of the de Rham differential

$$\Omega^{pG} \rightarrow \Omega^{p+1G}.$$

Thus, by Lemma 2.4,

$${}^{\parallel}E_2^{p,q} = \begin{cases} H^p(\mathfrak{g}, \mathfrak{g}_{\text{red}}, \mathfrak{a}), & \text{if } q = 0 \\ 0, & \text{otherwise.} \end{cases} \tag{13}$$

In particular, the spectral sequence ${}^{\parallel}E$ as well degenerates at the ${}^{\parallel}E_2$ -term. Moreover, we have a canonical isomorphism

$$H^p(\mathfrak{g}, \mathfrak{g}_{\text{red}}, \mathfrak{a}) \simeq H^p(C, \delta + d). \tag{14}$$

Comparing the above isomorphism with the isomorphism (8) of §2.2, we get a canonical isomorphism:

$$H_{\text{alg}}^p(G, \mathfrak{a}) \simeq H^p(\mathfrak{g}, \mathfrak{g}_{\text{red}}, \mathfrak{a}).$$

For the isomorphism $H^p(\mathfrak{g}, \mathfrak{g}_{\text{red}}, \mathfrak{a}) \simeq H^p(\mathfrak{g}_u, \mathfrak{a})^{\mathfrak{g}}$, see Theorem 1.8 and its proof. This proves Theorem 2.2. \square

2.6. Remark. Even though we took the field \mathbb{C} as our base field, all the results of this paper hold (by the same proofs) over any algebraically closed field of characteristic 0, if we replace the fundamental group π_1 by the algebraic fundamental group.

Acknowledgments

This work was done while the authors were visiting the Fields Institute, Toronto (Canada) in July, 2003, hospitality of which is gratefully acknowledged. The first author was partially supported from NSF. We thank J-P. Serre for a helpful advice regarding exposition.

References

- [B] A. Borel, *Linear Algebraic Groups*, Graduate Texts in Math. Vol. 126, second ed., Springer-Verlag, 1991.
- [CE] C. Chevalley and S. Eilenberg, Cohomology theory of Lie groups and Lie algebras, *Transactions Amer. Math. Soc.* **63** (1948), 85–124.
- [GH] P. Griffiths and J. Harris, *Principles of Algebraic Geometry*, John Wiley and Sons., Inc., 1978.
- [H] J. Humphreys, *Linear Algebraic Groups*, Graduate Texts in Math. Vol. 21, Springer-Verlag 1995.
- [HS] G. Hochschild and J-P. Serre, Cohomology of Lie algebras, *Annals of Math.* **57** (1953), 591–603.
- [K] S. Kumar, *Kac-Moody Groups, their Flag Varieties and Representation Theory*, Progress in Math. Vol. 204, Birkhäuser, 2002.
- [V] W.T. Van-Est, Une application d’une méthode de Cartan–Leray, *Indag. Math.* **18** (1955), 542–544.