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The generalized triangle inequalities for rank 3 symmetric spaces of noncompact type.

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Dedicated to Robert Greene on his sixtieth birthday.

ABSTRACT. We compute the generalized triangle inequalities explicitly for all rank 3 symmetric spaces of noncompact type. For $\mathrm{SL}(4,\mathbb{C})$ there are 50 inequalities none of them redundant by [**KTW**]. For both $\mathrm{Sp}(6,\mathbb{C})$ and $\mathrm{Spin}(7,\mathbb{C})$ there are 135 inequalities of which 24 are trivially redundant in the sense that they follow from the inequalities defining the Weyl chamber Δ. There are 9 more redundant inequalities for each of these two groups. One interesting feature is that these inequalities do not occur for the other system (and consequently must be redundant because the two polyhedral cones are the same by Theorem 1.8). The two equal polyhedral cones $D_3(B_3) = D_3(C_3)$ have precisely 102 facets and 51 generators (edges).

1. Introduction

Let G be a connected semisimple real Lie group with no compact factors and finite center and Lie algebra $\mathfrak{g}$, K be a maximal compact subgroup and $X = G/K$ be the associated symmetric space. By a symmetric space of noncompact type we will mean a symmetric space G/K where G is as above. We let $\mathfrak{k}$ denote the Lie algebra of K and let $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ be the Cartan decomposition. Let $\mathfrak{a}$ be a Cartan subspace in $\mathfrak{p}$ (i.e., a maximal subalgebra in $\mathfrak{p}$ which is necessarily abelian). Let A be the real points of the split torus in G corresponding to $\mathfrak{a}$. Choose an ordering of the restricted roots and let $\Delta \subset \mathfrak{a}$ be the corresponding (closed) Weyl chamber. Let A_Δ be the image of Δ under the exponential map $\exp : \mathfrak{g} \to G$. Let o be the point in X that is fixed by K. We will refer to o as the basepoint for X. We will need the following theorem, the Cartan decomposition for the group G, see [**He**], Theorem 1.1, pg. 402.

THEOREM 1.1. *We have*

$$G = KA_\Delta K.$$

Moreover, for any $g \in G$, the intersection of the double coset KgK with A_Δ consists of a single point to be denoted $a(g)$.

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Suppose $\overline{x_1x_2}$ is the oriented geodesic segment in X joining the point x_1 to the point x_2. Then there exists a unique element $g \in G$ which sends x_1 to o and x_2 to $y = \exp(\delta)$ where $\delta \in \Delta$. Note that the point δ is uniquely determined by $\overline{x_1x_2}$. We define a map σ from G-orbits of oriented geodesic segments to Δ by

$$\sigma(\overline{x_1x_2}) = \delta.$$

Clearly we have the following consequence of the Cartan decomposition.

LEMMA 1.2. *The map σ gives rise to a one-to-one correspondence between G-orbits of oriented geodesic segments in X and the points of Δ.*

REMARK 1.3. *In the real-rank 1 case $\sigma(\overline{x_1x_2})$ is just the length of the geodesic segment $\overline{x_1x_2}$. In general we will call $\sigma(\overline{x_1x_2})$ the Δ-length of $\overline{x_1x_2}$ or the Δ-distance between x_1 and x_2. We will write $d_\Delta(x_1, x_2) = \sigma(\overline{x_1x_2})$.*

We note the formula

$$d_\Delta(x_1, x_2) = \log a(g_1^{-1}g_2) \text{ where } x_1 = g_1K, x_2 = g_2K.$$

REMARK 1.4. *The delta distance is* symmetric *in the sense that*

$$d_\Delta(x_1, x_2) = -w_0 d_\Delta(x_2, x_1),$$

where w_0 is the longest element in the restricted Weyl group. However, the naive triangle inequality

$$d_\Delta(x_1, x_3) \leq d_\Delta(x_1, x_2) + d_\Delta(x_2, x_3)$$

does not hold [**KLM2**]. *Here the order is the one defined by the cone Δ. The naive triangle inequality has to be replaced by the inequalities below.*

We have the fundamental problem of finding the *generalized triangle inequalities* for X, precisely we have

PROBLEM 1.5. *Give conditions on a triple $(v_1, v_2, v_3) \in \Delta^3$ that are necessary and sufficient in order that there exist a triangle in X with vertices x_1, x_2, x_3 such that $d_\Delta(x_1, x_2) = v_1$, $d_\Delta(x_2, x_3) = v_2$ and $d_\Delta(x_3, x_1) = v_3$.*

We now describe a system of linear inequalities on Δ^3 which will give the required necessary and sufficient conditions. These inequalities will be called the *generalized triangle inequalities.*

REMARK 1.6. *We will include the inequalities defining Δ^3 in $\mathfrak{a}^3$ in the generalized triangle inequalities.*

We will need the following:

DEFINITION 1.7. *Suppose that W and W' are Coxeter groups acting on V and V' respectively by their natural reflection representations. We define a monomorphism of Coxeter systems to be a pair (f, ϕ) where $f : V \to V'$ is an isometric embedding and ϕ is a monomorphism $W \to W'$ satisfying $f(wx) = \phi(w)f(x)$.*

First we reduce to the case in which G is complex by the following:

THEOREM 1.8 ([**LM, KLM1**]). *1. The set $D_3(X) \subset \Delta^3$ of triples (v_1, v_2, v_3) for which a triangle in the Problem 1.5 (for X) exists, is a polyhedral cone.*

2. $D_3(X)$ depends only on the spherical Coxeter complex associated to X. More precisely, a monomorphism (f, ϕ) of the Coxeter system $(\mathfrak{a}, W)$ to the Coxeter system $(\mathfrak{a}', W')$ induces an affine embedding $D_3(X) \to D_3(X')$. In particular, if f and ϕ are also surjective, then the map $D_3(X) \to D_3(X')$ is an affine isomorphism.

Thus given G as above we can replace G by any complex semisimple group of the same rank as G whose Weyl group coincides with the restricted Weyl group W of G. Thus it suffices to find the generalized triangle inequalities for the case in which G is complex.

So from now on we will assume that G is a connected complex semisimple group. We will accordingly often rewrite $D_3(X)$ as $D_3(R)$ where R is the reduced root system associated to the restricted root system of X.

The system of inequalities for $D_3(R)$ breaks up into rank($\mathfrak{g}$) subsystems $*_P$ where P is a standard maximal parabolic subgroup. The subsystem $*_P$ is controlled by the Schubert calculus in the generalized Grassmannian G/P in the sense that there is one inequality $*_{w_1,w_2,w_3}$ for each triple of elements $w_1, w_2, w_3 \in W^P$ such that

$$X^P_{w_1} \cdot X^P_{w_2} \cdot X^P_{w_3} = [pt]$$

in $H_*(G/P)$. Here W^P is the set of minimal length coset representatives for the set of cosets W/W_P (W_P being the Weyl group of P), $\cdot$ is the intersection product and $X^P_w, w \in W^P$, is the Schubert class in G/P. To describe the inequality $*_{w_1,w_2,w_3}$, let P be the standard maximal parabolic corresponding to a fundamental weight λ. Then the action of W on the weight lattice of $\mathfrak{g}$ induces a one-to-one correspondence $f : W^P \to W\lambda$. Thus we may reparametrize the Schubert classes in G/P by elements of $\mathfrak{a}^*$. We let $\lambda_i = f(w_i), i = 1, 2, 3$. We will sometimes denote the Schubert cycle X^P_w as $X_{f(w)}$. Then the inequality $*_{w_1,w_2,w_3}$ is given by

$$\lambda_1(v_1) + \lambda_2(v_2) + \lambda_3(v_3) \leq 0, \ (v_1, v_2, v_3) \in \Delta^3.$$

REMARK 1.9. *The key point is that there is a basis of the algebra $H_*(G/P)$ parametrized by certain linear functionals on $\mathfrak{a}$.*

In order to give an accurate account of the history of work on the problem, we first need to describe the corresponding problem for the *infinitesimal* symmetric space $\mathfrak{p}$. The AdK orbits in $\mathfrak{p}$ are again parametrized by the points of the cone Δ. Thus given a triple of side-lengths $(v_1, v_2, v_3) \in \Delta^3$ as above we can look for a triple $e = (e_1, e_2, e_3) \in \mathfrak{p}^3$ such that $e_1 \in AdK \cdot v_1, \ e_2 \in AdK \cdot v_2, \ e_3 \in AdK \cdot v_3$ and

$$e_1 + e_2 + e_3 = 0.$$

We let $D_3(\mathfrak{p})$ be the subset of $(v_1, v_2, v_3) \in \Delta^3$ such that there is a solution to the above problem. The following theorem (for all connected semisimple groups with no compact factors) was proved in [**KLM1**] (however see below).

THEOREM 1.10. $D_3(\mathfrak{p}) = D_3(X)$.

Many people contributed to finding the generalized triangle inequalities. The reader is urged to consult [**Fu**] and [**KLM2**] for a more complete account. In fact the inequalities were computed for the *infinitesimal* symmetric space $\mathfrak{p}$ for G complex by [**BeSj**] and [**LM**] and for general G (real or complex) in [**LM**]. An intense interest in recent years in the generalized triangle inequalities was triggered by the paper of Klyachko, [**Kly98**]. Klyachko proved the generalized triangle inequalities for $GL(m, \mathbb{C})$ in the infinitesimal symmetric space case and a refinement was obtained by Belkale [**Bel**]. In a second paper, Klyachko, [**Kly99**] , proved the equality $D_3(\mathfrak{p}) = D_3(X)$ for certain symmetric spaces of complex simple groups (including $SL(m)$). In [**AMW**], Alekseev, Meinrenken and Woodward proved the corresponding equality for all complex simple G.

The above system of inequalities is not so explicit; in particular, the polyhedral cone $D_3(R)$ is not well understood. Thus it is important to compute explicit examples. We will see that the theorem of Knutson,Tao and Woodward [**KTW**] that the inequalities are irredundant for $GL(m)$ is highly exceptional.

DEFINITION 1.11. *The inequality* $*_{w_1,w_2,w_3}$ *will be said to be* trivially redundant *if it is a consequence of the inequalities defining the cone* Δ *in* $\mathfrak{a}$.

In the case G has rank one, the generalized triangle inequalities are precisely the ordinary triangle inequalities. The rank two examples were worked out in [**LM**]. For the cases of $B_2 = C_2$ and G_2 the above system of inequalities was not minimal. For the case of B_2 there was one trivially redundant inequality. Once it was removed the remaining inequalities were irredundant. For the case of G_2 there were no trivially redundant inequalities but three redundant inequalities. The point of this paper is to work out all the rank three examples. We will see that for C_3 and B_3 there are 24 trivially redundant inequalities. It is rather surprising that there are only 9 more redundant inequalities. Furthermore as explained in the abstract these redundancies were easy to find since the redundant inequalities occurred for one system and not for the other. Thus we might say that only "obvious" redundancies occurred. Another interesting consequence is that although the polyhedral cones $D_3(B_3)$ and $D_3(C_3)$ are isomorphic (i.e., there is an affine isomorphism from one to the other) by Theorem 1.8, the systems of inequalities are different.

In Chapter 3, see Theorem 3.10, we have given a self-contained account of how one uses the Demazure–B.G.G. operators to realize the duals of the Schubert homology classes in the Borel model. Our calculations in this paper of the products of Schubert classes aare based on this realization.

This paper is dedicated to Robert Greene on the occasion of his sixtieth birthday. The third author takes great pleasure in acknowledging many helpful conversations and some hard-fought tennis matches over the years. This paper depends on [**LM**] and the two papers [**KLM1**] and [**KLM2**]. We have used the computer program Porta written by Thomas Christof and Andreas Löbel. Finally we would like to thank George Stantchev for finding the computer program Porta and for much help and advice in implementing it.

2. Schubert cycles in generalized flag varieties

We continue to assume that G is a connected complex semisimple algebraic group. For convenience, we further assume that G is simply-connected. We fix a Borel subgroup B of G. Let $\mathfrak{b}$, resp. $\mathfrak{g}$, be the Lie algebra of B, resp. G. We also fix a Cartan subalgebra $\mathfrak{h} \subset \mathfrak{b}$. The choice of $(\mathfrak{h}, \mathfrak{b})$ determines the set $\Pi \subset \mathfrak{h}^*$ of positive roots and thus the set $\Phi = \{\alpha_1, \cdots, \alpha_l\} \subset \Pi$ of simple roots and also the fundamental weights $\{\omega_1, \cdots, \omega_l\}$, l being the rank of G. We consider the real form $\mathfrak{a}$ of $\mathfrak{h}$ which is the real span of the simple coroots $\{\alpha_1^\vee, \cdots, \alpha_l^\vee\}$. Then $\Pi \subset \mathfrak{a}^*$ and also $\omega_i \in \mathfrak{a}^*$. Any algebraic subroup P of G containing B is called a *standard parabolic subgroup.* Let $\Delta \subset \mathfrak{a}^*$ be the cone on the fundamental weights. By definition, the *dominant weights* are the elements of $\oplus_{i=1}^l \mathbb{Z}_+\omega_i$.

To each dominant weight λ we define the *associated standard parabolic subgroup* P_λ to be the (connected) subgroup of G with Lie algebra spanned by $\mathfrak{b}$ together with the root vectors $X_{-\alpha}$ (corresponding to the root $-\alpha$) such that $\lambda(H_\alpha) = 0$. Here α runs through the positive roots Π of $\mathfrak{g}$ and $H_\alpha \in \mathfrak{a}$ is the corresponding coroot which is equal to $\frac{2\alpha}{\langle\alpha,\alpha\rangle}$ under the Killing form. We let S be the set of (simple) reflections in the root hyperplanes defined by the simple roots and let $W \subset \operatorname{Aut}\mathfrak{h}$ be the Weyl group generated by S. Then W can canonically be identified with the group $N(T)/T$, where T is the maximal torus of G with Lie algebra $\mathfrak{h}$ and $N(T)$ is its normalizer in G.

For any standard parabolic subgroup P of G, let $W_P \subset W$ be the subgroup consisting of those w that have representatives in P and $S_P = S \cap W_P$. We let ℓ be the length function on W. Each coset wW_P in W/W_P has a unique representative of minimum length [**Hi82**], Ch. I, Corollary 5.4. We will denote this element by $w^* = w^*_P$. The set of such representatives will be denoted by W^P. We have the following criterion for the minimum length element in the coset wW_P, see [**Hi82**], Ch. I, Corollary 5.4.

LEMMA 2.1. *$w^* \in wW_P$ is the minimum length representative if and only if*

$$\ell(w^*s) = l(w^*) + 1, \text{ for all } s \in S_P.$$

We will also need the following result, see [**Hi82**], Ch. I, Theorem 5.3.

LEMMA 2.2. *Suppose that $w \in W^P$ and $v \in W_P$. Then*

$$\ell(wv) = \ell(w) + \ell(v).$$

We now recall the Bruhat decomposition for G, see [**He**], Theorem 1.4, pg. 403.

THEOREM 2.3. *For any standard parabolic subgroup P of G,*

$$G = \sqcup_{w\in W} BwB, \tag{1}$$

$$G = \sqcup_{w\in W^P} BwP. \tag{2}$$

As a consequence of (2) the generalized flag variety G/P is the disjoint union of the subsets $\{C^P_w := BwP/P\}_{w\in W^P}$. The subset C^P_w is biregular isomorphic to the affine space $\mathbb{C}^{\ell(w)}$ and is called a *Schubert cell.* The closure of C^P_w, to be denoted X^P_w, is an algebraic subvariety of G/P and is called a *Schubert variety.* We will use $[X^P_w]$ to denote the integral homology class in $H_*(G/P)$ carried by X^P_w but we will often abuse notation and use the same symbol X^P_w for the variety and its class in homology. We will use $PD(X^P_w)$ to denote the cohomology class (of complementary degree to the degree of X^P_w) associated to X^P_w by the Poincaré duality. In what follows we will let $N = dim(G/B) = l(w_0)$ and $N_P = dim(G/P)$.

We recall the following well known:

THEOREM 2.4. *The integral homology $H_*(G/P)$ is a free $\mathbb{Z}$–module with basis $\{X^P_w : w \in W^P\}$.*

In particular, the integral homology $H_*(G/B)$ is a free $\mathbb{Z}$-module with basis $\{X_w : w \in W\}$, where X^B_w is abbreviated by X_w.

Since $H_*(G/P)$ is free and we have a distinguished basis (the Schubert classes) it is reasonable to consider the basis for the corresponding cohomology groups that are dual under the Kronecker pairing $\langle\ ,\ \rangle$ between homology and cohomology. Let $\{\epsilon^P_w : w \in W^P\}$ denote the dual basis. Thus we have for $w, w' \in W^P$,

$$\langle\epsilon^P_w, X^P_{w'}\rangle = \delta_{w,w'}.$$

It suffices to study $H^*(G/B)$ because of the following well known theorem.

THEOREM 2.5. *Let $\pi_P : G/B \to G/P$ be the projection. Then the induced map $\pi_P^* : H^*(G/P) \to H^*(G/B)$ is injective with image precisely equal to the W_P-invariants of $H^*(G/B)$.*

Moreover, for $w \in W^P$,

$$\pi_P^*(\epsilon_w^P) = \epsilon_w,$$

where again we abbreviate ϵ_w^B by ϵ_w.

So, from now on, we will identify $H^*(G/P)$ as a subring of $H^*(G/B)$ and denote ϵ_w^P by ϵ_w itself.

2.1. Poincaré duality in G/P. In the following, for each $w \in W^P$ we will need to identify $PD(X_w^P)$ in terms of the basis $\{\epsilon_v^P\}_{v \in W^P}$.

Define the involutive map $\theta^P : W \to W$ by $\theta^P(w) = w_0 w w_{0,P}$, where w_0 (resp. $w_{0,P}$) is the longest element of W (resp. W_P).

For lack of a precise reference, we give a proof of the following:

PROPOSITION 2.6. *For $w \in W^P$, $\theta^P(w) \in W^P$ and we have*

$$PD(X_w^P) = \epsilon_{\theta^P(w)}^P.$$

The proposition will follow from the next two lemmas.

LEMMA 2.7. *The map θ^P carries W^P into itself. Moreover, we have*

$$\ell(\theta^P(w)) = N_P - \ell(w),$$

where N_P denotes the complex dimension of G/P. Thus, $N_P = \ell(w_0) - \ell(w_{0,P})$.

Proof. For $w \in W^P$ and any $v \in W_P$ we have

$$\ell(w_0 w w_{0,P} v) = \ell(w_0) - \ell(w w_{0,P} v).$$

But, by Lemma 2.2, since $w \in W^P$ and $w_{0,P} v \in W_P$ we get

$$\ell(w w_{0,P} v) = \ell(w) + \ell(w_{0,P} v) = \ell(w) + \ell(w_{0,P}) - \ell(v).$$

Thus,

$$\ell(w_0 w w_{0,P} v) = (\ell(w_0) - \ell(w) - \ell(w_{0,P})) + \ell(v).$$

But the above argument shows that the terms in parentheses equal $\ell(w_0 w w_{0,P})$. Thus multiplying $\theta^P(w)$ by any $v \in W_P$ increases the length of $\theta^P(w)$ and accordingly by Lemma 2.1 we have

$$\theta^P(w) \in W^P.$$

Taking v to be the identity in the above formula we have

$$\ell(w_0 w w_{0,P}) = \ell(w_0) - \ell(w_{0,P}) - \ell(w) = N_P - \ell(w).$$

□

Before proving the proposition we need a general result from algebraic topology. Let M be a compact connected oriented manifold of dimension n. For $a \in H_p(M)$ and $b \in H_{n-p}(M)$ we will let $a \cdot b$ denote the intersection pairing, [**Br**],Ch.V, Section 11. We then have see [**Br**], pg. 367,

$$\langle PD(a), b \rangle = b \cdot a.$$

REMARK 2.8. *In our case all the homology is even dimensional so we will not have to worry about the interchange of order.*

Suppose now that $H_*(M)$ is free over $\mathbb{Z}$ and accordingly we have $H^p(M) \cong \mathrm{Hom}_{\mathbb{Z}}(H_p(M), \mathbb{Z})$. Thus to prove the proposition we have to identify the element $PD(X_w^P)$ of $\mathrm{Hom}_{\mathbb{Z}}(H_{2N_P-2\ell(w)}(G/P), \mathbb{Z})$.

. The key point in the proof of the proposition is then

LEMMA 2.9. *For $v, w \in W^P$ with $\ell(v) = \ell(w)$,*

$$X_w^P \cdot X_{\theta^P(v)}^P = \delta_{w,v}.$$

Proof. Since the action of any element of G by left-multiplication on G/P induces the trivial action on $H^*(G/P)$, to prove the lemma it suffices to prove the following equalities at the cycle level:

$$X_w^P \cdot w_0 X_{\theta^P(v)}^P = \emptyset, \text{ if } v \neq w, \tag{3}$$

$$X_w^P \cdot w_0 X_{\theta^P(w)}^P = \{w\}. \tag{4}$$

Suppose that the above cycles X_w^P and $w_0 X_{\theta^P(v)}^P$ intersect in a nonempty set. Then (since each cycle is T–stable) the intersection is T-stable and projective and consequently will contain a T-fixed point, say $u \in W^P$. Since the Schubert cell BwP/P contains the unique T-fixed point w and $\overline{BwP/P} = \bigsqcup_{x \leq w} BxP/P$ we find $u \leq w$.

Similarly, since $u \in w_0 \overline{Bw_0 v w_{0,P} P/P}$ and w_0 is of order 2, we have $w_0 u w_{0,P} \in \overline{Bw_0 vP/P}$ and as above we find $w_0 u w_{0,P} \leq w_0 v w_{0,P}$ and hence $u w_{0,P} \geq v w_{0,P}$. But since $u, v \in W^P$, we get $u \geq v$ (cf. [Ku], Lemma 1.3.18).

Thus $v \leq u \leq w$. But since $\ell(v) = \ell(w)$, we obtain $v = u = w$. By the above argument the intersection $X_w^P \cap w_0 X_{\theta^P(w)}^P = \{w\}$ set theoretically. The proof of (4) is completed by observing that the intersection is tranverse at w or alternatively by using the Poincaré duality and (3). $\square$

Thus as operators on $H_{2N_P-2\ell(w)}(G/P)$ we have $PD(X_w^P) = \epsilon_{\theta^P(w)}$ and the proposition follows.

3. A formula for ϵ_w

3.1. The Borel model. We continue to assume that G is a connected, simply-connected complex semisimple algebraic group. As earlier let ω_i denote the i-th fundamental weight. Recall that the Borel model for $H^*(G/B)$ is obtained through the Borel homomorphism

$$\beta : \mathbb{Z}[\omega_1, \omega_2, \cdots, \omega_l] \to H^*(G/B),$$

which is the unique algebra homomorphism taking ω_i to the first Chern class of the line bundle on G/B associated to the character $-\omega_i$ of B. It is easy to see that the homomorphism β commutes with the Weyl group actions and thus for any standard parabolic subgroup P of G, the W_P-invariants $\mathbb{Z}[\omega_1, \omega_2, \cdots, \omega_l]^{W_P}$ is mapped to $H^*(G/P)$ under β.

Let $I \subset \mathbb{Z}[\omega_1, \omega_2, \cdots, \omega_l]$ be the ideal generated by the W–invariant polynomials with zero constant term. Then by extending the scalars to the real numbers $\mathbb{R}, \beta$ induces a surjective homomorphism (still denoted by) $\beta : \mathbb{R}[\omega_1, \omega_2, \cdots, \omega_l] \to H^*(G/B, \mathbb{R})$, with kernel precisely equal to $I_{\mathbb{R}} := I \otimes_{\mathbb{Z}} \mathbb{R}$.

In a subsequent subsection we will use the divided-difference operators of Demazure and Bernstein-Gelfand-Gelfand to find a polynomial $p_w \in \mathbb{R}[\omega_1, \cdots, \omega_l]$ such that $\beta(p_w)$ is the cohomology class ϵ_w for $w \in W$.

3.2. The Demazure–BGG operators. For more details on this subsection the reader is urged to consult [**Hi82**], Chapter IV and [**Ku**], Chapter XI. We will set $V = \mathfrak{a}^*$ henceforth. Let α_i be a simple root and let s_i be the corresponding simple reflection. We define the divided difference operator $A_{s_i} : S^k(V) \to S^{k-1}(V)$ by

$$A_{s_i}(f) = \frac{f - s_i f}{\alpha_i}.$$

We note that $A_{s_i} \circ A_{s_i} = 0$. It is also important to note (and simple to prove) that A_{s_i} is a twisted derivation in the following sense.

LEMMA 3.1. $A_{s_i}(pq) = A_{s_i}(p)q + (s_i p)A_{s_i}(q)$.

From the definition and the above lemma, it is easy to see that A_{s_i} keeps the integral form $\mathbb{Z}[\omega_1, \omega_2, \cdots, \omega_l] \subset S(V)$ stable and, moreover, it also keeps $I_{\mathbb{R}}$ stable.

For any $w \in W$ we further define A_w by

$$A_w := A_{s_1} \circ A_{s_2} \circ \cdots \circ A_{s_k}$$

where $w = s_1 \cdots s_k$ is a reduced decomposition of w as a product of simple reflections. We have [**Hi82**], Chapter IV, Proposition 1.7,

PROPOSITION 3.2. *The operators A_w are well-defined, i.e., they do not depend upon the choice of the reduced decomposition of w. Moreover, we have $A_w \circ A_v = A_{wv}$ if $\ell(wv) = \ell(w) + \ell(v)$ and $A_w \circ A_v = 0$ otherwise.*

3.3. The topological Demazure–BGG operators. As earlier, for any topological space X, $H^*(X)$ denotes the singular cohomology of X with integral coefficients.

There is an analogue of the Demazure–BGG operator D_{s_i} on $H^*(G/B)$ (for any simple root α_i) defined directly as follows.

Let $\pi_i : G/B \to G/P_i$ be the locally trivial fibration with fibre (over eP_i) $P_i/B \simeq \mathbb{P}^1$. It is easy to see that the restriction map $\gamma : H^*(G/B) \to H^*(P_i/B)$ is surjective. In fact, ϵ_{s_i} maps to the generator of $H^2(P_i/B)$. Choose a $\mathbb{Z}$-module splitting $\sigma : H^*(P_i/B) \to H^*(G/B)$ of γ. Then by the Leray-Hirsch Theorem, the map

$$\Phi : H^*(G/P_i) \otimes H^*(P_i/B) \to H^*(G/B),\ u \otimes v \mapsto (\pi_i^* u) \cup \sigma(v),$$

is an isomorphism. Hence, π_i^* is injective and $H^*(G/B)$ is a free module over $H^*(G/P_i)$ (under π_i^*) with basis 1 and $\sigma(\varepsilon)$, where $\varepsilon := \epsilon_{s_{i|(P_i/B)}} \in H^2(P_i/B)$ is the generator, i.e.,

$$H^n(G/B) \cong H^n(G/P_i) \oplus \sigma(\varepsilon)\, H^{n-2}(G/P_i), \qquad \text{for any } n \geq 0.$$

Write, for any $u \in H^*(G/B)$,

$$u = \pi_i^* u_1 + \sigma(\varepsilon)\, \pi_i^* u_2,$$

where $u_1 \in H^*(G/P_i)$ and $u_2 \in H^{*-2}(G/P_i)$ are uniquely determined by the above equation.

Now define

$$D_{s_i} u := \pi_i^* u_2 \in H^{*-2}(G/B).$$

It is easy to see that D_{s_i} does not depend upon the choice of the splitting σ. Clearly,

$$D_{s_i}^2 = 0.$$

PROPOSITION 3.3. *The Borel homomorphism β intertwines A_{s_i} and D_{s_i} for any simple reflection s_i.*

Proof. Let R be the ring $\mathbb{Z}[\omega_1, \omega_2, \cdots, \omega_l]$ and R_i the subring of s_i-invariants. Then R is generated as an R_i-module by 1 and ω_i. Further, $\beta(R_i) \subset H^*(G/P_i)$ (cf. subsection 3.1). By definition, A_{s_i} commutes with the multiplication by R_i and also D_{s_i} commutes with the multiplication by $H^*(G/P_i)$. Moreover, $D_{s_i}(1) = A_{s_i}(1) = 0$. Thus, to prove the proposition, it suffices to observe that

$$(*) \qquad D_{s_i}(\beta(\omega_i)) = \beta(A_{s_i}(\omega_i)) = 1.$$

Now, it is easy to see that $\beta(\omega_i)_{|(P_i/B)}$ is the generator ε, which proves $(*)$. □

Thus the operators D_{s_i} again satisfy the braid relations and we may extend D_{s_i} to D_w by taking a reduced decomposition of w. Moreover, D_{s_i} satisfies the twisted derivation property:

$$D_{s_i}(xy) = D_{s_i}(x)y + (s_i x)D_{s_i}(y), \text{ for } x, y \in H^*(G/B).$$

We also recall the following well-known result due to Chevalley, cf. [BGG], Theorem 3.14.

LEMMA 3.4. *For any simple reflection s_i and any $w \in W$, the cup product*

$$\epsilon_{s_i} \cdot \epsilon_w = \sum_{w \xrightarrow{\beta} v} \langle w\omega_i, \beta^\vee \rangle \, \epsilon_v, \tag{5}$$

where the notation $w \xrightarrow{\beta} v$ means $w \le v$, $\ell(v) = \ell(w) + 1$, $\beta \in \Pi$ and $v = s_\beta w$; and $s_\beta \in W$ is defined by $s_\beta \chi = \chi - \langle \chi, \beta^\vee \rangle \beta$, for $\chi \in \mathfrak{h}^*$.

The following result is of basic importance.

PROPOSITION 3.5. *For any simple reflection s_i and any $w \in W$,*

$$D_{s_i} \epsilon_w = \epsilon_{ws_i}, \text{ if } ws_i < w,$$

and

$$D_{s_i} \epsilon_w = 0, \text{ otherwise.}$$

Proof. We first consider the case $ws_i > w$ so $w \in W^{P_i}$. By Theorem 2.5, $\epsilon_w \in H^*(G/P_i)$, hence $D_{s_i} \epsilon_w = 0$.

So assume now that $ws_i < w$. By the Chevalley formula 3.4,

$$\epsilon_{s_i} \cdot \epsilon_{ws_i} = \epsilon_w + \sum_{ws_i \xrightarrow{\beta} v, v \ne w} \langle ws_i \omega_i, \beta^\vee \rangle \, \epsilon_v.$$

By a standard property of Coxeter groups [Ku], Corollary 1.3.19, any v appearing in the above sum satisfies $vs_i > v$. Hence applying D_{s_i} to the above equation and using the previous case, we get

$$D_{s_i}(\epsilon_{s_i} \cdot \epsilon_{ws_i}) = D_{s_i}(\epsilon_w).$$

By using the twisted derivation property and the previous case again, we get

$$D_{s_i}(\epsilon_{s_i} \cdot \epsilon_{ws_i}) = D_{s_i}(\epsilon_{s_i}) \epsilon_{ws_i} = \epsilon_{ws_i},$$

since $D_{s_i}(\epsilon_{s_i}) = 1$ (because ϵ_{s_i} restricted to P_i/B equals ε). Combining the above two equations, we get the proposition. $\square$

3.4. The polynomials p_w. In this section we will polynomials p_w such that

$$\beta(p_w) = \epsilon_w.$$

First we will find p_{w_0}. We define p_{w_0} for w_0 the longest element in W as follows. Let d be the product of the positive roots. Then define

$$p_{w_0} = \frac{d}{|W|}. \tag{6}$$

PROPOSITION 3.6.

$$\beta(p_{w_0}) = \epsilon_{w_0}.$$

As we will see that Proposition 3.6 will be an almost immediate consequence of Lemma 3.8. However we need a preliminary general lemma from algebra.

Let G be a finite group and $\rho : G \to \mathrm{Aut}(V)$ be a faithful representation. Let $\mathcal{R} = S(V^*)$ be the algebra of regular functions on V and $\mathcal{F} = Q(V^*)$ be the quotient field. The representation ρ induces a representation $\widetilde{\rho}$ from G into $\mathrm{Aut}(\mathcal{F})$ where we consider $\mathcal{F}$ as a vector space over the fixed field $\mathcal{L} := \mathcal{F}^G$. We have

$$\widetilde{\rho}(g)q(v) = q(\rho(g)^{-1}v), \; q(v) \in \mathcal{F}.$$

LEMMA 3.7. *The set $\{\widetilde{\rho}(g) : g \in G\}$ is an independent subset of the $\mathcal{F}$ vector space $\mathrm{End}_{\mathcal{L}}(\mathcal{F})$, where $\mathcal{F}$ acts on $\mathrm{End}_{\mathcal{L}}(\mathcal{F})$ via its multiplication on the range.*

Proof. Suppose we have a minimal dependence relation

$$\sum_{g\in G} q_g \widetilde{\rho}(g) = 0, q_g \in \mathcal{F}.$$

Write $q_g = \frac{a_g}{b_g}$ with $a_g, b_g \in \mathcal{R}$ and relatively prime. For $f \in \mathcal{R}$ we let $V(f)$ denote the zero locus of f. For each $g \in G$, let $F(g)$ be the fixed subspace of $\rho(g)$. For $g \neq I$ the subspace $F(g)$ is proper hence $\bigcup_{g\neq I} F(g) \subsetneq V$. Choose $v_0 \in V \backslash \left(\bigcup_{g\neq I} F(g) \cup \bigcup_{g\in G} V(a_g) \cup \bigcup_{g \in G} V(b_g)\right)$. Then $G \to G\cdot v_0$ is an embedding. Fix $g_0 \in G$. Find a polynomial p_{g_0} such that

(1) $p_{g_0}(\rho(g_0)^{-1}(v_0)) = 1$

(2) $p_{g_0}(v) = 0, v \in G \cdot v_0 \setminus \{\rho(g_0)^{-1}(v_0)\}$.

Now $0 = \sum_{g\in G} q_g(v_0)\widetilde{\rho}(g)p_{g_0}(v_0) = \sum_{g \in G} q_g(v_0) p_{g_0}(\rho(g)^{-1}v_0) = q_{g_0}(v_0) \neq 0$. This is a contradiction. $\square$

LEMMA 3.8. $A_{w_0} p_{w_0} = 1$.

Proof. Take a reduced decomposition $w_0 = s_{i_1} \cdots s_{i_N}$. Then it is standard that $\{\alpha_{i_1}, s_{i_1}\alpha_{i_2}, \cdots, s_{i_1} \cdots s_{i_{N-1}}\alpha_{i_N}\}$ is an enumeration of the positive roots. For $q \in \mathcal{F}$, let M_q be the operation of multiplication by q. Write $A_{w_0} = M_{\alpha_{i_1}^{-1}}(I - s_{i_1}) \circ \cdots \circ M_{\alpha_{i_N}^{-1}}(I - s_{i_N})$. It is then easy to see that for any $p \in \mathcal{R}$ we may write (see the next paragraph)

$$A_{w_0} p = \sum_{w\in W} q_w w \cdot p. \tag{7}$$

Moreover we have

$$q_{w_0} = (-1)^N \frac{1}{d} \tag{8}$$

Let s_i be a simple reflection. Since $\ell(s_i w_0) < \ell(w_0)$ we have, by Proposition 3.2, $A_{s_i} A_{w_0} p = 0$. From this we see that for all $p \in \mathcal{R}$ we have $\sum_w q_w w \cdot p = \sum_w (s_i \cdot q_w)(s_i w \cdot p)$. Apply Lemma 3.7 to conclude

$$s_i \cdot q_{s_i w} = q_w. \tag{9}$$

Combining this with (8) we obtain

$$q_w = (-1)^{\ell(w)+N} q_{w_0} = (-1)^{\ell(w)} \frac{1}{d}.$$

Hence by (7) we obtain

$$A_{w_0} p_{w_0} = \sum_{w \in W} (-1)^{\ell(w)} \frac{1}{d} w \cdot p_{w_0} = 1.$$

□

Now we can complete the proof of Proposition 3.6. From $dim(G/B) = N$ we deduce that the N-th graded component of $S(V^*)/I_{\mathbb{R}}$ is a one-dimensional vector space over $\mathbb{R}$ with basis $\{\epsilon_{w_0}\}$. We will now prove that d (or p_{w_0}) is another basis element.

We prove that $d \notin I_{\mathbb{R}}$. Suppose $d \in I_{\mathbb{R}}$. Then $d = \sum_i f_i g_i$ where $g_i \in S(V^*)^W$ are homogeneous with zero constant term. By applying the alternator $\sum_w (-1)^{\ell(w)} w$ to both sides we see that we may assume that the f_i are antiinvariant elements of $S(V^*)$. But any antiinvariant element vanishes on all the root hyperplanes and consequently is divisible by d. We conclude that all the f_i's are zero. This implies that $d = 0$, a contradiction.

Since d has degree N we find that d (and hence p_{w_0}) is another basis vector for the N-graded component of $S(V^*)/I_{\mathbb{R}}$. Hence there exists $c \in \mathbb{R}$ such that $p_{w_0} = c\epsilon_{w_0} \pmod{I_{\mathbb{R}}}$. But, by Proposition 3.5, we have $A_{w_0} \epsilon_{w_0} = \epsilon_1 = 1$. Hence $c = A_{w_0} p_{w_0}$. Thus, by Lemma 3.8, we have $c = 1$ and Proposition 3.6 is proved.

We obtain as a first consequence the following

LEMMA 3.9. *The Weyl group W acts on the top graded component (of degree $\ell(w_0)$) of the graded ring $S(V)/I_{\mathbb{R}}$ by the sign representation.*

We next define p_w for a general w. Express w as a left segment of w_0. Precisely we find v such that $w_0 = wv$ and $\ell(w_0) = \ell(w) + \ell(v)$. Then we define

$$p_w = A_v p_{w_0} = A_{w^{-1} w_0} p_{w_0}.$$

As a consequence of Propositions 3.3, 3.6 and 3.5, we get the desired realization of the duals of the Schubert homology classes in the Borel model.

THEOREM 3.10.

$$\beta(p_w) = \epsilon_w.$$

We will need the following lemma.

LEMMA 3.11. *Suppose that we have a factorization $w = w_1 v_1$ with $\ell(w) = \ell(w_1) + \ell(v_1)$. Then*

$$A_{v_1} p_w = p_{w_1}.$$

Proof. Suppose we have realized w as a left segment of w_0 by $w_0 = wv$. Then $w_0 = w_1(v_1 v)$ realizes w_1 as a left segment of w_0 with $w_1^{-1} w_0 = v_1 v$ and $A_{v_1 v} = A_{v_1} \circ A_v$ by Proposition 3.2. □

4. The inequalities for the rank 3 root systems

In this section we describe the inequalities for the rank 3 root systems A_3, C_3 and B_3. Since there are many inequalities in each case we will give only a system of representatives modulo the action of the symmetric group S_3 and leave to the reader the task of symmetrizing the inequalities. We note that the polytopes for C_3 and B_3 are isomorphic by Theorem 1.8, though we will see below that the systems are different. We will also see that there are many trivially redundant inequalities labelled as $(*)$.

In each of the three cases there are 3 standard maximal parabolics, hence the system breaks up into three subsystems. We let r,s and t be the simple reflections associated to the nodes from left to right of the Dynkin diagram following the Bourbaki convention (so t corresponds to the long simple root in the case of C_3 and the short simple root in the case of B_3). In what follows w_0 will denote the longest element in W and w_0^P will denote the longest element in W^P. Let λ be a fundamental weight. We will also use the notation $X_{w\lambda}$ for the Schubert cycle X_w^P with $w \in W^P$ and P the standard maximal parabolic subgroup associated to λ.

In general the Weyl chamber Δ is a simplicial cone, hence in rank 3 it is defined by 3 linear inequalities. These inequalities will contribute 9 inequalities in (v_1, v_2, v_3) after symmetrization.

4.1. The inequalities for A_3. In this case the quotients G/P for maximal parabolics P are Grassmannians and the cohomology rings are well-known. In particular, all the structure constants are 1 or 0. We will merely record the inequalities for the three subsystems. The Weyl chamber Δ is given by

$$\Delta = \{(x, y, z, w) : x + y + z + w = 0, x \geq y \geq z \geq w\}.$$

We give below the inequalities in terms of triples $(v_1, v_2, v_3) \in \Delta^3$ with $v_i = (x_i, y_i, z_i, w_i), i = 1, 2, 3$. But, to get the full set of inequalities, we need to symmetrize these with respect to the action of S_3 diagonally permuting the variables $x_1, x_2, x_3; y_1, y_2, y_3; z_1, z_2, z_3; w_1, w_2, w_3$.

4.1.1. *The subsystem associated to $H^*(G/P_1)$.* In this case the quotient G/P_1 is $\mathbb{CP}^3$. Thus, we obtain the subsystem (before symmetrization):

$$x_1 + w_2 + w_3 \leq 0$$
$$y_1 + z_2 + w_3 \leq 0$$
$$z_1 + z_2 + z_3 \leq 0.$$

Hence there are 10 *inequalities after symmetrization.*

4.1.2. *The subsystem associated to* $H^*(G/P_2)$. In this case the quotient G/P_2 is the Grassmannian of 2- planes in $\mathbb{C}^4$. We obtain the subsystem (before symmetrization):

$$x_1 + y_1 + z_2 + w_2 + z_3 + w_3 \le 0$$
$$x_1 + z_1 + y_2 + w_2 + z_3 + w_3 \le 0$$
$$x_1 + w_1 + x_2 + w_2 + z_3 + w_3 \le 0$$
$$y_1 + z_1 + y_2 + z_2 + z_3 + w_3 \le 0$$

$$x_1 + w_1 + y_2 + w_2 + y_3 + w_3 \le 0$$
$$y_1 + z_1 + y_2 + w_2 + y_3 + w_3 \le 0$$

Hence, there are 21 *inequalities after symmetrization.*

4.1.3. *The subsystem associated to* $H^*(G/P_3)$. This subsystem is dual to the first subsystem. In this case the quotient is the Grassmannian of 3-planes in $\mathbb{C}^4$. We obtain the subsystem (dual to the first)

$$x_1 + y_1 + z_1 + y_2 + z_2 + w_2 + y_3 + z_3 + w_3 \le 0$$
$$x_1 + y_1 + w_1 + x_2 + z_2 + w_2 + y_3 + z_3 + w_3 \le 0$$

$$x_1 + z_1 + w_1 + x_2 + z_2 + w_2 + x_3 + z_3 + w_3 \le 0$$

Hence again there are 10 *inequalities after symmetrization.*

Thus there are altogether $50 = 41 + 9$ *inequalities defining* $D_3(A_3)$ *and the system is minimal,* [**KTW**], *where* 9 *in* $41 + 9$ *accounts for* 9 *inequalities defining* Δ^3 *in* $\mathfrak{a}^3$.

4.2. The inequalities for C_3. In this subsection we take simply-connected G of type C_3, i.e., $G = \mathrm{Sp}\ (6)$.

We note that the Weyl chamber Δ is given by the triples x, y, z of real numbers satisfying

$$x \ge y \ge z \ge 0.$$

Here x, y, z are the coordinates relative to the standard basis $\epsilon_1, \epsilon_2, \epsilon_3$ in the notation of [**Bo**], pg. 254 - 255. The inequalities will now be in terms of $(v_1, v_2, v_3) \in \Delta^3$ with $v_i = (x_i, y_i, z_i), i = 1, 2, 3$. We will need to symmetrize the inequalities with respect to the action of S_3 diagonally permuting the variables $x_1, x_2, x_3; y_1, y_2, y_3; z_1, z_2, z_3$.

In what follows one will often need to verify that an expression of an element $w \in W$ in the generators r, s, t is of minimal length. One can do this by finding the word as a connected subword of a minimal length expression of the longest word w_0. Thus one needs a plentiful supply of such expressions. The first part of the following lemma follows from [**Bo**], Proposition 1.2, pg. 121. Also, in the reflection representation, the coordinate sign changes are given by $rstsr$ (first coordinate), sts (second coordinate) and t (third coordinate) and $w_0 = -1$. From this the second part of the following lemma follows easily.

LEMMA 4.1. (1) *Let* uvw *be a product of the simple generators (in any order). Then* $(uvw)^3 = w_0$.

(2) *A product of the three sign-changes* $rstsr, sts, t$ *in any order is equal to* w_0.

We will also need the following proposition.

PROPOSITION 4.2.
$$p_{w_0} = x^4y^2(xyz) \, mod \, I,$$
where I *is the ideal generated by the* W*-invariant polynomials in* $\mathbb{Z}[x, y, z]$ *with zero constant term.*

Proof. By the definition of p_{w_0} (cf. equation (6)), we have
$$p_{w_0} = \frac{(x^2 - y^2)(x^2 - z^2)(y^2 - z^2)(2x)(2y)(2z)}{48}.$$
In the ring $\mathbb{Z}[X, Y, Z]$ we have $(X-Y)(X-Z)(Y-Z) \equiv 6X^2Y \bmod J$, where J is the ideal of $\mathbb{Z}[X, Y, Z]$ generated by the symmetric polynomials with zero constant term. Now the proposition follows from the above by taking $X = x^2, Y = y^2, Z = z^2$. □

We will also need the following simple fact about the Weyl group W of C_3. Let $w \in W$ and choose a reduced decomposition of w. Let $n(w, t)$ be the number of times t appears in this decomposition.

LEMMA 4.3. $n(w, t)$ *is independent of the reduced decomposition of* w.

Proof. From the Coxeter group property of W, any one reduced decomposition of $w \in W$ can be obtained from another by using the Artin (i.e. generalized braid) relations, that is by replacing rsr by srs or rt by tr or $stst$ by $tsts$. None of these change the number of t's. □

4.2.1. *The subsystem associated to* $H^*(G/P_1)$. We have $G/P_1 \cong \mathbb{CP}^5$ so all structure constants are 1 and we get one inequality for each ordered partition of 5. We note that W_{P_1} is the group generated by s and t.

We leave the proof of the following lemma to the reader

LEMMA 4.4. $W^{P_1} = \{e, r, sr, tsr, stsr, rstsr\}$.

We will abbreviate the classes ϵ_w for $w \in W^{P_1}$ by a_i, $i = 1, 2, 3, 4, 5$ according to the following table. Moreover, in the following table we list the elements w of W^{P_1} , their lengths, the *maximally singular weight* $\lambda_w := w\omega_1$ associated to the Schubert cycle $X_{\lambda_w} = X_w^{P_1}$ and the notation for the cohomology class $PD(X_{\lambda_w})$, the cohomology class that is Poincaré dual to X_{λ_w}.

w	$\ell(w)$	λ_w	ϵ_w	$PD(X_{\lambda_w})$
e	0	$(1, 0, 0)$	1	a_5
r	1	$(0, 1, 0)$	a_1	a_4
sr	2	$(0, 0, 1)$	a_2	a_3
tsr	3	$(0, 0, -1)$	a_3	a_2
$stsr$	4	$(0, -1, 0)$	a_4	a_1
$rstsr$	5	$(-1, 0, 0)$	a_5	1

We now give the corresponding subsytem leaving to the reader the task of symmetrizing the inequalities below. We label each inequality with the ordered partition of 5 that it corresponds to. For example the label $(3, 2, 0)$ means the inequality corresponds to the formula $a_3 \cdot a_2 \cdot 1 = a_5 =$ top class in $H^*(G/P_1)$.

We then refer to the above chart to obtain $PD(a_3) = X^{P_1}_{sr} = X_{(0,0,1)}, PD(a_2) = X^{P_1}_{tsr} = X_{(0,0,-1)}$ and $PD(1) = X^{P_1}_{rstsr} = X_{(-1,0,0)}$. We have

$$X_{(0,0,1)} \cdot X_{(0,0,-1)} \cdot X_{(-1,0,0)} = [pt].$$

Applying the linear functionals $(0,0,1), (0,0,-1)$ and $(-1,0,0)$ to v_1, v_2, v_3 respectively we get the inequality $z_1 - z_2 - x_3 \leq 0$.

So the system of inequalities (before symmetrization) is given by:

$$\begin{array}{rll} & x_1 \leq x_2 + x_3 & (5,0,0) \\ & y_1 \leq y_2 + x_3 & (4,1,0) \\ & z_1 \leq z_2 + x_3 & (3,2,0) \\ \\ & z_1 \leq y_2 + y_3 & (3,1,1) \\ (*) & -z_1 - z_2 - y_3 \leq 0 & (2,2,1) \end{array}$$

The three inequalities in this subsystem generated by $(*)$ corresponding to the ordered partition $(2,2,1)$ are trivially redundant and do not occur in the system for B_3.

Thus, there are 21 *inequalities in this subsystem after symmetrization which includes* 3 *trivially redundant inequalities. There are no other redundant inequalities in this subsytem.*

4.2.2. *The subsystem associated to* $H^*(G/P_2)$. The space G/P_2 is the space of totally-isotropic 2-planes. The group W_{P_2} is generated by the commuting simple reflections r and t. We leave the proof of the following lemma to the reader.

LEMMA 4.5. $W^{P_2} = \{e, s, rs, ts, rts, sts, srts, rsts, tsrts, rstrs, rtsrts, srtsrts\}$.

We will abbreviate the classes ϵ_w for $w \in W^{P_2}$ to a_i or a'_i or a''_i as indicated in the next table. For the benefit of the reader, we also list for the elements w in W^{P_2}, the corresponding maximally singular weight $w \cdot \omega_2 = w \cdot (1,1,0)$ and the Poincaré dual class $PD(X^{P_2}_w)$.

w	λ_w	$\ell(w)$	ϵ_w	$\mathrm{PD}(X_{\lambda_w})$
e	$(1,1,0)$	0	1	a_7
s	$(1,0,1)$	1	a_1	a_6
rs	$(0,1,1)$	2	a'_2	a'_5
ts	$(1,0,-1)$	2	a''_2	a''_5
rts	$(0,1,-1)$	3	a'_3	a'_4
sts	$(1,-1,0)$	3	a''_3	a''_4
$srts$	$(0,-1,1)$	4	a'_4	a'_3
$rsts$	$(-1,1,0)$	4	a''_4	a''_3
$tsrts$	$(0,-1,-1)$	5	a'_5	a'_2
$rstrs$	$(-1,0,1)$	5	a''_5	a''_2
$rtsrts$	$(-1,0,-1)$	6	a_6	a_1
$srtsrts$	$(-1,-1,0)$	7	a_7	1

We list the polynomials $p_w \pmod{I}$ in the next table, which is obtained by applying Lemma 3.11 and observing that both of $x^4y^2 + x^2y^4$ and $x^4 + y^4 + x^2y^2$ belong to I.

In the following table it will be convenient to use g_n to denote the finite geometric series

$$g_n = \sum_{i=0}^{n} x^i y^{n-i}.$$

w	v	$\tilde{w} = wv$	$p_w (\mathrm{mod}\, I)$
$srtsrts$	rt	w_0	$(xy)^3(x+y)$
$rtsrts$	rts	w_0	$(xy)^3$
$tsrts$	$rtsr$	w_0	$xy(x+y)(xy+z^2)$
$rstrs$	$trst$	w_0	xyg_3
$srts$	$rtsrt$	w_0	$(xy)^2$
$rsts$	$rstst$	w_0	g_4
rts	rts	$rtsrts$	$xy(x+y)$
sts	$trstsr$	w_0	g_3
rs	ts	$rsts$	xy
ts	trs	$tstrs$	g_2
s	ts	sts	$x+y$

Now computing the products of p_w (mod I) and using the Chevalley formula Lemma 3.4, we obtain the following.

THEOREM 4.6. *The cohomology ring $H^*(G/P_2)$ is given by the following table.*

$H^*(G/P_2)$	a_1	a_2'	a_2''	a_3'	a_3''
a_1	$a_2'+a_2''$	a_3'	$a_3'+a_3''$	$2a_4'+a_4''$	$a_4'+2a_4''$
a_2'		a_4'	$a_4'+a_4''$	$a_5'+a_5''$	a_5''
a_2''			$2a_4'+2a_4''$	$a_5'+2a_5''$	$a_5'+2a_5''$
a_3'				$2a_6$	a_6
a_3''					$2a_6$

$H^*(G/P_2)$	a_4'	a_4''	a_5'	a_5''	a_6	a_7
a_1	$a_5'+a_5''$	a_5''	a_6	a_6	a_7	0
a_2'	a_6	0	a_7	0	0	0
a_2''	a_6	a_6	0	a_7	0	0
a_3'	a_7	0	0	0	0	0
a_3''	0	a_7	0	0	0	0

We now write down the subsystem (before symmetrization).

$$
\begin{array}{lll}
 & x_1 + y_1 \leq x_2 + y_2 + x_3 + y_3 & (7,0,0) \\
 & x_1 + z_1 \leq x_2 + z_2 + x_3 + y_3 & (6,1,0) \\
 & y_1 + z_1 \leq y_2 + z_2 + x_3 + y_3 & (5',2',0) \\
 & x_1 + z_2 \leq z_1 + x_2 + x_3 + y_3 & (5'',2'',0) \\
 & y_1 + z_2 \leq z_1 + y_2 + x_3 + y_3 & (4',3',0) \\
 & x_1 + y_2 \leq y_1 + x_2 + x_3 + y_3 & (4'',3'',0) \\
 & & \\
 & y_1 + z_1 \leq x_2 + z_2 + x_3 + z_3 & (5',1,1) \\
(**) & x_1 \leq z_1 + x_2 + z_2 + x_3 + z_3 & (5'',1,1) \\
(**) & y_1 \leq z_1 + y_2 + z_2 + x_3 + z_3 & (4',2',1) \\
 & y_1 + z_2 \leq z_1 + x_2 + x_3 + z_3 & (4',2'',1) \\
 & x_1 + z_2 \leq y_1 + x_2 + x_3 + z_3 & (4'',2'',1,) \\
(*) & y_1 + z_2 \leq x_1 + y_2 + x_3 + z_3 & (3'',3',1) \\
 & & \\
(*) & z_1 \leq y_1 + y_2 + z_2 + y_3 + z_3 & (3',2',2') \\
(*) & z_1 + z_2 \leq y_1 + x_2 + y_3 + z_3 & (3',2'',2') \\
(*) & y_1 + z_2 \leq x_1 + x_2 + y_3 + z_3 & (3'',2'',2')
\end{array}
$$

We explain how an inequality corresponds to a decorated ordered partition by the example of $(4',2'',1)$. The decorated partition corresponds to the three-fold product $a_4' \cdot a_2'' \cdot a_1 = a_7 =$ top class in $H^*(G/P_2)$. Taking the Poincaré dual cycles, the above product corresponds to the intersection product $X_{rts}^{P_2} \cdot X_{rstrs}^{P_2} \cdot X_{rtsrts}^{P_2} =$ $[pt]$. Using the correspondence between Weyl group elements in W^{P_2} and linear functionals arising from the Weyl group orbit of $(1,1,0)$, we find that the three Weyl group elements indexing the cycles in the intersection product correspond to the linear functionals $(0,1,-1),(-1,0,1),(-1,0,-1)$. Applying these linear functionals to v_1, v_2, v_3 respectively and collecting terms one obtains the inequality $y_1 - z_1 - x_2 + z_2 - x_3 - z_3 \leq 0$ or equivalently $y_1 + z_2 \leq z_1 + x_2 + x_3 + z_3$.

The inequalities in the subsystem labelled $(*)$ corresponding to the ordered partitions $(3'',3',1),(3',2',2'),(3',2'',2'),(3'',2'',2')$ are trivially redundant. They generate (after symmetrization) respectively 6,3,6,6 trivially redundant inequalities.

The inequalities corresponding to the ordered partitions $(5'',1,1)$, $(4',2',1)$ and $(3',2',2')$ do not occur in the system for B_3. Consequently they too must be redundant. We now check this directly.

First the three inequalities corresponding to the decorated ordered partition $(3',2',2')$ are trivially redundant. In order to check that the three inequalities corresponding to $(5'',1,1)$ are redundant, we observe that we have $x_1 \leq x_2 + x_3$ from the first subsystem (corresponding to G/P_1). As a consequence we have $x_1 \leq x_2 + x_3 + z_1 + z_2 + z_3$. Finally to check that the six inequalities corresponding to $(4',2',1)$ are redundant, we observe that we have $y_1 \leq y_2 + x_3$ from the first subsystem again. Hence $y_1 \leq y_2 + x_3 + z_1 + z_2 + z_3$.

This subsystem (corresponding to G/P_2) after symmetrization consists of 78 *inequalities of which* 21 *are trivially redundant (marked by* $(*)$*). There are* 9 *more redundant inequalities marked by* $(**)$*. These* 9 *inequalities do not occur in the system for* B_3*.*

4.2.3. *The subsystem associated to $H^*(G/P_3)$.* The space G/P_3 is the space of totally-isotropic 3-planes, i.e., the Lagrangian Grassmannian. The group W_{P_3} is generated by the simple reflections r and s. We leave the proof of the following lemma to the reader.

LEMMA 4.7.

$$W^{P_3} = \{e, t, st, rst, tst, trst, strst, tstrst\}.$$

We again abbreviate the classes ϵ_w to a_i or a_i' or a_i'' according to the following table.

w	$\ell(w)$	λ_w	ϵ_w	$PD(X_{\lambda_w})$
e	0	$(1,1,1)$	1	a_6
t	1	$(1,1,-1)$	a_1	a_5
st	2	$(1,-1,1)$	a_2	a_4
rst	3	$(-1,1,1)$	a_3'	a_3''
tst	3	$(1,-1,-1)$	a_3''	a_3'
$trst$	4	$(-1,1,-1)$	a_4	a_2
$strst$	5	$(-1,-1,1)$	a_5	a_1
$tstrst$	6	$(-1,-1,-1)$	a_6	1

We again give the formulas for $p_w \pmod I$. In the following table we define a symmetric cubic polynomial $f(x,y,z)$ by $f(x,y,z) = x^2y + xy^2 + x^2z + xz^2 + y^2z + yz^2$.

w	v	$\tilde{w} = wv$	$p_w (\mathrm{mod}\, I)$
$w_0^{P_3} = tstrst$	rsr	w_0	$(xyz)f$
$strst$	$rstr$	w_0	$(xyz)(xy+xz+yz)$
$trst$	$rstrs$	w_0	$(xyz)(x+y+z)$
tst	rst	$w_0^{P_3} = tstrst$	$f + xyz$
rst	$rstrst$	w_0	xyz
st	rst	$strst$	$xy+xz+yz$
t	st	tst	$x+y+z$

We now have the following table constructed by multiplying the polynomials p_w modulo I.

THEOREM 4.8. *The cohomology ring $H^*(G/P_3)$ is given by the following table.*

$H^*(G/P_3)$	a_1	a_2	a_3'	a_3''	a_4	a_5	a_6
a_1	$2a_2$	$2a_3' + a_3''$	a_4	$2a_4$	$2a_5$	a_6	0
a_2		$2a_4$	a_5	$2a_5$	a_6	0	0
a_3'			0	a_6	0	0	0
a_3''				0	0	0	0

Using the above tables, the reader can easily verify that we have the following inequalities (which have to be symmetrized).

$$x_1 + y_1 + z_1 \leq x_2 + y_2 + z_2 + x_3 + y_3 + z_3 \qquad (6,0,0)$$
$$x_1 + y_1 + z_2 \leq z_1 + x_2 + y_2 + x_3 + y_3 + z_3 \qquad (5,1,0)$$
$$x_1 + z_1 + y_2 \leq y_1 + x_2 + z_2 + x_3 + y_3 + z_3 \qquad (4,2,0)$$
$$x_1 + y_2 + z_2 \leq y_1 + z_1 + x_2 + x_3 + y_3 + z_3 \qquad (3',3'',0)$$

$$x_1 + y_2 + z_3 \leq y_1 + z_1 + x_2 + z_2 + x_3 + y_3 \qquad (3',2,1)$$

This gives that the subsystem corresponding to G/P_3 consists of 27 inequalities. None of them are trivially redundant.

The 27 inequalities above can be rewritten in a very simple way. Let $S = \sum_{i=1}^{3} x_i + y_i + z_i$. Then the 27 inequalities are just the inequalities

$$x_i + y_j + z_k \leq \frac{S}{2}, \; i,j,k = 1,2,3.$$

Thus finally we see that for C_3 there are $135 = 126 + 9$ inequalities of which 24 are trivially redundant (9 in $126+9$ coming from the inequalities defining Δ^3 inside $\mathfrak{a}^3$). There are 9 more redundant inequalities. These 9 inequalities do not occur in the system for B_3. Hence the subsystem for C_3 can be brought down to altogether 102 inequalities. Moreover, a computer calculation shows that the polyhedral cone $D_3(C_3)$ has exactly 102 faces and thus these 102 inequalities are irredundant.

4.3. The inequalities for B_3. In this subsection we take simply-connected G of type B_3, i.e., $G = \text{Spin }(7)$.

We note that the Weyl chamber Δ is given by triples x, y, z of real numbers satisfying

$$x \geq y \geq z \geq 0.$$

The inequalities will now be in terms of $(v_1, v_2, v_3) \in \Delta^3$ with $v_i = (x_i, y_i, z_i), i = 1,2,3$.

The Weyl groups for Sp(6) and Spin(7) are isomorphic. In fact, in the standard coordinates (x, y, z) they are identical. Hence the sets $W^{P_i}, i = 1,2,3$ will be identical. However the operators A_w and the polynomials p_w will be different (but proportional) as we now see. To distinguish, we will denote them by $p_w^{\mathrm{Sp}(6)}$ and $p_w^{\mathrm{Spin}(7)}$ respectively.

We will need the following proposition.

PROPOSITION 4.9.

$$p_{w_0}^{\mathrm{Spin}(7)} = \frac{x^4y^2(xyz)}{8} \; (\mathrm{mod}\, I).$$

Proof. By equation (6) we have

$$p_{w_0}^{\mathrm{Spin}(7)} = \frac{xyz(x^2-y^2)(x^2-z^2)(y^2-z^2)}{48}.$$

Hence the proposition follows from the corresponding result for Sp(6). □

The next lemma tells us how to read off the polynomials p_w for the case of Spin(7) from the corresponding polynomials for Sp(6). We will temporarily use the notation $A_w^{\mathrm{Spin}(7)}$ and $A_w^{\mathrm{Sp}(6)}$ for the Demazure-BGG operators A_w for the groups Spin(7) and Sp(6) respectively.

LEMMA 4.10. *Let $v \in W$ and let $n(v,t)$ be the number of times the simple reflection t occurs in some reduced decomposition of v. Then we have*

$$A_v^{\mathrm{Spin}(7)} = 2^{n(v,t)} A_v^{\mathrm{Sp}(6)}.$$

Proof. $A_t^{\mathrm{Spin}(7)} = 2\, A_t^{Sp(6)}$ and $A_r^{\mathrm{Spin}(7)} = A_r^{Sp(6)}$, $A_s^{\mathrm{Spin}(7)} = A_s^{Sp(6)}$. □

COROLLARY 4.11. *For any $w \in W$, $p_w^{\mathrm{Spin}(7)} = \frac{2^{n(w^{-1}w_0,t)}}{8}\, p_w^{\mathrm{Sp}(6)} = 2^{-n(w,t)}\, p_w^{\mathrm{Sp}(6)}$.*

Proof. Note that $n(w,t) = 8 - n(v,t)$, for $v = w^{-1}w_0$. □

4.3.1. *The subsystem associated to $H^*(G/P_1)$.* We note that W_{P_1} is the group generated by s and t. We have (since this is what we had for Sp(6))

LEMMA 4.12. $W^{P_1} = \{e, r, sr, tsr, stsr, rstsr\}$.

We have $G/P_1 \cong Q_5$, the smooth quadric hypersurface in $\mathbb{CP}^6$ so the inequalities will be parametrized by a subset of the ordered partitions of 5. We will abbreviate the classes ϵ_w to a_i according to the following table. In addition, we list the elements w of W^{P_1}, their lengths, the maximally singular weight λ_w associated to the Schubert cycle $X_w^{P_1} = X_{\lambda_w}$ and the notation for the cohomology class $PD(X_{\lambda_w})$, the cohomology class that is Poincaré dual to X_{λ_w}.

w	$\ell(w)$	λ_w	ϵ_w	$PD(X_{\lambda_w})$
e	0	$(1,0,0)$	1	a_5
r	1	$(0,1,0)$	a_1	a_4
sr	2	$(0,0,1)$	a_2	a_3
tsr	3	$(0,0,-1)$	a_3	a_2
$stsr$	4	$(0,-1,0)$	a_4	a_1
$rstsr$	5	$(-1,0,0)$	a_5	1

The following theorem follows easily from Corollary 4.11 by using the corresponding theorem for Sp(6).

THEOREM 4.13. *The multiplication table for $H^*(G/P_1)$ is given by the following table.*

$H^*(G/P_1)$	a_1	a_2	a_3	a_4	a_5
a_1	a_2	$2a_3$	a_4	a_5	0
a_2		$2a_4$	a_5	0	0

We now give the corresponding subsystem, leaving to the reader the task of symmetrizing the inequalities below.

$$\begin{aligned} x_1 &\le x_2 + x_3 &\quad& (5,0,0)\\ y_1 &\le y_2 + x_3 && (4,1,0)\\ z_1 &\le z_2 + x_3 && (3,2,0)\\ z_1 &\le y_2 + y_3 && (3,1,1) \end{aligned}$$

After symmetrizing there are 18 *inequalities, none are trivially redundant.*

4.3.2. *The subsystem associated to $H^*(G/P_2)$.* The space G/P_2 is the space of totally-isotropic 2-planes. We have that W_{P_2} is the group generated by the commuting simple reflections r and t. We have, as for Sp(6),

LEMMA 4.14. $W^{P_2} = \{e, s, rs, ts, rts, sts, srts, rsts, tsrts, rstrs, rtsrts, srtsrts\}$.

We will abbreviate the classes ϵ_w by b_i or b'_i or b''_i according to the following table.

The following table follows easily from the corresponding tables for Sp(6) and Corollary 4.11.

w	λ_w	$\ell(w)$	ϵ_w	$p_w \pmod I$	$PD(X_{\lambda_w})$
e	$(1,1,0)$	0	1	1	b_7
s	$(1,0,1)$	1	b_1	$x+y$	b_6
rs	$(0,1,1)$	2	b'_2	xy	b'_5
ts	$(1,0,-1)$	2	b''_2	$1/2\ g_2$	b''_5
rts	$(0,1,-1)$	3	b'_3	$1/2\ xy(x+y)$	b'_4
sts	$(1,-1,0)$	3	b''_3	$1/2\ g_3$	b''_4
$srts$	$(0,-1,1)$	4	b'_4	$1/2\ (xy)^2$	b'_3
$rsts$	$(-1,1,0)$	4	b''_4	$1/2\ g_4$	b''_3
$tsrts$	$(0,-1,-1)$	5	b'_5	$1/4\ xy(x+y)(xy+z^2)$	b'_2
$rstrs$	$(-1,0,1)$	5	b''_5	$1/2\ xyg_3$	b''_2
$rtsrts$	$(-1,0,-1)$	6	b_6	$1/4\ (xy)^3$	b_1
$srtsrts$	$(-1,-1,0)$	7	b_7	$1/4\ (xy)^3(x+y)$	1

From these formulae one can easily construct the multiplication table for $H^*(G/P_2)$.

THEOREM 4.15. *The cohomology ring $H^*(G/P_2)$ is given by the following table.*

$H^*(G/P_2)$	b_1	b'_2	b''_2	b'_3	b''_3
b_1	$b'_2+2b''_2$	$2b'_3$	$b'_3+b''_3$	$2b'_4+b''_4$	$b'_4+2b''_4$
b'_2		$2b'_4$	$b'_4+b''_4$	$2b'_5+b''_5$	b''_5
b''_2			$b'_4+b''_4$	$b'_5+b''_5$	$b'_5+b''_5$
b'_3				$2b_6$	b_6
b''_3					$2b_6$

$H^*(G/P_2)$	b'_4	b''_4	b'_5	b''_5	b_6	b_7
b_1	$2b'_5+b''_5$	b''_5	b_6	$2b_6$	b_7	0
b'_2	$2b_6$	0	b_7	0	0	0
b''_2	b_6	b_6	0	b_7	0	0
b'_3	b_7	0	0	0	0	0
b''_3	0	b_7	0	0	0	0

We then read off the inequalities.

$$
\begin{array}{lll}
 & x_1 + y_1 \le x_2 + y_2 + x_3 + y_3 & (7,0,0) \\
 & x_1 + z_1 \le x_2 + z_2 + x_3 + y_3 & (6,1,0) \\
 & y_1 + z_1 \le y_2 + z_2 + x_3 + y_3 & (5',2',0) \\
 & x_1 + z_2 \le z_1 + x_2 + x_3 + y_3 & (5'',2'',0) \\
 & y_1 + z_2 \le z_1 + y_2 + x_3 + y_3 & (4',3',0) \\
 & x_1 + y_2 \le y_1 + x_2 + x_3 + y_3 & (4'',3'',0) \\
 & & \\
 & y_1 + z_1 \le x_2 + z_2 + x_3 + z_3 & (5',1,1) \\
 & y_1 + z_2 \le z_1 + x_2 + x_3 + z_3 & (4',2'',1,) \\
 & x_1 + z_2 \le y_1 + x_2 + x_3 + z_3 & (4''2'',1) \\
(*) & y_1 + z_2 \le x_1 + y_2 + x_3 + z_3 & (3'',3',1) \\
 & & \\
(*) & y_1 + z_2 \le x_1 + x_2 + y_3 + z_3 & (3'',2'',2') \\
(*) & z_1 + z_2 \le y_1 + x_2 + y_3 + z_3 & (3',2'',2') \\
 & & \\
(*) & z_1 + z_2 + z_3 \le y_1 + x_2 + x_3 & (3',2'',2'') \\
(*) & y_1 + z_2 + z_3 \le x_1 + x_2 + x_3 & (3'',2'',2'')
\end{array}
$$

The inequalities corresponding to the ordered partitions $(3'',3',1)$ $(3'',2'',2')$, $(3',2'',2')$, $(3',2'',2'')$ and $(3'',2'',2'')$ are trivially redundant. After symmetrizing they give rise to 24 trivially redundant inequalities. The trivially redundant inequalities corresponding to the decorated ordered partitions $(3',2'',2'')$ and $(3'',2'',2'')$ do not occur in the system for Sp(6).

There are 72 *inequalities after symmetrizing, of which* 24 *are trivially redundant.*

4.3.3. *The subsystem associated to* $H^*(G/P_3)$. The space G/P_3 is the space of totally-isotropic 3-planes in $\mathbb{C}^7$. The group W_{P_3} is generated by the simple reflections r and s. We have the following from the corresponding result for Sp(6).

LEMMA 4.16.

$$W^{P_3} = \{e, t, st, rst, tst, trst, strst, tstrst\}.$$

We again abbreviate the classes ϵ_w to b_i's or b_i''s or b_i'''s according to the following table.

w	$\ell(w)$	λ_w	ϵ_w	$PD(X_{\lambda_w})$
e	0	$(1,1,1)$	1	b_6
t	1	$(1,1,-1)$	b_1	b_5
st	2	$(1,-1,1)$	b_2	b_4
rst	3	$(-1,1,1)$	b_3'	b_3''
tst	3	$(1,-1,-1)$	b_3''	b_3'
$trst$	4	$(-1,1,-1)$	b_4	b_2
$strst$	5	$(-1,-1,1)$	b_5	b_1
$tstrst$	6	$(-1,-1,-1)$	b_6	1

THEOREM 4.17. *The cohomology ring is given by the following table.*

$H^*(G/P_3)$	b_1	b_2	b_3'	b_3''	b_4	b_5	b_6
b_1	b_2	$b_3'+b_3''$	b_4	b_4	b_5	b_6	0
b_2		$2b_4$	b_5	b_5	b_6	0	0
b_3'			0	b_6	0	0	0
b_3''				0	0	0	0

In this case we find the following subsystem of linear inequalities.

$$x_1+y_1+z_1 \le x_2+y_2+z_2+x_3+y_3+z_3 \qquad (6,0,0)$$
$$x_1+y_1+z_2 \le z_1+x_2+y_2+x_3+y_3+z_3 \qquad (5,1,0)$$
$$x_1+z_1+y_2 \le y_1+x_2+z_2+x_3+y_3+z_3 \qquad (4,2,0)$$
$$x_1+y_2+z_2 \le y_1+z_1+x_2+x_3+y_3+z_3 \qquad (3',3'',0)$$

$$(**) \qquad x_1+z_1+z_2+z_3 \le y_1+x_2+y_2+x_3+y_3 \qquad (4,1,1)$$
$$x_1+y_2+z_3 \le y_1+z_1+x_2+z_2+x_3+y_3 \qquad (3',2,1)$$
$$(**) \qquad y_1+z_1+y_2+z_3 \le x_1+x_2+z_2+x_3+y_3 \qquad (3'',2,1)$$

After symmetrizing there are 36 *inequalities. None are trivially redundant. However the* 9 *inequalities corresponding to the ordered partitions* $(4,1,1)$ *and* $(3'',2,1)$ *and marked by (**) above do not occur for* $Sp(6)$ *and consequently they must be redundant.*

We now check this directly.

In order to see that the 3 inequalities corresponding to $(4,1,1)$ are redundant, we observe (from the first subsystem correspoding to G/P_1) that $x_1 \le x_2+x_3$. Furthermore, we have the inequalities for Δ given by $z_i \le y_i, 1 \le i \le 3$. Hence $x_1+z_1+z_2+z_3 \le x_2+x_3+y_1+y_2+y_3$. As for the 6 inequalities corresponding to $(3'',2,1)$, we have (from the first subsystem) $z_1 \le z_2+x_3$ and the inequalities (for Δ) $y_1 \le x_1$, $y_2 \le x_2$, and $z_3 \le y_3$. Hence $z_1+y_1+y_2+z_3 \le z_2+x_3+x_1+x_2+y_3$.

To summarize, for B_3*, there are altogether* $135 = 126+9$ *inequalities (including* 9 *needed to define* Δ^3 *in* $\mathfrak{a}^3$*) of which* 24 *are trivially redundant and there are* 9 *more redundant inequalities. These* 9 *inequalities do no occur in the system for* $\mathrm{Sp}(6)$*. Hence the subsystem for* B_3 *can be brought down to altogether* 102 *inequalities. Moreover, a computer calculation shows that the polyhedral cone* $D_3(B_3)$ *has exactly* 102 *faces and thus these* 102 *inequalities are irredundant.* (Of course, by Theorem 1.8, $D_3(B_3) = D_3(C_3)$.)

5. Generators of the cone

In the previous section we have described the irredundant system of linear inequalities defining the polyhedral cones $D_3(A_3)$ and $D_3(C_3) = D_3(B_3)$. We now give a system of generators for the cone $D_3(C_3)$. The components of each of the 51 generators are arranged in the order $x_1, x_2, x_3, y_1, y_2, y_3, z_1, z_2, z_3$ whereas the coordinates of the corresponding generators $(v_1, v_2, v_3) \in D_3(C_3)$ satisfy

$$v_i = (x_i, y_i, z_i), 1 \leq i \leq 3.$$

THEOREM 5.1. *The following* 51 *vectors are a set of generators of the polyhedral cone* $D_3(C_3) = D_3(B_3)$ *in the* 9 *dimensional space* $\mathfrak{a}^3$.

(1) 1 1 2 1 1 0 0 0 0
(2) 1 1 2 1 1 0 1 1 0
(3) 1 1 2 1 1 1 0 0 1
(4) 1 1 2 1 1 1 0 1 0
(5) 1 1 2 1 1 1 1 0 0
(6) 1 2 1 1 0 1 0 0 0
(7) 1 2 1 1 0 1 1 0 1
(8) 1 2 1 1 1 1 0 0 1
(9) 1 2 1 1 1 1 0 1 0
(10) 1 2 1 1 1 1 1 0 0
(11) 2 1 1 0 1 1 0 0 0
(12) 2 1 1 0 1 1 0 1 1
(13) 2 1 1 1 1 1 0 0 1
(14) 2 1 1 1 1 1 0 1 0
(15) 2 1 1 1 1 1 1 0 0
(16) 0 1 1 0 0 0 0 0 0
(17) 1 1 2 1 1 2 1 1 0
(18) 1 2 1 1 2 1 1 0 1
(19) 1 2 3 1 2 1 1 0 1
(20) 1 3 2 1 1 2 1 1 0
(21) 1 0 1 0 0 0 0 0 0
(22) 2 1 1 2 1 1 0 1 1
(23) 2 1 3 2 1 1 0 1 1
(24) 1 1 0 0 0 0 0 0 0
(25) 2 3 1 2 1 1 0 1 1
(26) 3 1 2 1 1 2 1 1 0
(27) 3 2 1 1 2 1 1 0 1
(28) 1 2 2 1 1 2 0 1 0
(29) 1 2 2 1 2 1 0 0 1
(30) 2 1 2 1 1 2 1 0 0
(31) 2 1 2 2 1 1 0 0 1
(32) 2 2 1 1 2 1 1 0 0
(33) 2 2 1 2 1 1 0 1 0
(34) 0 1 1 0 1 1 0 0 0
(35) 1 0 1 1 0 1 0 0 0
(36) 1 1 0 1 1 0 0 0 0
(37) 1 1 1 0 0 1 0 0 0
(38) 1 1 1 0 1 0 0 0 0
(39) 1 1 1 1 0 0 0 0 0
(40) 0 1 1 0 1 1 0 1 1
(41) 1 0 1 1 0 1 1 0 1
(42) 1 1 0 1 1 0 1 1 0
(43) 1 1 1 0 1 1 0 0 1
(44) 1 1 1 0 1 1 0 1 0
(45) 1 1 1 1 0 1 0 0 1
(46) 1 1 1 1 0 1 1 0 0
(47) 1 1 1 1 1 0 0 1 0
(48) 1 1 1 1 1 0 1 0 0
(49) 1 1 1 1 1 1 0 0 1
(50) 1 1 1 1 1 1 0 1 0
(51) 1 1 1 1 1 1 1 0 0

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