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# Frobenius splitting of Hilbert schemes of points on surfaces

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**Abstract.** Let $X$ be a quasiprojective smooth surface defined over an algebraically closed field of positive characteristic. In this note we show that if $X$ is Frobenius split then so is the Hilbert scheme $\mathrm{Hilb}^n(X)$ of $n$ points in $X$. In particular, we get the higher cohomology vanishing for ample line bundles on $\mathrm{Hilb}^n(X)$ when $X$ is projective and Frobenius split.

## Introduction

Let $X$ be a quasiprojective smooth surface defined over an algebraically closed field $k$ of positive characteristic $p$. For an integer $n \geq 1$, let $X^{(n)}$ be the $n$-th symmetric product of $X$ and let $X^{[n]}$ denote the Hilbert scheme of $n$ points in $X$ (parametrizing the zero dimensional closed subschemes of $X$ of length $n$). Recall that $X^{[n]}$ is smooth and there is a birational 'Hilbert-Chow' morphism $\psi : X^{[n]} \to X^{(n)}$, which to each zero dimensional closed subscheme in $X$ of length $n$ associates its support (with multiplicities). Let $X_*^{(n)}$ denote the open locus of $X^{(n)}$ corresponding to the set of n-tuples with at least $n-1$ distinct points and let $X_*^{[n]}$ denote its inverse image under $\psi$. We show that $\psi : X_*^{[n]} \to X_*^{(n)}$ is a crepant resolution if $p > 2$, in the sense that $X_*^{(n)}$ is Gorenstein such that its dualizing line bundle $\omega_{X_*^{(n)}}$ pulls back to the canonical bundle $\omega_{X_*^{[n]}}$ on $X_*^{[n]}$ under $\psi$ (cf. Theorem 1). In fact, if $p > n$, $\psi : X^{[n]} \to X^{(n)}$ itself is a crepant resolution (cf. Corollary 1). (This generalizes the corresponding result in char. 0 due to Beauville.) We make crucial use of our Theorem 1 to prove the following main result of this paper:

Let $X$ be as above and $p > 2$ . Assume that $X$ is Frobenius split. Then, for any $n \geq 1$, the Hilbert scheme $X^{[n]}$ is Frobenius split (cf. Theorem 2). In particular, if $X$, in addition, is projective and $L$ is an ample line bundle on $X^{[n]}$, then $L$ has vanishing higher cohomology (cf. Corollary 2).

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The contents of the paper are as follows: Section 1 is devoted to recalling the definition of Hilbert schemes, and Sect. 2 is devoted to the basic definitions of Frobenius splitting. Sects. 3 and 4 are devoted to proving that $\psi$ is a crepant resolution. We prove our main theorem (Theorem 2) in Sect. 5.

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## 1. Hilbert schemes of points

Let $X$ be a quasiprojective variety defined over an algebraically closed field $k$. Fix an integer $n \geq 1$. The Hilbert scheme $X^{[n]} = \mathrm{Hilb}^n(X)$ of $n$ points in $X$ parameterizes zero dimensional closed subschemes of $X$ of length $n$. The scheme $\mathrm{Hilb}^n(X)$ is quasiprojective and in fact projective when $X$ is so.

### *1.1. Symmetric products*

Let $X^n = X \times \cdots \times X$ denote the n-fold product of $X$, and let $S_n$ denote the symmetric group on $n$ letters. Then $S_n$ acts on $X^n$ by permuting the factors. As $X^n$ is quasiprojective and $S_n$ is finite, the geometric quotient of this action exists (cf. [10], Chap. III, Sect. 14). The quotient is denoted by $X^{(n)}$ and is called the $n$-th *symmetric product* of $X$. Let $\Phi : X^n \to X^{(n)}$ denote the quotient map.

Points in $X^{(n)}$ correspond to unordered tuples of (not necessarily distinct) $n$ points in $X$. The open subset of $X^{(n)}$ consisting of the tuples of $n$ distinct points is denoted by $X^{(n)}_{**}$. If $X$ is smooth, the variety $X^{(n)}$ is smooth along $X^{(n)}_{**}$ and moreover it is singular along the complement of $X^{(n)}_{**}$ if $\dim X \geq 2$ (cf. [3], Sect. 2). Clearly, the codimension of $X^{(n)} \setminus X^{(n)}_{**}$ in $X^{(n)}$ is equal to $\dim X$. Let $X^{(n)}_{*}$ denote the open locus of $X^{(n)}$ corresponding to the set of n-tuples with at least $n-1$ distinct points.

### *1.2. Hilbert-Chow morphism ([2], Sect. 2)*

Let $X^{[n]}_{red}$ denote the underlying reduced subscheme of $X^{[n]}$. The *Hilbert-Chow morphism* is the map $\psi : X^{[n]}_{red} \to X^{(n)}$, which to each zero dimensional closed subscheme in $X$ of length $n$ associates its support (with multiplicities). The Hilbert-Chow morphism is birational, being an isomorphism over the open set $X^{(n)}_{**}$.

When $X$ is a smooth surface, the Hilbert scheme $X^{[n]}$ is also smooth (in particular reduced). Hence, in this case, $\psi$ is a desingularization of the symmetric product $X^{(n)}$.

## 2. Frobenius splitting – basic definitions

Let $\pi : X \to \mathrm{Spec}(k)$ be a scheme defined over an algebraically closed field $k$ of positive characteristic $p$. The *absolute Frobenius morphism* on $X$ is the identity on point spaces and raising to the $p$-th power locally on functions. The absolute Frobenius morphism is *not* a morphism of $k$-schemes. Let $X'$ be the scheme obtained from $X$ by base change with the absolute Frobenius morphism on $\mathrm{Spec}(k)$, i.e., the underlying topological space of $X'$ is that of $X$ with the same structure sheaf $\mathcal{O}_X$ of rings, only the underlying $k$-algebra structure on $\mathcal{O}_{X'}$ is twisted as $\lambda \odot f = \lambda^{1/p} f$, for $\lambda \in k$ and $f \in \mathcal{O}_{X'}$. Using this description of $X'$, *the relative Frobenius morphism* $F : X \to X'$ is defined in the same way as the absolute Frobenius morphism and it is a morphism of $k$-schemes.

### *2.1. Frobenius splitting [8]*

Recall that a variety $X$ is called *Frobenius split* if the homomorphism $\mathcal{O}_{X'} \to F_*\mathcal{O}_X$ of $\mathcal{O}_{X'}$-modules is split. A homomorphism $\sigma : F_*\mathcal{O}_X \to \mathcal{O}_{X'}$ is a splitting of $\mathcal{O}_{X'} \to F_*\mathcal{O}_X$ (called a *Frobenius splitting*) if and only if $\sigma(1) = 1$.

When $X$ is a smooth variety with canonical bundle $\omega_X$, there is a natural isomorphism of $\mathcal{O}_{X'}$-modules ([8]):

$$F_*(\omega_X^{1-p}) \cong \mathcal{H}om_{\mathcal{O}_{X'}}(F_*\mathcal{O}_X, \mathcal{O}_{X'}).$$

In this way global sections of $\omega_X^{1-p}$ correspond to homomorphisms $F_*\mathcal{O}_X \to \mathcal{O}_{X'}$. A section of $\omega_X^{1-p}$ which corresponds to a Frobenius splitting in this way, is called a *splitting section*. Checking whether a section of $\omega_X^{1-p}$ is a splitting section can be done locally. More precisely, we have the following result.

**Lemma 1** ([8]). Let $U$ be an open dense subset of a smooth variety $X$. If a section $s \in \mathrm{H}^0(X, \omega_X^{1-p})$ restricts to a splitting section $s|_U \in \mathrm{H}^0(U, \omega_U^{1-p})$ on $U$, then $s$ is a splitting section.

An immediate consequence of the definition of Frobenius splitting is

**Lemma 2** ([8]). Let $X$ be a Frobenius split variety and let $L$ be a line bundle on $X$ such that $H^i(X, L^m) = 0$ for all large $m$ (for a fixed $i$). Then $H^i(X, L) = 0$.

*Proof.* This follows from the fact that if $X$ is Frobenius split and $L$ is a line bundle on $X$, then there is an injective map

$$\mathrm{H}^i(X, L) \hookrightarrow \mathrm{H}^i(X, L^p)$$

of abelian groups. □

In particular, Lemma 2.1 implies that ample line bundles on projective Frobenius split varieties have vanishing higher cohomology.

## 3. Ramification

In this section we have the following setup. By $H = \{e, \sigma\}$ we denote the group of order 2, acting nontrivially on a smooth quasiprojective variety $Y$ over a field of characteristic $p \neq 2$. We denote the quotient of $Y$ under this action by $X$ with the corresponding quotient map $\pi$. We will assume, in addition, that $X$ is smooth.

**Lemma 3.** *Assume that $Y = \mathrm{Spec}(B)$ and $X = \mathrm{Spec}(A)$ are affine. Let $E$ denote an irreducible subvariety of $X$ of codimension 1 corresponding to a prime ideal $\mathfrak{p}$ in $A$. Let $s \in A$ generate $\mathfrak{p}$ in the local ring $A_{\mathfrak{p}}$. If $\pi$ is bijective over $E$, then there exist a unique prime ideal $\mathfrak{p}'$ in $B$ over $\mathfrak{p}$. Furthermore, if $v$ denotes the valuation on the discrete valuation ring $B_{\mathfrak{p}'}$, then $v(s) = 2$.*

*Proof.* Assume that $\mathfrak{p}'$ and $\mathfrak{p}''$ are two different prime ideals in $B$ lying over the prime ideal $\mathfrak{p}$ in $A$. Let $E'$ and $E''$ denote the corresponding subvarieties of $Y$. Then $\sigma(E') = E''$. Choose $y \in E' \setminus E''$. Since $\sigma(y) \in E''$, $\sigma(y) \neq y$. But $y$ and $\sigma(y)$ both map to the same point in $E$, which is a contradiction. This proves the first part of the statement.

Let $E'$ denote the irreducible subvariety of $Y$ corresponding to the prime ideal $\mathfrak{p}'$ in $B$ lying over $\mathfrak{p}$. Let $t \in B$ be an element generating the maximal ideal in the local ring $B_{\mathfrak{p}'}$. Choose $b, b' \in B \setminus \mathfrak{p}'$ such that

$$s = t^{v(s)} \frac{b}{b'}.$$

As the product $\sigma(t)t$ is $H$-invariant, we can find $a, a' \in A \setminus \mathfrak{p}$ and a positive integer $l$ such that

$$\sigma(t)t = s^l \frac{a}{a'}.$$

Hence, we get

$$s^2 = \sigma(s)s = (\sigma(t)t)^{v(s)} \frac{\sigma(b)b}{\sigma(b')b'} = s^{lv(s)} (\frac{a}{a'})^{v(s)} \frac{\sigma(b)b}{\sigma(b')b'},$$

from which we obtain $lv(s) = 2$ (observe that $\sigma(b)b$ and $\sigma(b')b' \in A \setminus \mathfrak{p}$). Assume, if possible, that $v(s) = 1$. Then replacing $t$ by $tb\sigma(b')$ and $s$ by $sb'\sigma(b')$, we can assume that $s = t$. Take a nonzero $f \in B$ such that $\sigma(f) = -f$ (e.g. $f = g - \sigma(g)$ for an element $g$ not invariant under $H$). Since $H$ is acting trivially on $E'$, it acts trivially on $B/\mathfrak{p}'$ and hence $f$ belongs to $\mathfrak{p}'$ (here we are using the assumption that $p \neq 2$).

Write

$$f = t^{v(f)} \frac{c}{c'},$$

for $c, c' \in B \setminus \mathfrak{p}'$. Applying $\sigma$ we get,

$$\sigma(f) = t^{v(f)} \frac{\sigma(c)}{\sigma(c')}.$$

But, by choice, $\sigma(f) = -f$ and hence $\sigma(c\sigma(c')) = -(c\sigma(c'))$. In particular, $c\sigma(c') \in \mathfrak{p}'$. A contradiction, proving that $v(s) = 2$. □

**Proposition 1.** *Let $E$ be an irreducible reduced divisor of $X$ and assume that $\pi$ is bijective over $E$. Then there exist a unique irreducible reduced divisor $E'$ of $Y$ mapping onto $E$. Furthermore, $\pi^*(\mathcal{O}(E)) = \mathcal{O}(2E')$.*

*Proof.* That there exists a unique (reduced and irreducible) divisor $E'$ in $Y$ mapping onto $E$ follows from the corresponding local statement in Lemma 3. Let $s$ be a section of $\mathcal{O}(E)$ with scheme theoretic divisor of zeros $(s)_0$ equal to $E$. We want to show that $\pi^*(s)$ has divisor of zeros equal to $2E'$. But this can be checked locally, and the local statement follows from Lemma 3. □

*Remark 1.* The above proposition is false, in general, for $p = 2$ and so is the next lemma.

The following lemma is well known.

**Lemma 4.** *Let $V$ be a closed $H$-invariant subvariety of $Y$. Then $\pi(V)$ (with the reduced closed subscheme structure) is the quotient of $V$ by $H$. (For this lemma, it is not necessary to assume $Y$ or $X$ to be smooth.)*

The following is an analogue of Hurwitz theorem.

**Proposition 2.** *Let $E = \{y \in Y : \sigma(y) = y\}$ denote the fixed point (reduced) subvariety of the action of $H$ on $Y$. If $E$ is a (closed) irreducible divisor in $Y$, then $\pi^*(\omega_X) = \omega_Y \otimes \mathcal{O}(-E)$.*

*Proof.* Let $(d\pi)^n : \pi^*(\omega_X) \to \omega_Y$ denote the $n$-th (where $n := dim(Y)$) exterior power of the differential $d\pi : \pi^*(\Omega_X) \to \Omega_Y$ of $\pi$, and let $\rho$ denote the corresponding global section of the line bundle $\omega_Y \otimes \pi^*(\omega_X)^{-1}$. We want to show that the scheme theoretic divisor of zeros $(\rho)_0$ of $\rho$ is equal to $E$:

Let $U$ denote the complement of $E$ in $Y$. Then $U$ is an open subset of $Y$ on which $H$ acts freely. The restriction of the quotient map $\pi$ to $U$ is hence étale. In particular, the support of $(\rho)_0$ must be contained in $E$. As $E$ is irreducible and $(\rho)_0$ is effective, there exists a non-negative integer $l$ such that $(\rho)_0 = lE$. We have to show that $l = 1$: This can be done locally around a point in $E$, so we may assume that $X$ and $Y$ are affine with coordinate rings $A \subset B$ respectively.

By Lemma 4, the image $\pi(E)$ (with the reduced closed subscheme structure) is isomorphic to $E$. We may therefore think of $E$ as a closed (irreducible) subvariety of both $X$ and $Y$ (of codim. 1). Let $\mathfrak{p}$ (resp. $\mathfrak{p}'$) denote the prime ideal of height 1 in $A$ (resp. $B$) corresponding to $E$. Choose $s \in A$ (resp. $t \in B$) generating $\mathfrak{p}$ (resp. $\mathfrak{p}'$) in the local ring $A_{\mathfrak{p}}$ (resp. $B_{\mathfrak{p}'}$).

By Lemma 3, we know that there exist $b, b' \in B \setminus \mathfrak{p}'$ such that

$$s = t^2 \frac{b}{b'}.$$

Replacing $s$ by $sb'\sigma(b')$ and $b$ by $b\sigma(b')$, we may assume that $b' = 1$. Hence $s = t^2 b$. Now choose a point $z$ in $E$ such that

– $E$ is smooth at $z$.
– $b(z) \neq 0$.
– $\mathfrak{p}$ (resp. $\mathfrak{p}'$) is generated by $s$ (resp. $t$) in the local ring $A_{m_z}$ (resp. $B_{m'_z}$), where $m_z$ (resp. $m'_z$) is the maximal ideal corresponding to $z$ in $X$ (resp. $Y$).

(Since all these three conditions are separately valid on dense open sets in $E$, such a $z$ indeed exists.) As $E$ (by the choice of $z$) is smooth at $z$, the local ring $A_{m_z}/\mathfrak{p} = B_{m'_z}/\mathfrak{p}'$ is regular. We can therefore choose elements $s_2, \ldots, s_n \in A$ generating the maximal ideal in this local ring. Hence $ds \wedge ds_2 \wedge \cdots \wedge ds_n$ (resp. $dt \wedge ds_2 \wedge \cdots \wedge ds_n$) is a generator of $\pi^*(\omega_X)$ (resp. $\omega_Y$) at $z$. Let $c \in B_{m'_z}$ be the element such that

$$db \wedge ds_2 \wedge \cdots \wedge ds_n = c \cdot (dt \wedge ds_2 \wedge \cdots \wedge ds_n).$$

Then

$$ds \wedge ds_2 \wedge \cdots \wedge ds_n = t(ct + 2b) \cdot (dt \wedge ds_2 \wedge \cdots \wedge ds_n). \tag{1}$$

Noticing that $ct + 2b$ is a unit in $B_{m'_z}$ (by the choice of $z$), it follows that $l = 1$ (since $l$ is the exponent of $t$ on the right side of Equation (1) above). □

*Remark 2.* All the results in this section are apparently known, but we did not find an appropriate reference. Also one can formulate and prove the analogues of all these results for $H$ replaced by any finite group $G$, provided that $p$ is coprime to the order of $G$.

## 4. Crepant resolutions

In this section $X$ will denote a smooth quasiprojective surface over an algebraically closed field $k$ of char. $p \neq 2$. For any positive integer $n$, as in Sect. 1, let $X^{[n]}$ denote the Hilbert scheme of $n$ points in $X$ and $\psi : X^{[n]} \to X^{(n)}$ the Hilbert-Chow morphism. Whenever $Z$ is a smooth variety, we denote by $\omega_Z$ the canonical bundle on $Z$.

### *4.1. A fibre diagram*

As in Sect. 1, let $\Phi : X^n \to X^{(n)}$ denote the quotient map. Restrictions of $\Phi$ and $\psi$ to $X^n_* := \Phi^{-1}(X^{(n)}_*)$ and $X^{[n]}_* := \psi^{-1}(X^{(n)}_*)$ respectively, yields the fibre product diagram:

$$\begin{array}{ccc} \tilde{X}^n_* & \xrightarrow{\tilde{\psi}} & X^n_* \\ {\scriptstyle \tilde{\Phi}}\downarrow & & \downarrow{\scriptstyle \Phi} \\ X^{[n]}_* & \xrightarrow{\psi} & X^{(n)}_* \end{array}$$

It is well known that $\tilde{X}^n_*$ is the blow-up of $X^n_*$ along the big diagonals $\Delta_{ij} := \{(x_1, \ldots, x_n) \in X^n_* : x_i = x_j\}$ $(i < j)$, and that the map $\tilde{\Phi}$ is the quotient map by the induced $S_n$-action (cf. [3], Lemma 4.4). Let $\tilde{E}_{ij}$ denote the exceptional (reduced) divisor in $\tilde{X}^n_*$ corresponding to the diagonal $\Delta_{ij}$, and let $\tilde{E}$ denote the union of the $\tilde{E}_{ij}$. Let $X^{[n]}_{**}$ denote the open subset $\psi^{-1}(X^{(n)}_{**})$ in $X^{[n]}_*$, and let $E$ denote the complement of $X^{[n]}_{**}$ in $X^{[n]}_*$ with the reduced scheme structure. The variety $E$ is called the *exceptional locus* of $X^{[n]}_*$. Clearly, $E$ is the image of $\tilde{E}_{ij}$ under $\tilde{\Phi}$ for any $i < j$. In particular, $E$ is an irreducible variety.

### *4.2. Factorization of $\tilde{\Phi}$*

As mentioned above, the map $\tilde{\Phi} : \tilde{X}^n_* \to X^{[n]}_*$ is the quotient of a certain $S_n$-action on $\tilde{X}^n_*$. We may divide this quotient into two parts. Let $A_n$ be the alternating (normal) subgroup of $S_n$, and let $H$ denote the quotient $S_n/A_n$. Let $\tilde{X}^{[n]}_*$ denote the quotient of $\tilde{X}^n_*$ by $A_n$, and let $\tilde{\Phi}_1$ denote the corresponding quotient map. Clearly $X^{[n]}_*$ is then the quotient of $\tilde{X}^{[n]}_*$ by $H$, and we denote the corresponding quotient map by $\tilde{\Phi}_2$. Then $\tilde{\Phi} = \tilde{\Phi}_2 \circ \tilde{\Phi}_1$.

*4.2.1. Description of $\tilde{\Phi}_1$ and $\tilde{\Phi}_2$* It is easily seen that $A_n$ is acting freely on $X^n_*$ and hence also on $\tilde{X}^n_*$. As $\tilde{X}^n_*$ is smooth, this implies that the quotient $\tilde{X}^{[n]}_*$ is also smooth, and that the quotient map is étale. In particular, we get

**Lemma 5.** $\tilde{\Phi}_1^*(\omega_{\tilde{X}^{[n]}_*}) = \omega_{\tilde{X}^n_*}$.

All the divisors $\tilde{E}_{ij}$ map to the same divisor $E'$ in $\tilde{X}^{[n]}_*$. Clearly $H$ acts trivially on $E'$, hence it follows from Lemma 4 that (reduced) $E'$ is isomorphic to $E$. We will however keep the notation $E'$ to emphasize that $E'$ is thought of as a subvariety of $\tilde{X}^{[n]}_*$. By Proposition 1, we get

**Lemma 6.** $\tilde{\Phi}_2^*(\mathcal{O}(E)) = \mathcal{O}(2E')$.

We also need the following similar result.

**Lemma 7.** $\tilde{\Phi}_1^*(\mathcal{O}(E')) = \mathcal{O}(\tilde{E})$.

*Proof.* This follows easily since $\tilde{\Phi}_1$ is an étale map (in particular, a smooth morphism) and the set theoretic inverse image of $E'$ under $\tilde{\Phi}_1$ is exactly equal to $\tilde{E}$. □

Finally, we need the following result which follows immediately from Proposition 2.

**Lemma 8.** $\tilde{\Phi}_2^*(\omega_{X_*^{[n]}}) = \omega_{\tilde{X}_*^{[n]}} \otimes \mathcal{O}(-E')$.

### 4.3. *Crepant resolution*

In this section we will prove the following crucial result.

**Theorem 1.** *Let char. $k \neq 2$. Then $\psi : X_*^{[n]} \to X_*^{(n)}$ is a crepant resolution, meaning that $X_*^{(n)}$ is Gorenstein such that its dualizing line bundle $\omega_{X_*^{(n)}}$ pulls back to the canonical bundle $\omega_{X_*^{[n]}}$ on $X_*^{[n]}$ under $\psi$.*

First we need the following preparatory lemmas.

Recall that two cycles $Z = \sum m_i Z_i$ and $Y = \sum n_j Y_j$ in an irreducible scheme $X$ are said to meet *properly* if $codim(Z_i \cap Y_j) = codim(Z_i) + codim(Y_j)$, whenever $m_i$ and $n_j$ are non-zero (cf. [4], Sect. 11.4).

**Lemma 9.** *Let $L$ be a line bundle on any quasiprojective smooth variety $X$ defined over an algebraically closed field $k$, and $p_1, p_2, \ldots, p_n$ be a finite set of points in $X$. Then there exist an open subset $U$ in $X$ containing $p_1, p_2, \ldots, p_n$ such that the restriction of $L$ to $U$ is trivial.*

*Proof.* As any line bundle on a smooth variety is the quotient of two effective line bundles, we may assume that $L$ is effective. Let $s$ be a global section of $L$, and let $(s)_0$ denote the divisor of zeros of $s$. By the Moving Lemma ([4], Sect. 11.4), there exist a divisor $Z$ rationally equivalent to $(s)_0$ such that $Z$ meets properly with $\sum p_i$. In other words, the complement $U$ of the support of $Z$ contains $p_1, \ldots, p_n$. Since rationally equivalent divisors give rise to isomorphic line bundles (cf. [4], Example 2.1.1), $L_{|U}$ is trivial. This proves the lemma. □

Let now $X$ be a smooth quasiprojective even dimensional variety of dimension $m$, and let $\omega$ denote the canonical bundle on $X$. Then the canonical bundle on $X^n$ is isomorphic to $\omega_n := \otimes_{i=1}^n p_i^*(\omega)$, where $p_i$ is the projection $X^n \to X$ on the $i$-th factor. We regard $\omega_n$ as a $S_n$-equivariant sheaf on $X^n$ in the obvious way. The sheaf $\omega_n^{S_n}$ of $S_n$-invariant sections of $\omega_n$ can then naturally be thought of as a sheaf on $X^{(n)}$. We claim

**Lemma 10.** *The sheaf $\omega_n^{S_n}$ is a line bundle on $X^{(n)}$.*

*Proof.* Let $p = (p_1, \dots, p_n)$ be a point of $X^{(n)}$. By the above Lemma, there exist an open subset $U$ of $X$ containing $p_1, \dots, p_n$ and such that the line bundle $\omega_{|U}$ is trivial. As the fibre over $p$ (under the quotient map) is contained in $U^n$ and as the assertion of the lemma is local, we may assume that $X = U$. In particular, we can assume that $\omega$ is trivial.

Let $dX$ be a generating global section of $\omega$. Then $dX_n = \boxtimes_{i=1}^n dX$ is a generating global section of $\omega_n$. As $dX$ is an even form, the section $dX_n$ is $S_n$-invariant, and hence also a global generating section of $\omega_n^{S_n}$. This proves that $\omega_n^{S_n}$ is a line bundle. □

**Lemma 11.** *As above, let $X$ be a smooth quasiprojective even dimensional variety. Then, there exists a unique line bundle $L$ on $X^{(n)}$ which restricts to the canonical bundle on $X_{**}^{(n)}$.*

*In particular, if char. $k \neq 2$, $X_*^{(n)}$ is Gorenstein with the dualizing line bundle $L_{|X_*^{(n)}}$. (We denote $L_{|X_*^{(n)}}$ by $\omega_{X_*^{(n)}}$.)*

*Proof.* Taking $L = \omega_n^{S_n}$, the existence of line bundle $L$ follows from the above lemma. Since the map $\Phi$ restricted to $X_{**}^n$ is étale, the canonical bundle $\omega_{X_{**}^{(n)}}$ pulls back to the canonical bundle $\omega_{X_{**}^n}$. Hence, $L$ restricts to the canonical bundle on $X_{**}^{(n)}$. The uniqueness of $L$ follows since the codimension of $X^{(n)} \setminus X_{**}^{(n)}$ in the normal variety $X^{(n)}$ is $m \geq 2$.

Since $A_n$ acts freely on (smooth) $X_*^n$, the quotient $\tilde{X}_*^{(n)}$ is smooth (and hence Cohen-Macaulay). Further, $X_*^{(n)} = \tilde{X}_*^{(n)}/H$ and hence it is Cohen-Macaulay (since $p \neq 2$). Now, the assertion that $X_*^{(n)}$ is Gorenstein, follows from [6], Lemma (2.7). □

*Remark 3.* Let $X$ be a normal and Gorenstein variety $X$ of even dimension over an algebraically closed field of char. $p$. Then $X^{(n)}$ is Gorenstein (and normal) provided $p > n$. (This is a result due to Aramova [1].) To prove this, apply the 'descent' lemma (cf., e.g., [7]) to the canonical bundle $\omega_{X^n}$ of the $S_n$-variety $X^n$ to get a line bundle $L$ on $X^{(n)}$. Moreover, $L_{|U_{**}^{(n)}}$ is the canonical bundle (where $U \subset X$ is the smooth locus), since $\Phi_{|U^n}$ is an étale map. But the complement of $U_{**}^{(n)}$ in $X^{(n)}$ has codim. $\geq 2$ and $X^{(n)}$ is Cohen-Macaulay. Hence, by [6], Lemma (2.7), $L$ is the dualizing line bundle of $X^{(n)}$. This proves that $X^{(n)}$ is Gorenstein.

*From now on, we revert to the assumption that $X$ is a smooth quasiprojective surface and $p \neq 2$.*

**Lemma 12.** *Let $\omega_{X_*^{(n)}}$ be the dualizing line bundle on $X_*^{(n)}$ guaranteed by the above lemma. Then there exist an integer $t$ such that*

$$\psi^*(\omega_{X_*^{(n)}}) \backsimeq \omega_{X_*^{[n]}} \otimes \mathcal{O}(tE).$$

*Proof.* As $\psi$ is an isomorphism over $X_{**}^{(n)}$ and the restriction of $\omega_{X_*^{(n)}}$ to $X_{**}^{(n)}$ is isomorphic to the canonical bundle, we see that

$$(\psi^*(\omega_{X_*^{(n)}}))_{|X_{**}^{[n]}} = \omega_{X_{**}^{[n]}}.$$

As $E$ is irreducible, this clearly implies the result. □

**Lemma 13.** *The canonical bundle on $\tilde{X}_*^n$ is given by*

$$\omega_{\tilde{X}_*^n} = \tilde{\psi}^*(\omega_{X_*^n}) \otimes \mathcal{O}(\tilde{E}),$$

*where $\omega_{X_*^n}$ denotes the canonical bundle on $X_*^n$.*

*Proof.* Follows from [5], Exercise II.8.5. □

Now we can prove Theorem 1.

*Proof.* (*of Theorem 1*) Choose $t \in \mathbb{Z}$ with the property as given in Lemma 12. We need to show that $t = 0$: By Lemmas 12, 5 - 8, we know that

$$\begin{aligned} \tilde{\Phi}^*(\psi^*(\omega_{X_*^{(n)}})) &= \tilde{\Phi}^*(\omega_{X_*^{[n]}} \otimes \mathcal{O}(tE)) \\ &= \tilde{\Phi}_1^*(\omega_{\tilde{X}_*^{[n]}} \otimes \mathcal{O}((2t-1)E')) \\ &= \omega_{\tilde{X}_*^n} \otimes \mathcal{O}((2t-1)\tilde{E}). \end{aligned} \tag{2}$$

We want to compare this with an alternative way of calculating the left side of the equation above. Since $\psi \circ \tilde{\Phi} = \Phi \circ \tilde{\psi}$,

$$\tilde{\Phi}^*(\psi^*(\omega_{X_*^{(n)}})) = \tilde{\psi}^*(\Phi^*(\omega_{X_*^{(n)}})). \tag{3}$$

As $\Phi$ is étale over $X_{**}^{(n)}$, the canonical bundle on $X_{**}^{(n)}$ pulls back to the canonical bundle on $X_{**}^n$. In particular, $\Phi^*(\omega_{X_*^{(n)}})$ restricts to the canonical bundle on $X_{**}^n$. But the complement of $X_{**}^n$ in $X_*^n$ has codimension 2, which forces $\Phi^*(\omega_{X_*^{(n)}})$ to be the canonical bundle on $X_*^n$ (as $X_*^n$ is smooth, in particular, normal). By Lemma 13, we therefore get

$$\tilde{\psi}^*(\Phi^*(\omega_{X_*^{(n)}})) = \omega_{\tilde{X}_*^n} \otimes \mathcal{O}(-\tilde{E}). \tag{4}$$

Combining (2) – (4), we get $(2t-1) = -1$ (since $\mathcal{O}(\tilde{E})$ is a nontorsion element of Pic $\tilde{X}_*^n$), which forces $t$ to be equal to zero as desired. □

The following result in char. 0 is due to Beauville.

**Corollary 1.** *Let char. $k > n$. Then $X^{(n)}$ is Gorenstein and $\psi : X^{[n]} \to X^{(n)}$ is a crepant resolution.*

*Proof.* The assertion that $X^{(n)}$ is Gorenstein follows by the same argument as for $X_*^{(n)}$ (cf. the proof of Lemma 11) . Now, since the codim. of $X^{[n]} \setminus X_*^{[n]}$ in $X^{[n]}$ is at least two, the corollary follows from the above theorem. □

## 5. Frobenius splitting of Hilbert schemes

Let $X$ be a quasiprojective smooth surface over an algebraically closed field $k$ of positive char. $p$. In this section we will prove that $X^{[n]}$ is Frobenius split if $X$ is Frobenius split. First we need

**Lemma 14.** *Let $Y$ be a quasiprojective Frobenius split variety over $k$. Then the $n$-th symmetric product $Y^{(n)}$ of $Y$ is Frobenius split.*

*Proof.* Let $\sigma : F_*\mathcal{O}_Y \to \mathcal{O}_{Y'}$ be a Frobenius splitting of $Y$. Then $\sigma^{\boxtimes n} : F_*\mathcal{O}_{Y^n} \to \mathcal{O}_{(Y^n)'}$ is a Frobenius splitting of the $n$-fold product of $Y$. As $\sigma^{\boxtimes n}$ is equivariant with respect to the natural actions of the symmetric group $S_n$, it takes $S_n$-invariant functions on $Y^n$ to $S_n$-invariant functions on $(Y^n)'$. As $\mathcal{O}_{Y^{(n)}}$ is the subsheaf of $\mathcal{O}_{Y^n}$ consisting of $S_n$-invariant functions, $\sigma^{\boxtimes n}$ induces a Frobenius splitting of $Y^{(n)}$. □

**Theorem 2.** *Let $X$ be a quasiprojective Frobenius split smooth surface over an algebraically closed field $k$ of char. $p > 2$ . Then, for any $n \geq 1$, the Hilbert scheme $X^{[n]}$ of $n$ points in $X$ is Frobenius split.*

*Proof.* By Lemma 14, the $n$-th symmetric product $X^{(n)}$ is Frobenius split. In particular, $X^{(n)}_{**}$ is Frobenius split. Let $\sigma'$ be a splitting section of $\omega^{1-p}_{X^{(n)}_{**}}$ on $X^{(n)}_{**}$. Thinking of $\sigma'$ as a section of $\omega^{1-p}_{X^{(n)}_{*}}$ over $X^{(n)}_{**}$, as $X^{(n)}_{*}$ is normal and codim. of $X^{(n)}_{*} \setminus X^{(n)}_{**}$ in $X^{(n)}_{*}$ is two, we can extend $\sigma'$ to a global section $\sigma$ of $\omega^{1-p}_{X^{(n)}_{*}}$ over $X^{(n)}_{*}$ (cf. Lemma 11). Consider the section $\tilde{\sigma} = \psi^*(\sigma)$ of $\psi^*(\omega^{1-p}_{X^{(n)}_{*}}) = \omega^{1-p}_{X^{[n]}_{*}}$ over $X^{[n]}_{*}$ (cf. Theorem 1), and extend it to a section $\hat{\sigma}$ of $\omega^{1-p}_{X^{[n]}}$ over $X^{[n]}$. (This is possible since $X^{[n]}$ is smooth, in particular, normal and the codim. of $X^{[n]} \setminus X^{[n]}_{*}$ in $X^{[n]}$ is at least two.)

We claim that $\hat{\sigma}$ is a splitting section of $\omega^{1-p}_{X^{[n]}}$ over $X^{[n]}$. To see this, it is enough to prove that the restriction $\hat{\sigma}'$ of $\hat{\sigma}$ to $X^{[n]}_{**}$ is a splitting section over $X^{[n]}_{**}$. But $X^{[n]}_{**}$ is isomorphic to $X^{(n)}_{**}$ under $\psi$, and moreover $\hat{\sigma}'$ corresponds to $\sigma'$ under this isomorphism. As $\sigma'$, by definition, Frobenius splits $X^{(n)}_{**}$, the result follows. □

**Corollary 2.** *Let $X$ be a smooth projective Frobenius split surface over a field of characteristic $p > 2$, and let $L$ be an ample line bundle on the Hilbert scheme $X^{[n]}$. Then $L$ has vanishing higher cohomology.*

*Remark 4.* (a) One can use Corollary 2 and the Semicontinuity Theorem to get a similar vanishing result in characteristic 0.

(b) As mentioned by V. Mehta, the known list of Frobenius split smooth surfaces includes

(1) Projective examples : toric surfaces (in particular $\mathbb{P}^1 \times \mathbb{P}^1$ and $\mathbb{P}^2$), ordinary K3 surfaces and ordinary abelian surfaces. Furthermore, if $s$ is a splitting section of a smooth surface $X$ which vanishes to order $(p-1)$ along a point $x$ on $X$, then the blow-up of $X$ along $x$ is also Frobenius split.
(2) Affine examples : any smooth affine surface is Frobenius split. (In fact, any smooth affine variety is Frobenius split.)

It is furthermore known that any projective surface, with Kodaira dimension $\geq 1$, is not Frobenius split. Also non-ordinary K3 and abelian surfaces are not Frobenius split.

(c) If the punctual Hilbert scheme $H^{[n]}$ (i.e. the fibre of the Hilbert-Chow morphism $\psi$ at $(x, \cdots, x)$ for some $x \in X$) has $H^i(H^{[n]}, \mathcal{O}_{H^{[n]}}) = 0$ for all $i > 0$, then (under the assumptions of Theorem 2 ) $\psi$ is a rational resolution. In particular, for any smooth quasi projective surface $X$ (not necessarily Frobenius split), $X^{(n)}$ would be Cohen-Macaulay.

*Remark 5.* T. Ekedahl has informed us that the following vanishing theorem can be deduced from the results in this paper : Let $X$ be an abelian surface over a field of characteristic zero and let $L$ be an ample line bundle on $X^{[n]}$. Using that $X$ has infinitely many ordinary reductions to fields of positive characteristics ([9], Cor. 2.9) one gets : $H^i(X^{[n]}, L) = 0\ ,\ i > 0$.

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