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Homology of Certain Truncated Lie Algebras

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1. Introduction

This paper arose from some conjectures of P. Hanlon [H_1]. For any complex Lie algebra $\mathfrak{s}$ and positive integer k, let $\mathfrak{s}_k$ be the truncated Lie algebra $\mathfrak{s} \otimes_{\mathbb{C}} \mathbb{C}[t]/\langle t^k \rangle$. Following Hanlon, the Lie algebra $\mathfrak{s}$ is said to satisfy the property M_k if the Lie algebra homology with trivial coefficients $H_*(\mathfrak{s}_k) \simeq H_*(\mathfrak{s})^{\otimes k}$, as graded vector spaces. Hanlon conjectured that the following two classes of $\mathfrak{s}$ satisfy the property M_k (for any k):

a) $\mathfrak{s} = \mathfrak{g}$ any semisimple Lie algebra;

b) $\mathfrak{s} = \mathfrak{u}$ the nilradical of any parabolic subalgebra in a semisimple algebra.

The conjecture in the first case, i.e., $\mathfrak{s} = \mathfrak{g}$ is still open (to my knowledge). (Hanlon proved it for $\mathfrak{g} = sl_n$ and also for the limit of so_n and sp_n. Any reductive Lie algebra $\mathfrak{g}$ satisfies the property M_2 is well known and easy to see (cf. Corollary 3.5).)

The present paper is largely devoted to the second case $\mathfrak{s} = \mathfrak{u}$ and, in fact, $k = 2$. We show that $\mathfrak{s} = T_n$, where T_n is the Lie algebra of $n \times n$ strictly upper triangular matrices, does *not* satisfy the property M_2 (for $n = 4, 5$) and so is the nilradical of a Borel subalgebra of B_2 (and G_2). I believe that among the nilradicals $\mathfrak{n}$ of Borel subalgebras of simple Lie algebras $\mathfrak{g}$, only $\mathfrak{g} = A_1$ and A_2 satisfy the property M_2 (cf. Remark 5.8(2)).

Now we describe the contents of this paper:

Section 2. By a simple argument using the Universal Coefficient Theorem, we show that a finite-dimensional Lie algebra $\mathfrak{s}$ satisfies the property M_k iff the Lie algebra $\hat{\mathfrak{s}}_k$ over $\mathbb{C}[z]$ (obtained by deforming the original $\mathfrak{s}_k$) has $H_*(\hat{\mathfrak{s}}_k)$ free over $\mathbb{C}[z]$ (cf. Proposition 2.4).

Section 3. We study the Lie algebra homology $H_*(\mathfrak{s}_2)$ for any $\mathfrak{s}$. We construct a spectral sequence ${}''E^r$ converging to $H_*(\mathfrak{s})^{\otimes 2}$ with ${}''E^1_{p,q} = H_q(\mathfrak{s}, \wedge^p(\mathfrak{s}))$, where $\wedge^p(\mathfrak{s})$ is an $\mathfrak{s}$-module under the adjoint action (cf. §3.3 and Proposition 3.4). We further show that any finite-dimensional Lie algebra satisfies the property M_2 iff ${}''E^1 = {}''E^\infty$ (cf. Corollary 3.5). We next show that ${}''E^1 = {}''E^2$ for an arbitrary $\mathfrak{s}$ (cf. Theorem 3.8). We analyze the property M_2 for any three-dimensional $\mathfrak{s}$

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using the spectral sequence ${}''E^r$ (cf. Lemma 3.11 and Remark 3.12). It may be mentioned that the study of $H_*(\mathfrak{s}, \wedge(\mathfrak{s}))$ was initiated by Kostant for $\mathfrak{s} = \mathfrak{n}$ in the sixties, but still (as is fair to say) it is a poorly understood object.

Section 4. For the case where $\mathfrak{s} = \mathfrak{u}$ as in (b) above, we give an interpretation of the Lie algebra cohomology $H^*(\mathfrak{u}, \wedge^p(\mathfrak{u}))$ (equivalently of the homology $H_*(\mathfrak{u}, \wedge^{\dim \mathfrak{u}-p}(\mathfrak{u})))$ in terms of Dolbeault cohomology of the flag variety G/P (cf. Theorem 4.1 and Remark 4.2).

Section 5. We introduce a certain generating function $\chi(\mathfrak{u}, q)$ for the Lie algebra $\mathfrak{u}$ as in (b) in terms of the Euler-Poincaré characteristic of $H_*(\mathfrak{u}, \wedge(\mathfrak{u}))$ and determine it explicitly in Proposition (5.2). We show that any such $\mathfrak{u}$ does not satisfy the property M_2 if $|\chi(\mathfrak{u}, q)| > (\dim H_*(\mathfrak{u}))^2$ (cf. Theorem 5.7). Using this result, we obtain that $T_4, T_5, \mathfrak{n}(B_2), \mathfrak{n}(G_2)$ (where $\mathfrak{n}(B_2)$ denotes the nilradical of a Borel subalgebra of B_2) do *not* satisfy the property M_2, thereby disproving Hanlon's conjecture for these algebras (also see Remark 5.8(2)).

Section 6. For $\mathfrak{s} = \mathfrak{g}$ as in (a), we give an expression for the Laplacian Δ acting on $\wedge^*(\mathfrak{s}_k)$ (cf. Theorem 6.5). But we are unable to calculate the homology $H_*(\mathfrak{s}_k)$ from this expression of Δ.

2. The property M_k and its equivalent formulation

For any positive integer k, let $\mathcal{R}_k := \mathbb{C}[t]/\langle t^k \rangle$ be the truncated polynomial ring in one variable t. Also consider the $\mathbb{C}[z]$-algebra $\mathbb{C}[z] \hookrightarrow \mathcal{A}_k := \mathbb{C}[t,z]/\langle t^k - z^k \rangle$, $z \mapsto z$. Then clearly $\mathcal{A}_k$ is a free $\mathbb{C}[z]$-module and moreover

$$\mathcal{R}_k = \mathbb{C}_0 \otimes_{\mathbb{C}[z]} \mathcal{A}_k,$$

where, for any $z_0 \in \mathbb{C}$, $\mathbb{C}_{z_0}$ denotes the $\mathbb{C}[z]$-algebra $\mathbb{C}$ under $e_{z_0} : \mathbb{C}[z] \to \mathbb{C}$, $z \mapsto z_0$.

DEFINITION 2.1. For any Lie algebra $\mathfrak{s}$ over $\mathbb{C}$ and any commutative $\mathbb{C}$-algebra R, define the Lie algebra over R (in particular, Lie algebra over $\mathbb{C}$)

$$\mathfrak{s}_R := \mathfrak{s} \otimes_{\mathbb{C}} R,$$

under the bracket

$$[x \otimes a, y \otimes b] = [x,y] \otimes ab, \quad \text{for } x, y \in \mathfrak{s} \text{ and } a, b \in R.$$

In particular, we have the Lie algebra over $\mathbb{C}$ (for any positive integer k)

$$\mathfrak{s}_k := \mathfrak{s}_{\mathcal{R}_k},$$

and Lie algebra over $\mathbb{C}[z]$

$$\hat{\mathfrak{s}}_k := \mathfrak{s}_{\mathcal{A}_k}.$$

Following Hanlon [H_1, Definition 2.1.2], $\mathfrak{s}$ *is said to have property* M_k for $k > 0$, if the Lie algebra homology (with trivial coefficients) of the Lie algebra $\mathfrak{s}_k$ over $\mathbb{C}$,

$$H_*(\mathfrak{s}_k) \simeq (H_*(\mathfrak{s}))^{\otimes k},$$

as graded vector spaces.

If $\mathfrak{s}$ has property M_k for all $k > 0$, $\mathfrak{s}$ *is said to have property* M.

We recall the following conjecture due to Hanlon [H_1, Conjecture 2.1.4] (also see [H_2, Conjecture 6G]).

CONJECTURE 2.2. *Let $\mathfrak{s}$ be the nilradical of a parabolic subalgebra of a semisimple Lie algebra $\mathfrak{g}$. Then $\mathfrak{s}$ has property M.*

Since $\hat{\mathfrak{s}}_k$ is a Lie algebra that is a free module over $\mathbb{C}[z]$, its homology over $\mathbb{C}[z]$ is defined (cf. [CE, Chap. 13]). To highlight the coefficient $\mathbb{C}[z]$, we denote it by $H_*^{\mathbb{C}[z]}(\hat{\mathfrak{s}}_k)$.

The following lemma is easy to verify.

LEMMA 2.3. *Let $P \in \mathbb{C}[t]$ be such that all its roots are distinct. Then, for any Lie algebra $\mathfrak{s}$, $\mathfrak{s} \otimes_{\mathbb{C}} \mathbb{C}[t]/\langle P\rangle \simeq \mathfrak{s}^{\oplus d}$ as Lie algebras, under the map*

$$x \otimes Q \mapsto (Q(\beta_1)x, \cdots, Q(\beta_d)x),$$

for $x \in \mathfrak{s}$ and $Q \in \mathbb{C}[t]$; where $d := \deg P$ and $\beta_1, \cdots, \beta_d$ are the (distinct) roots of P.

Similarly, as Lie algebras over the function field $\mathbb{C}(z)$,

$$\mathbb{C}(z) \otimes_{\mathbb{C}[z]} \hat{\mathfrak{s}}_k \simeq (\mathbb{C}(z) \otimes_{\mathbb{C}} \mathfrak{s})^{\oplus k}.$$

□

PROPOSITION 2.4. *A finite-dimensional Lie algebra $\mathfrak{s}$ satisfies the property M_k (for $k > 0$) iff $H_*^{\mathbb{C}[z]}(\hat{\mathfrak{s}}_k)$ is a free $\mathbb{C}[z]$-module.*

PROOF. The standard chain complex to calculate the Lie algebra homology $H_*^{\mathbb{C}[z]}(\hat{\mathfrak{s}}_k)$ is given by:

$$0 \longleftarrow \wedge_0 \xleftarrow{\partial_1} \wedge_1 \xleftarrow{\partial_2} \cdots \xleftarrow{\partial_p} \wedge_p \longleftarrow \cdots ,$$

where $\wedge_p := \wedge^p_{\mathbb{C}[z]}(\hat{\mathfrak{s}}_k)$ (the notation $\wedge^p_{\mathbb{C}[z]}$ denotes the p-th exterior power over the ring $\mathbb{C}[z]$) and ∂_p are certain $\mathbb{C}[z]$-module maps.

Further, the standard chain complex to calculate the homology of the complex Lie algebra $\mathbb{C}_{z_0} \otimes_{\mathbb{C}[z]} \hat{\mathfrak{s}}_k$ (for any $z_0 \in \mathbb{C}$) is nothing but

$$0 \leftarrow \mathbb{C}_{z_0} \otimes_{\mathbb{C}[z]} \wedge_0 \xleftarrow{1\otimes\partial_1} \mathbb{C}_{z_0} \otimes_{\mathbb{C}[z]} \wedge_1 \leftarrow \cdots \xleftarrow{1\otimes\partial_p} \mathbb{C}_{z_0} \otimes_{\mathbb{C}[z]} \wedge_p \leftarrow \cdots .$$

Hence, by the Universal Coefficient Theorem [S, Chap. 5, §2, Theorem 8], for any $p \geq 0$, we have the split short exact sequence

$$(*) \quad 0 \to \mathbb{C}_{z_0} \otimes_{\mathbb{C}[z]} H_p^{\mathbb{C}[z]}(\hat{\mathfrak{s}}_k) \to H_p(\mathbb{C}_{z_0} \otimes_{\mathbb{C}[z]} \hat{\mathfrak{s}}_k) \to \operatorname{Tor}_1^{\mathbb{C}[z]}\big(H_{p-1}^{\mathbb{C}[z]}(\hat{\mathfrak{s}}_k), \mathbb{C}_{z_0}\big) \to 0 .$$

Decompose $H_p^{\mathbb{C}[z]}(\hat{\mathfrak{s}}_k) = F_p \oplus T_p$, where F_p (resp. T_p) is the free (resp. torsion) part of the finitely generated $\mathbb{C}[z]$-module $H_p^{\mathbb{C}[z]}(\hat{\mathfrak{s}}_k)$. (Of course, F_p is not canonical.)

Considering the analogue of $(*)$ with $\mathbb{C}_{z_0}$ replaced by $\mathbb{C}(z)$, we get the isomorphism

$$\mathbb{C}(z) \otimes_{\mathbb{C}[z]} F_p \simeq H_p\big(\mathbb{C}(z) \otimes_{\mathbb{C}[z]} \hat{\mathfrak{s}}_k\big). \tag{2.1}$$

But by Lemma (2.3), as graded vector spaces,

$$H_*\big(\mathbb{C}(z) \otimes_{\mathbb{C}[z]} \hat{\mathfrak{s}}_k\big) \simeq \mathbb{C}(z) \otimes_{\mathbb{C}} H_*(\mathfrak{s}^{\oplus k}). \tag{2.2}$$

In particular,

$$\operatorname{rank} F_p = \dim H_p(\mathfrak{s}^{\oplus k}). \tag{2.3}$$

Observe that $\mathbb{C}_{z_0} \otimes_{\mathbb{C}[z]} \mathcal{A}_k \simeq \mathbb{C}[t]/\langle t^k - z_0^k\rangle$, as $\mathbb{C}$-algebras (under $t \mapsto t$, $z \mapsto z_0$). Hence by $(*)$, Lemma (2.3) and identity (2.3), we get for any $z_0 \neq 0$,

$$\mathbb{C}_{z_0} \otimes_{\mathbb{C}[z]} T_p = 0. \tag{2.4}$$

Further, by $(*)$, the Lie algebra $\mathfrak{s}$ satisfies the property M_k iff for all p

$$\mathbb{C}_0 \otimes_{\mathbb{C}[z]} T_p = 0 \quad \text{and} \quad \operatorname{Tor}_1^{\mathbb{C}[z]}(T_{p-1}, \mathbb{C}_0) = 0. \tag{2.5}$$

In view of (2.4), (2.5) is equivalent to $T_p = 0$, for all $p \geq 0$. This proves the proposition. □

COROLLARY 2.5. *For any finite-dimensional Lie algebra $\mathfrak{s}$, and positive integer k,*

$$\dim H_p(\mathfrak{s}_k) \geq \dim H_p(\mathfrak{s}^{\oplus k}), \textit{ for all } p \geq 0.$$

PROOF. This follows immediately by combining $(*)$ and (2.3) of the above proof. □

3. A spectral sequence connecting $H_*(\mathfrak{s}_2)$ with $H_*(\mathfrak{s}^{\oplus 2})$

We follow the notation from §2 and take $k = 2$. We are interested in calculating the homology of the complex Lie algebra

$$\begin{aligned} \mathfrak{s}_2 &:= \mathfrak{s} \otimes_{\mathbb{C}} \mathbb{C}[t]/\langle t^2 \rangle \\ &\simeq \mathfrak{s} \oplus t\mathfrak{s}, \text{ as } \mathbb{C}\text{-vector spaces.} \end{aligned}$$

We define t-deg of any element in $\wedge^p(\mathfrak{s}) \otimes \wedge^q(t\mathfrak{s})$ to be q. It is easy to see that the boundary maps ∂_n in the standard chain complex

$$0 \leftarrow \wedge^0(\mathfrak{s}_2) \xleftarrow{\partial_1} \wedge^1(\mathfrak{s}_2) \leftarrow \cdots \xleftarrow{\partial_n} \wedge^n(\mathfrak{s}_2) \leftarrow \cdots \tag{$*$}$$

preserve the t-deg under the decomposition

$$\wedge^n(\mathfrak{s}_2) \simeq \bigoplus_{p+q=n} \wedge^p(\mathfrak{s}) \otimes \wedge^q(t\mathfrak{s}).$$

For any $q \geq 0$, let $H_*^{(q)}(\mathfrak{s}_2)$ be the homology of the subcomplex of $(*)$ consisting of elements of t-deg q. Then

LEMMA 3.1.

$$H_p(\mathfrak{s}_2) \simeq \oplus_{q \geq 0} H_p^{(q)}(\mathfrak{s}_2). \tag{a}$$

$$H_p^{(q)}(\mathfrak{s}_2) \simeq H_{p-q}(\mathfrak{s}, \wedge^q(\mathfrak{s})), \tag{b}$$

where $H_*(\mathfrak{s}, \wedge^q(\mathfrak{s}))$ denotes the Lie algebra homology of $\mathfrak{s}$ with coefficients in $\wedge^q(\mathfrak{s})$ under the adjoint action.

PROOF. (a) follows since the chain complex for $\mathfrak{s}_2$ itself breaks up as a direct sum of chain subcomplexes according to the t-deg. The assertion (b) follows easily from the explicit description of the boundary maps for $\mathfrak{s}_2$ and for $\mathfrak{s}$ with coefficients in $\wedge^q(\mathfrak{s})$. □

DEFINITION 3.2. For any Lie algebra $\mathfrak{s}$, let $\mathfrak{s} \oplus \mathfrak{s}$ be the direct product Lie algebra, i.e., $[(x, y), (x', y')] := ([x, x'], [y, y'])$. To distinguish the second component from the first, we denote it (i.e. the second component) by $\bar{\mathfrak{s}}$.

Define a vector space isomorphism

$$\theta : \mathfrak{s} \oplus \bar{\mathfrak{s}} \to \mathfrak{s} \oplus \bar{\mathfrak{s}} \quad \text{by} \quad \theta(x, y) = (x, x + y).$$

Put a Lie algebra structure on the domain $\mathfrak{s} \oplus \bar{\mathfrak{s}}$ so that θ is a Lie algebra isomorphism (where the range is equipped with the product Lie algebra structure). With this

Lie algebra structure, $\mathfrak{s} \oplus \bar{\mathfrak{s}}$ is denoted by $\mathfrak{s}^{td}$ (*td* stands for *twisted double*), and the product Lie algebra continues to be denoted by $\mathfrak{s} \oplus \bar{\mathfrak{s}}$.

Explicitly, the bracket in $\mathfrak{s}^{td}$ is given by

$$[(x,y),(x',y')]_{td} = \big([x,x'],[x,y'] + [y,x'] + [y,y']\big).$$

Set, for any $p, q \in \mathbb{Z}_+$,

$$\wedge^{p,q} = \wedge^p(\mathfrak{s}) \otimes \wedge^q(\bar{\mathfrak{s}}).$$

Define $d : \wedge^{p,q} \to \wedge^{p-1,q}$ as the boundary map in the Lie algebra complex for the Lie algebra $\mathfrak{s}$ with coefficients in $\wedge^q(\bar{\mathfrak{s}})$ under the adjoint action. Similarly, define $\partial : \wedge^{p,q} \to \wedge^{p,q-1}$ by

$$\partial(a \otimes b) = (-1)^p a \otimes \partial_{\bar{\mathfrak{s}}}(b), \text{ for } a \in \wedge^p(\mathfrak{s}),\ b \in \wedge^q(\bar{\mathfrak{s}}),$$

where $\partial_{\bar{\mathfrak{s}}}$ denotes the boundary map for the Lie algebra $\bar{\mathfrak{s}} = \mathfrak{s}$ (with trivial coefficients).

The following lemma is easy to verify from an explicit calculation.

LEMMA 3.3. *$D := d + \partial$ is the standard boundary map for the Lie algebra $\mathfrak{s}^{td}$. In particular, $d\partial + \partial d = 0$.* □

The boundary maps d and ∂ make $\{\wedge^{p,q}\}_{p,q}$ into a double complex (cf. [HS, Chap. 5, §1]), giving rise to two standard filtrations (cf. [HS, Chap. 8, §9]) of the chain complex $(\wedge^\bullet(\mathfrak{s}^{td}), D)$:

$$'\mathcal{F}_p := \bigoplus_{p' \le p} \wedge^{p',*}$$
$$''\mathcal{F}_p := \bigoplus_{p' \le p} \wedge^{*,p'}.$$

Let the corresponding spectral sequences be denoted by $'E^r$ and $''E^r$ respectively. Then, by the above lemma, both of

$$'E^r \Rightarrow H_*(\mathfrak{s}^{td}) \simeq H_*(\mathfrak{s} \oplus \bar{\mathfrak{s}}), \text{ and}$$
$$''E^r \Rightarrow H_*(\mathfrak{s}^{td}).$$

PROPOSITION 3.4. *With the notation as above, for an arbitrary Lie algebra $\mathfrak{s}$,*

(a) $'E^1_{p,q} = \wedge^p(\mathfrak{s}) \otimes H_q(\bar{\mathfrak{s}})$,

(b) $'E^2_{p,q} = H_p(\mathfrak{s}) \otimes H_q(\bar{\mathfrak{s}})$.

(c) *If $\mathfrak{s}$ is finite dimensional,* $'E^2_{p,q} = {'E^\infty_{p,q}}$.

(d) *For arbitrary $\mathfrak{s}$,* $''E^1_{p,q} = H_q(\mathfrak{s}, \wedge^p(\bar{\mathfrak{s}}))$,

where the latter is the Lie algebra homology of $\mathfrak{s}$ with coefficients in $\wedge^p(\bar{\mathfrak{s}})$ under the adjoint action.

PROOF. (a) and (d) follow easily from [HS, Chap. 8, Proposition 9.1]. To prove (b), use (a) and the fact that $\mathfrak{s}$ acts trivially on $H_*(\bar{\mathfrak{s}})$ (induced from the adjoint action). We now prove (c):

Since $'E^r \Rightarrow H_*(\mathfrak{s}^{td})$, and $H_*(\mathfrak{s}^{td}) \simeq H_*(\mathfrak{s} \oplus \bar{\mathfrak{s}}) \simeq H_*(\mathfrak{s}) \otimes H_*(\bar{\mathfrak{s}})$, it suffices to prove that

$$\sum_{p,q} \dim {'E^2_{p,q}} = \dim\big(H_*(\mathfrak{s}) \otimes H_*(\bar{\mathfrak{s}})\big),$$

but this follows from the (b)-part. □

COROLLARY 3.5. *Let $\mathfrak{s}$ be any finite-dimensional Lie algebra. Then $\mathfrak{s}$ has property $M_2 \Leftrightarrow {}''E^1_{p,q} = {}''E^\infty_{p,q}$, for all p, q. In particular, any finite-dimensional reductive Lie algebra $\mathfrak{g}$ has property M_2.*

PROOF. By the above proposition and Lemma 3.1(b),

$$''E^1_{p,q} = H^{(p)}_{p+q}(\mathfrak{s}_2).$$

So the first part of the corollary follows from Lemma 3.1(a) since

$$''E^1 \Rightarrow H_*(\mathfrak{s} \oplus \bar{\mathfrak{s}}).$$

To prove the 'In particular' statement, by Proposition 3.4(d),

$$\begin{aligned} ''E^1_{p,q} = H_q(\mathfrak{g}, \wedge^p \bar{\mathfrak{g}}) &\simeq H_q(\mathfrak{g}, (\wedge^p \bar{\mathfrak{g}})^{\mathfrak{g}}) \\ &\simeq H_q(\mathfrak{g}) \otimes H_p(\mathfrak{g}), \text{ since } H_p(\mathfrak{g}) \simeq \wedge^p(\bar{\mathfrak{g}})^{\mathfrak{g}}, \end{aligned}$$

by [Ko, Theorem 9.3]. Since $''E^1 \Rightarrow H_*(\mathfrak{g} \oplus \bar{\mathfrak{g}})$, $''E^1_{p,q} \simeq {}''E^\infty_{p,q}$ (from dimensional considerations). So $\mathfrak{g}$ has property M_2 from the first part. □

For any $p \geq 1$, let $\partial_p : \wedge^p(\bar{\mathfrak{s}}) \to \wedge^{p-1}(\bar{\mathfrak{s}})$ be the standard boundary map. Then ∂_p is an $\mathfrak{s}$-module map (under the adjoint action of $\mathfrak{s}$ on $\wedge^*(\bar{\mathfrak{s}})$), where $\bar{\mathfrak{s}}$ is nothing but a copy of $\mathfrak{s}$ as earlier.

PROPOSITION 3.6. *For any $q \geq 0$ and $p \geq 1$, the canonical map*

$$\hat{\partial}_p : H_q(\mathfrak{s}, \wedge^p(\bar{\mathfrak{s}})) \to H_q(\mathfrak{s}, \wedge^{p-1}(\bar{\mathfrak{s}})),$$

induced from the map ∂_p, is the zero map.

PROOF. As in §3.2, let d denote the boundary map in the chain complex of the Lie algebra $\mathfrak{s}$ with coefficients in $\wedge^*(\bar{\mathfrak{s}})$ (under the adjoint action). Let $\{e_i\}_{i\in\Gamma}$ be a basis of $\bar{\mathfrak{s}}$ parametrized by a well ordered set Γ and let $\{e_I = e_{i_1} \wedge \cdots \wedge e_{i_p}\}_{I=(i_1<\cdots<i_p)}$ be the induced basis of $\wedge^p(\bar{\mathfrak{s}})$.

As is easy to see,

$$\partial_p\, e_I = \frac{1}{2}\sum_{j=1}^{p}(-1)^j\,(ad\, e_{i_j})\, e_I^{(j)}, \tag{3.1}$$

where $e_I^{(j)} := e_{i_1} \wedge \cdots \wedge \hat{e}_{i_j} \wedge \cdots \wedge e_{i_p}$, and the sign convention for ∂_p is as in [CE, Chap. XIII, § 8].

Take a d-cycle $\omega \in \wedge^q(\mathfrak{s}) \otimes \wedge^p(\bar{\mathfrak{s}})$ (i.e. $d\omega = 0$) and write $\omega = \sum_I \omega_I \otimes e_I$, for some $\omega_I \in \wedge^q(\mathfrak{s})$. Then,

$$\begin{aligned} \hat{\partial}_p\omega &= \sum_I \omega_I \otimes \partial_p e_I \\ &= \frac{1}{2}\sum_I \omega_I \otimes \sum_{j=1}^{p}(-1)^j\,(ad\, e_{i_j})\, e_I^{(j)}, \qquad \text{by (3.1)}. \end{aligned} \tag{3.2}$$

Write $\omega_I = \sum_\beta \omega_I(\beta)$, where $\omega_I(\beta) := \omega_{i_1}(\beta) \wedge \cdots \wedge \omega_{i_q}(\beta)$ is decomposable. Then, by the definition of d,

$$0 = d\omega = \sum_I (\partial_q\omega_I) \otimes e_I + \sum_{I,\beta}\sum_{\ell=1}^{q}(-1)^\ell\, \omega_I(\beta)^{(\ell)} \otimes (ad\, \omega_{i_\ell}(\beta))\, e_I, \tag{3.3}$$

where $\omega_I(\beta)^{(\ell)}$ has the same meaning as that of $e_I^{(\ell)}$. Define

$$\tilde{\omega} = \sum_I \sum_{j=1}^{p} (-1)^{j+1} e_{i_j} \wedge \omega_I \otimes e_I^{(j)} \in \wedge^{q+1}(\mathfrak{s}) \otimes \wedge^{p-1}(\bar{\mathfrak{s}}).$$

Then, by [Ku, I3],

(3.4)

$$d\tilde{\omega} = \sum_I \sum_{j=1}^{p} (-1)^j \big((\operatorname{ad} e_{i_j})\omega_I + e_{i_j} \wedge \partial_q \omega_I\big) \otimes e_I^{(j)} + \sum_I \sum_{j=1}^{p} (-1)^j \omega_I \otimes (\operatorname{ad} e_{i_j})\, e_I^{(j)}$$
$$+ \sum_{I,\beta} \sum_{j=1}^{p} \sum_{\ell=1}^{q} (-1)^{j+\ell} e_{i_j} \wedge \omega_I(\beta)^{(\ell)} \otimes (\operatorname{ad} \omega_{i_\ell}(\beta))\, e_I^{(j)}.$$

For any fixed I and $1 \le j \le p$,

$$\sum_\beta \sum_{\ell=1}^{q} (-1)^{j+\ell} e_{i_j} \wedge \omega_I(\beta)^{(\ell)} \otimes (\operatorname{ad} \omega_{i_\ell}(\beta))\, e_I^{(j)} + (-1)^j \big((\operatorname{ad} e_{i_j})\omega_I\big) \otimes e_I^{(j)}$$

$$\begin{aligned} &= \sum_\beta \sum_{\ell=1}^{q} (-1)^{j+\ell+q+1} \omega_I(\beta)^{(\ell)} \wedge \big(\operatorname{ad} \omega_{i_\ell}(\beta)(e_{i_j} \otimes e_I^{(j)})\big) \\ &\quad + \sum_\beta \sum_{\ell=1}^{q} (-1)^{j+\ell+q+1} \omega_I(\beta)^{(\ell)} \wedge \Big(\big((\operatorname{ad} e_{i_j})\, \omega_{i_\ell}(\beta)\big) \otimes e_I^{(j)}\Big) \\ &\quad + (-1)^j \big((\operatorname{ad} e_{i_j})\omega_I\big) \otimes e_I^{(j)} \\ &= \sum_\beta \sum_{\ell=1}^{q} (-1)^{j+\ell+q+1} \omega_I(\beta)^{(\ell)} \wedge \big(\operatorname{ad} \omega_{i_\ell}(\beta)(e_{i_j} \otimes e_I^{(j)})\big), \end{aligned} \tag{3.5}$$

since

$$\sum_{\ell=1}^{q} (-1)^\ell \omega_I(\beta)^{(\ell)} \wedge (\operatorname{ad} e_{i_j}) \omega_{i_\ell}(\beta) = (-1)^q (\operatorname{ad} e_{i_j})\, \omega_I(\beta).$$

Applying the next lemma to the identity (3.3), we get

$$\begin{aligned} 0 &= \sum_I \sum_{j=1}^{p} (-1)^j (\partial_q \omega_I) \wedge e_{i_j} \otimes e_I^{(j)} \\ &\quad + \sum_{I,\beta} \sum_{\ell=1}^{q} \sum_{j=1}^{p} (-1)^{\ell+j} \omega_I(\beta)^{(\ell)} \wedge \big(\operatorname{ad} \omega_{i_\ell}(\beta)(e_{i_j} \otimes e_I^{(j)})\big). \end{aligned} \tag{3.6}$$

Substituting (3.2), (3.5) and (3.6) in (3.4), we get $d\tilde{\omega} = 2\,\hat{\partial}_p \omega$. This proves the proposition modulo the following lemma.

□

LEMMA 3.7. *For any Lie algebra* $\mathfrak{s}$ *and* $p \ge 1$, *the map*

$$\theta : \wedge^p(\mathfrak{s}) \to \mathfrak{s} \otimes \wedge^{p-1}(\mathfrak{s})$$

$$e = e_1 \wedge \cdots \wedge e_p \mapsto \sum_{j=1}^{p} (-1)^j\, e_j \otimes e^{(j)}$$

is well defined and is a $\mathfrak{s}$*-module map with respect to the adjoint actions (where, as above,* $e^{(j)} := e_1 \wedge \cdots \wedge \hat{e}_j \wedge \cdots \wedge e_p$*).*

PROOF. Consider the corresponding map

$$\otimes^p(\mathfrak{s}) \to \mathfrak{s} \otimes \left(\otimes^{p-1}(\mathfrak{s})\right) \to \mathfrak{s} \otimes \wedge^{p-1}(\mathfrak{s}).$$

It is easy to see that it descends to $\wedge^p(\mathfrak{s})$, proving the lemma. □

As a corollary of Proposition (3.6), we get the following.

THEOREM 3.8. *For any Lie algebra* $\mathfrak{s}$*, and any* $p, q \geq 0$

$$''E^1_{p,q} = ''E^2_{p,q},$$

where $''E^r$ *is the spectral sequence of* §*3.3.*

PROOF. By Proposition 3.4(d),

$$''E^1_{p,q} = H_q(\mathfrak{s}, \wedge^p(\bar{\mathfrak{s}})).$$

Moreover, the differential $d_1 : ''E^1_{p,q} \to ''E^1_{p-1,q}$ under the above identification is nothing but the map $\hat{\partial}_p$ of Proposition (3.6). In particular, by Proposition (3.6), $d_1 = 0$. This proves the theorem. □

LEMMA 3.9. *For any Lie algebra* $\mathfrak{s}$*,* $r \geq 2$ *and* $p \geq 0$*, the differential*

$$d_r : ''E^r_{r,p} \to ''E^r_{0,p+r-1} \quad \textit{is zero.}$$

PROOF. We follow the notation, without explanation, from [S, Chap. 9, Sec. 1, Proof of Theorem 2]. Take $x \in Z^r_r$ and write $x = x_r + x_{r-1} + \cdots + x_0$, with $x_i \in \wedge^{*,i}$. Then $Dx \in \wedge^{*,0}$, i.e., $dx_r = dx_{r-1} + \partial x_r = \cdots = dx_1 + \partial x_2 = 0$. Hence $d_r x = Dx = dx_0 = Dx_0$ (since $\partial x_1 = \partial x_0 = 0$). But $x_0 \in Z^{r-1}_{r-1}$ (clearly). This proves the lemma. □

LEMMA 3.10. *Let* $\mathfrak{s} = \mathfrak{n}$ *be the nilradical of a Borel subalgebra of a simple Lie algebra* $\mathfrak{g}$*. Then*

$$''E^2_{p,q} = ''E^\infty_{p,q} \qquad \textit{for } p + q \leq 2.$$

PROOF. Using Lemma (3.9), it is easy to see that $''E^2_{p,q} = ''E^\infty_{p,q}$, for $p+q \leq 1$. So take $p + q = 2$. By Proposition (3.4)(d) and Theorem (3.8),

$$\begin{aligned} \dim ''E^2_{0,2} + \dim ''E^2_{1,1} + \dim ''E^2_{2,0} \\ = \dim H_2(\mathfrak{n}) + \dim H_1(\mathfrak{n}, \mathfrak{n}) + \dim H_0(\mathfrak{n}, \wedge^2\mathfrak{n}). \end{aligned} \tag{3.7}$$

Now let ℓ be the rank of $\mathfrak{g}$. Then

$$\dim H_2(\mathfrak{n}) = \frac{(\ell - 1)(\ell + 2)}{2}, \tag{3.8}$$

by Kostant's $\mathfrak{n}$-homology result, and

$$\dim H_1(\mathfrak{n}, \mathfrak{n}) = \ell^2 + \ell - 1, \tag{3.9}$$

by a result of Kostant, cf. [LL, Theorem 5.4] or [W, Theorem 8.7.13].

Further, by the same argument as that of the proof of [LL, Theorem 5.1], we have

$$\wedge^2\mathfrak{n}/(\operatorname{ad}\mathfrak{n} \cdot (\wedge^2\mathfrak{n})) \simeq \wedge^2\mathfrak{n}/(\mathfrak{n} \wedge [\mathfrak{n}, \mathfrak{n}]). \tag{3.10}$$

(But, to start the induction in loc. cit., we need to prove that $e_{\alpha_i} \wedge e_\theta \in \operatorname{ad} \mathfrak{n} \cdot (\wedge^2 \mathfrak{n})$ for any simple root α_i and highest root θ such that $\mu := \theta - \alpha_i$ is a root as well; which follows since $\operatorname{ad}(e_{\alpha_i}) \cdot (e_{\alpha_i} \wedge e_\mu) = e_{\alpha_i} \wedge e_\theta$.) By (3.10),

$$\dim H_0(\mathfrak{n}, \wedge^2 \mathfrak{n}) = \frac{\ell(\ell-1)}{2}. \tag{3.11}$$

Further,

$$\begin{aligned} \dim {}''E^\infty_{0,2} + \dim {}''E^\infty_{1,1} + \dim {}''E^\infty_{2,0} &= \dim H_2(\mathfrak{n} \oplus \bar{\mathfrak{n}}) \\ &= 2 \dim H_2(\mathfrak{n}) + (\dim H_1(\mathfrak{n}))^2 \\ &= (\ell-1)(\ell+2) + \ell^2, \text{ by (3.8).} \end{aligned} \tag{3.12}$$

Combining (3.7)–(3.9), and (3.11)–(3.12), we get the lemma. □

LEMMA 3.11. *Let $\mathfrak{s}$ be any three-dimensional Lie algebra such that $H_3(\mathfrak{s}) \neq 0$. Then ${}''E^2_{p,q} = {}''E^\infty_{p,q}$ for all p, q and hence $\mathfrak{s}$ satisfies the property M_2 by Corollary (3.5) and Theorem (3.8).*

In particular, the Lie algebra T_3 of strictly upper triangular 3×3 matrices satisfies the property M_2.

PROOF. Since $\dim \mathfrak{s} = 3$, by Lemma (3.9), the only possible nonzero differentials d_r $(r \geq 2)$ are

$$d_2 : {}''E^2_{3,p} \to {}''E^2_{1,p+1}, \qquad 0 \leq p \leq 2.$$

Again following the notation from [S, Chap. 9, Sec. 1, Proof of Theorem 2], take $x \in Z^2_3$ and write $x = x_3 + x_2 + x_1 + x_0$, with $x_i \in \wedge^{*,i}$. Then $Dx \in \wedge^{*,1} \oplus \wedge^{*,0}$, i.e., $dx_3 = \partial x_3 + dx_2 = 0$. But $\partial x_3 = 0$ (by assumption) and hence $dx_2 = 0$. So

$$\begin{aligned} Dx &= \partial x_2 + dx_1 + dx_0 \qquad (\text{since } \partial x_1 = \partial x_0 = 0) \\ &= D(x_1 + x_2) + dx_0 \in DZ^1_2 + Z^1_0. \end{aligned}$$

Hence $d_2 x = 0$. □

REMARK 3.12. As given by Hanlon [H_1, Proposition 2.1.6], there are examples of three dimensional solvable Lie algebras $\mathfrak{s}$ such that $\mathfrak{s}$ does *not* satisfy the property M_2. In particular, by the argument in the above Lemma (3.11), for such an algebra $\mathfrak{s}$, $d_2 : {}''E^2_{3,p} \to {}''E^2_{1,p+1}$ is nonzero for some $0 \leq p \leq 2$.

In fact, fix a positive integer k and consider the three-dimensional Lie algebra $\mathfrak{s} = \mathfrak{s}(k)$ with basis X, Y, H and the brackets given by

$$[X,Y] = 0, \ [H,X] = (k+1)X, \ [H,Y] = -Y.$$

Then (as observed by M. Duflo), it is easy to see, by considering the H-weights, that $\mathfrak{s}$ satisfies the property M_i (for all $i \leq k$) but **not** M_{k+1}.

QUESTION 3.13. *Is it true that all the weights in $H_*(\mathfrak{n}, \wedge \mathfrak{n})$ are of the form $v\rho - w\rho$ for some $v, w \in W$, where ρ is half the sum of positive roots.*

4. Interpretation of $H_*(\mathfrak{u}_2)$ in terms of the Dolbeault cohomology of G/P

Let G be a semisimple simply-connected complex algebraic group and let P be a parabolic subgroup with Lie algebra $\mathfrak{p}$ and let $\mathfrak{u} = \mathfrak{u}_P$ be its nilradical. Fix a Levi component $R = R_P$ of P.

The adjoint action of R on $\mathfrak{u}$ induces a natural R-module structure on the Lie algebra cohomology $H^p(\mathfrak{u}, \wedge^q(\mathfrak{u}))$ (and also on the Lie algebra homology $H_p(\mathfrak{u}, \wedge^q(\mathfrak{u}))$), where $\mathfrak{u}$ acts on $\wedge^q(\mathfrak{u})$ via the adjoint action. For any irreducible R-module V and R-module W, let W_V denote the space $\mathrm{Hom}_R(V, W)$ of R-module maps. Then $\dim W_V$ is nothing but the number of copies of V in W. Also, let $\mathcal{L}(W)$ denote the homogeneous vector bundle on the flag variety G/P associated to the representation W of P (where we extend the R-module structure on W to a P-module structure by demanding its unipotent radical to act trivially on W). For any $q \geq 0$, let Ω^q be the locally free sheaf of holomorphic q-forms on G/P.

With this notation, we have the following theorem.

THEOREM 4.1. *For any $p, q \geq 0$, and irreducible R-module V,*

$$\begin{aligned} H^p(\mathfrak{u}, \wedge^q(\mathfrak{u}))_V &\simeq H^p\big(G/P, \mathcal{L}(\wedge^q(\mathfrak{u}) \otimes V^*)\big)^G \\ &\simeq H^p(G/P, \Omega^q \otimes \mathcal{L}(V^*))^G \\ &= H^{q,p}(G/P, \mathcal{L}(V^*))^G, \end{aligned}$$

where V^* denotes the dual representation and superscript G refers to the space of G-invariants.

PROOF. By Bott's theorem (cf. [B, Theorem I]), for any P-module M,

$$H^p(G/P, \mathcal{L}(M)) \simeq \bigoplus_{\theta \in D} V(\theta)^* \otimes H^p(\mathfrak{u}, V(\theta) \otimes M)^R, \tag{4.1}$$

where D is the dominant chamber for G and $V(\theta)$ is the irreducible G-module with highest weight θ. Taking $M = \wedge^q(\mathfrak{u}) \otimes V^*$ in (4.1) and taking the G-invariants, we get

$$\begin{aligned} H^p\big(G/P, \mathcal{L}(\wedge^q(\mathfrak{u}) \otimes V^*)\big)^G &\simeq H^p\big(\mathfrak{u}, \wedge^q(\mathfrak{u}) \otimes V^*\big)^R \\ &= \mathrm{Hom}_R\big(V, H^p(\mathfrak{u}, \wedge^q(\mathfrak{u}))\big), \end{aligned}$$

since V is a trivial $\mathfrak{u}$-module (by definition).

The identification of Ω^q with $\mathcal{L}(\wedge^q(\mathfrak{u}))$ is well known (and easy to see). Further, by the definition of Dolbeault cohomology,

$$H^{q,p}(G/P, \mathcal{L}(V^*)) = H^p(G/P, \Omega^q \otimes \mathcal{L}(V^*)).$$

This proves the theorem. □

REMARK 4.2. An important special case of the above theorem is when P is the Borel subgroup B of G. We denote $\mathfrak{u}$ in this case by $\mathfrak{n}$. In this case, R is any maximal torus T of B, so any irreducible R-module V is nothing but a character λ of T and $H^p(\mathfrak{n}, \wedge^q(\mathfrak{n}))_V$ is the λ-th weight space. So the above theorem, in this case, asserts :

$$H^p(\mathfrak{n}, \wedge^q(\mathfrak{n}))_\lambda \simeq H^{q,p}(G/B, \mathcal{L}(\lambda))^G,$$

where $\mathcal{L}(\lambda)$ is the line bundle on G/B associated to the character λ^{-1} of T.

Also recall that

$$\begin{aligned} H^p(\mathfrak{u}, \wedge^q(\mathfrak{u})) &\simeq H_p(\mathfrak{u}, \wedge^q(\mathfrak{u}^*))^* \\ &\simeq H_p(\mathfrak{u}, \wedge^{N_P - q}(\mathfrak{u}))^*, \end{aligned}$$

where $N_P = \dim \mathfrak{u}$ (since the $\mathfrak{u}$-invariant pairing $\wedge^q(\mathfrak{u}) \otimes \wedge^{N_P - q}(\mathfrak{u}) \to \wedge^{N_P}(\mathfrak{u}) \simeq \mathbb{C}$ is nondegenerate).

5. Counterexamples to Hanlon's conjecture for $H_*(\mathfrak{n} \otimes \mathbb{C}[t]/\langle t^2 \rangle)$

As in §4, let $\mathfrak{u}$ be the nilradical of a parabolic subalgebra $\mathfrak{p}$ of a semisimple Lie algebra $\mathfrak{g}$.

The following generating function plays a crucial role in providing counterexamples to Hanlon's conjecture for $H_*(\mathfrak{n} \otimes \mathbb{C}[t]/\langle t^2 \rangle)$.

DEFINITION 5.1. We introduce the following generating function in the variable q

$$\chi(\mathfrak{u}, q) := \sum_{n \geq 0} q^n \sum_{i \geq 0} (-1)^i \operatorname{ch} H_i(\mathfrak{u}, \wedge^n \mathfrak{u}),$$

where ch denotes the character under the action of a fixed maximal torus $T \subset P$ (P being the parabolic subgroup with Lie $P = \mathfrak{p}$).

PROPOSITION 5.2. *For any* $\mathfrak{u}$ *as above,*

$$\chi(\mathfrak{u}, q) = \prod_{\alpha \in \Delta(\mathfrak{u})} (1 - e^\alpha)(1 + q e^\alpha),$$

where $\Delta(\mathfrak{u})$ *is the set of roots of* $\mathfrak{u}$ *(with respect to the adjoint action of* T*).*

PROOF. Taking the alternating sum of the characters of the standard chain complex to calculate $H_*(\mathfrak{u}, \wedge^n(\mathfrak{u}))$ we get, for any $n \geq 0$,

$$\sum_{i \geq 0} (-1)^i \operatorname{ch} H_i(\mathfrak{u}, \wedge^n(\mathfrak{u})) = \Big(\sum_{i \geq 0} (-1)^i \operatorname{ch} \wedge^i (\mathfrak{u}) \Big) \cdot \operatorname{ch} \wedge^n (\mathfrak{u}).$$

So

$$\begin{aligned} \chi(\mathfrak{u}, q) &= \Big(\sum_{i \geq 0} (-1)^i \operatorname{ch} \wedge^i (\mathfrak{u}) \Big) \cdot \Big(\sum_{n \geq 0} q^n \operatorname{ch} \wedge^n (\mathfrak{u}) \Big) \\ &= \Big(\prod_{\alpha \in \Delta(\mathfrak{u})} (1 - e^\alpha) \Big) \prod_{\alpha \in \Delta(\mathfrak{u})} (1 + q e^\alpha). \end{aligned}$$

This proves the proposition. □

REMARK 5.3. Notice the similarity (and differences) with the Macdonald's Δ function Δ_2 (cf. [M, (1.3)]).

DEFINITION 5.4. For any formal character $\chi = \sum n_\lambda e^\lambda$, $n_\lambda \in \mathbb{Z}[q]$, $e^\lambda \in X(T)$ (where $X(T)$ is the group of characters of T), define

$$|\chi| = \sum_\lambda |n_\lambda|,$$

where, for $n_\lambda = \sum_{i \geq 0} n_\lambda^i q^i$ with $n_\lambda^i \in \mathbb{Z}$, define

$$|n_\lambda| = \sum_{i \geq 0} |n_\lambda^i|.$$

The following lemma follows immediately from the definition of $\chi(\mathfrak{u}, q)$.

LEMMA 5.5. *For any* $\mathfrak{u}$ *as in the beginning of the section*

$$\sum_{n, i \geq 0} \dim H_i(\mathfrak{u}, \wedge^n(\mathfrak{u})) \geq |\chi(\mathfrak{u}, q)|. \quad \square$$

EXAMPLE 5.6. Take $\mathfrak{g} = sl(N)$ and let $\mathfrak{u}$ be the set T_N of strictly upper triangular $N \times N$ matrices (so that $\mathfrak{u}$ is the nilradical of a Borel subalgebra of $sl(N)$). Let T be the maximal torus consisting of the diagonal matrices of $SL(N)$. Let $x_i : T \to \mathbb{C}$ be the function $\begin{pmatrix} x_1 & & \\ & \ddots & \\ & & x_N \end{pmatrix} \mapsto x_i$. Then $\Delta(\mathfrak{u})$ are precisely the characters $\{x_i/x_j : i < j\}$. In particular, by Proposition (5.2),

$$\begin{aligned} \chi(T_N, q) &= \prod_{i<j} \left(1 - \frac{x_i}{x_j}\right)\left(1 + \frac{q\,x_i}{x_j}\right) \\ &= \prod_{i<j} \left[\frac{(x_j - x_i)(x_j + qx_i)}{x_j^2}\right] \qquad (5.1) \\ &= \frac{\prod_{i<j}(x_j - x_i)(x_j + qx_i)}{(x_1^0 x_2^1 \cdots x_N^{N-1})^2}. \end{aligned}$$

THEOREM 5.7. *Let $\mathfrak{u}$ be the nilradical of a parabolic subalgebra $\mathfrak{p}$ of a semisimple Lie algebra $\mathfrak{g}$. Then $\mathfrak{u}$ does* **not** *satisfy the property M_2 if*

$$|\chi(\mathfrak{u}, q)| > (\dim H_*(\mathfrak{u}))^2. \qquad (5.2)$$

In particular, if $\mathfrak{u} = \mathfrak{n}$ is the nilradical of a Borel subalgebra of $\mathfrak{g}$, then $\mathfrak{n}$ does **not** *satisfy the property M_2 if*

$$|\chi(\mathfrak{n}, q)| > |W|^2, \qquad (5.3)$$

where W is the Weyl group of $\mathfrak{g}$.

For $\mathfrak{n} = T_4$ (resp. $\mathfrak{n} = T_5, \mathfrak{n}(B_2), \mathfrak{n}(G_2)$) the value of $|\chi(\mathfrak{n}, q)|$ is 592 (resp. 16040, 68, 212), where $\mathfrak{n}(B_2)$ denotes the nilradical of a Borel subalgebra of B_2. In particular, none of $T_4, T_5, \mathfrak{n}(B_2), \mathfrak{n}(G_2)$ satisfies the property M_2, thereby disproving Hanlon's conjecture (2.2).

PROOF. By Corollary (3.5), $\mathfrak{u}$ satisfies M_2 iff

$$''E^1_{n,i} = {}''E^\infty_{n,i} \qquad \text{for all } n, i. \qquad (5.4)$$

By Proposition 3.4(d),

$$\sum_{n,i} \dim {}''E^1_{n,i} = \sum_{n,i} \dim H_i(\mathfrak{u}, \wedge^n(\mathfrak{u})) \geq |\chi(\mathfrak{u}, q)| \quad \text{(by Lemma 5.5)}. \qquad (5.5)$$

By §3.3,

$$\begin{aligned} \sum_{n,i} \dim {}''E^\infty_{n,i} &= \sum_p \dim H_p(\mathfrak{u} \oplus \mathfrak{u}) \qquad (5.6) \\ &= (\dim H_*(\mathfrak{u}))^2. \end{aligned}$$

So, under the assumption (5.2), by (5.5) and (5.6), for some n and i, $''E^1_{n,i} \neq {}''E^\infty_{n,i}$, violating (5.4). This proves the first part of the theorem.

The 'In particular' assertion follows since $\dim H_*(\mathfrak{n}) = |W|$, by the $\mathfrak{n}$-homology result of Kostant [K, Cor. 5.14].

For $\mathfrak{n} = T_4$, T_5, $\mathfrak{n}(B_2), \mathfrak{n}(G_2)$, the values of $|\chi(\mathfrak{n}, q)|$ were calculated by computer* using Proposition (5.2) (and formula (5.1) in Example (5.6)). The order of

*I thank Robert Proctor for running this computer program for me. Later it was reverified for T_4, T_5 by P. Hanlon as well.

the Weyl groups in these cases is of course 24, 120, 8 and 12 respectively. In particular, by Lemma (3.1) and identities (5.5), (5.6), $\dim H_*(\mathfrak{n}_2) - \dim H_*(\mathfrak{n} \oplus \mathfrak{n}) \geq 16$ (for T_4), ≥ 1640 (for T_5), ≥ 4 (for $\mathfrak{n}(B_2)$) and ≥ 68 (for $\mathfrak{n}(G_2)$). □

REMARK 5.8. (1) By Kostant's theorem, $H_*(\mathfrak{u})$ is completely known as a module for a Levi subalgebra $\mathfrak{r}$ of $\mathfrak{p}$. In particular, by Weyl Dimension Formula, $\dim H_*(\mathfrak{u})$ can be determined.

(2) By Lemma (3.11), T_3 satisfies the property M_2. By the above, none of T_4, T_5, $\mathfrak{n}(B_2)$ and $\mathfrak{n}(G_2)$ satisfies the property M_2 and almost certainly any T_n $(n \geq 4)$ does *not* satisfy the property M_2. In fact, for simple $\mathfrak{g}$, I believe that the nilradical $\mathfrak{n}$ of a Borel subalgebra satisfies M_2 iff $\mathfrak{g} = sl(2)$ or $\mathfrak{g} = sl(3)$.

6. Computation of Laplacian for $\mathfrak{g}_k$ for semisimple $\mathfrak{g}$

In this section $\mathfrak{g}$ is any complex simple Lie algebra and $k \geq 1$ any integer.

Take a compact real form $\mathfrak{f} \subset \mathfrak{g}$ and choose an orthonormal basis $\{e_1, \cdots, e_N\}$ of $\mathfrak{f}$ with respect to a positive definite $\mathfrak{f}$-invariant inner product $\{,\}$ on $\mathfrak{f}$. Now we put the positive definite inner product (again denoted by $\{,\}$) on the Lie algebra $\mathfrak{f}_k := \mathfrak{f} \otimes_{\mathbb{R}} \mathbb{R}[t]/\langle t^k \rangle$ so that $\{e_p \otimes t^i\}_{\substack{1 \leq p \leq N \\ 0 \leq i \leq k-1}}$ is an orthonormal basis. Of course, $\mathfrak{f}_k$ is a real form of the Lie algebra $\mathfrak{g}_k$.

Let $\partial_j : \wedge^j(\mathfrak{f}_k) \to \wedge^{j-1}(\mathfrak{f}_k)$ be the standard boundary map for the Lie algebra $\mathfrak{f}_k$ and let $\partial_j^* : \wedge^{j-1}(\mathfrak{f}_k) \to \wedge^j(\mathfrak{f}_k)$ be its adjoint with respect to the above inner product. Then recall that the Laplacian $\Delta_j : \wedge^j(\mathfrak{f}_k) \to \wedge^j(\mathfrak{f}_k)$ is defined by $\Delta_j := \partial_j^* \partial_j + \partial_{j+1} \partial_{j+1}^*$.

We often drop the subscript from ∂ (and ∂^*).

LEMMA 6.1. *For any finite dimensional real Lie algebra $\mathfrak{s}$ and any positive definite inner product on $\mathfrak{s}$ (not necessarily $\mathfrak{s}$-invariant), $\partial^* : \wedge(\mathfrak{s}) \to \wedge(\mathfrak{s})$ is a graded derivation, i.e.,*

$$\partial^*(a \wedge b) = (\partial^* a) \wedge b + (-1)^{\deg a} a \wedge \partial^* b, \quad \textit{for } a, b \in \wedge(\mathfrak{s}).$$

PROOF. The inner product $\{,\}$ gives rise to the identification

$$\theta_p : \wedge^p(\mathfrak{s}) \xrightarrow{\sim} \wedge^p(\mathfrak{s}^*).$$

Moreover, it is easy to see that the following diagram is commutative:

$$\begin{array}{ccc} \wedge^p(\mathfrak{s}) & \xrightarrow[\sim]{\theta_p} & \wedge^p(\mathfrak{s}^*) \\ \big\downarrow \partial^* & & \big\downarrow d \\ \wedge^{p+1}(\mathfrak{s}) & \xrightarrow[\sim]{\theta_{p+1}} & \wedge^{p+1}(\mathfrak{s}^*), \end{array}$$

where d is the coboundary map in the cochain complex for the Lie algebra $\mathfrak{s}$. Since d is a graded derivation, the lemma follows. □

Denote $x \otimes t^m$ by $x(m)$. For any $m \geq 0$ and $x \in \mathfrak{f}$, define the linear map $\operatorname{ad}^+ x(-m) : \mathfrak{f}_k \to \mathfrak{f}_k$ by

$$\begin{aligned} (\operatorname{ad}^+ x(-m))\, y(\ell) &= [x, y](\ell - m), && \text{if } \ell \geq m \\ &= 0, && \text{otherwise,} \end{aligned}$$

(for $y \in \mathfrak{f}$ and $0 \leq \ell \leq k-1$). Extend $\mathrm{ad}^+ x(-m)$ as a derivation (again denoted by)

$$\mathrm{ad}^+ x(-m) : \wedge(\mathfrak{f}_k) \to \wedge(\mathfrak{f}_k).$$

The following lemma follows immediately from the $\mathfrak{f}$-invariance of $\{,\}$ on $\mathfrak{f}$.

LEMMA 6.2. *For any $a, b \in \mathfrak{f}_k$, $x \in \mathfrak{f}$ and $0 \leq m \leq k-1$,*

$$\{a, [x(m), b]\} = -\{(\mathrm{ad}^+ x(-m))\, a, b\}. \quad \square$$

LEMMA 6.3. *For any $a \in \wedge(\mathfrak{f}_k)$,*

$$\partial^* a = \frac{1}{2} \sum_{p=1}^{N} \sum_{m=0}^{k-1} e_p(m) \wedge (\mathrm{ad}^+ e_p(-m)\, a).$$

PROOF. First assume that $a \in \mathfrak{f}_k$. Then

$$\begin{aligned} \partial^* a &= \frac{1}{2} \sum_{p_1, p_2} \sum_{m_1, m_2 = 0}^{k-1} \{\partial^* a, e_{p_1}(m_1) \wedge e_{p_2}(m_2)\}\, e_{p_1}(m_1) \wedge e_{p_2}(m_2) \\ &= -\frac{1}{2} \sum_{p_1, p_2} \sum_{m_1, m_2 = 0}^{k-1} \{a, [e_{p_1}(m_1), e_{p_2}(m_2)]\}\, e_{p_1}(m_1) \wedge e_{p_2}(m_2) \\ &= \frac{1}{2} \sum_{p_1} \sum_{m_1 = 0}^{k-1} e_{p_1}(m_1) \wedge \Big(\sum_{p_2} \sum_{m_2 = 0}^{k-1} \big\{ \big(\mathrm{ad}^+ e_{p_1}(-m_1)\big) a, e_{p_2}(m_2) \big\}\, e_{p_2}(m_2) \Big), \\ &\qquad \text{by Lemma (6.2)} \\ &= \frac{1}{2} \sum_{p_1} \sum_{m_1 = 0}^{k-1} e_{p_1}(m_1) \wedge \big(\mathrm{ad}^+ e_{p_1}(-m_1)\big) a. \end{aligned}$$

So, the lemma is proved in the case $a \in \mathfrak{f}_k$. For arbitrary $a \in \wedge(\mathfrak{f}_k)$, the lemma follows since $\mathrm{ad}^+ e_p(-m)$ is a derivation and ∂^* is a graded derivation (by Lemma 6.1). $\square$

DEFINITION 6.4. Denote the Lie algebra bracket in the loop algebra $L(\mathfrak{f}) := \mathfrak{f} \otimes_{\mathbb{R}} \mathbb{R}[t, t^{-1}]$ by $[,]_L$. Define the vector space embedding $\psi : \mathfrak{f}_k \to L(\mathfrak{f})$ by $x \otimes t^i \mapsto x \otimes t^i$ for $0 \leq i \leq k-1$ and $x \in \mathfrak{f}$. Also, define the vector space projection $\pi : L(\mathfrak{f}) \to \mathfrak{f}_k$ by $\pi(x \otimes t^i) = x \otimes t^i$ for $0 \leq i \leq k-1$ and zero otherwise.

For any $m \geq 0$ and $x \in \mathfrak{f}$, define the *curvature operator* $\Omega_{x(-m)} : \wedge^2 \mathfrak{f}_k \to \mathfrak{f}_k$ by

$$a \wedge b \mapsto (\mathrm{ad}^+ x(-m))[a, b] - \beta_{x(-m)}(a \wedge b),$$

where $\beta_{x(-m)} : \wedge^2(\mathfrak{f}_k) \to \mathfrak{f}_k$ is the operator

$$a \wedge b \mapsto \pi\big(\mathrm{ad}_L\; x(-m)\, [\psi(a), \psi(b)]_L\big).$$

It is easy to see that

$$\Omega_{x(-m)}(a \wedge b) = -\pi\big(\big[x(-m), [\psi(a), \psi(b)]_L^{\geq k}\big]_L\big),$$

where, for an element $x = \sum_{i \in \mathbb{Z}} x_i t^i \in L(\mathfrak{f})$, $x^{\geq k} := \sum_{i \geq k} x_i t^i$.

Let $d : \wedge(\mathfrak{f}_k) \to \wedge(\mathfrak{f}_k)$ be the linear map

$$x_1(n_1) \wedge \cdots \wedge x_q(n_q) \mapsto (n_1 + \cdots + n_q)\, x_1(n_1) \wedge \cdots \wedge x_q(n_q),$$

for $x_i \in \mathfrak{f}$ and $0 \leq n_i \leq k-1$. Also, let $\Omega_{\mathfrak{f}} := \sum_{p=1}^{N} e_p^2$ be the Casimir operator for the simple Lie algebra $\mathfrak{f}$. Then $\Omega_{\mathfrak{f}}$ acts by a scalar (say) h on $\mathfrak{f}$ (for the adjoint action).

THEOREM 6.5. *For any $a = a_1 \wedge \cdots \wedge a_q \in \wedge^q(\mathfrak{f}_k)$,*

$$\Delta a = -\frac{1}{2}\Omega_{\mathfrak{f}} a - \frac{h}{2} d(a) + \frac{1}{2}\sum_{p,m} \sum_{1\leq j<\ell\leq q} (-1)^{j+\ell} e_p(m) \wedge \Omega_{e_p(-m)}(a_j \wedge a_\ell) \wedge a_1 \wedge \cdots \wedge \hat{a_j} \wedge \cdots \wedge \hat{a_\ell} \wedge \cdots \wedge a_q,$$

where $\{e_p(m)\}_{\substack{1\leq p\leq N\\ 0\leq m\leq k-1}}$ is the orthonormal basis of $\mathfrak{f}_k$.

PROOF. By Lemma (6.3) and [Ku, I_3]

$$2\,\Delta a = \sum_{p,m}\Big(-e_p(m) \wedge \partial(\mathrm{ad}^+ e_p(-m)\, a) - \mathrm{ad}\; e_p(m)(\mathrm{ad}^+ e_p(-m)\, a) + e_p(m) \wedge (\mathrm{ad}^+ e_p(-m)\, \partial a)\Big), \tag{6.1}$$

where summation is over $1 \leq p \leq N$ and $0 \leq m \leq k-1$. Now,

$$\begin{aligned} &\partial(\mathrm{ad}^+ e_p(-m)\, a) - (\mathrm{ad}^+ e_p(-m)\, \partial a) \\ &= \sum_{1\leq j<\ell\leq q} (-1)^{j+\ell}\Big[\mathrm{ad}\, a_j(\mathrm{ad}^+ e_p(-m)\, a_\ell) - \mathrm{ad}\, a_\ell(\mathrm{ad}^+ e_p(-m)\, a_j) \\ &\quad - (\mathrm{ad}^+ e_p(-m)[a_j, a_\ell])\Big] \wedge a_1 \wedge \cdots \wedge \hat{a_j} \wedge \cdots \wedge \hat{a_\ell} \wedge \cdots \wedge a_q. \end{aligned} \tag{6.2}$$

Also

$$\begin{aligned} &\sum_{p,m} e_p(m) \wedge \sum_{j<\ell}(-1)^{j+\ell}\big[\mathrm{ad}\, a_j(\mathrm{ad}^+ e_p(-m)\, a_\ell) - \mathrm{ad}\, a_\ell(\mathrm{ad}^+ e_p(-m)\, a_j)\big] \\ &\quad \wedge a_1 \wedge \cdots \wedge \hat{a_j} \wedge \cdots \wedge \hat{a_\ell} \wedge \cdots \wedge a_q \\ &= \sum_{p,m} e_p(m) \wedge \sum_{\ell}\sum_{j<\ell}(-1)^{\ell}\, a_1 \wedge \cdots \wedge \big(\mathrm{ad}(\mathrm{ad}^+ e_p(-m)\, a_\ell)\, a_j\big) \wedge \cdots \wedge \hat{a_\ell} \\ &\quad \wedge \cdots \wedge a_q + \sum_{p,m} e_p(m) \wedge \sum_{j}\sum_{\ell>j}(-1)^{j}\, a_1 \wedge \cdots \wedge \hat{a_j} \wedge \cdots \\ &\quad \wedge \big(\mathrm{ad}(\mathrm{ad}^+ e_p(-m)\, a_j)\, a_\ell\big) \wedge \cdots \wedge a_q \end{aligned}$$

$$\begin{aligned} &= \sum_{p,m} e_p(m) \wedge \sum_{\ell}(-1)^{\ell}\, \mathrm{ad}(\mathrm{ad}^+ e_p(-m)\, a_\ell)(a^{(\ell)}) \qquad (6.3)\\ &= \sum_{\ell}(-1)^{\ell}\sum_{j,n}\sum_{p,m} e_p(m) \wedge \{\mathrm{ad}^+ e_p(-m)\, a_\ell, e_j(n)\}\; \mathrm{ad}\, e_j(n)(a^{(\ell)}), \\ &\qquad \text{where } 1 \leq j \leq N,\ 0 \leq n \leq k-1 \\ &= -\sum_{p,m}\sum_{\ell=1}^{q}(-1)^{\ell}(\mathrm{ad}^+ e_p(-m)\, a_\ell) \wedge (\mathrm{ad}\, e_p(m)\, a^{(\ell)}), \\ &\qquad \text{where } a^{(\ell)} := a_1 \wedge \cdots \wedge \hat{a_\ell} \wedge \cdots \wedge a_q. \end{aligned}$$

Further,

$$\sum_{p,m} e_p(m) \wedge \sum_{j<\ell} (-1)^{j+\ell} \big(\beta_{e_p(-m)}(a_j \wedge a_\ell)\big) \wedge a_1 \wedge \cdots \wedge \hat{a_j} \wedge \cdots \wedge \hat{a_\ell} \wedge \cdots \wedge a_q$$
$$= \sum_{p,m} \sum_{\ell} (-1)^{\ell} e_p(m) \wedge \sum_{j,n} \{\operatorname{ad}^+ e_p(-m)\, a_\ell, e_j(n)\} \ \operatorname{ad} e_j(n)\, a^{(\ell)}$$
$$+ \sum_{p,m} \sum_{\ell} (-1)^{\ell+1} e_p(m) \wedge \sum_{j,\ 0<n' \leq k-1} \{[e_j(n'), a_\ell], e_p(m)\} \ \operatorname{ad}^+ e_j(-n')\, a^{(\ell)}$$
$$= \sum_{j,n} \sum_{\ell} (-1)^{\ell+1} \, (\operatorname{ad}^+ e_j(-n)\, a_\ell) \wedge \operatorname{ad} e_j(n)\, a^{(\ell)}$$
$$+ \sum_{j,\ 0<n'\leq k-1} \sum_{\ell} (-1)^{\ell+1} (\operatorname{ad} e_j(n')\, a_\ell) \wedge \operatorname{ad}^+ e_j(-n')\, a^{(\ell)}.$$

Combining the above with (6.1)–(6.3), we get

$$2\Delta a = \sum_{p,m} - \operatorname{ad} e_p(m) (\operatorname{ad}^+ e_p(-m)\, a)$$
$$+ \sum_{p,\ 0<m\leq k-1} \sum_{\ell} (-1)^{\ell+1} (\operatorname{ad} e_p(m)\, a_\ell) \wedge (\operatorname{ad}^+ e_p(-m)\, a^{(\ell)})$$
$$+ \sum_{p,m} e_p(m) \wedge \sum_{j<\ell} (-1)^{j+\ell} \Omega_{e_p(-m)}(a_j \wedge a_\ell) \wedge a_1 \wedge \cdots \wedge \hat{a_j} \wedge \cdots \wedge \hat{a_\ell} \wedge \cdots \wedge a_q$$
$$= - \sum_{p} \operatorname{ad} e_p (\operatorname{ad} e_p\, a) - \sum_{p,\ 0<m\leq k-1} \sum_{\ell} a_1 \wedge \cdots \wedge \big(\operatorname{ad} e_p(m) (\operatorname{ad}^+ e_p(-m)\, a_\ell)\big)$$
$$\wedge \cdots \wedge a_q + \sum_{p,m} e_p(m) \wedge \sum_{j<\ell} (-1)^{j+\ell} \Omega_{e_p(-m)}(a_j \wedge a_\ell) \wedge a_1 \wedge \cdots \wedge \hat{a_j}$$
$$\wedge \cdots \wedge \hat{a_\ell} \wedge \cdots \wedge a_q$$
$$= -\Omega_{\mathfrak{f}} a - h\, d(a) + \sum_{p,m} e_p(m) \wedge \sum_{j<\ell} (-1)^{j+\ell}\, \Omega_{e_p(-m)}(a_j \wedge a_\ell) \wedge a_1 \wedge \cdots \wedge \hat{a_j}$$
$$\wedge \cdots \wedge \hat{a_\ell} \wedge \cdots \wedge a_q.$$

This proves the theorem. □

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