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*Inventiones mathematicae*


# Frobenius splitting of cotangent bundles of flag varieties

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*To the memory of A. Ramanathan*

Let $G$ be a semisimple and simply connected algebraic group over an algebraically closed field of characteristic $p > 0$. Let $U$ be the unipotent radical of a Borel subgroup $B \subset G$ and $\mathfrak{u}$ the Lie algebra of $U$. Springer [16] has shown for good primes, that there is a $B$-equivariant isomorphism $U \to \mathfrak{u}$, where $B$ acts through conjugation on $U$ and through the adjoint action on $\mathfrak{u}$ (for $G = \mathrm{SL}_n$ one has the well known equivariant isomorphism $A \mapsto A - I$ between unipotent and nilpotent upper triangular matrices). Let $p$ be a good prime for $G$. Then there is an isomorphism of homogeneous bundles $X = G \times^B U \to G \times^B \mathfrak{u}$, where the latter can be identified with the cotangent bundle $T^*(G/B)$ of $G/B$.

Motivated in part by [12] we establish a link between the $G$-invariant form $\chi$ on the Steinberg module $\mathrm{St} = \mathrm{H}^0(G/B, (p-1)\rho)$ (cf. §1.8) and Frobenius splittings [15] of the cotangent bundle of $G/B$: the representation $\mathrm{H}^0(G/B,\ 2(p-1)\rho)$ is a quotient of the space of functions $\mathrm{H}^0(X, \mathcal{O}_X)$ on $X$ (here $\mathrm{H}^0(G/B, M)$ denotes the $G$-module induced from the $B$-module $M$ and $\rho$ half the sum of the roots $R^+$ opposite to the roots

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of $B$) (cf. Corollary 1). There is a natural map

$$\varphi' : \mathrm{St} \otimes \mathrm{St} \to \mathrm{H}^0(X, \mathcal{O}_X)$$

such that the multiplication $\mu : \mathrm{St} \otimes \mathrm{St} \to \mathrm{H}^0(G/B, 2(p-1)\rho)$ factors through the projection $\mathrm{H}^0(X, \mathcal{O}_X) \to \mathrm{H}^0(G/B, 2(p-1)\rho)$. In the notation of Corollary 1, $\varphi' = H^0(\varphi)$. Surprisingly the simple situation of [12] generalizes in that $\varphi'(v)$ is a Frobenius splitting of $X$ if and only if $\chi(v) \neq 0$ (if and only if $\mu(v)$ is a Frobenius splitting of $G/B$) (cf. Theorem 1). In particular, the cotangent bundle $T^*(G/B)$ is Frobenius split (cf. Corollary 2).

Frobenius splitting of the cotangent bundle in this setup has a number of interesting consequences. By filtering the differential forms via a morphism to a suitable partial flag variety and using diagonality of Hodge cohomology and Koszul resolutions, we obtain the vanishing theorem (cf. Theorem 2)

$$\mathrm{H}^i(G/B, S\mathfrak{u}^* \otimes \lambda) = 0, i > 0$$

where $\lambda$ is any dominant weight and $S\mathfrak{u}^*$ denotes the symmetric algebra of $\mathfrak{u}^*$. This was proved in [1] for large dominant weights and for all dominant weights for groups of classical type and $G_2$ (and large primes). The simple key lemma in the very simple proof of the Borel–Bott–Weil theorem [6] implies that the above vanishing theorem can be extended to weights $\mathcal{C} = \{\lambda \mid \langle \lambda, \alpha^\vee \rangle \geq -1, \forall \alpha \in R^+\}$. This vanishing theorem was proved in characteristic zero by Broer [3] using complete reducibility and the Borel–Bott–Weil theorem. As in characteristic zero ([3], Theorem 4.4) it follows that the subregular nilpotent variety is normal, Gorenstein and has rational singularities (cf. Theorem 6).

In the parabolic case we prove the above vanishing theorem for $P$-regular dominant weights (after proving that the cotangent bundle of partial flag varieties $G/P$ is also Frobenius split) (cf. Corollary 3 and Theorem 5).

By using the Koszul resolution, the vanishing theorem also gives the Dolbeault vanishing:

$$\mathrm{H}^i\left(G/B, \Omega^j_{G/B} \otimes \mathcal{L}(\lambda)\right) = 0$$

for $i > j$ and $\lambda \in \mathcal{C}$ (cf. Theorem 3). Another consequence is the conjectured isomorphism in ([9], II.12.15) between the group cohomology $\mathrm{H}^i(G_1, \mathrm{H}^0(G/B, \mu))^{[-1]}$ of the first Frobenius kernel of $G$ and the space of sections of a homogeneous line bundle on $T^*(G/B)$ (cf. Theorem 8 for a precise statement). Furthermore, by using the $B$-module structure of $\mathrm{St} \otimes \mathrm{St}$, it follows easily that $T^*(G/B)$ carries a canonical Frobenius splitting [13, 10]. This implies that

$$\mathrm{H}^0(G/B, S\mathfrak{u}^* \otimes \lambda)$$

has a good filtration [10] for any weight $\lambda$ (cf. Theorem 7). One obtains, in particular, that the cohomology of induced representations $\mathrm{H}^i(G_1, \mathrm{H}^0(G/B, \mu))^{[-1]}$ has a good filtration [1] (for $\mu$ dominant and $p$ bigger than the Coxeter number of $G$).

All of our proofs (and results) work for all groups in a uniform manner. Our canonical splitting relates to the splitting of Mehta and van der Kallen in the $\mathrm{GL}_n$-case [14] by taking a certain homogeneous component. For now we have ignored the more combinatorial aspects of the methods in this paper, like analyzing compatible Frobenius splitting.

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## 1. Notation and preliminaries

The following notation is used throughout the paper. Fix an algebraically closed field $k$ of characteristic $p > 0$. All schemes and morphisms will be over $k$.

**1.1. Group data.** Let $G$ be a connected, simply connected, semisimple algebraic group, $B$ a Borel subgroup of $G$, $T \subset B$ a maximal torus and $U$ the unipotent radical of $B$. The Lie algebras of $G$, $B$ and $U$ are denoted $\mathfrak{g}$, $\mathfrak{b}$ and $\mathfrak{u}$ respectively. In the following $B$ will act on $U$ by conjugation and on $\mathfrak{u}$ by the adjoint action. Let $B^+$ be the opposite Borel subgroup with unipotent radical $U^+$, $R = R(T, G)$ the root system of $G$ with respect to $T$, $R^- = R(T, U)$ (the negative roots), $R^+ = R(T, U^+) = \{\alpha_1, \ldots, \alpha_N\}$ (the positive roots), $S \subset R^+$ the simple roots and $h$ the Coxeter number of $G$. For a parabolic subgroup $P \supset B$ we let $U_P$ denote the unipotent radical of $P$, $U_P^+$ the opposite unipotent radical of $P$, $\mathfrak{u}_P$ the Lie algebra of $U_P$, $\mathfrak{p}$ the Lie algebra of $P$ and $R_P \supset T$ the Levi factor of $P$. By $\langle \cdot, \cdot \rangle$ we denote the natural pairing $X(T) \times Y(T) \to \mathbb{Z}$, where $X(T)$ is the group of characters (also identified with the weight lattice) and $Y(T)$ the group of one parameter subgroups of $T$ (also identified with the coroot lattice). A simple root $\alpha \in R^+$ defines the (simple) reflection $s_\alpha(\lambda) = \lambda - \langle \lambda, \alpha^\vee \rangle \alpha$, where $\lambda \in X(T)$ and $\alpha^\vee \in Y(T)$ is the coroot associated with $\alpha$. For a subset $I \subset S$ we let $P = P_I$ denote the associated parabolic subgroup. Recall that the group of characters $X(P)$ of $P$ can be identified with $\{\lambda \in X(T) | \langle \lambda, \alpha^\vee \rangle = 0$, for all $\alpha \in I\}$. In particular, $X(B) = X(T)$. A weight $\lambda \in X(B)$ is called *dominant* if $\langle \lambda, \alpha^\vee \rangle \geq 0$ for all $\alpha \in S$. A dominant weight $\lambda \in X(P)$ is called *$P$-regular* if $\langle \lambda, \alpha^\vee \rangle > 0$ for all $\alpha \notin I$, where $P = P_I$ is a parabolic subgroup. A $B$-regular dominant weight is called *regular*. The Weyl group $W$ of $G$ is generated by the simple reflections. The "dot" action of $W$ on $X(T)$ is given by $w \cdot \lambda = w(\lambda + \rho) - \rho$, where $\langle \rho, \alpha^\vee \rangle = 1$ for every simple root $\alpha \in S$. On the weight lattice $X(T)$ the integral cone $\mathbb{Z}_+ R^+ \subseteq X(T)$ defines the partial order: $\lambda \geq \mu$ iff $\lambda - \mu \in \mathbb{Z}_+ R^+$.

Recall that the prime $p$ is defined to be a *good prime* for $G$ if $p$ is coprime to all the coefficients of the highest root of $G$ written in terms of the simple roots. For simple $G$, $p$ is a good prime if $p \geq 2$ for type $A$; $p \geq 3$ for the types $B$, $C$ and $D$; $p \geq 5$ for the types $F_4$, $E_6$, $E_7$ and $G_2$; $p \geq 7$ for the type $E_8$.

**1.2. Homogeneous bundles.** A $P$-scheme $X$ gives rise to an associated locally trivial fibration $G \times^P X$ over $G/P$ ([9], I.5.14, II.4.1). If $M$ is a finite dimensional $P$-representation, we let $\mathcal{L}(M)$ denote the sheaf of sections of the vector bundle $G \times^P M$ on $G/P$.

**1.3. The relative Frobenius morphism.** The *absolute Frobenius morphism* on a scheme is the identity on point spaces and raising to the $p$-th power locally on functions. The absolute Frobenius morphism is not a morphism of $k$-schemes. Let $\pi : X \to \mathrm{Spec}(k)$ be a scheme. Let $X'$ be the scheme obtained from $X$ by base change with the absolute Frobenius morphism on $\mathrm{Spec}(k)$, i.e., the underlying topological space of $X'$ is that of $X$ with the same structure sheaf $\mathcal{O}_X$ of rings, only the underlying $k$-algebra structure on $\mathcal{O}_{X'}$ is twisted as $\lambda \odot f = \lambda^{1/p} f$, for $\lambda \in k$ and $f \in \mathcal{O}_{X'}$. Using this description of $X'$, *the relative Frobenius morphism* $F : X \to X'$ is defined in the same way as the absolute Frobenius morphism and it is a morphism of $k$-schemes.

**1.4. Frobenius splitting.** Following Mehta and Ramanathan [15] a variety $X$ is called *Frobenius split* if the homomorphism $\mathcal{O}_{X'} \to F_*\mathcal{O}_X$ of $\mathcal{O}_{X'}$-modules is split. A homomorphism $\sigma : F_*\mathcal{O}_X \to \mathcal{O}_{X'}$ is a splitting of $\mathcal{O}_{X'} \to F_*\mathcal{O}_X$ if and only if $\sigma(1) = 1$. By abuse of terminology we will call an $\mathcal{O}_{X'}$-module homomorphism $\sigma : F_*\mathcal{O}_X \to \mathcal{O}_{X'}$ *a Frobenius splitting* if $\sigma(1) \in k \setminus \{0\}$ (so that $\sigma$ is a splitting up to a constant).

A splitting $\sigma : F_*\mathcal{O}_X \to \mathcal{O}_{X'}$ is said to *split the subvariety* $Y \subseteq X$ *compatibly* if $\sigma(F_*\mathcal{I}_Y) \subseteq \mathcal{I}_{Y'}$, where $\mathcal{I}_Y$ denotes the ideal sheaf of $Y$.

If $X$ is a smooth variety with canonical line bundle $\omega_X$, the Cartier operator gives an isomorphism ([15], Proposition 5)

$$\mathcal{H}om_{\mathcal{O}_{X'}}(F_*\mathcal{O}_X, \mathcal{O}_{X'}) \cong F_*\left(\omega_X^{1-p}\right).$$

In this way global sections of $\omega_X^{1-p}$ correspond to homomorphisms $F_*\mathcal{O}_X \to \mathcal{O}_{X'}$. A section of $\omega_X^{1-p}$ which corresponds to a Frobenius splitting in this way, is called *a splitting section*. The above isomorphism can be described quite explicitly in local coordinates ([15], Proposition 5).

**Proposition 1.** *Let $P$ be a closed point of a smooth variety $Y$ over $k$ of dimension $n$. Choose a system $x_1, \ldots, x_n$ of regular parameters in the (regular) local ring $\mathcal{O}_{Y,P}$. Then the isomorphism*

$$F_*\left(\omega_Y^{1-p}\right) \to \mathcal{H}om_{\mathcal{O}_{Y'}}(F_*\mathcal{O}_Y, \mathcal{O}_{Y'})$$

*is locally described as*

$$x^\alpha/(dx)^{p-1} : x^\beta \mapsto x^{((\alpha+\beta+\underline{1})/p)-\underline{1}},$$

*for any* $\alpha = (\alpha_1, \ldots, \alpha_n)$, $\beta \in \mathbb{Z}_+^n$. *Here we use the multinomial notation* $x^\alpha$ *for the element* $x_1^{\alpha_1} \ldots x_n^{\alpha_n} \in \mathcal{O}_{Y,P}$, *and* $\underline{m} = (m, \ldots, m) \in \mathbb{Z}_+^n$ *for an integer* $m$. *If* $\gamma = (\gamma_1, \ldots, \gamma_n)$ *with at least one* $\gamma_i$ *nonintegral, we interpret* $x^\gamma$ *as zero. Furthermore* $dx$ *denotes the element* $dx_1 \wedge \cdots \wedge dx_n$, *and* $x^\alpha/(dx)^{p-1}$ *denotes the local section of* $\omega_Y^{1-p}$ *with value* $x^\alpha$ *on* $(dx)^{p-1}$.

We also have the following well known [15]

**Lemma 1.** *Let* $U$ *be an open dense subset of a smooth variety* $X$. *If a section* $s \in \mathrm{H}^0(X, \omega_X^{1-p})$ *restricts to a splitting section* $s|_U \in \mathrm{H}^0(U, \omega_U^{1-p})$, *then* $s$ *is a splitting section.*

**Lemma 2.** *Let* $X$ *be a Frobenius split variety and* $\mathcal{L}$ *a line bundle on* $X$. *Then there is for each* $i \geq 0$ *an injection*

$$\mathrm{H}^i(X, \mathcal{L}) \hookrightarrow \mathrm{H}^i(X, \mathcal{L}^p)$$

*of abelian groups.*

**1.5. Volume forms.** Let $X$ be a smooth variety with trivial canonical bundle $\omega_X$. A *volume form* is a nowhere vanishing section $\theta_X$ of $\omega_X$ (necessarily unique up to scalar multiples if $\mathrm{H}^0(X, \mathcal{O}_X)^* = k$). *A function* $f$ *on* $X$ *is said to Frobenius split* $X$ (with respect to $\theta_X$) if $f\,\theta_X^{1-p}$ is a splitting section of $\omega_X^{1-p}$.

**Proposition 2.** *Let* $X = \operatorname{Spec} k[x_1, \ldots, x_n]$ *be affine* $n$*-space. A volume form on* $X$ *is given by* $\theta_X = dx_1 \wedge \cdots \wedge dx_n$ *and a function* $f \in k[X]$ *Frobenius splits* $X$ *if and only if the coefficient of* $x^{\underline{p-1}}$ *in* $f$ *is nonzero and the coefficients of the terms* $x^{\underline{p-1}+p\alpha}$ *are zero for* $\alpha \in \mathbb{Z}_{\geq 0}^n \setminus \{0\}$ *(in the multinomial notation of Proposition 1).*

*Proof.* An element $\sigma \in \mathrm{Hom}_{\mathcal{O}_{X'}}(F_*\mathcal{O}_X, \mathcal{O}_{X'})$ is a Frobenius splitting if and only if $\sigma(1)$ is a nonzero constant. The proposition now follows from Proposition 1. □

**1.6. Filtration of differentials.** Let $f : X \to Y$ be a smooth morphism between smooth varieties $X$ and $Y$. Let $\Omega_{X/k}$ (resp. $\Omega_{X/Y}$) be the sheaf of differentials of $X$ (resp. the sheaf of relative differentials of $X$ over $Y$). Then we have the following

**Lemma 3.** *There is a short exact sequence*

$$0 \to f^*\Omega_{Y/k} \to \Omega_{X/k} \to \Omega_{X/Y} \to 0,$$

*giving a natural filtration of the sheaf of* $m$*-forms* $\Omega_{X/k}^m$ *for* $m \geq 1$ *with associated graded object*

$$\mathrm{Gr}\, \Omega_{X/k}^m = \bigoplus_{i=0}^{m} f^*\Omega_{Y/k}^i \otimes \Omega_{X/Y}^{m-i}.$$

**1.7. The induction functor.** Let $P$ be a parabolic subgroup, $M$ a $P$-module and $\mathrm{H}^0(G/P, M)$ the induced $G$-module. Recall that $\mathrm{H}^0(G/P, M) = (k[G] \otimes M)^P$, where $P$ acts on $k[G]$ by right multiplication (it is a $G$-module with $G$ acting trivially on $M$ and by left multiplication on $k[G]$). This translates into the more familiar

$$\mathrm{H}^0(G/P, M) = \left\{ f : G \to M \mid f(g\,p) = p^{-1}.\,f(g)\,\forall g \in G,\, p \in P \right\}.$$

In this formulation $\mathrm{H}^0(G/P, M)$ is simply the global sections of the homogeneous vector bundle $\mathcal{L}(M)$ on $G/P$. The sheaf cohomology $\mathrm{H}^i(G/P, \mathcal{L}(M))$ will also be denoted $\mathrm{H}^i(G/P, M)$ for $i \geq 0$. For $P = B$, the functor $\mathrm{H}^0(G/B, -)$ is also denoted $\mathrm{H}^0(-)$. If $M$ is a $G$-module, then $i : M \to \mathrm{H}^0(G/P, M)$ given by $i(m)(g) = g^{-1}.m$ is an isomorphism of $G$-modules.

**1.8. The Steinberg module.** The *Steinberg module* $\mathrm{St} = H^0(G/B, (p-1)\rho)$ is irreducible and selfdual. Fix an isomorphism $\mathrm{St} \to \mathrm{St}^*$ and denote the image of $v \in \mathrm{St}$ in $\mathrm{St}^*$ by $v^*$. This defines a $G$-invariant form given by $\chi(v \otimes w) = \langle v, w\rangle = v^*(w)$. Let $v^+$ and $v^-$ denote highest and lowest weight vectors of St.

Let $G$ act on itself by conjugation. Then the map $\mathrm{St} \otimes \mathrm{St} \to k[G]$ given by $(v \otimes w)(g) = \langle v, g\,w\rangle$ is a $G$-homomorphism. We get, in particular, by restriction a $B$-homomorphism

$$\varphi : \mathrm{St} \otimes \mathrm{St} \to k[U].$$

The global functions on $G \times^B U$ can be identified with $\mathrm{H}^0(G/B, k[U])$. In this setting we have $\mathrm{H}^0(\varphi)(v \otimes w)(g, u) = \langle v, gug^{-1}w\rangle$ using the identification $i$ from §1.7.

**1.9. The Frobenius kernel.** The relative Frobenius morphism $U \to U'$ is a homomorphism of group schemes. The kernel $U_1$ is called the (first) *Frobenius kernel* and is a normal (one point) subgroup scheme of $U$ ([9], I.9). If we fix a $T$-equivariant isomorphism (such that $x_i$ has weight $\alpha_i$)

$$k[U] \to k[x_1, \dots, x_N],$$

then $k[U_1] \cong k[x_1, \dots, x_N]/(x_1^p, \dots, x_N^p)$. Let $\gamma$ denote the (restriction) homomorphism $k[U] \to k[U_1]$. Notice that $k[U_1]$ is a finite dimensional $B$-representation with all weights $\leq 2(p-1)\rho$ and that $\gamma$ is $B$-equivariant. The $T$-equivariant projection on the highest weight space spanned by the vector $\bar{x}_1^{p-1} \dots \bar{x}_N^{p-1}$ is a $B$-homomorphism $\psi : k[U_1] \to 2(p-1)\rho$, where the bar denotes the corresponding element in $k[U_1]$.

## 2. Frobenius splitting of $G \times^B U$

We begin with the following elementary lemma.

**Lemma 4.** *For any parabolic subgroup $P$, the canonical line bundle of the varieties $G \times^P U_P$ and $G \times^P \mathfrak{u}_P$ is $G$-equivariantly trivial.*

*Proof.* We give the proof in the case $G \times^P U_P$. The argument for $G \times^P \mathfrak{u}_P$ is similar (in fact this is, for good primes, isomorphic to the cotangent bundle of $G/P$). Let $n = \dim U_P$. The restriction of the locally free sheaf of relative differentials $\Omega = \Omega_{(G \times^P U_P)/(G/P)}$ on $G \times^P U_P$ to $U_P = P \times^P U_P$ is the sheaf of differentials of $U_P$, and hence $\Omega^n|_{U_P} = \omega_{U_P}$. Let $\theta_{U_P}$ be a volume form on $U_P$. Since $k[U_P]$ has no nonconstant units, the canonical action of $P$ on $\theta_{U_P}$ gives rise to a character $\beta$ of $P$, which can be determined by considering the action of $P$ on $\omega_{U_P}|_e$, as the identity $e \in U_P$ is fixed under $P$. The cotangent space at $e$ is canonically isomorphic to $\mathfrak{M}_e/\mathfrak{M}_e^2$, where $\mathfrak{M}_e$ denotes the maximal ideal of functions in $k[U_P]$ vanishing at $e$. Hence $\beta = \sum_{\alpha \in R(T, U_P^+)} \alpha$. Since $\Omega^n$ is a $G$-sheaf, it is the pull back of the line bundle induced by $\beta$ on $G/P$. As the canonical line bundle of $G/P$ is induced by $-\beta$, the result follows from Lemma 3. □

Fix $T$-eigenfunctions $y_1, \ldots, y_N$ of weights $-\alpha_1, \ldots, -\alpha_N$ respectively, such that $k[U^+] \cong k[y_1, \ldots, y_N]$. By Lemma 4, $X = G \times^B U$ carries a volume form $\theta_X$ restricting to $dy_1 \wedge \cdots \wedge dy_N \wedge dx_1 \wedge \cdots \wedge dx_N$ on the open subset $U^+ \times U \hookrightarrow G \times^B U$. The following lemma is instrumental in proving Frobenius splitting of $G \times^B U$.

**Lemma 5.** *The map $\psi \circ \gamma \circ \varphi : \mathrm{St} \otimes \mathrm{St} \to 2(p-1)\rho$ is non-zero.*

*Proof.* It suffices to prove that the monomial $x_1^{p-1} \ldots x_N^{p-1}$ occurs with non-zero coefficient in $f \in k[U]$, where $f(x) = \langle v^+, x\, v^+ \rangle$. The functions $x \mapsto \langle v^+, x\, v^- \rangle$ and $x \mapsto \langle v^-, x\, v^- \rangle$ from $G$ to $k$ are highest and lowest weight vectors in $\mathrm{St} = \mathrm{H}^0(G/B, (p-1)\rho)$ respectively. By Theorem 2.3 in [12] the function $\sigma$

$$x \mapsto \langle v^+, x\, v^- \rangle \langle v^-, x\, v^- \rangle \in \mathrm{H}^0(G/B, 2(p-1)\rho)$$

is a splitting section of $G/B$. The restriction of $\sigma$ to $U^+ \subset G/B$ is given by $x \mapsto \langle v^-, x\, v^- \rangle$. Since $f$ corresponds to this function (which Frobenius splits $U^+$) under conjugation with $w_0$ (the longest element in $W$), the coefficient of $x_1^{p-1} \ldots x_N^{p-1}$ in $f$ must be nonzero by Proposition 2. □

If $M$ is a $G$-module and $N$ a $B$-module, then by Frobenius reciprocity, restriction followed by evaluation at $e \in G$ is an isomorphism ([9], Proposition I.3.4)

$$\mathrm{Hom}_G\left(M, \mathrm{H}^0(G/B, N)\right) \to \mathrm{Hom}_B(M, N).$$

Let $\mu : \mathrm{St} \otimes \mathrm{St} \to \mathrm{H}^0(G/B, 2(p-1)\rho)$ denote the multiplication map.

**Corollary 1.** *There is a commutative diagram*

$$\begin{array}{ccc} \mathrm{H}^0(G/B, k[U]) & \xrightarrow[\mathrm{H}^0(\gamma)]{} & \mathrm{H}^0(G/B, k[U_1]) \\ {\scriptstyle \mathrm{H}^0(\varphi)}\uparrow & & \downarrow{\scriptstyle \mathrm{H}^0(\psi)} \\ \mathrm{St} \otimes \mathrm{St} & \xrightarrow{\mu} & \mathrm{H}^0(G/B, 2(p-1)\rho) \end{array}$$

*of $G$-equivariant homomorphisms.*

*Proof.* By applying the induction functor we get a homomorphism

$$\mathrm{H}^0(\psi) \circ \mathrm{H}^0(\gamma) \circ \mathrm{H}^0(\varphi) : \mathrm{St} \otimes \mathrm{St} \to \mathrm{H}^0(G/B, 2(p-1)\rho),$$

which is non-zero by Lemma 5 (and Frobenius reciprocity). By Frobenius reciprocity $\mu$ is (up to a constant) the unique $G$-homomorphism $\mu : \mathrm{St} \otimes \mathrm{St} \to \mathrm{H}^0(G/B, 2(p-1)\rho)$. Adjusting constants this gives that the diagram is commutative. □

**Theorem 1.** *Let $v = \sum_i v_i \otimes w_i$ be an element of* $\mathrm{St} \otimes \mathrm{St}$. *The function $f_v = \mathrm{H}^0(\varphi)(v)$ Frobenius splits $G \times^B U$ if and only if $\mu(v)$ is a splitting section of $\omega_{G/B}^{1-p}$.*

*In particular, the function $f_v : G \times^B U \to k$ given by*

$$f_v(g, u) = \sum_i \langle v_i, gug^{-1} w_i \rangle$$

*for $g \in G$, $u \in U$, Frobenius splits $G \times^B U$ if and only if $\chi(v)$ is nonzero.*

*Proof.* Suppose that $\mu(v)$ is a splitting section of $\omega_{G/B}^{1-p}$. Let $f = \mathrm{H}^0(\varphi)(v)$. We prove that $f$ Frobenius splits $X = G \times^B U$ with respect to the volume form $\theta_X$. Restrict $f\,\theta_X^{1-p}$ to the open subset $U^+ \times U \hookrightarrow G \times^B U$. This leads to a form $f'(dy_1 \wedge \cdots \wedge dy_N \wedge dx_1 \wedge \cdots \wedge dx_N)^{1-p}$ on $U^+ \times U$. By Proposition 2 and Lemma 1, we are done if we prove that the monomial $y^{\underline{p-1}} x^{\underline{p-1}}$ occurs with nonzero coefficient in $f'$ and the monomials $y^{\underline{p-1}+p\alpha} x^{\underline{p-1}+p\beta}$ occur with zero coefficient where $\alpha, \beta \in \mathbb{Z}_{\geq 0}^N$ not simultaneously zero (in the multinomial notation of Proposition 1). We have the following commutative diagram

$$\begin{array}{ccccc} k[U^+] \otimes k[U] & \xrightarrow{1 \otimes \gamma} & k[U^+] \otimes k[U_1] & \xrightarrow{1 \otimes \psi} & k[U^+] \otimes 2(p-1)\rho \\ \uparrow & & \uparrow & & \uparrow \\ (k[G] \otimes k[U])^B & \xrightarrow{\mathrm{H}^0(\gamma)} & (k[G] \otimes k[U_1])^B & \xrightarrow{\mathrm{H}^0(\psi)} & (k[G] \otimes 2(p-1)\rho)^B \end{array}$$

with natural $T$-equivariant maps. A monomial $y^{\underline{p-1}+p\alpha} x^{\underline{p-1}+p\beta}$ occuring in $f'$ must have $\beta = 0$, as it is the restriction of an element in the image of

$(k[G] \otimes \mathrm{St} \otimes \mathrm{St})^B \to (k[G] \otimes k[U])^B$ and since any weight in $\mathrm{St} \otimes \mathrm{St}$ is $\leq 2(p-1)\rho$. Furthermore, by Corollary 1, $(\mathrm{H}^0(\psi) \circ \mathrm{H}^0(\gamma))(f)$ restricted to $U^+$ is a Frobenius splitting. Chasing through the above diagram this means (using $\beta = 0$) that $\alpha = 0$ and the monomial $y^{\underline{p-1}} x^{\underline{p-1}}$ occurs with nonzero coefficient in $f'$, so that $f$ Frobenius splits $G \times^B U$. On the other hand if $\mathrm{H}^0(\varphi)(v)$ is a Frobenius splitting it is easy to read off the diagram that $\mu(v)$ is a splitting section. The last part of the theorem follows from Theorem 2.3 in [12]. □

Recall that the cotangent bundle $T^*(G/P)$ of $G/P$ is the $G$-fibration associated to the $P$-module $(\mathfrak{g}/\mathfrak{p})^*$ under the adjoint action. It is well known that there is an isomorphism $(\mathfrak{g}/\mathfrak{p})^* \cong \mathfrak{u}_P$ of $P$-modules in good characteristics ([16], Lemma 4.4). Hence in this case $T^*(G/P) \cong G \times^P \mathfrak{u}_P$. We have the following crucial result due to Springer ([16], Proposition 3.5).

**Proposition 3.** *Let* char $k$ *be a good prime for $G$. Then there exists a $B$-equivariant isomorphism $\zeta : U \to \mathfrak{u}$. Moreover for any parabolic subgroup $P$, $\zeta$ restricts to give a $P$-equivariant isomorphism $\zeta_P : U_P \to \mathfrak{u}_P$.*

**Corollary 2.** *Let* char $k$ *be a good prime for $G$. Then the cotangent bundle $T^*(G/B)$ of $G/B$ is Frobenius split.*

*Proof.* By Proposition 3 we get a $G$-isomorphism $G \times^B U \to G \times^B \mathfrak{u}$, where the latter can be identified with the cotangent bundle of $G/B$. The result now follows from Theorem 1. □

*Remark 1.* For $v \in \mathrm{St} \otimes \mathrm{St}$ define $\tilde{f}_v : G \times^B B \to k$ as in Theorem 1 (where $B$ acts on itself by cojugation). Then $\tilde{f}_v$ Frobenius splits $G \times^B B$ if and only if $\chi(v) \neq 0$. Also the function $g \mapsto \langle v^-, gv^+\rangle\langle v^-, g^{-1}v^+\rangle$ splits $G$. Furthermore, if char $k$ is a good prime for $G$, any such $v$ gives rise to a Frobenius splitting of $G \times^B \mathfrak{b}$, which descends via the map $(g, X) \mapsto Ad(g)X$ to the Lie algebra $\mathfrak{g}$. Since we have no nontrivial applications of these results we do not give any proofs.

## 3. Vanishing

Let

$$\mathfrak{C} = \left\{\mu \in X(T) | \langle \mu, \alpha^\vee \rangle \geq -1, \forall \alpha \in R^+\right\}.$$

It is easy to see ([4], Proposition 2) that $\mathfrak{C}$ is the set of weights $\lambda$ such that if $\mu$ is a dominant weight with $\lambda \leq \mu \leq \lambda^+$, then $\mu = \lambda^+$ (here $\lambda^+$ denotes the dominant weight in the $W$-orbit of $\lambda$). The set $\mathfrak{C}$ is precisely the weights of line bundles on $G/B$ in characteristic zero, which have vanishing higher cohomology when pulled back to the cotangent bundle ([3], Theorem 2.4). In this section we prove the analogous vanishing theorem in good prime characteristics.

Andersen and Jantzen ([1], Theorem 3.6) proved the following vanishing theorem under the assumption that $p > h$ and either $\lambda = 0$ or $\lambda$ strongly dominant (i.e. $\langle\lambda, \alpha^\vee\rangle \geq h-1$ for all $\alpha \in S$). For $p \geq h-1$ and all components of $G$ classical or $G_2$ they proved the vanishing theorem for $\lambda$ dominant ([1], Proposition 5.4). Actually the condition $\lambda+\rho$ dominant in ([1], Proposition 5.4) is not sufficient for vanishing as noticed by Graham and Broer – this is also revealed using Lemma 6 in §3.2 coupled with Bott's theorem. Let $\pi : T^*(G/B) \to G/B$ denote the projection.

**Theorem 2.** *Let* char $k$ *be a good prime for* $G$ *and suppose that* $\lambda \in \mathcal{C}$. *Then*

$$\mathrm{H}^i(T^*(G/B), \pi^*\mathcal{L}(\lambda)) = \mathrm{H}^i(G/B, S\mathfrak{u}^* \otimes \lambda) = 0$$

*when* $i > 0$.

*Remark 2.* By the semicontinuity theorem our result implies the same vanishing theorem over fields of characteristic zero.

**3.1. The Koszul resolution.** Let

$$0 \to V' \to V \to V'' \to 0$$

be a short exact sequence of vector spaces. For any $n > 0$ one obtains a functorial exact sequence (called the *Koszul resolution*, ([9], II.12.12))

$$\cdots \to S^{n-i}V \otimes \wedge^i V' \to \cdots \to S^{n-1}V \otimes V' \to S^n V \to S^n V'' \to 0.$$

**3.2. A simple lemma.** Let $P_\alpha$ be the minimal parabolic subgroup corresponding to a simple root $\alpha$. If $\lambda \in X(T)$ is a weight with $\langle\lambda, \alpha^\vee\rangle = -1$ and $V$ a $P_\alpha$-module, then

$$\mathrm{H}^i(G/B, V \otimes \lambda) = 0$$

for $i \geq 0$. This result is the simple key lemma in Demazure's very simple proof of the Borel–Bott–Weil theorem [6]. It has the following consequence (a similar approach has been used by Broer in [5]).

**Lemma 6.** *Suppose that* $\lambda \in \mathcal{C}$ *and* $\langle\lambda, \alpha^\vee\rangle = -1$ *for a simple root* $\alpha$. *Then* $s_\alpha(\lambda) \in \mathcal{C}$ *and*

$$\mathrm{H}^i\left(G/B, S^n\mathfrak{u}^* \otimes \lambda\right) \cong \mathrm{H}^i\left(G/B, S^{n-1}\mathfrak{u}^* \otimes s_\alpha(\lambda)\right)$$

*for* $i \geq 0$ *and* $n > 0$.

*Proof.* As $s_\alpha$ permutes $R^+ \setminus \{\alpha\}$ and maps $\alpha$ to $-\alpha$, we get that $s_\alpha(\lambda) \in \mathcal{C}$. The isomorphism follows by applying §3.1 to the short exact sequence of $B$-modules

$$0 \to \alpha \to \mathfrak{u}^* \to \mathfrak{u}^*_{P_\alpha} \to 0,$$

and then tensoring with $\lambda$. □

**3.3. Large dominant weights.** This section contains a proof of a lemma enabling us to turn Frobenius splitting into vanishing for weights, which are not necessarily regular. The key lies in filtering differentials using the fibration $G/B \to G/P$ for a suitable parabolic subgroup $P \supset B$.

**Lemma 7.** *Let* $\lambda$ *be a dominant weight. Then*

$$\mathrm{H}^i\left(G/B, \Omega^j_{G/B} \otimes \mathcal{L}(m\lambda)\right) = 0$$

*for* $i > j$ *and all* $m$ *sufficiently big.*

*Proof.* If $\lambda = 0$, we are done by the fact that $H^i(G/B, \Omega^j_{G/B}) = 0$ for $i \neq j$ ([9], II.6.18). This is usually referred to as diagonality of Hodge cohomology. If $\lambda \neq 0$, there exists a (unique) parabolic subgroup $P \neq G$, such that $\lambda$ is a ($P$-regular) character of $P$ and the induced line bundle $\mathcal{L}(\lambda)$ is ample on $G/P$. Let $f$ denote the smooth $(P/B)$-fibration $G/B \to G/P$. Using Lemma 3, we see that it is enough to prove that the cohomology groups

$$\mathrm{H}^i\left(G/B,\ f^*\Omega^r_{G/P} \otimes \Omega^{j-r}_{(G/B)/(G/P)} \otimes \mathcal{L}(m\lambda)\right)$$

vanish for all sufficiently big $m$, where $0 \leq r \leq j$. The $E_2$-terms in the Leray spectral sequence for $f$ are (using the projection formula)

$$\begin{aligned} E_2^{pq} &= \mathrm{H}^p\left(G/P, \mathcal{L}(m\lambda) \otimes \Omega^r_{G/P} \otimes R^q f_*\Omega^{j-r}_{(G/B)/(G/P)}\right) \\ &= \mathrm{H}^p\left(G/P, \mathcal{L}(m\lambda) \otimes \Omega^r_{G/P} \otimes \mathcal{L}\left(H^q\left(P/B, \Omega^{j-r}_{P/B}\right)\right)\right). \end{aligned}$$

For all $m$ sufficiently big we get $E_2^{pq} = 0$ for $p > 0$ by Serre vanishing. Diagonality of Hodge cohomology for $P/B$ gives that $E_2^{pq} = 0$ unless $q = j - r$. In particular, for $m$ sufficiently big, combining the two, we get $E_2^{pq} = 0$ unless $p = 0$ and $q = j - r$. Now the result follows by the Leray spectral sequence, since $i > j$ by assumption. □

**3.4. Proof of Theorem 2.** The first isomorphism follows since $\pi : T^*(G/B) \to G/B$ is an affine morphism and $\pi_*\mathcal{O}_{T^*(G/B)} = \mathcal{L}(S\mathfrak{u}^*)$. To prove the vanishing part we may assume that $\lambda$ is dominant, because of the following argument: Assume by induction on $n$ that $\mathrm{H}^i(G/B, S^j\mathfrak{u}^* \otimes \lambda) = 0$ for $j < n$, $i > 0$ and $\lambda \in \mathcal{C}$. We wish to prove the same result for $j = n$. Take a non dominant weight $\lambda \in \mathcal{C}$. Then there is a simple root $\alpha$ such that $\langle \lambda, \alpha^\vee \rangle = -1$. By Lemma 6, $s_\alpha(\lambda) \in \mathcal{C}$ and

$$\mathrm{H}^i\left(G/B, S^n\mathfrak{u}^* \otimes \lambda\right) = \mathrm{H}^i\left(G/B, S^{n-1}\mathfrak{u}^* \otimes s_\alpha(\lambda)\right),$$

where the latter group vanishes by induction.

So assume that $\lambda$ is dominant. Since $(\mathfrak{b}/\mathfrak{u})^*$ is a trivial $B$-module, it follows from §3.1 (applied to the sequence $0 \to (\mathfrak{b}/\mathfrak{u})^* \to \mathfrak{b}^* \to \mathfrak{u}^* \to 0$, and breaking the resulting Koszul resolution up into short exact sequences) that the vanishing of $\mathrm{H}^i(G/B, S\mathfrak{b}^* \otimes \lambda)$ implies the vanishing of $\mathrm{H}^i(G/B, S\mathfrak{u}^* \otimes \lambda)$ for $i > 0$. Again using §3.1 for the short exact sequence $0 \to (\mathfrak{g}/\mathfrak{b})^* \to \mathfrak{g}^* \to \mathfrak{b}^* \to 0$ we get for $n \geq 1$ an exact sequence

$$\cdots \to \wedge^1(\mathfrak{g}/\mathfrak{b})^* \otimes S^{n-1}\mathfrak{g}^* \otimes \lambda \to S^n\mathfrak{g}^* \otimes \lambda \to S^n\mathfrak{b}^* \otimes \lambda \to 0$$

after tensoring with $\lambda$. By breaking this up into short exact sequences, we see that the vanishing $\mathrm{H}^i(G/B, S\mathfrak{b}^* \otimes \lambda) = 0$ for any fixed $i > 0$ follows from the vanishing

$$\mathrm{H}^{i+j}\left(G/B, \wedge^j(\mathfrak{g}/\mathfrak{b})^* \otimes \lambda\right) = 0$$

for all $j \geq 0$. The $B$-representation $\wedge^j(\mathfrak{g}/\mathfrak{b})^*$ induces the bundle of $j$-forms $\Omega^j_{G/B}$ on $G/B$. By Lemma 7, we get for all large enough $r$ that $H^{i+j}(G/B, \wedge^j(\mathfrak{g}/\mathfrak{b})^* \otimes (p^r\lambda)) = 0$ for $j \geq 0$ and hence $\mathrm{H}^i(G/B, S\mathfrak{u}^* \otimes (p^r\lambda)) = 0$ for $i > 0$. But by Corollary 2 and Lemma 2, we have an injection of abelian groups

$$\mathrm{H}^i(T^*(G/B), \pi^*\mathcal{L}(\lambda)) \hookrightarrow \mathrm{H}^i(T^*(G/B), \pi^*\mathcal{L}(p^r\lambda))$$

which translates into an injection $\mathrm{H}^i(G/B, S\mathfrak{u}^* \otimes \lambda) \hookrightarrow \mathrm{H}^i(G/B, S\mathfrak{u}^* \otimes (p^r\lambda))$ for any $r > 0$ (this is where the assumption that $p$ is good for $G$ is used). This proves the theorem.

**3.5. Dolbeault vanishing.** Theorem 2 is in fact equivalent to the following (Dolbeault) vanishing (see [4] for results in characteristic zero and the parabolic case).

**Theorem 3.** *Let* char $k$ *be a good prime for* $G$ *and* $\lambda \in \mathcal{C}$. *Then*

$$\mathrm{H}^i\left(G/B, \Omega^j_{G/B} \otimes \mathcal{L}(\lambda)\right) = 0$$

*for* $i > j$.

*Proof.* Theorem 2 implies that $\mathrm{H}^i(G/B, S^n\mathfrak{b}^* \otimes \lambda) = 0$ for $i > 0$, using induction on $n$ in the Koszul resolution (tensored with $\lambda$) coming from the short exact sequence $0 \to (\mathfrak{b}/\mathfrak{u})^* \to \mathfrak{b}^* \to \mathfrak{u}^* \to 0$. This vanishing now fits in a similar induction on $n$ in the Koszul resolution (tensored with $\lambda$) coming from the short exact sequence $0 \to (\mathfrak{g}/\mathfrak{b})^* \to \mathfrak{g}^* \to \mathfrak{b}^* \to 0$. This gives the desired vanishing. □

## 4. The parabolic case

In this section we prove that the cotangent bundle of $G/P$, where $P$ is a parabolic subgroup, is Frobenius split when char $k$ is a good prime for $G$.

**4.1. Frobenius splitting of $G \times^P U_P$.** Let $P = P_I \supset B$ be the parabolic subgroup given by a subset $I \subset S$. Let $R_I$ denote the root system generated by $I$. The space of functions $k[(U_P)_1]$ on the Frobenius kernel of $U_P$ is a finite dimensional $P$-representation with all weights $\leq (p-1)\delta_P$, where $\delta_P := \sum_{\alpha \in R^+ \setminus R_I^+} \alpha \in X(P)$. Observe that $-\delta_P$ is the weight inducing the canonical line bundle of $G/P$. The canonical line bundle of $G \times^P U_P$ is trivial by Lemma 4. The space of global functions $k[G \times^P U_P]$ can be identified with $\mathrm{H}^0(G/P, k[U_P])$. As in the case of a Borel subgroup $B \subset P$, we have a natural $P$-equivariant map $\varphi_P : \mathrm{St} \otimes \mathrm{St} \to k[U_P]$. The natural map

$$\mathrm{H}^0(G/P, k[U_P]) \to \mathrm{H}^0(G/P, k[(U_P)_1]) \to \mathrm{H}^0(G/P, (p-1)\delta_P),$$

(where the last map is induced by the $P$-module map $k[(U_P)_1] \to (p-1)\delta_P$) composed with $\mathrm{H}^0(G/P, \varphi_P)$ gives the $G$-eqivariant map $\mu_P : \mathrm{St} \otimes \mathrm{St} \to \mathrm{H}^0(G/P, (p-1)\delta_P)$.

**Theorem 4.** *Let $v = \sum_i v_i \otimes w_i$ be an element of* St ⊗ St. *The function $f = \mathrm{H}^0(G/P, \varphi_P)(v)$ Frobenius splits $G \times^P U_P$ if and only if $\mu_P(v)$ is a splitting section of $\omega_{G/P}^{1-p}$.*

*The function $f = f_v^P : G \times^P U_P \to k$ given by*

$$f_v^P(g, u) = \sum_i \langle v_i, gug^{-1} w_i \rangle$$

*for $g \in G$, $u \in U_P$, Frobenius splits $G \times^P U_P$ if and only if $\chi(v)$ is nonzero.*

*Proof.* It follows by analogous weight considerations for the restriction of $f$ to $U_P^+ \times U_P$ as in the $B$-case, that $f = \mathrm{H}^0(G/P, \varphi_P)(v)$ Frobenius splits $G \times^P U_P$ if and only if $\mu_P(v)$ is a splitting section of $G/P$ (since any $\alpha \in R^+ \setminus R_I^+$ contains a simple root outside $I$ with nonzero coefficient when written as a sum of simple roots and $(p-1)\delta_P + \sum_{\beta_i \in R^+ \setminus R_I^+} n_i \beta_i$ can not be a weight of St ⊗ St for $n_i \geq 0$ unless each $n_i$ is 0).

In order to prove the last part of the theorem, we need to exhibit an element $w \in \mathrm{St} \otimes \mathrm{St}$ such that $\mathrm{H}^0(\varphi_P)(w)$ Frobenius splits $G \times^P U_P$ (because this implies that $\mu_P(w)$ is a Frobenius splitting of $G/P$, so that $\mu_P$ followed by the $G$-equivariant "evaluation" map [12] $\mathrm{H}^0(G/P, (p-1)\delta_P) \to k$ is a non-zero $G$-homomorphism $\mathrm{St} \otimes \mathrm{St} \to k$ and hence equals $\chi$ up to a non-zero scalar multiple).

As proved in Theorem 1, the function $f(g, u) = \langle v^-, gug^{-1} v^+ \rangle$, Frobenius splits $G \times^B U$. The restriction of this function to $U^+ \times U$ therefore Frobenius splits $U^+ \times U$. Observe that this restriction is given by

$$f(g, u) = \langle v^-, guv^+ \rangle, \; g \in U^+, u \in U.$$

Let $w_0'$ be the longest element of the Weyl group of $R_P$ and let $v_0^+ = w_0' v^+$, $v_0^- = w_0' v^-$. Index the set of positive roots $\{\alpha_1, \ldots, \alpha_N\}$ in such a manner that the first $n := |R_I^+|$ roots are the positive roots of $R_P$. Let $\mathbf{y}_i : k \to U$ (resp. $\mathbf{x}_i : k \to U^+$) be the root homomorphism corresponding to the root $-\alpha_i$ (resp. $\alpha_i$). Write $u = \mathbf{y}_N(t_N) \dots \mathbf{y}_1(t_1)$ and $g = \mathbf{x}_1(s_1) \dots \mathbf{x}_N(s_N)$. Then

$$uv^+ = \mathbf{y}_N(t_N) \dots \mathbf{y}_{n+1}(t_{n+1}) \left( \sum_{l \neq \underline{p-1}} c_l t_n^{l_n} \dots t_1^{l_1} v_l + c t_n^{p-1} \dots t_1^{p-1} v_0^+ \right),$$

for some $c, c_l \in k$ and $v_l$ weight vectors in St, where $l = (l_1, \dots, l_n)$.

As $f$ Frobenius splits $U^+ \times U$, we see that the coefficient of $t_{n+1}^{p-1} \dots t_N^{p-1} s_N^{p-1} \dots s_{n+1}^{p-1}$ in

$$\left\langle v_0^-, \mathbf{x}_{n+1}(s_{n+1}) \dots \mathbf{x}_N(s_N) \mathbf{y}_N(t_N) \dots \mathbf{y}_{n+1}(t_{n+1}) v_0^+ \right\rangle$$

is nonzero. By weight considerations, it therefore easily follows that the function

$$f' : U_P^+ \times U_P \to k, \quad f'(g, u) = \left\langle v_0^-, guv_0^+ \right\rangle$$

Frobenius splits $U_P^+ \times U_P$. But $f'$ extends to the function (again denoted by) $f' : G \times^P U_P \to k$ given by $(g, u) \mapsto \langle v_0^-, gug^{-1} v_0^+ \rangle$. (To see this, it suffices to observe that $U_P^+$ fixes $v_0^+$.) Hence $f'$ Frobenius splits $G \times^P U_P$. □

**Corollary 3.** *Let* char $k$ *be a good prime for* $G$. *Then the cotangent bundle* $T^*(G/P)$ *of* $G/P$ *is Frobenius split.*

*Proof.* This follows from Theorem 4 and Proposition 3. □

**Theorem 5.** *Assume that* char $k$ *is a good prime for* $G$. *Let* $\lambda \in X(P)$ *be a* $P$*-regular weight. Then*

$$\mathrm{H}^i(T^*(G/P), \pi^* \mathcal{L}(\lambda)) = \mathrm{H}^i \left( G/P, S\mathfrak{u}_P^* \otimes \lambda \right) = 0$$

*for* $i > 0$, *where* $\pi = \pi_P : T^*(G/P) \to G/P$ *is the projection.*

*Proof.* The proof follows §3.4. One applies the Koszul resolution for the short exact sequence of $P$-modules $0 \to (\mathfrak{g}/\mathfrak{u}_P)^* \to \mathfrak{g}^* \to \mathfrak{u}_P^* \to 0$. We get for $n \geq 1$ an exact sequence

$$\cdots \to S^{n-1}\mathfrak{g}^* \otimes \wedge^1 (\mathfrak{g}/\mathfrak{u}_P)^* \otimes \lambda \to S^n \mathfrak{g}^* \otimes \lambda \to S^n \mathfrak{u}_P^* \otimes \lambda \to 0$$

after tensoring with $\lambda$. Again the vanishing $\mathrm{H}^i(G/P, S\mathfrak{u}_P^* \otimes \lambda) = 0$ for any fixed $i > 0$ follows from the vanishing

$$\mathrm{H}^{i+j}(G/P, \wedge^j (\mathfrak{g}/\mathfrak{u}_P)^* \otimes \lambda) = 0$$

for all $j \geq 0$. Since $\lambda$ induces an ample line bundle on $G/P$ this vanishing follows when $\lambda$ is replaced by $n\lambda$ for all sufficiently large $n$. In particular, we get the vanishing of $\mathrm{H}^i(T^*(G/P), \pi^* \mathcal{L}(p^r \lambda)) = \mathrm{H}^i(G/P, S\mathfrak{u}_P^* \otimes p^r \lambda)$ for any $i > 0$ and all sufficiently large $r$. Now the result follows from Corollary 3 and Lemma 2. □

## 5. The subregular nilpotent variety

Throughout this section we assume that $G$ is simple (and simply connected) and that char $k$ is good for $G$.

Let $\mathcal{U}$ be the unipotent variety in $G$, i.e., the closed subvariety of $G$ consisting of all the unipotent elements. Then the map

$$\varphi : G \times^B U \to \mathcal{U}$$

mapping $(g, u)$ to $gug^{-1}$ is a resolution of singularities ([8], Theorem 6.3) for all prime characteristics. If $P = P_\alpha$ is the minimal parabolic subgroup associated with a short simple root $\alpha$, then $\varphi$ restricted to $G \times^B U_P$ factors through

$$\varphi_\alpha : G \times^P U_P \to \mathcal{U}.$$

**Lemma 8.** *The map*

$$\varphi_\alpha : G \times^P U_P \to S$$

*is birational onto its image $S$, which consists of the closed subvariety of irregular elements (called the subregular unipotent variety).*

*Proof.* It follows by an argument of Tits that $\varphi_\alpha$ has connected fibres (see [3], Proposition 4.2), so we need to show that $\varphi_\alpha$ is separable (since $\dim G \times^P U_P = \dim S$). By Richardson's theorem ([17], I 5.1-5.6) the orbit maps for the conjugation action of $G$ on itself are separable for very good primes. This implies the separability of $\varphi_\alpha$ for good primes, when $G$ is not of type $A$. In type $A$ the separability of $\varphi_\alpha$ follows from the $\mathrm{GL}_n$-case, where the orbit maps for the conjugation action are separable for all primes. □

By ([2], Corollary 9.3.4) there is a (Springer) $G$-isomorphism between the unipotent variety $\mathcal{U}$ and the nilpotent cone $\mathcal{N}$, i. e., the closed subvariety of $\mathfrak{g}$ consisting of all the nilpotent elements. In particular, we get that $\mathcal{N}$ is normal by the normality of $\mathcal{U}$ ([8], Theorem 4.24(iii)). As in the unipotent case, the Springer resolution

$$\tilde{\varphi} : G \times^B \mathfrak{u} \to \mathcal{N}, (g, X) \mapsto \operatorname{Ad} g(X),$$

is a resolution of singularities, which gives a resolution (Lemma 8)

$$\tilde{\varphi}_\alpha : G \times^P \mathfrak{u}_P \to \mathcal{S}$$

of singularities of the subregular nilpotent variety $\mathcal{S}$, where $\mathfrak{u}_P$ is the nilpotent radical of the Lie algebra of $P$. Moreover, all the morphisms $\varphi, \tilde{\varphi}, \varphi_\alpha, \tilde{\varphi}_\alpha$ are projective morphisms. Let $\pi : T^*(G/B) \to G/B$ denote the projection.

**Theorem 6.** *The subregular nilpotent variety $\mathcal{S}$ is a normal Gorenstein variety with rational singularities.*

*Proof.* The characteristic zero proof ([3], Theorem 4.4) carries over: The closed subvariety $G \times^B \mathfrak{u}_P$ of the cotangent bundle $G \times^B \mathfrak{u}$ is the zero scheme of a section of the pull back $\pi^*\mathcal{L}(-\alpha)$. So we get an exact sequence

$$0 \to \pi^*\mathcal{L}(\alpha) \to \mathcal{O}_{G\times^B\mathfrak{u}} \to \mathcal{O}_{G\times^B\mathfrak{u}_P} \to 0.$$

By Theorem 2 (since $\alpha \in \mathcal{C}$) and the normality of $\mathcal{N}$, we get a short exact sequence

$$0 \to \mathrm{H}^0(T^*(G/B), \pi^*\mathcal{L}(\alpha)) \to k[\mathcal{N}] \to k\left[G \times^P \mathfrak{u}_P\right] \to 0.$$

Let $\tilde{\mathcal{S}}$ denote the normalization of $\mathcal{S}$. The surjection $k[\mathcal{N}] \to k[G \times^P \mathfrak{u}_P]$ factors through the injection $k[\mathcal{S}] \to k[\tilde{\mathcal{S}}]$ (followed by the map $k[\tilde{\mathcal{S}}] \to k[G \times^P \mathfrak{u}_P]$ induced by the normalization) via the restriction map $k[\mathcal{N}] \to k[\mathcal{S}]$. This proves that $k[\mathcal{S}] = k[\tilde{\mathcal{S}}]$ so that $\mathcal{S}$ is normal. By Theorem 2 the higher cohomologies of $\mathcal{O}_{G\times^B\mathfrak{u}}$ and $\pi^*\mathcal{L}(\alpha)$ vanish. It follows that $\mathrm{H}^i(G \times^B \mathfrak{u}_P, \mathcal{O}_{G\times^B\mathfrak{u}_P}) = \mathrm{H}^i(G \times^P \mathfrak{u}_P, \mathcal{O}_{G\times^P\mathfrak{u}_P}) = 0$ for $i > 0$, giving that $\mathcal{S}$ has rational singularities (since $\tilde{\varphi}_\alpha$ is birational by Lemma 8). As the canonical line bundle of $G \times^P \mathfrak{u}_P$ is trivial, $\mathcal{S}$ is Gorenstein ([11], p. 49–50). □

## 6. Good filtrations

Let $X$ be a smooth $B$-variety. A splitting section (or Frobenius splitting) $\sigma \in \mathrm{H}^0(X, \omega_X^{1-p})$ is called *canonical* [13], ([10], Definition 4.3.5) if $\sigma$ is $T$-invariant and for all $\alpha \in S$ and $t \in k$

$$x_\alpha(t).\sigma = \sum_{i=0}^{p-1} t^i \sigma_{i,\alpha}$$

for suitable $\sigma_{i,\alpha} \in \mathrm{H}^0(X, \omega_X^{1-p})$ (of weight $i\,\alpha$), where $x_\alpha : k \to B$ is the root homomorphism corresponding to the root $-\alpha$.

Recall that a filtration $0 = V_0 \subset V_1 \subset \ldots$ of a $G$-module $V$ is called a *good filtration* if $V$ is the union of the $G$-submodules $V_0, V_1, \ldots$ and $V_i/V_{i-1} \cong \mathrm{H}^0(G/B, \lambda_i)$ for $\lambda_i$ dominant. We have the following weaker version of a result due to Mathieu ([10], Lemma 4.4.2) sufficient for our purposes.

**Lemma 9.** *Let $X$ be a smooth $B$-variety and $\mathcal{L}$ a $G$-equivariant line bundle on $G \times^B X$. Assume that $G \times^B X$ admits a canonical splitting, then the $G$-module $\mathrm{H}^0(G \times^B X, \mathcal{L})$ has a good filtration.*

For good primes there is a $G$-equivariant map

$$\varphi' : \mathrm{St} \otimes \mathrm{St} \to \mathrm{H}^0\left(T^*(G/B), \mathcal{O}_{T^*(G/B)}\right)$$

such that $\varphi'(a \otimes b)$ is a splitting section if $\chi(a \otimes b) \neq 0$ ( where $\varphi' := H^0(\varphi)$, cf. §2). Consider the splitting section of the cotangent bundle $T^*(G/B)$ given by $\varphi'(v^+ \otimes v^-)$. It is easy to see that $\varphi'(v^+ \otimes v^-)$ is a canonical Frobenius splitting of $T^*(G/B) = G \times^B \mathfrak{u}$, since the definition can be checked for $v^+ \otimes v^- \in \mathrm{St} \otimes \mathrm{St}$.

**Theorem 7.** *Suppose that* char $k$ *is a good prime for* $G$. *Let* $\lambda \in X(T)$ *be a weight (not necessarily dominant). Then*

$$\mathrm{H}^0(G/B, S^n\mathfrak{u}^* \otimes \lambda)$$

*has a good filtration for* $n \geq 0$.

*Proof.* By the above $T^*(G/B) = G \times^B \mathfrak{u}$ admits a canonical Frobenius splitting. Hence

$$\mathrm{H}^0(T^*(G/B), \pi^*\mathcal{L}(\lambda)) = \mathrm{H}^0(G/B, S\mathfrak{u}^* \otimes \lambda)$$

has a good filtration by Lemma 9, where $\pi : T^*(G/B) \to G/B$ denotes the projection. □

*Remark 3.* Using Theorem 4 it follows in the same way that the cotangent bundle $T^*(G/P) = G \times^P \mathfrak{u}_P$ of $G/P$ admits a canonical Frobenius splitting for any parabolic subgroup $P \supset B$. Mathieu has informed us that $H^0(X, \mathcal{L})$ has a good filtration if $X$ is a smooth $G$-variety with a canonical Frobenius splitting and $\mathcal{L}$ a $G$-equivariant line bundle on $X$. In our case one can prove directly that $G \times^B (G \times^P \mathfrak{u}_P) \cong G/B \times (G \times^P \mathfrak{u}_P)$ admits a canonical Frobenius splitting, so that Lemma 9 implies that $\mathrm{H}^0(G/P, S\mathfrak{u}_P^* \otimes \lambda)$ has a good filtration for (arbitrary) weights $\lambda \in X(P)$.

**Theorem 8.** *Suppose that* $p > h$ *and let* $\lambda$ *be a dominant weight. Then we have an isomorphism for any* $w \in W$ *such that* $w \cdot 0 + p\lambda$ *is dominant*

$$\mathrm{H}^i\left(G_1, \mathrm{H}^0(G/B, w \cdot 0 + p\,\lambda)\right)^{[-1]} \cong \begin{cases} \mathrm{H}^0\left(G/B, S^{(i-\ell(w))/2}\mathfrak{u}^* \otimes \lambda\right) & \textit{if } i \equiv \ell(w) \mod 2, \\ 0 & \textit{otherwise}, \end{cases}$$

*where* $()^{[-1]}$ *denotes Frobenius (un)twist of a representation. In particular,* $\mathrm{H}^i(G_1, \mathrm{H}^0(G/B, w \cdot 0 + p\,\lambda))^{[-1]}$ *admits a good filtration.*

*Proof.* The key ingredient in the proof (in [1], §3.3) of the isomorphism is the vanishing Theorem 2, which makes the spectral sequence ([1], 3.3(2)) degenerate. The good filtrations follow from Theorem 7. □

*Remark 4.* Andersen and Jantzen proved Theorem 8 for groups not having any components of types $E$ and $F$([1], §5). For arbitrary $G$ they proved Theorem 8 under the assumption that $\lambda$ is strongly dominant ([1], Corollary 3.7(b)).

*Remark 5.* It follows from the linkage principle that the only dominant $\mu$ with

$$\mathrm{H}^\bullet\left(G_1, \mathrm{H}^0(G/B, \mu)\right) \neq 0$$

are of the form $w \cdot 0 + p\,\lambda$ for some $\lambda$ dominant and $w \in W$.

## 7. Homogeneous Frobenius splittings

We assume that char $k$ is a good prime for $G$. The space of functions $(k[G] \otimes k[\mathfrak{u}_P])^P = (k[G] \otimes S\mathfrak{u}_P^*)^P$ on the cotangent bundle $T^*(G/P)$ has a natural grading. Let $\pi_d : (k[G] \otimes S\mathfrak{u}_P^*)^P \to (k[G] \otimes S^d\mathfrak{u}_P^*)^P$ be the projection on the $d$-th homogeneous factor. Let $N_P$ denote the dimension of $G/P$. Then a function $f$ Frobenius splits $T^*(G/P)$ implies that $\pi_{N_P(p-1)}(f)$ Frobenius splits $T^*(G/P)$. A homogeneous splitting function (of degree $N_P(p-1)$) descends to give a Frobenius splitting of the projectivization $\mathbb{P}(T^*(G/P))$ (lines in $T^*(G/P)$) of the cotangent bundle. These splittings are in some sense better behaved than the splittings coming directly from $\mathrm{St} \otimes \mathrm{St}$ via Corollaries 2 and 3.

**7.1. The $A_n$-case.** In type $A_n$ ($G = \mathrm{SL}_{n+1}(k)$) we have the $B$-equivariant isomorphism $\sigma : A \mapsto I + A$ between the upper triangular nilpotent matrices $\mathfrak{u}$ and the upper triangular unipotent matrices $U$. In this way we see that the element $v^+ \otimes v^-$ in $\mathrm{St} \otimes \mathrm{St}$ maps to the (splitting) function $f$

$$(g, A) \mapsto \left\langle v^+, g(A + I)g^{-1}v^- \right\rangle$$

on the cotangent bundle $T^*(G/B) = G \times^B \mathfrak{u}$ via $\mathrm{H}^0(\varphi)$ and $\sigma$. The function $g \mapsto \langle v^+, gv^- \rangle$ is a highest weight vector in St and equals the $(p-1)$-st power of the highest weight function $f_\rho : g \mapsto \langle w^+, gw^- \rangle$, where $w^+$ and $w^-$ are highest and lowest weight vectors in $\mathrm{H}^0(G/B, \rho)$. The function $f_\rho$ is a product of certain highest weight functions $f_{\omega_1}, \dots, f_{\omega_n}$, with weight of $f_{\omega_i} = \omega_i$, where $\omega_i$ denotes the $i$-th fundamental dominant weight. Let $A = (a_{i,j})_{1 \le i,j \le n+1}$ be a matrix in $G$, then it is well known that

$$f_{\omega_s}(A) = \det\left((a_{i,j})_{1 \le i,j \le s}\right)$$

for $1 \le s \le n$. In this way the (magical) splitting function of Mehta and van der Kallen [14] on $T^*(G/B)$ is exactly $\pi_{N(p-1)}(f)$, where $N = n(n+1)/2$. One interesting aspect of the Mehta–van der Kallen splitting is that it compatibly splits all $G \times^B \mathfrak{u}_P$, for any parabolic subgroup $P \supseteq B$. Finding a suitable splitting in this context for the other groups would be very interesting.

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