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The nil Hecke ring and singularity of Schubert varieties

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Introduction

Let G be a semisimple simply-connected complex algebraic group and $T \subset B$ a maximal torus and a Borel subgroup respectively. Let $\mathfrak{h} = \text{Lie } T$ be the Cartan subalgebra of the Lie algebra Lie G, and $W := N(T)/T$ the Weyl group associated to the pair (G, T), where $N(T)$ is the normalizer of T in G. We can view any element $w = \bar{w} \bmod T \in W$ as the element (denoted by the corresponding German character) $\mathfrak{w}$ of G/B, defined as $\mathfrak{w} = \bar{w}B$. For any $w \in W$, there is associated the Schubert variety $X_w := \overline{B\bar{w}B/B} \subset G/B$ and the T-fixed points of X_w (under the canonical left action) are precisely the elements of $I_w := \{\mathfrak{v} : v \in W \text{ and } v \leqq w\}$.

We (together with B. Kostant) have defined a certain ring $Q_W(T)$ (which is the smash product of the group algebra $\mathbb{Z}[W]$ with the W-field $Q(T)$ of rational functions on the torus T) and certain elements $y_w \in Q_W(T)$ (for any $w \in W$). Expressing the elements y_w in the $\{\delta_v\}_{v \in W}$ basis:

$$y_w = \sum b_{w^{-1}, v^{-1}} \delta_v ,$$

we get the matrix $B = (b_{w^{-1}, v^{-1}})_{w, v \in W}$ with entries in $Q(T)$ (cf. Definition 2.1(d)). Analogously, we defined the nil Hecke ring Q_W (which is the smash product of the group algebra $\mathbb{Z}[W]$ with the W-field $Q(\mathfrak{h})$ of rational functions on the Cartan subalgebra $\mathfrak{h}$) and certain elements $x_w \in Q_W$. Writing

$$x_w = \sum c_{w^{-1}, v^{-1}} \delta_v ,$$

we get another matrix $C = (c_{w^{-1}, v^{-1}})_{w, v \in W}$ with entries in $Q(\mathfrak{h})$ (cf. Definition 3.1(b)).

We prove that the formal T-character of the ring of functions on the scheme theoretic tangent cone $T_{\mathfrak{v}}(X_w)$ (for any $\mathfrak{v} \in I_w$) is nothing but $*b_{w^{-1}, v^{-1}}$ (cf. Theorem 2.2), where $*$ is the involution of $Q(T)$ given by $e^\lambda \mapsto e^{-\lambda}$. This sharpens a result due to Rossmann [R]. In fact this work of Rossmann, and our own work with B. Kostant on the equivariant K-theory of flag varieties, motivated our current work. The proof of Theorem (2.2) requires the Demazure character formula, and occupies Sect. 2 of this paper. We use this theorem to prove that $b_{w^{-1},v^{-1}} \neq 0$ if and only if $v \leqq w$, and in this case it has a pole of order exactly equal to $\ell(w)$. Similarly $c_{w^{-1},v^{-1}} \neq 0$ if and only if $v \leqq w$ (cf. Corollaries 3.2).

We study the graded algebra structure on the space of functions $\mathrm{gr}(\mathcal{O}_{\mathfrak{v},X_w})$ of the scheme theoretic tangent cone $T_{\mathfrak{v}}(X_w)$ in Sect. 4. Our principal result in this direction is Theorem (4.4), which roughly asserts that the graded algebra Gr $(\mathcal{O}_{\mathfrak{v},X_w})$ arises from the natural filtration of the Demazure module $v^{-1}V_w(\lambda)$ induced from the standard filtration of the universal enveloping algebra $U(\mathfrak{u}^-)$, where $\mathfrak{u}^-$ is the nil-radical of the opposite Borel subalgebra and $V_w(\lambda)$ is defined in Sect. 1. We use this theorem to derive a result due to Carrell–Peterson asserting that for simply-laced G, a point $\mathfrak{v} \in X_w$ is rationally smooth if and only if the reduced tangent cone $T_{\theta}^{\mathrm{red}}(X_w)$ is an affine space for all $v \leqq \theta \leqq w$ (cf. Corollary 4.11).

The main result of our paper is a necessary and sufficient condition for a point $\mathfrak{v} \in X_w$ to be smooth, in terms of the matrix entry $c_{w^{-1},v^{-1}}$ (cf. Theorem 5.5 (b)). This result asserts that for any $v \leqq w \in W$, the point $\mathfrak{v} \in X_w$ is smooth $\Leftrightarrow$

$$c_{w^{-1},v^{-1}} = (-1)^{\ell(w)-\ell(v)} \prod_{\beta \in S(w^{-1},v^{-1})} \beta^{-1},$$

where $S(w^{-1},v^{-1}) := \{\alpha \in \Delta_+ : v^{-1}r_\alpha \leqq w^{-1}\}$, and r_α is the reflection corresponding to the root α.

There is a very similar criterion for a point $\mathfrak{v} \in X_w$ to be rationally smooth (cf. Theorem 5.5(a)). This criterion of rational smoothness can be easily deduced by combining some results of Dyer and Carrell–Peterson, but we give a different geometric proof as that proof is used crucially to prove our criterion of smoothness mentioned above (i.e. Theorem 5.5(b)).

It should be mentioned that the elements $c_{w^{-1},v^{-1}}$ (as well as $b_{w^{-1},v^{-1}}$) are defined combinatorially and admit closed expressions (cf. Lemma 3.5).

The nil Hecke ring approach to singularity, developed in this paper, is applied to some specific examples discussed in Sects. 6 and 7. In Sect. 6, we determine the precise singular locus of any Schubert variety in any rank-2 group (cf. Proposition 6.1). I believe this result should be well known, but I did not find it explicitly written down in the literature. In Sect. 7, we use our Theorem (5.5) to study the smoothness (and rational smoothness) of codimension one Schubert varieties X_i in any G/B. Proposition (7.5) (resp. Corollary 7.8) gives a criterion for a point $\mathfrak{v} \in X_i$ to be smooth (resp. rationally smooth). This criterion is applied to give a complete list of codimension one smooth (as well as rationally smooth) Schubert varieties in any G/B (cf. Proposition 7.10).

Finally in Sect. 8, we extend our main result giving the criterion of smoothness to arbitrary (not even symmetrizable) Kac–Moody groups (cf. Theorem 8.9). We also extend our result determining the formal character of the ring of functions on the scheme theoretic tangent cone at any $\mathfrak{v} \in X_w$ to arbitrary Kac–Moody groups (cf. Theorem 8.6). The proofs in the Kac–Moody case are similar to the finite case, and hence we have been brief and outlined only the necessary changes.

There are other criteria for smoothness, due to Lakshmibai–Seshadri (for classical groups) [LS] [L], Ryan (for $SL(n)$) [Ry], Wolper (again for $SL(n)$) [W1] [W2], ...; and for rational smoothness due to Jantzen [J], Kazhdan–Lusztig [KL], Carrell–Peterson [C1], ...; and by works of Deodhar and Peterson rational smoothness implies smoothness for simply-laced groups. It may be mentioned that our criterion for smoothness (as in Theorem 5.5(b)) is applicable to all G uniformly, in contrast to the above mentioned criteria for smoothness. We refer the reader to two survey articles, one by Carrell [C2], and the other by Deodhar [D2].

The main results of this paper were announced in [Ku2].

1. Notation

For a complex vector space V (possibly infinite dimensional), V^* denotes its full vector space dual. For a finite set S, $\#S$ denotes its cardinality. Unless otherwise stated, we take the base field to be the field of complex numbers.

Let G be a semisimple simply-connected (in particular connected) complex algebraic group, and let B be a fixed Borel subgroup and $T \subset B$ a maximal torus. Let B^- be the (opposite) Borel subgroup such that $B^- \cap B = T$. We denote by U (resp. U^-) the unipotent radical of B (resp. B^-). Let $\mathfrak{g}, \mathfrak{b}, \mathfrak{b}^-, \mathfrak{u}, \mathfrak{u}^-, \mathfrak{h}$ be the Lie algebras of the groups G, B, B^-, U, U^-, T respectively. Let $\Delta \subset \mathfrak{h}^*$ (resp. Δ_+) denote the set of roots for the pair (G, T) (resp. (B, T)). Let $\{\alpha_1, \ldots, \alpha_n\}$ be the set of simple roots in Δ_+ and let $\{\alpha_1^\vee, \ldots, \alpha_n^\vee\}$ be the corresponding (simple) coroots (where $n = \text{rank } G$).

Let $W := N(T)/T$ be the Weyl group (where $N(T)$ is the normalizer of T in G) of G. Then W is a Coxeter group, generated by the simple reflections $\{r_1, \ldots, r_n\}$ (where r_i is the reflection corresponding to the simple root α_i). In particular, we can talk of the length $\ell(w)$ of any element $w \in W$. We denote the identity element of W by e.

Let $\mathfrak{h}_{\mathbb{Z}}^* := \{\lambda \in \mathfrak{h}^* : \lambda(\alpha_i^\vee) \in \mathbb{Z}, \text{ for all } i\}$ be the set of integral weights and $D := \{\lambda \in \mathfrak{h}_{\mathbb{Z}}^* : \lambda(\alpha_i^\vee) \geqq 0, \text{ for all } i\}$ (resp. $D^o := \{\lambda \in \mathfrak{h}_{\mathbb{Z}}^* : \lambda(\alpha_i^\vee) > 0, \text{ for all } i\}$) the set of dominant (resp. dominant regular) integral weights. For any $\lambda \in D$ and $w \in W$, we denote by $V(\lambda)$ the irreducible representation of G with highest weight λ, and $V_w(\lambda)$ is the smallest B-submodule of $V(\lambda)$ containing the extremal weight vector $e_{w\lambda}$ (of weight $w\lambda$). Let $R(T) := \mathbb{Z}[X(T)]$ be the group algebra of the character group $X(T)$ of the torus T. Then $\{e^\lambda\}_{\lambda \in \mathfrak{h}_{\mathbb{Z}}^*}$ are precisely the elements of $X(T)$. Let $Q(T)$ be the quotient field of $R(T)$. Clearly W acts on $Q(T)$ and moreover $Q(T)$ admits an involution $*$ (i.e. a field automorphism of order 2) taking $e^\lambda \mapsto e^{-\lambda}$.

For any $w \in W$, the Schubert variety X_w is by definition the closure $\overline{BwB/B}$ of BwB/B in G/B under the Zariski topology (where the notation BwB/B means $B\bar{w}B/B$ for any representative $\bar{w}$ of w in $N(T)$). Then X_w is an irreducible (projective) subvariety of G/B of dimension $\ell(w)$. We can view any element $w = \bar{w} \bmod T \in W$ as the element (denoted by the corresponding German character) $\mathfrak{w}$ of G/B, defined as $\mathfrak{w} = \bar{w}B$. By the Bruhat decomposition, any $\mathfrak{v}$ such that $v \leqq w$ belongs to X_w, where $\leqq$ is the Bruhat (or Chevalley) partial order in W. The Schubert variety X_w is clearly B-stable (in particular T-stable) under the left multiplication of B on G/B. The T-fixed points of X_w are precisely $I_w := \{\mathfrak{v} : v \in W \text{ and } v \leqq w\}$. For any variety X over $\mathbb{C}$, we denote by $\mathbb{C}[X]$ the ring of global regular functions $X \to \mathbb{C}$. For any $\lambda \in \mathfrak{h}^*_{\mathbb{Z}}$, let $\mathbb{C}_\lambda$ be the 1-dimensional representation of B given by the character e^λ and let $\mathfrak{L}(\lambda)$ be the line bundle on G/B associated to the principal B-bundle $G \to G/B$ via the representation $\mathbb{C}_{-\lambda}$ of B.

2. Character of the ring of functions on the tangent cones of X_w

We follow the notation as in Sect. 1.

(2.1) Definitions. (a) *For any local ring R with maximal ideal* $\mathfrak{m}$, *define the graded $R/\mathfrak{m}$-algebra*:

$$\operatorname{gr} R := \sum_{n \geqq 0} \mathfrak{m}^n/\mathfrak{m}^{n+1} .$$

Let X be a scheme of finite type over an algebraically closed field and let x be a closed point of X. Then the tangent cone $T_x(X)$ *of X at x is, by definition* (cf. [M, Chapter 3, Sect. 3]), Spec $(\operatorname{gr} \mathcal{O}_x)$, *where $\mathcal{O}_x = \mathcal{O}_{x,X}$ is the local ring at $x \in X$.*

(b) *Let $\widetilde{R(T)}$ be the set of all the formal sums $\sum_{e^\lambda \in X(T)} n_\lambda e^\lambda$, with arbitrary $n_\lambda \in \mathbb{Z}$ (we allow infinitely many of the n_λ's to be non-zero). Even though $\widetilde{R(T)}$ is not a ring, it has a canonical $R(T)$-module structure (got by the multiplication). We define the $Q(T)$-module $\widetilde{Q(T)}$ as $Q(T) \otimes_{R(T)} \widetilde{R(T)}$. Since $Q(T)$ is a flat $R(T)$-module, $Q(T)$ canonically embeds in $\widetilde{Q(T)}$.*

(c) *A T-module M is said to be a* weight module *if $M = \bigoplus_{e^\lambda \in X(T)} M_\lambda$, where $M_\lambda := \{m \in M : tm = e^\lambda(t)m\}$ is the λ-th weight space. A weight module M is said to be an* admissible T-module *if* dim $M_\lambda < \infty$, *for all $e^\lambda \in X(T)$.*

For any admissible T-module M, one can define its formal character ch $M := \sum_{e^\lambda \in X(T)}$ (dim M_λ) e^λ *as an element of $\widetilde{R(T)}$.*

(d) The ring $Q(T)_W$ ([KK2, Section 2]): *Let $Q(T)_W$ be the smash product of the W-field $Q(T)$ with the group algebra $\mathbb{Z}[W]$, i.e., $Q(T)_W$ is a free right $Q(T)$-module with basis $\{\delta_w\}_{w \in W}$ and the multiplication is given by*:

(1) $$(\delta_{w_1} q_1) \cdot (\delta_{w_2} q_2) = \delta_{w_1 w_2} (w_2^{-1} q_1) q_2, \quad \textit{for } q_1, q_2 \in Q(T) \textit{ and } w_1, w_2 \in W .$$

For any simple reflection r_i, $1 \leqq i \leqq n$, define the element $y_{r_i} \in Q(T)_W$ by:

(2) $$y_{r_i} = (\delta_e + \delta_{r_i}) \frac{1}{(1 - e^{-\alpha_i})} .$$

Now, for any $w \in W$, define $y_w \in Q(T)_W$ by

$$y_w = y_{r_{i_1}} \cdots y_{r_{i_p}} , \tag{3}$$

where $w = r_{i_1} \cdots r_{i_p}$ is a reduced decomposition. By [KK2, Proposition 2.4], *y_w is well defined. Write*

$$y_w = \sum_v b_{w^{-1}, v^{-1}} \delta_v , \tag{4}$$

for some (unique) $b_{w^{-1},v^{-1}} \in Q(T)$. It can be easily seen that $b_{w^{-1},v^{-1}} = 0$ unless $v \leqq w$ (cf. [KK2, Proposition 2.6]).

The ring $Q(T)_W$ has a canonical representation in $Q(T)$ defined by

$$(\delta_w q_1) \cdot q_2 = w(q_1 q_2) . \tag{5}$$

It is easy to see that for any r_i, $y_{r_i} \cdot R(T) \subset R(T)$, in particular, $y_w \cdot R(T) \subset R(T)$ for any $w \in W$.

Since $\mathfrak{v} \in X_w$ is fixed under the action of T (cf. Sect. 1), the local ring $\mathcal{O}_{\mathfrak{v},X_w}$ at $\mathfrak{v} \in X_w$ is canonically a T-module.

(2.2) Theorem. *Take any $v \leqq w \in W$. Then* $\mathrm{gr}\, \mathcal{O}_{\mathfrak{v},X_w}$ *is an admissible T-module and moreover*

$$\mathrm{ch}(\mathrm{gr}\ \mathcal{O}_{\mathfrak{v},X_w}) = *b_{w^{-1},v^{-1}} ,$$

as elements of $\widetilde{Q(T)}$, where ch *(which is an element of $\widetilde{R(T)}$) is to be thought of as the element $1 \otimes$* ch *of $\widetilde{Q(T)} := Q(T) \otimes_{R(T)} \widetilde{R(T)}$.*

In particular, $\mathrm{ch}(\mathrm{gr}\ \mathcal{O}_{\mathfrak{v},X_w}) \in Q(T)$.

Before we come to the proof of Theorem (2.2), we need the following preparation.

We recall the following simple lemma without proof.

(2.3) Lemma. *Let Y be an irreducible projective variety with an ample line bundle $\mathfrak{L}$ on Y, together with a non-zero $\sigma \in H^0(Y, \mathfrak{L})$. Define the variety $Y^o := Y \backslash Z(\sigma)$, where $Z(\sigma)$ is the zero-set of σ. Then Y^o is affine and moreover for any $f \in \mathbb{C}[Y^o]$, there exists some $n > 0$ (depending upon f) such that the section $f \cdot \sigma^n$ (of $H^0(Y^o, \mathfrak{L}^{\otimes n})$) extends to an element of $H^0(Y, \mathfrak{L}^{\otimes n})$.*

(2.4) Lemma. *Given any $f \in \mathbb{C}[U^-]$, there exists a large enough $\lambda \in D$ (i.e. $\lambda(\alpha_i^\vee) \gg 0$, for all the simple coroots $\alpha_i^\vee$) and $\theta \in V(\lambda)^*$ such that*

$$f(g) = \langle \theta, g e_\lambda \rangle, \quad \text{for } g \in U^- ,$$

where e_λ is a non-zero highest weight vector of $V(\lambda)$.

Moreover, for any $v \leqq w \in W$,

$$f \text{ vanishes on } (v^{-1}\overline{BwB}) \cap U^- \Leftrightarrow \theta \in (V(\lambda)/v^{-1} V_w(\lambda))^* .$$

Proof. The first part is due to Andersen and also Cline–Parshall–Scott [CPS, Sect. 5]. However, for completeness, we give a proof.

By the Borel–Weil theorem (for any $\lambda \in D$), there is an isomorphism of G-modules $\chi : V(\lambda)^* \xrightarrow{\sim} H^0(G/B, \mathfrak{L}(\lambda))$, where for any $\phi \in V(\lambda)^*, \chi(\phi)$ is given by the section $\chi(\phi)(gB) = (g, (g^{-1}\phi)_{|\mathbb{C}e_\lambda})$ mod B. (Observe that $\mathbb{C}e_\lambda \subset V(\lambda)$ is a one-dimensional representation of B corresponding to the character e^λ and hence $(\mathbb{C}e_\lambda)^*$ corresponds to the character $e^{-\lambda}$.) Let $\phi_\lambda \in V(\lambda)^*$ be the element defined by $\phi_\lambda(e_\lambda) = 1$ and $\phi_\lambda(v) = 0$, for any weight vector $v \in V(\lambda)$ of weight $\mu \neq \lambda$. Consider $U^- \approx U^- \cdot \mathfrak{e} \subset G/B$ as an open subset and take any (ample) line bundle $\mathfrak{L}(\lambda_o)$ on G/B for $\lambda_o \in D^o$. Taking the section $\sigma = \chi(\phi_{\lambda_o})$ of $\mathfrak{L}(\lambda_o)$ and applying Lemma (2.3), we get the first part of the lemma for $\lambda = n\lambda_o$ (for some $n > 0$). (Observe that $Z(\sigma) = G/B \setminus U^- \cdot \mathfrak{e}$, since λ_o is regular.)

Let $\tilde{f} : G \to \mathbb{C}$ be the extension of f given by $\tilde{f}(g) = \langle \theta, ge_\lambda \rangle$. Then, since $v^{-1}B\bar{w}B$ is an irreducible subvariety of G, and by Bruhat decomposition $v^{-1}B\bar{w}B \cap U^-B$ is non-empty open subset of $v^{-1}B\bar{w}B$,

$$\begin{aligned} f \text{ vanishes on } v^{-1}B\bar{w}B \cap U^- &\Leftrightarrow \tilde{f} \text{ vanishes on } v^{-1}B\bar{w}B \cap (U^-B) \\ &\Leftrightarrow \tilde{f} \text{ vanishes on } v^{-1}B\bar{w}B \\ &\Leftrightarrow \tilde{f} \text{ vanishes on } v^{-1}BwB \\ &\Leftrightarrow \langle \theta, \ v^{-1}Bwe_\lambda \rangle = 0 \\ &\Leftrightarrow \langle \theta, \ v^{-1}V_w(\lambda) \rangle = 0 \,. \end{aligned}$$

This proves the lemma. □

For any $\lambda \in D$, define the map

$$\varphi_\lambda : V(\lambda)^* \otimes \mathbb{C}_\lambda \to \mathbb{C}[U^-]$$

by $\varphi_\lambda(\theta \otimes e_\lambda)(g) = \langle \theta, ge_\lambda \rangle$, for $\theta \in V(\lambda)^*$, $g \in U^-$ and $e_\lambda \in \mathbb{C}_\lambda$; where $\mathbb{C}_\lambda \subset V(\lambda)$ is identified as the highest weight space.

(2.5) Lemma. *The map φ_λ is T-equivariant with respect to the adjoint action of T on U^-, and is injective.*

Proof. For any $t \in T$,

$$\begin{aligned} \varphi_\lambda(t\theta \otimes te_\lambda)(g) &= \langle t\theta, gte_\lambda \rangle \\ &= \langle \theta, t^{-1}gte_\lambda \rangle \\ &= (t \cdot (\varphi_\lambda(\theta \otimes e_\lambda)))(g) \,. \end{aligned}$$

This proves the T-equivariance of φ_λ.

To prove the injectivity of φ_λ, take $\theta \otimes e_\lambda \in \ker \varphi_\lambda$, i.e., $\langle \theta, ge_\lambda \rangle = 0$, for all $g \in U^-$. Hence $\langle \theta, gbe_\lambda \rangle = 0$ for all $g \in U^-$ and $b \in B$. In particular, by the density of U^-B in G and the irreducibility of $V(\lambda)$, we get $\langle \theta, \ V(\lambda) \rangle = 0$, i.e., $\theta = 0$, proving the injectivity of φ_λ. □

§ **(2.6)** For any $\gamma \in D$, let us choose a highest weight vector $e_\gamma \in V(\gamma)$, and define (for any $\lambda, \mu \in D$)

$$V(\lambda + \mu) \overset{i_{\lambda,\mu}}{\hookrightarrow} V(\lambda) \otimes V(\mu) \xrightarrow{\mathrm{Id} \otimes \pi_\mu} V(\lambda) \otimes \mathbb{C}_\mu \,,$$

where $i_{\lambda,\mu}$ is the unique G-module map taking $e_{\lambda+\mu} \mapsto e_\lambda \otimes e_\mu$, Id is the identity map and $\pi_\mu : V(\mu) \to \mathbb{C}_\mu$ is the T-equivariant projection onto the highest weight space $\mathbb{C}_\mu = \mathbb{C}e_\mu \subset V(\mu)$. We denote the composite map $(\mathrm{Id} \otimes \pi_\mu) \circ i_{\lambda,\mu} : V(\lambda+\mu) \to V(\lambda) \otimes \mathbb{C}_\mu$ by $\hat{\delta}_{\lambda,\mu}$. Dualizing the above, we get the map

$$\bar{\delta}_{\lambda,\mu} : V(\lambda)^* \otimes \mathbb{C}_{-\mu} \to V(\lambda+\mu)^* \quad ,$$

and hence the map

$$\delta_{\lambda,\mu} = \bar{\delta}_{\lambda,\mu} \otimes \mathrm{Id} : V(\lambda)^* \otimes \mathbb{C}_\lambda \approx V(\lambda)^* \otimes \mathbb{C}_{-\mu} \otimes \mathbb{C}_{\lambda+\mu} \to V(\lambda+\mu)^* \otimes \mathbb{C}_{\lambda+\mu} .$$

It is easy to see that $\delta_{\lambda,\mu}$ is injective. Moreover, the following diagram is commutative:

$$\begin{tikzcd} V(\lambda)^* \otimes \mathbb{C}_\lambda \arrow[rr, hook, "\delta_{\lambda,\mu}"] \arrow[dr, "\varphi_\lambda"'] & & V(\lambda+\mu)^* \otimes \mathbb{C}_{\lambda+\mu} \arrow[dl, "\varphi_{\lambda+\mu}"] \\ & \mathbb{C}[U^-] & \end{tikzcd} .$$

By virtue of Lemma (2.4), for any $\lambda \in D$ and $v \leqq w \in W$, we get the injective map

$$\varphi_\lambda(v,w) : (v^{-1}V_w(\lambda))^* \otimes \mathbb{C}_\lambda \hookrightarrow \mathbb{C}[(v^{-1}\overline{BwB}) \cap U^-] ,$$

by restricting the map φ_λ.

(2.7) Lemma. *For $\lambda, \mu \in D$ and $v \leqq w \in W$, $\hat{\delta}_{\lambda,\mu}(v^{-1}V_w(\lambda+\mu)) = v^{-1}V_w(\lambda) \otimes \mathbb{C}_\mu$. In particular, there exists a unique map $\delta_{\lambda,\mu}(v,w)$ making the following diagram commutative*:

$$\begin{tikzcd} V(\lambda)^* \otimes \mathbb{C}_\lambda \arrow[r] \arrow[d, "\delta_{\lambda,\mu}"] & (v^{-1}V_w(\lambda))^* \otimes \mathbb{C}_\lambda \arrow[d, "\delta_{\lambda,\mu}(v,w)"] \\ V(\lambda+\mu)^* \otimes \mathbb{C}_{\lambda+\mu} \arrow[r] & (v^{-1}V_w(\lambda+\mu))^* \otimes \mathbb{C}_{\lambda+\mu}, \end{tikzcd}$$

where the horizontal maps are the canonical restriction maps. Moreover, $\delta_{\lambda,\mu}(v,w)$ is injective.

Proof. For $b \in B$,

$$(1) \qquad \hat{\delta}_{\lambda,\mu}(\bar{v}^{-1}b\bar{w}e_{\lambda+\mu}) = \bar{v}^{-1}b\bar{w}e_\lambda \otimes [\bar{v}^{-1}b\bar{w}e_\mu]_\mu ,$$

where $[x]_\mu$ denotes the component of $x \in V(\mu)$ in the μ^{th} weight space, and $\bar{v}$ is a representative of v in $N(T)$. Define the closed subvariety $Y \subset B$ by $Y = \{b \in B : [\bar{v}^{-1}b\bar{w}e_\mu]_\mu = 0\}$. Then $Y \neq B$, for otherwise $e_\mu \notin v^{-1}V_w(\mu)$, which is a contradiction (since $v \leqq w$ by assumption). Now for $b \in B \backslash Y$, $\hat{\delta}_{\lambda,\mu}(\bar{v}^{-1}b\bar{w}e_{\lambda+\mu}) = \bar{v}^{-1}b\bar{w}e_\lambda \otimes e_\mu$, up to a non-zero scalar. But since $\hat{\delta}_{\lambda,\mu}(v^{-1}V_w(\lambda+\mu))$ is a (closed) linear subspace and $B \backslash Y$ is dense in B,

$$v^{-1}V_w(\lambda) \otimes \mathbb{C}_\mu \subset \hat{\delta}_{\lambda,\mu}(v^{-1}V_w(\lambda+\mu)) .$$

The inverse inclusion is clear from (1). This proves the first part of the lemma. The 'in particular' statement follows immediately from dualizing the map

$$\hat{\delta}_{\lambda,\mu}|_{v^{-1}V_w(\lambda+\mu)} : v^{-1}V_w(\lambda+\mu) \longrightarrow v^{-1}V_w(\lambda) \otimes \mathbb{C}_\mu .$$

The injectivity of $\delta_{\lambda,\mu}(v,w)$ follows from the surjectivity of $\hat{\delta}_{\lambda,\mu}|_{v^{-1}V_w(\lambda+\mu)}$. $\square$

By virtue of the above lemma, we get the following commutative diagram:

$$\begin{tikzcd}
(v^{-1}V_w(\lambda))^* \otimes \mathbb{C}_\lambda \arrow[rr, hook, "\delta_{\lambda,\mu}(v,w)"] \arrow[dr, "\varphi_\lambda(v,w)"'] & & (v^{-1}V_w(\lambda+\mu))^* \otimes \mathbb{C}_{\lambda+\mu} \arrow[dl, "\varphi_{\lambda+\mu}(v,w)"] \\
& \mathbb{C}[v^{-1}\overline{BwB} \cap U^-] . &
\end{tikzcd}$$

(2.8) Definition. *Define a partial order $\prec$ in D as follows:*

$$\lambda \prec \mu \Leftrightarrow \mu - \lambda \in D .$$

Taking the limit of the maps $\varphi_\lambda(v,w)$, we get the T-equivariant map

$$\varphi(v,w) \colon \operatorname*{limit}_{\lambda \in D} {}_{\rightarrow}((v^{-1}V_w(\lambda))^* \otimes \mathbb{C}_\lambda) \longrightarrow \mathbb{C}[v^{-1}\overline{BwB} \cap U^-] .$$

(2.9) Proposition. *The above map $\varphi(v,w)$ is an isomorphism, for all $v \leqq w \in W$.*

Proof. Injectivity of the map $\varphi(v,w)$ is clear from the injectivity of the maps $\varphi_\lambda(v,w)$. Surjectivity of $\varphi(v,w)$ follows from Lemma (2.4). $\square$

(2.10) Definition. *For any directed set Λ and any sequence $\theta : \Lambda \to \widetilde{R(T)}$, given as $\theta(\alpha) = \sum_{e^\lambda \in X(T)} n_\lambda(\alpha) e^\lambda$ with $n_\lambda(\alpha) \in \mathbb{Z}$, we say that $\operatorname*{limit}_{\alpha \in \Lambda} \theta(\alpha) = \sum n_\lambda e^\lambda$, if for any $e^\lambda \in X(T)$, there exists $\alpha_\lambda \in \Lambda$ such that $n_\lambda(\alpha) = n_\lambda$ for all $\alpha \geqq \alpha_\lambda$. Of course $\operatorname*{limit}_{\alpha \in \Lambda} \theta(\alpha)$ may not exist in general.*

Observe that if $\operatorname*{limit}_{\alpha \in \Lambda} \theta(\alpha)$ exists, then so is $\operatorname*{limit}_{\alpha \in \Lambda} (p\theta(\alpha))$, for any fixed $p \in R(T)$. Moreover

$$\operatorname*{limit}_{\alpha \in \Lambda} (p\theta(\alpha)) = p \operatorname*{limit}_{\alpha \in \Lambda} \theta(\alpha) \quad . \tag{1}$$

(2.11) Corollary. ch $\mathbb{C}[v^{-1}\overline{BwB} \cap U^-] = \operatorname*{limit}_{\lambda \in D} (\delta_{v^{-1}} \cdot (e^{v\lambda} * (y_w \cdot e^\lambda)))$.

Proof. By the previous proposition and the Demazure character formula (cf. [A], [Jo2], [Ra2, Remarks 4.4], [Se], [Ku1, Theorem 3.4], [Ma]),

$$\begin{aligned}
\operatorname{ch} \mathbb{C}[v^{-1}\overline{BwB} \cap U^-] &= \operatorname*{limit}_{\lambda \in D} (\delta_{v^{-1}} \cdot (e^{v\lambda} \operatorname{ch} (V_w(\lambda)^*))) \\
&= \operatorname*{limit}_{\lambda \in D} (\delta_{v^{-1}} \cdot (e^{v\lambda} * (y_w \cdot e^\lambda))) .
\end{aligned}$$

Observe that the existence of the above limit is guaranteed by Proposition (2.9) and the fact that $\mathbb{C}[v^{-1}\overline{BwB} \cap U^-]$ is an admissible T-module (being quotient of $\mathbb{C}[U^-]$). $\square$

Finally we come to the proof of Theorem (2.2).

§ **(2.12)** *Proof of Theorem (2.2).* Write (cf. (4) of Sect. 2.1)

$$y_w = \sum_{u \leqq w} b_{w^{-1}, u^{-1}} \delta_u .$$

Then

$$(1)\qquad e^{v\lambda} * (y_w \cdot e^{\lambda}) = \sum_u (*b_{w^{-1},u^{-1}})\, e^{v\lambda-u\lambda}$$
$$= *b_{w^{-1},v^{-1}} + \sum_{\substack{u \neq v\\ u \leqq w}} (*b_{w^{-1},u^{-1}})\, e^{v\lambda-u\lambda}\,.$$

For any (regular) weight $\lambda_o \in D^o$, $v\lambda_o - u\lambda_o \neq 0$ for $u \neq v$. From the definition of $b_{w^{-1},u^{-1}}$, it is easy to see that there exist positive roots $\{\beta_1, \ldots, \beta_\ell\}$ depending on w (possibly with repetitions) such that

$$(2)\qquad P * b_{w^{-1},u^{-1}} \in R(T) \quad \text{for all } u \leqq w\;,$$

where $P := \prod_{k=1}^{\ell}(1 - e^{-\beta_k})$.

Fix $\lambda_o \in D^o$. Then the subset $\{n\lambda_o\}_{n \geqq 1} \subset D$ being cofinal in D under $\prec$,

$$(3)\qquad \operatorname*{limit}_{\lambda \in D} (e^{v\lambda} * (y_w \cdot e^{\lambda})) = \operatorname*{limit}_{n \to \infty} (e^{nv\lambda_o} * (y_w \cdot e^{n\lambda_o}))\,.$$

Then by (1) of Definition (2.10) and (1)–(3) as above, we get

$$P \operatorname*{limit}_{\lambda \in D} (e^{v\lambda} * (y_w \cdot e^{\lambda})) = \operatorname*{limit}_{n \to \infty} (P(e^{nv\lambda_o} * (y_w \cdot e^{n\lambda_o})))$$
$$= \operatorname*{limit}_{n \to \infty} (P * b_{w^{-1},v^{-1}} + \sum_{\substack{u \neq v\\ u \leqq w}} (P * b_{w^{-1},u^{-1}})\, e^{n(v\lambda_o - u\lambda_o)})$$
$$= P * b_{w^{-1},v^{-1}} + \sum_{\substack{u \neq v\\ u \leqq w}} (P * b_{w^{-1},u^{-1}}) \operatorname*{limit}_{n \to \infty} (e^{n(v\lambda_o - u\lambda_o)})$$
$$= P * b_{w^{-1},v^{-1}}\,.$$

So, we get (in the $Q(T)$-module $\widetilde{Q(T)}$)

$$(4)\qquad 1 \otimes \operatorname*{limit}_{\lambda \in D} (e^{v\lambda} * (y_w \cdot e^{\lambda})) = *b_{w^{-1},v^{-1}}\,.$$

So, by Corollary (2.11) and Identity (4), we get

$$\operatorname{ch} \mathbb{C}[v^{-1}\overline{BwB} \cap U^-] = \delta_{v^{-1}} \cdot (*b_{w^{-1},v^{-1}})\,.$$

But the variety $v^{-1}\overline{BwB} \cap U^-$ provides an affine open neighborhood of the point $\mathrm{e} \in v^{-1}X_w$. In particular,

$$\operatorname{gr} \mathcal{O}_{\mathrm{e},\, v^{-1}X_w} \cong \operatorname{gr} \mathbb{C}[v^{-1}\overline{BwB} \cap U^-]\,.$$

The theorem now follows from the complete reducibility of the T-module $\mathbb{C}[v^{-1}\overline{BwB} \cap U^-]$, by translating the variety $v^{-1}X_w$ under $\bar{v}$. □

(2.13) *Remarks.* (1) This theorem was obtained by the author in 1987 and privately circulated in the preprint "A connection of equivariant K-theory with the singularity of Schubert varieties".

(2) A different proof of the theorem was subsequently given by Bressler. M. Brion mentioned to me that he also obtained a proof of this theorem (unpublished), by using some results of Baum–Fulton–Quart.

3. Some consequences of Theorem (2.2)

After the following definitions, we give some of the corollaries of Theorem (2.2).

(3.1) Definitions. (a) *For any* $\ell \in \mathbb{Z}_+ := \{0,1,2,\ldots\}$ *and any* $a = \sum n_\lambda e^\lambda \in R(T)$, *denote by* $(a)_\ell = \sum n_\lambda(\lambda^\ell/\ell!) \in S^\ell(\mathfrak{h}^*)$, *where* $S^\ell(\mathfrak{h}^*)$ *is the space of homogeneous polynomials of degree* ℓ *on* $\mathfrak{h}$. *Further, denote by* $[a] = (a)_{\ell_0}$ *where* ℓ_0 *is the smallest element of* $\mathbb{Z}_+$ *such that* $(a)_{\ell_0} \neq 0$. *(If a itself is* 0, *we define* $[a] = 0$.) *Now for* $q = \frac{a}{b} \in Q(T)$, *where* $a,b \in R(T)$, *we define* $[q] = \frac{[a]}{[b]} \in Q(\mathfrak{h})$ *(the quotient field of the symmetric algebra* $S(\mathfrak{h}^*)$). *Clearly* $[q]$ *is well defined.*

When $q \neq 0$ *and* $\deg [a] \leqq \deg [b]$, *we say that* q *has a pole (at the identity* e) *of order* $= \deg [b] - \deg [a]$. *It is easy to see that* $b_{w^{-1},v^{-1}}$ (cf. (4) *of* Sect. 2.1), *when non-zero, has a pole of order* $\leqq \ell(w)$.

(b) The nil Hecke ring Q_W ([KK1, Sect. 4]): *Let* Q_W *be the smash product of the* W*-field* $Q(\mathfrak{h})$ *with the group algebra* $\mathbb{Z}[W]$, *with the product given by the same formula* (1) *in* Sect. 2.1. *For any simple reflection* r_i, $1 \leqq i \leqq n$, *define* $x_{r_i} \in Q_W$ *by*

$$x_{r_i} = -(\delta_{r_i} + \delta_e)\frac{1}{\alpha_i}.$$

Now, for any $w \in W$, *define* $x_w = x_{r_{i_1}} \ldots x_{r_{i_p}}$, *where* $w = r_{i_1} \ldots r_{i_p}$ *is a reduced decomposition. The element* x_w *is well defined by* [KK1, Proposition 4.2]. *Write, as in* [KK1, Proposition 4.3],

$$x_w = \sum_v c_{w^{-1},v^{-1}}\ \delta_v, \text{ for some (unique) } c_{w^{-1},v^{-1}} \in Q(\mathfrak{h}) .$$

(3.2) Corollaries (of Theorem 2.2). *For any* $v, w \in W$:

(a) $b_{w^{-1},v^{-1}} \neq 0$ *if and only if* $v \leqq w$; *and in this case it has a pole of order exactly equal to* $\ell(w)$. *Further,*

$$\left(\prod_{\beta\in\Delta_+} (1 - e^\beta)\right) b_{w^{-1},v^{-1}} \in R(T) . \tag{1}$$

(b) $[*b_{w^{-1},v^{-1}}] = c_{w^{-1},v^{-1}}$; *and hence for any* $v \leqq w$,

$$[\operatorname{ch}(\operatorname{gr} \mathcal{O}_{v,X_w})] = c_{w^{-1},v^{-1}} ,$$

as elements of $Q(\mathfrak{h})$.

In particular, $c_{w^{-1},v^{-1}} \neq 0$ *if and only if* $v \leqq w$.

Further

$$\left(\prod_{\beta\in\Delta_+} \beta\right) c_{w^{-1},v^{-1}} \in S(\mathfrak{h}^*) . \tag{2}$$

(For a strenghthening of (1) and (2), see Remark 3.4(3) and Lemma 5.4.)

Proof. As observed in Sect. 2.1(d), $b_{w^{-1},v^{-1}} = 0$ unless $v \leqq w$. So let us assume that $v \leqq w$. Set $\mathscr{A}^v = vU^-e \subset G/B$. Since $\mathscr{A}^v \cap X_w$ is a closed subvariety of the affine space $\mathscr{A}^v$, $\mathrm{Tor}_p^{\mathbb{C}[\mathscr{A}^v]}(\mathbb{C}[\mathscr{A}^v \cap X_w], \mathbb{C})$ is a finite dimensional vector space over $\mathbb{C}$ (cf. proof of Lemma 5.4) for any p and moreover ($\mathscr{A}^v$ being smooth) is 0 for large enough p. Set

$$F = \sum_p (-1)^p \operatorname{ch}(\mathrm{Tor}_p^{\mathbb{C}[\mathscr{A}^v]}(\mathbb{C}[\mathscr{A}^v \cap X_w], \mathbb{C})) \in R(T) .$$

Then from the Koszul complex we get,

$$\left(\prod_{\beta \in \Delta_+} (1 - e^{v\beta})\right) \operatorname{ch} \mathbb{C}[\mathscr{A}^v \cap X_w] = F, \text{ as elements of } \widetilde{R(T)} . \tag{3}$$

It can be easily seen that the coefficient of e^0 in the left side of the above identity is non-zero, in particular, $F \neq 0$. From (3) we obtain

(4)

$$\begin{aligned} 1 \otimes \operatorname{ch} \mathbb{C}[\mathscr{A}^v \cap X_w] &= F \prod_{\beta \in \Delta_+} (1 - e^{v\beta})^{-1} \\ &= F\left(\prod_{\gamma \in \Delta_+ \cap v\Delta_-} (-e^{\gamma})\right) \prod_{\beta \in \Delta_+} (1 - e^{\beta})^{-1}, \text{ as elements of } \widetilde{Q(T)} . \end{aligned}$$

From (4) it is clear that $1 \otimes \operatorname{ch} \mathbb{C}[\mathscr{A}^v \cap X_w] \neq 0$ as an element of $\widetilde{Q(T)}$. Moreover, since $\mathscr{A}^v \cap X_w$ is an affine neighborhood of $\mathfrak{v}$ in X_w, we get

$$\operatorname{ch} \mathbb{C}[\mathscr{A}^v \cap X_w] = \operatorname{ch} (\operatorname{gr} \mathcal{O}_{\mathfrak{v}, X_w}) . \tag{5}$$

But then by (5) and Theorem (2.2), we get that $b_{w^{-1},v^{-1}} \neq 0$. The assertion that $b_{w^{-1},v^{-1}}$ has a pole of order exactly equal to $\ell(w)$ (whenever $v \leqq w$) follows from a lemma of Joseph [Jo1, Sect. 2.3]. This proves the first part of Corollary (a). Assertion (1) of part (a) follows immediately from (4), (5) (and Theorem 2.2).

To prove part (b), in view of Theorem (2.2), we only need to show that

$$[{*}b_{w^{-1},v^{-1}}] = c_{w^{-1},v^{-1}} \ : \tag{6}$$

By induction (on $\ell(w)$) we assume the validity of (6) for any w with $\ell(w) \leqq k$ and any $v \in W$, and take $w' = wr_i$ of length $k+1$, where r_i is a simple reflection such that $\ell(w') > \ell(w)$. (The case $w = e$ is obviously true.) By Definition 2.1 (d),

$$\begin{aligned} y_{w'} &:= y_w y_{r_i} \\ &= \left(\sum_v b_{w^{-1},v^{-1}} \delta_v\right) (\delta_e + \delta_{r_i}) \left(\frac{1}{1 - e^{-\alpha_i}}\right) \\ &= \sum_v \frac{b_{w^{-1},v^{-1}} + b_{w^{-1},r_i v^{-1}}}{1 - e^{-v\alpha_i}} \delta_v . \end{aligned}$$

This gives, for any $v \in W$,

$$b_{r_i w^{-1},v^{-1}} = \frac{b_{w^{-1},v^{-1}} + b_{w^{-1},r_i v^{-1}}}{1 - e^{-v\alpha_i}} . \tag{7}$$

Exactly the same way, using the definitions from Sect. 3.1(b), we obtain:

$$c_{r_iw^{-1},v^{-1}} = \frac{c_{w^{-1},v^{-1}} + c_{w^{-1},r_iv^{-1}}}{-v\alpha_i}. \tag{8}$$

By (7) and part (a) of the corollary we get:

$$[*b_{r_iw^{-1},v^{-1}}] = \frac{[*b_{w^{-1},v^{-1}}] + [*b_{w^{-1},r_iv^{-1}}]}{-v\alpha_i}.$$

Hence by the induction hypothesis (using (8)), (6) follows for $w' = wr_i$. This completes the proof of Corollaries (3.2). □

(3.3) Definition. *For any $v \leqq w \in W$, define $S(w,v) = \{\alpha \in \Delta_+ : vr_\alpha \leqq w\}$, where $r_\alpha \in W$ is the reflection defined by $r_\alpha(\lambda) = \lambda - \langle\lambda,\alpha^\vee\rangle\alpha$. Then as is easy to see $\#S(w,v) = \#S(w^{-1},v^{-1})$.*

(3.4) *Remarks.* (1) The (b)-part of the above corollaries (3.2) is due to Rossmann [R, Sect. 3.2]. In fact, this motivated our Theorem (2.2).

(2) The assertions (1) and (2) as in corollaries (3.2) can be derived purely algebraically (cf. [KK2, Corollary 4.18 and Remark 4.17(b)]).

(3) (due to Referee) The assertion (in the first sentence of 3.2(a) and its generalization to the Kac–Moody case as in Theorem 8.6(b)) that '$b_{w^{-1},v^{-1}} \neq 0$ if and only if $v \leqq w$ and in this case it has a pole of order exactly equal to $\ell(w)$,' and also the assertion '$[*b_{w^{-1},v^{-1}}] = c_{w^{-1},v^{-1}}$ (in 3.2(b) and 8.6(c))' follows easily by induction using (3.2) (7)–(8) and [Dy, Proposition (1)]. Further, it follows from loc. cit. that (for any $v \leqq w$)

$$\left(\prod_{\beta\in S(w^{-1},v^{-1})} \beta\right) c_{w^{-1},v^{-1}} \in S(\mathfrak{h}^*),$$

thereby strengthening 3.2(2). Also, essentially the same proof as for [Dy, Proposition (1)] can be used to obtain the following strengthening of 3.2(1) (including in the Kac–Moody case):

$$\left(\prod_{\beta\in S(w^{-1},v^{-1})} (1-e^\beta)\right) b_{w^{-1},v^{-1}} \in R(T).$$

(See Sect. 5.4 for a geometric proof of this.)

The following lemma gives an expression for $b_{w,v}$ (and $c_{w,v}$) and can be easily proved by using the definitions.

(3.5) Lemma. *Fix any $v \leqq w \in W$, and take a reduced decomposition $w = r_{i_1}\dots r_{i_p}$. Then*

$$b_{w^{-1},v^{-1}} = \sum((1-e^{-r_{i_1}^{\varepsilon_1}\alpha_{i_1}})(1-e^{-r_{i_1}^{\varepsilon_1}r_{i_2}^{\varepsilon_2}\alpha_{i_2}})\cdots(1-e^{-r_{i_1}^{\varepsilon_1}\cdots r_{i_p}^{\varepsilon_p}\alpha_{i_p}}))^{-1}.$$

Similarly

$$c_{w^{-1},v^{-1}} = (-1)^p\sum((r_{i_1}^{\varepsilon_1}\alpha_{i_1})(r_{i_1}^{\varepsilon_1}r_{i_2}^{\varepsilon_2}\alpha_{i_2})\cdots(r_{i_1}^{\varepsilon_1}\cdots r_{i_p}^{\varepsilon_p}\alpha_{i_p}))^{-1},$$

where both the sums run over all those $(\varepsilon_1, \ldots, \varepsilon_p) \in \{0,1\}^p$ *satisfying* $r_{i_1}^{\varepsilon_1} \ldots r_{i_p}^{\varepsilon_p} = v$. *(The notation* r_i^0 *means the identity element.)*

4. Ring of functions on the tangent cone–the graded algebra structure

§ **(4.1)** For any $\lambda \in D$, the (finite dimensional) G-module $V(\lambda)$ admits a filtration $\{\mathscr{F}_p(\lambda)\}_{p \geqq 0}$ as follows:

Let $\{U_p(\mathfrak{u}^-)\}_{p \geqq 0}$ be the standard filtration of the universal enveloping algebra $U(\mathfrak{u}^-)$, where we recall that $U_p(\mathfrak{u}^-)$ is the span of the monomials $X_1 \ldots X_m$ for $X_i \in \mathfrak{u}^-$ and $m \leqq p$. Now set

$$\mathscr{F}_p(\lambda) = U_p(\mathfrak{u}^-) \cdot e_\lambda,$$

where e_λ is any non-zero highest weight vector in $V(\lambda)$.

Fix $\lambda \in D$, $v \leqq w \in W$, $\theta \in V(\lambda)^*$, and a highest weight vector $e_\lambda \in V(\lambda)$. Recall the definition of the function φ_λ from Sect. 2.4. We abbreviate $\varphi_\lambda(\theta \otimes e_\lambda)$ by φ^θ. Thus $\varphi^\theta : U^- \to \mathbb{C}$ is the function

$$\varphi^\theta(g) = \langle \theta, g e_\lambda \rangle, \quad \text{for } g \in U^-.$$

By Lemma (2.4), φ^θ vanishes on $v^{-1}\overline{BwB} \cap U^- \Leftrightarrow \langle \theta, v^{-1} V_w(\lambda) \rangle = 0$. Identify U^- with the affine space $\mathfrak{u}^-$ under the exponential map. This gives rise to a gradation on $\mathbb{C}[U^-]$. Now let φ_d^θ be the dth graded component of φ^θ (for any $d \geqq 0$), i.e.,

$$(*) \qquad \varphi_d^\theta(X) = \frac{1}{d!} \langle \theta, X^d e_\lambda \rangle, \quad \text{for } X \in \mathfrak{u}^- .$$

The following lemma follows immediately from $(*)$, if we use the fact that for any vector space V, its pth symmetric power $S^p(V)$ is spanned by $\{v^p\}_{v \in V}$.

(4.2) Lemma. *Fix* $p \geqq 1$. *Then for any* θ *as above (i.e.* $\langle \theta, v^{-1} V_w(\lambda) \rangle = 0$ *),* $\varphi_d^\theta \equiv 0$ *for all* $0 \leqq d < p$ *if and only if* $\langle \theta, v^{-1} V_w(\lambda) + \mathscr{F}_{p-1}(\lambda) \rangle = 0$.

For any $p \geqq 0$ and any closed subvariety $0 \in Y \subset \mathbb{A}^n$, let $\mathscr{I}_p(Y)$ denote the set of degree pth components of all those functions f in the ideal $\mathscr{I}(Y)$ of $Y \subset \mathbb{A}^n$, such that the dth homogeneous component f_d of f is 0 for all $d < p$.

As an immediate consequence of the above lemma, we get the following.

(4.3) Corollary. *For any* $p \geqq 0$, *the map* $\theta \otimes e_\lambda \mapsto (\varphi_\lambda(\theta \otimes e_\lambda))_p$ *induces a* T*-equivariant injective map*

$$\left(\frac{v^{-1} V_w(\lambda) + \mathscr{F}_p(\lambda)}{v^{-1} V_w(\lambda) + \mathscr{F}_{p-1}(\lambda)} \right)^* \otimes \mathbb{C}_\lambda \overset{f_p(v,w;\lambda)}{\hookrightarrow} \mathscr{I}_p(v^{-1}\overline{BwB} \cap U^-),$$

where $\mathbb{C}_\lambda \subset V(\lambda)$ *is the highest weight subspace, and* $\mathscr{F}_{-1}(\lambda)$ *is defined to be* 0.

It is easy to see that under the map $\hat{\delta}_{\lambda,\mu} : V(\lambda+\mu) \longrightarrow V(\lambda) \otimes \mathbb{C}_\mu$ (of Sect. 2.6), the image $\hat{\delta}_{\lambda,\mu}(\mathscr{F}_p(\lambda+\mu)) = \mathscr{F}_p(\lambda) \otimes \mathbb{C}_\mu$. Moreover, by Lemma (2.7),

$$\hat{\delta}_{\lambda,\mu}(v^{-1} V_w(\lambda+\mu)) = v^{-1} V_w(\lambda) \otimes \mathbb{C}_\mu.$$

In particular, $\hat{\delta}_{\lambda,\mu}$ gives rise to a T-module map $\delta_{\lambda,\mu}(v,w;p)$ making the following diagram commutative (for any $\lambda,\mu \in D$, $v \leqq w \in W$ and $p \geqq 0$):

$$\begin{tikzcd}[column sep=tiny]
\left(\dfrac{v^{-1}V_w(\lambda)+\mathscr{F}_p(\lambda)}{v^{-1}V_w(\lambda)+\mathscr{F}_{p-1}(\lambda)}\right)^* \otimes \mathbb{C}_\lambda \arrow[rr, "\delta_{\lambda,\mu}(v,w;p)"] \arrow[dr, hook, "f_p(v,w;\lambda)"'] & & \left(\dfrac{v^{-1}V_w(\lambda+\mu)+\mathscr{F}_p(\lambda+\mu)}{v^{-1}V_w(\lambda+\mu)+\mathscr{F}_{p-1}(\lambda+\mu)}\right)^* \otimes \mathbb{C}_{\lambda+\mu} \arrow[dl, hook', "f_p(v,w;\lambda+\mu)"] \\
& \mathscr{I}_p(v^{-1}\overline{BwB} \cap U^-) . &
\end{tikzcd}$$

Thus

$$\left\{\left(\frac{v^{-1}V_w(\lambda)+\mathscr{F}_p(\lambda)}{v^{-1}V_w(\lambda)+\mathscr{F}_{p-1}(\lambda)}\right)^* \otimes \mathbb{C}_\lambda\right\}_{\lambda\in D}$$

forms a directed system of T-modules and there is an induced T-module map

$$f_p(v,w): \underset{\lambda\in D}{\operatorname{limit}_{\rightarrow}}\left(\left(\frac{v^{-1}V_w(\lambda)+\mathscr{F}_p(\lambda)}{v^{-1}V_w(\lambda)+\mathscr{F}_{p-1}(\lambda)}\right)^* \otimes \mathbb{C}_\lambda\right) \to \mathscr{I}_p(v^{-1}\overline{BwB} \cap U^-) .$$

By the injectivity of the map $f_p(v,w;\lambda)$, we see that $\delta_{\lambda,\mu}(v,w;p)$ is injective.

(4.4) Theorem. *The above map $f_p(v,w)$ is a T-equivariant isomorphism for all $p \geqq 0$ and all $v \leqq w \in W$. In particular, there is a T-equivariant isomorphism*

$$\text{(1)} \qquad \operatorname{gr}_p(\mathscr{O}_{e,v^{-1}X_w}) \approx \underset{\lambda\in D}{\operatorname{limit}_{\rightarrow}}\left(\left(\frac{v^{-1}V_w(\lambda)\cap\mathscr{F}_p(\lambda)}{v^{-1}V_w(\lambda)\cap\mathscr{F}_{p-1}(\lambda)}\right)^* \otimes \mathbb{C}_\lambda\right),$$

where $\operatorname{gr}_p(\mathscr{O}_{e,v^{-1}X_w})$ is the pth graded component of $\operatorname{gr}(\mathscr{O}_{e,v^{-1}X_w})$.

Proof. Since $f_p(v,w;\lambda)$ is injective for all $\lambda \in D$, $f_p(v,w)$ is clearly injective. The surjectivity of $f_p(v,w)$ follows from Lemma (2.4) and Lemma (4.2).

We now come to the proof of (1): Observe first that by [Ha, Lecture 20],

$$\text{(2)} \qquad \operatorname{gr}_p(\mathscr{O}_{e,v^{-1}X_w}) \approx S^p((\mathfrak{u}^-)^*)/\mathscr{I}_p(v^{-1}\overline{BwB} \cap U^-) ,$$

where S^p is the pth symmetric power.

Now for any (fixed) p, if we take λ to be sufficiently large, then the map

$$U_p(\mathfrak{u}^-) \otimes \mathbb{C}_\lambda \to \mathscr{F}_p(\lambda) \quad \text{given by } X \otimes e_\lambda \mapsto Xe_\lambda ,$$

for $e_\lambda \in \mathbb{C}_\lambda \subset V(\lambda)$, is a T-module isomorphism. In particular, by the Poincaré–Birkhoff–Witt theorem,

$$\text{(3)} \qquad \mathscr{F}_p(\lambda)/\mathscr{F}_{p-1}(\lambda) \approx S^p(\mathfrak{u}^-) \otimes \mathbb{C}_\lambda \quad \text{(for large enough } \lambda) .$$

Consider the exact sequence

$$0 \to \frac{v^{-1}V_w(\lambda)\cap\mathscr{F}_p(\lambda)}{v^{-1}V_w(\lambda)\cap\mathscr{F}_{p-1}(\lambda)} \to \frac{\mathscr{F}_p(\lambda)}{\mathscr{F}_{p-1}(\lambda)} \to \frac{v^{-1}V_w(\lambda)+\mathscr{F}_p(\lambda)}{v^{-1}V_w(\lambda)+\mathscr{F}_{p-1}(\lambda)} \to 0 .$$

Dualizing this sequence and using (3) we get (for large enough λ)

$$0 \leftarrow \left(\frac{v^{-1}V_w(\lambda) \cap \mathscr{F}_p(\lambda)}{v^{-1}V_w(\lambda) \cap \mathscr{F}_{p-1}(\lambda)}\right)^* \otimes \mathbb{C}_\lambda \leftarrow S^p((\mathfrak{u}^-)^*)$$
$$\leftarrow \left(\frac{v^{-1}V_w(\lambda) + \mathscr{F}_p(\lambda)}{v^{-1}V_w(\lambda) + \mathscr{F}_{p-1}(\lambda)}\right)^* \otimes \mathbb{C}_\lambda \leftarrow 0\,.$$

Now the isomorphism (1) is established from (2) and the isomorphism $f_p(v,w)$. □

§ **(4.5)** For any variety X and a closed point $x \in X$, let $Z_x(X)$ denote the Zariski tangent space of X at x. For any closed subvariety $Y \subset G/B$ containing the base point $\mathfrak{e}$, we get the induced inclusion $Z_\mathfrak{e}(Y) \hookrightarrow Z_\mathfrak{e}(G/B)$. But $Z_\mathfrak{e}(G/B)$ can be canonically identified with $\mathfrak{u}^-$ (since U^- is an open neighborhood around $\mathfrak{e}$ in G/B), in particular, $Z_\mathfrak{e}(Y)$ can be canonically viewed as a subspace of $\mathfrak{u}^-$.

The following result is due to Polo [P, Theorem 3.2]. It may be recalled that a different description of the Zariski tangent space in the case of classical groups was given by Lakshmibai–Seshadri (cf. [LS] [L]). Observe that by virtue of the automorphism of G/B, given by $gB \mapsto \bar{v}gB$ (for $g \in G$), $Z_\mathfrak{e}(v^{-1}X_w)$ is isomorphic with $Z_\mathfrak{v}(X_w)$.

The first part of the following result follows immediately from Theorem (4.4) and the second part follows from the fact that $X_w \subset G/B$ is defined by linear equations.

(4.6) Corollary. *For any* $v \leqq w$,

(1) $$Z_\mathfrak{e}(v^{-1}X_w) = \{X \in \mathfrak{u}^- : Xe_\lambda \in v^{-1}V_w(\lambda),\ \textit{for all}\ \lambda \in D\}\,,$$

where e_λ *is a non-zero highest weight vector of* $V(\lambda)$.
In fact (fixing any regular $\lambda_o \in D^o$*)*

(2) $$Z_\mathfrak{e}(v^{-1}X_w) = \{X \in \mathfrak{u}^- : Xe_{\lambda_o} \in v^{-1}V_w(\lambda_o)\}.$$

Proof. The identity (1) follows from Theorem (4.4) immediately, since $Z_\mathfrak{e}(v^{-1}X_w) = \mathrm{gr}_1(\mathscr{O}_{\mathfrak{e},v^{-1}X_w})^*$. However, we give the following direct proof:

Fix $\lambda \in D$ and take $\theta \in (V(\lambda)/v^{-1}V_w(\lambda))^*$, and consider the corresponding function $\varphi^\theta : U^- \to \mathbb{C}$ defined by

$$\varphi^\theta(\exp X) = \langle \theta, (\exp X)e_\lambda \rangle = \langle \theta, Xe_\lambda \rangle + \text{order two and higher terms.}$$

(Observe that $\langle \theta, e_\lambda \rangle = 0$ by assumption.) So the linear part $L(\varphi^\theta) \in (\mathfrak{u}^-)^*$ (under the identification $\exp : \mathfrak{u}^- \to U^-$) of φ^θ is given by

(3) $$L(\varphi^\theta)X = \langle \theta, Xe_\lambda \rangle, \quad \text{for } X \in \mathfrak{u}^-.$$

Let $\mathscr{I}(v^{-1}\overline{BwB}\cap U^-)$ denote the ideal of the closed subvariety $v^{-1}\overline{BwB}\cap U^-$ of U^-. Then, by the definition of the Zariski tangent space,

$$Z_e(v^{-1}X_w) = \{X \in \mathfrak{u}^- : L(f)X = 0, \text{ for all } f \in \mathscr{I}(v^{-1}\overline{BwB}\cap U^-)\},$$
$$= \{X \in \mathfrak{u}^- : Xe_\lambda \in v^{-1}V_w(\lambda), \text{ for all } \lambda \in D\},$$
$$\text{by (3) and Lemma (2.4)}.$$

This proves (1).

We now prove (2): The tensor product of sections gives rise to an algebra structure on the space $R := \bigoplus_{m\geqq 0} H^0(G/B, \mathfrak{L}(m\lambda_o))$. Let K_m be the kernel of the restriction map $H^0(G/B, \mathfrak{L}(m\lambda_o)) \to H^0(X_w, \mathfrak{L}(m\lambda_o)|_{X_w})$. Then, by a result of Ramanathan [Ra2, Theorem 3.11], the kernel $K := \sum_{m\geqq 0} K_m$ of the surjective map $\bigoplus_{m\geqq 0} H^0(G/B, \mathfrak{L}(m\lambda_o)) \to \bigoplus_{m\geqq 0} H^0(X_w, \mathfrak{L}(m\lambda_o)|_{X_w})$ is generated (as an ideal in the ring R) by K_1 (i.e. X_w is linearly defined in G/B with respect to $\mathfrak{L}(\lambda_o)$). This, in particular, implies (by translating via $\bar{v}^{-1}$ and using Lemma 2.3) that the ideal $\mathscr{I}(v^{-1}\overline{BwB}\cap U^-)$ is generated by the functions $\{\varphi^\theta\}$, where θ ranges over $(V(\lambda_o)/v^{-1}V_w(\lambda_o))^*$. Now by an argument identical to the proof of (1), we get (2). □

(4.7) Lemma. *Let* $\mathfrak{g}$ *be simply-laced. Assume that there exist integers* $p, p_1, \ldots, p_k \geqq 1$ *and roots* $\beta, \beta_1, \ldots, \beta_k \in \Delta_+$ *such that*

$$(1) \qquad p\beta = \sum_{j=1}^{k} p_j \beta j$$

and

$$(2) \qquad \sum p_j \leqq p\,.$$

Then $\beta j = \beta$, *for all* $1 \leqq j \leqq k$.

Proof. We can assume without loss of generality that no $\beta j = \beta$. Now by (1) we get

$$(3) \qquad p\langle \beta, \beta^\vee\rangle = \sum_{j=1}^{k} p_j \langle \beta j, \beta^\vee\rangle\,.$$

But $\mathfrak{g}$ being simply-laced, $\langle \beta j, \beta^\vee\rangle \leqq 1$ (since $\beta j \neq \beta$), and hence by (2) and (3) we get

$$2p \leqq \sum_{j=1}^{k} p_j \leqq p\,.$$

This contradiction proves the lemma. □

(4.8) Proposition. *Let* $\mathfrak{g}$ *be simply-laced. Fix* $v \leqq w \in W$. *Then for any* $\alpha \in \Delta_+$ *such that* $\alpha \notin S(w,v)$ *but* $E_{-\alpha} \in Z_e(v^{-1}X_w)$, *there exists a non-zero element* $\theta_\alpha \in \mathrm{gr}_1(\mathcal{O}_{e,v^{-1}X_w})$ *of weight* α *satisfying* $\theta_\alpha^{\langle\rho,\alpha^\vee\rangle} = 0$ *as an element of* $\mathrm{gr}_{\langle\rho,\alpha^\vee\rangle}(\mathcal{O}_{e,v^{-1}X_w})$, *where* $E_{-\alpha}$ *is a non-zero root vector of* $\mathfrak{g}$ *corresponding to the negative root* $-\alpha$, $S(w,v)$ *is as in Definition* (3.3), *and* ρ *is the half sum of positive roots. In particular, the tangent cone* $T_e(v^{-1}X_w)$ *is non-reduced in this case.*

Proof. By Lemma (4.7), the weight space of $U_p(\mathfrak{u}^-)$ corresponding to the weight $-p\alpha$ (for any $p \geqq 1$) is one dimensional, and is spanned by $E_{-\alpha}^p$. Since $\mathrm{gr}_1(\mathcal{O}_{y,Y})$ is canonically isomorphic with the dual space $Z_y(Y)^*$ (for any variety Y and closed point $y \in Y$), and $E_{-\alpha} \in Z_{\mathfrak{e}}(v^{-1}X_w)$, there exists a non-zero element $\theta_\alpha \in \mathrm{gr}_1(\mathcal{O}_{\mathfrak{e},v^{-1}X_w})$ of weight α. Under the embedding $Z_{\mathfrak{e}}(v^{-1}X_w) \hookrightarrow \mathfrak{u}^-$ (cf. Sect. 4.5), we can identify the element θ_α with the element of $(\mathfrak{u}^-)^*$ defined by $\theta_\alpha(E_{-\beta}) = \delta_{\alpha,\beta}$, for all $\beta \in \Delta_+$.

By virtue of Theorem (4.4), to prove that $\theta_\alpha^p = 0$ (where $p := \langle\rho, \alpha^\vee\rangle$), it suffices to show that (for all large enough $\lambda \in D$)

$$\theta^p_{\alpha|_{(v^{-1}V_w(\lambda)\cap\mathcal{F}_p(\lambda))}} \equiv 0 :$$

Since θ_α^p is of weight $p\alpha$ and the weight space of $U_p(\mathfrak{u}^-)$ corresponding to the weight $-p\alpha$ is spanned by $E_{-\alpha}^p$, it suffices to show that $E_{-\alpha}^p e_\lambda \notin v^{-1}V_w(\lambda)$ (for all large enough $\lambda \in D$):

For otherwise, assume that $E_{-\alpha}^p e_{\lambda_0} \in v^{-1}V_w(\lambda_0)$ (for some $\rho \prec \lambda_0$). Then by Lemma (2.7), $E_{-\alpha}^p e_\rho \in v^{-1}V_w(\rho)$. But since $\bar{r}_\alpha e_\rho = E_{-\alpha}^p e_\rho$ (up to a non-zero scalar multiple), $\bar{r}_\alpha e_\rho \in v^{-1}V_w(\rho)$ and hence by [BGG, Theorem 2.9] $vr_\alpha \leqq w$, which contradicts the assumption and proves the proposition. □

(4.9) *Remarks.* (a) The 'in particular' statement of the above Proposition can also be deduced from [C1, Theorem G(2)].

(b) Lakshmibai had earlier given an example (private communication) to show that in the group Spin (8), a certain tangent cone is non-reduced.

For a closed point x of a scheme X, recall the definition of the tangent cone $T_x(X)$ as Spec $(\mathrm{gr}\,\mathcal{O}_x)$ from Sect. 2.1. Define the *reduced tangent cone* $T_x^{\mathrm{red}}(X)$ as Spec $(\mathrm{gr}^{\mathrm{red}}\mathcal{O}_x)$, where $\mathrm{gr}^{\mathrm{red}}\mathcal{O}_x = (\mathrm{gr}\,\mathcal{O}_x)/N$ and N is the ideal consisting of all the nilpotent elements in gr $\mathcal{O}_x$.

We recall the definition of a rationally smooth point in a variety Y (cf. [KL, Appendix]).

(4.10) Definition. *An irreducible variety Y of* dim d *is said to be* rationally smooth *if for all $y \in Y$, the singular cohomology $H^i(Y, Y\backslash y, \mathbb{Q}) = 0$ if $i \neq 2d$ and $H^{2d}(Y, Y\backslash y, \mathbb{Q})$ is one-dimensional. A point $y_0 \in Y$ is said to be rationally smooth if there exists an open (in the Zariski topology) rationally smooth neighborhood of $y_0 \in Y$.*

A smooth point $y_0 \in Y$ is clearly rationally smooth.

The following result is due to Carrell–Peterson [C1, Theorem EG], proved by different methods.

(4.11) Corollary (of Proposition 4.8). *Let $\mathfrak{g}$ be an arbitrary semisimple Lie algebra and fix $v \leqq w \in W$. Assume that $T_{\mathfrak{e}}^{\mathrm{red}}(\theta^{-1}X_w)$ is an affine space for all $v \leqq \theta \leqq w$. Then the point $\mathfrak{v} \in X_w$ is rationally smooth.*

Conversely, in the case when $\mathfrak{g}$ is simply-laced, if the point $\mathfrak{v} \in X_w$ is rationally smooth, then $T_{\mathfrak{e}}^{\mathrm{red}}(\theta^{-1}X_w)$ is an affine space for all $v \leqq \theta \leqq w$.

Proof. As follows from Corollary (4.6) (cf. also [C1, Theorem F], [P, Proposition 4.2]), for any $\alpha \in S(w, \theta)$, $E_{-\alpha} \in Z_{\mathfrak{e}}(\theta^{-1}X_w) \cong \mathrm{gr}_1(\mathcal{O}_{\mathfrak{e},\theta^{-1}X_w})^*$. Choose a non-zero element θ_α of weight α in $\mathrm{gr}_1(\mathcal{O}_{\mathfrak{e},\theta^{-1}X_w})$. Then $\theta_\alpha^p \neq 0$ in $\mathrm{gr}_p(\mathcal{O}_{\mathfrak{e},\theta^{-1}X_w})$ (for any $p \geqq 1$): To prove this, it suffices to show that $E_{-\alpha}^p e_\lambda \in \theta^{-1}V_w(\lambda)$, for any $\lambda \in D$ such that $p \leqq \langle \lambda, \alpha^\vee \rangle$ (cf. proof of Proposition 4.8):

By the $sl(2)$-theory, $E_{-\alpha}^{\langle \lambda, \alpha^\vee \rangle} e_\lambda = \bar{r}_\alpha e_\lambda$ (up to non-zero scalar multiples). If $\theta\alpha \in \Delta_-$, clearly $E_{-\alpha}^p e_\lambda \in \theta^{-1}V_w(\lambda)$. So assume that $\theta\alpha \in \Delta_+$. Then (up to non-zero scalar multiples)

$$E_{-\alpha}^p e_\lambda = E_\alpha^{\langle \lambda, \alpha^\vee \rangle - p} E_{-\alpha}^{\langle \lambda, \alpha^\vee \rangle} e_\lambda = E_\alpha^{\langle \lambda, \alpha^\vee \rangle - p} \bar{r}_\alpha e_\lambda \in \theta^{-1} V_w(\lambda),$$

thereby proving the claim.

We come to the proof of the first part of the corollary. Since the dimension of the tangent cone is the same as the local dimension of the variety at that point (cf. [Ha, Lecture 20]), and (by assumption) $T_\mathfrak{e}^{\mathrm{red}}(\theta^{-1}X_w)$ is an affine space, $\dim T_\mathfrak{e}^{\mathrm{red}}(\theta^{-1}X_w) = \ell(w) \geqq \#S(w,\theta)$. But, by Deodhar's conjecture (see Theorem 5.1), $\ell(w) \leqq \#S(w,\theta)$. Hence $\ell(w) = \#S(w,\theta)$, for all $v \leqq \theta \leqq w$. So the first part of the corollary follows from [C1, Theorem E]. (Observe that for any $\theta \in W$, $\#\{\alpha \in \Delta_+ : r_\alpha\theta < \theta\} = \ell(\theta)$.)

In the simply-laced case, by Proposition (4.8) and the above argument,

$$\text{(1)} \qquad \dim(\mathrm{gr}_1^{\mathrm{red}}(\mathcal{O}_{\mathfrak{e},\theta^{-1}X_w})) = \#S(w,\theta) = \ell(w),$$

since $\mathfrak{v} \in X_w$ is assumed to be rationally smooth. But since $\mathrm{gr}^{\mathrm{red}}$ is generated (as an algebra) by $\mathrm{gr}_1^{\mathrm{red}}$, we get a surjective map $\gamma : S(\mathrm{gr}_1^{\mathrm{red}}(\mathcal{O}_{\mathfrak{e},\theta^{-1}X_w})) \twoheadrightarrow \mathrm{gr}^{\mathrm{red}}(\mathcal{O}_{\mathfrak{e},\theta^{-1}X_w})$ (where S is the symmetric algebra). But since $T_\mathfrak{e}^{\mathrm{red}}(\theta^{-1}X_w)$ is of dim $\ell(w)$, surjectivity of γ and (1) force γ to be an isomorphism. This proves the corollary. □

(4.12) *Remark.* The converse statement of the above corollary is not true in general for non simply-laced $\mathfrak{g}$. Take, e.g., $\mathfrak{g}$ to be of type C_2 or G_2 and $w = r_1r_2r_1, v = e$. Since $\mathfrak{g}$ is of rank 2 (as is well known; and can also be proved by using Lemma 6.2 and Theorem 5.5 (a)), $\mathfrak{e} \in X_w$ is rationally smooth. But it can be easily seen that $T_\mathfrak{e}^{\mathrm{red}}(X_w)$ is not an affine space.

5. Smoothness criterion of Schubert varieties

For any $v \leqq w \in W$, recall the definition of $S(w,v)$ from Definition (3.3). We recall the following very interesting conjecture of Deodhar [D1], which was proved by Carrell–Peterson [C1], Dyer [Dy], and Polo [P].

(5.1) Theorem. *For any $v \leqq w \in W$, $\#S(w,v) \geqq \ell(w)$.*

Even though the following proposition follows immediately by combining our Corollary 3.2(b) with [Dy, Proposition, Sect. 3], we give a different (geometric) proof (as that proof is crucially used in the proof of Theorem 5.5(b)).

(5.2) Proposition. *Let $v \leqq w \in W$. Then*

$$\#S(w^{-1},v^{-1}) = \ell(w) \Leftrightarrow [\mathrm{ch}\,(\mathrm{gr}\,\mathcal{O}_{\mathfrak{v},X_w})] = d(-1)^{\ell(w)-\ell(v)} \prod_{\beta \in S(w^{-1},v^{-1})} \beta^{-1},$$

for some $d \in \mathbb{C}$.

Proof. By Corollary 3.2 (b), $[\mathrm{ch}\,(\mathrm{gr}\,\mathcal{O}_{\mathfrak{v},X_w})] = c_{w^{-1},v^{-1}} \neq 0$ and, moreover, it can be easily seen from the definition of $c_{w^{-1},v^{-1}}$ that $\deg c_{w^{-1},v^{-1}} = -\ell(w)$, where $\deg\frac{P}{Q} := \deg P - \deg Q$ for non-zero $P, Q \in S(\mathfrak{h}^*)$. Hence the implication '$\Leftarrow$' of the above proposition follows.

Now we come to the implication '$\Rightarrow$':

Let $\exp : \mathfrak{u}^- \tilde{\to} U^-$ be the exponential map, where $\mathfrak{u}^-$ is the Lie algebra of U^-. (Observe that U^- being a unipotent group, exp is an algebraic morphism.) Let $Y := \exp^{-1}(U^-\mathfrak{e} \cap v^{-1}X_w)$ be the closed irreducible subvariety of $\mathfrak{u}^-$, where we identify U^- with $U^-\mathfrak{e}$. Fix non-zero root vectors $E_{-\beta}$ (corresponding to the negative root $-\beta$) for $\beta \in \Delta_+$. For any $\alpha \in \Delta_+$, let $f_\alpha : \mathfrak{u}^- \to \mathbb{C}$ be the linear map defined by $\sum_{\beta\in\Delta_+} t_\beta E_{-\beta} \mapsto t_\alpha$, and let f_α^Y be the restriction of f_α to Y.[1] Define the closed subvariety (with the reduced structure) of Y

$$Z_S = \{x \in Y : f_\alpha^Y(x) = 0, \text{ for all } \alpha \in S := S(w,v)\}\,.$$

Clearly $0 \in Z_S$. We claim that any irreducible component Z_S^o of Z_S through 0 is 0-dimensional:

The varieties $Z_S^o \subset Z_S$ are clearly T-stable under the adjoint action of the maximal torus T on $\mathfrak{u}^-$. Further, Z_S^o does not contain any 1-dimensional T-stable closed irreducible subvariety R: It is easy to see that any 1-dimensional T-stable closed irreducible subvariety of $\mathfrak{u}^-$ is of the form $\mathbb{C}E_{-\beta}$, for some $\beta \in \Delta_+$. In particular, $R = \mathbb{C}E_{-\beta_0}$ (for some $\beta_0 \in \Delta_+$). This gives that $\exp(\mathbb{C}E_{-v\beta_0})\mathfrak{v} \subset X_w$. Now if $-v\beta_0 \in \Delta_+$, then by [BGG, Corollary 2.3] $vr_{\beta_0} < v \leqq w$, so $\beta_0 \in S$. If $v\beta_0 \in \Delta_+$, then clearly $\exp(\mathbb{C}E_{v\beta_0})\exp(\mathbb{C}E_{-v\beta_0})\mathfrak{v} \subset X_w$. In particular, for the subgroup $\mathscr{S}_{vr_{\beta_0}v^{-1}} \subset G$ generated by $\exp(\mathbb{C}E_{-v\beta_0})$ and $\exp(\mathbb{C}E_{v\beta_0})$, by density $\mathscr{S}_{vr_{\beta_0}v^{-1}}\mathfrak{v} \subset X_w$. Again this gives (since $\mathscr{S}_{vr_{\beta_0}v^{-1}}$ contains some representative of the Weyl group element $vr_{\beta_0}v^{-1}$) that $\beta_0 \in S$. So, in either case, $R = \mathbb{C}E_{-\beta_0}$, for some $\beta_0 \in S$. But, by the definition of Z_S, such a R is not contained in Z_S. This contradiction establishes the claim that Z_S^o does not contain any 1-dimensional T-stable closed irreducible subvariety.

Embed $i : \mathfrak{u}^- \hookrightarrow G/B$ via the map $X \mapsto (\exp X)\mathfrak{e}$. The map i is clearly T-equivariant open immersion. Take the Zariski closure $\overline{Z_S^o}$ of $i(Z_S^o)$ in G/B. Now applying [C1, Lemma of Sect. 2] to the T-stable projective variety $\overline{Z_S^o} \subset G/B$, we get that $\dim Z_S^o = 0$ (since Z_S^o does not contain any 1-dimensional T-stable closed irreducible subvarieties). Since any irreducible component of Z_S is T-stable (and closed) in $\mathfrak{u}^-$ and any closed T-stable subset of $\mathfrak{u}^-$ contains 0, we get that any irreducible component of Z_S passes through 0. In particular, $Z_S = \{0\}$.

View f_γ^Y as elements of the local ring $\mathcal{O}_{0,Y}$ and let I be the ideal generated by $\{f_\gamma^Y\}_{\gamma\in S}$ (inside the local ring $\mathcal{O}_{0,Y}$). Then there exists an integer $d > 0$ such that $\mathfrak{m}^d \subset I \subset \mathfrak{m}$, where $\mathfrak{m} \subset \mathcal{O}_{0,Y}$ is the maximal ideal (since $Z_S = \{0\}$). Since the variety X_w is Cohen–Macaulay (cf. [Ra1], [Ku1,

[1]Note added at the time of revision: Polo informed me that he considered a different set (than our f_α^Y) of functions on Y and proved that they generate a primary ideal and this was crucially used in his proof of Deodhar's conjecture [P, Sect. 2].

Theorem 2.23], [Ma]), the variety Y is Cohen–Macaulay. Assume now that $\#S(w^{-1},v^{-1}) = \#S(w,v) = \ell(w) = \dim Y$, and enumerate the elements of $S(w,v)$ as $\{\gamma_1,\ldots,\gamma_\ell\}$, where $\ell = \ell(w)$. By [F, Lemma (a), Sect. 2.4] (since $\dim Z_S = 0$), the elements $\{f^Y_{\gamma_j}\}_{1\leqq j\leqq \ell}$ form a regular sequence in $\mathcal{O}_{0,Y}$. Moreover, by [F, Lemma (b), Sect. 2.4], the canonical ring homomorphism

$$(1)\qquad \frac{\mathcal{O}_{0,Y}}{I}[X_1,X_2,\ldots,X_\ell] \xrightarrow{\sim} \sum_{m\geqq 0} I^m/I^{m+1},$$

which takes X_j to the image of $f^Y_{\gamma_j}$ in I/I^2, is an isomorphism. In particular,

$$\begin{aligned}(2)\qquad \mathrm{ch}\,(\mathrm{gr}\,(\mathcal{O}_{\mathfrak{e},v^{-1}X_w})) &= \mathrm{ch}\,(\mathrm{gr}\,(\mathcal{O}_{0,Y}))\\ &= \mathrm{ch}(\mathbb{C}[Y]) \quad (Y \text{ being affine})\\ &= \mathrm{ch}\left(\sum_{m\geqq 0} I^m/I^{m+1}\right)\\ &= \mathrm{ch}\,(\mathcal{O}_{0,Y}/I)\prod_{j=1}^{\ell}(1-e^{\gamma_j})^{-1}, \text{ by (1).}\end{aligned}$$

But since $\mathcal{O}_{0,Y}/I$ corresponds to the 0-dimensional variety, it is finite dimensional vector space over $\mathbb{C}$ and hence

$$(3)\qquad [\mathrm{ch}\,(\mathcal{O}_{0,Y}/I)] = \dim(\mathcal{O}_{0,Y}/I)\,.$$

By (2) and (3) we get

$$(4)\qquad [\mathrm{ch}\,(\mathrm{gr}\,(\mathcal{O}_{\mathfrak{e},v^{-1}X_w}))] = (-1)^\ell d\prod_{j=1}^{\ell}\gamma_j^{-1},$$

where $d := \dim(\mathcal{O}_{0,Y}/I)$. Thus

$$\begin{aligned}(5)\qquad [\mathrm{ch}\,(\mathrm{gr}\,(\mathcal{O}_{\mathfrak{v},X_w}))] &= (-1)^\ell\, d\prod_{j=1}^{\ell}(v\gamma_j)^{-1}\\ &= (-1)^\ell d(-1)^{\#\{\gamma_j\,:\,v\gamma_j\in\Delta_-\}}\prod_{\beta\in S(w^{-1},v^{-1})}\beta^{-1}\\ &= (-1)^{\ell(w)-\ell(v)}d\prod_{\beta\in S(w^{-1},v^{-1})}\beta^{-1}\,.\end{aligned}$$

This proves the proposition. □

(5.3) *Remark.* When the equivalent condition as in the above Proposition (5.2) is satisfied, d in fact is an integer > 0 (as is clear from the above proof). Moreover, if G does not contain any factors of type G_2, using [C3, Theorem 1] and [Wh, Theorems 4D and 7P, Chapter 7], it can be seen that d is the multiplicity of the point $\mathfrak{v} \in X_w$.

As mentioned in Remark 3.4(3), the following result was pointed out by the referee. We give below a geometric proof of the result.

(5.4) Lemma. *For any $v \leqq w \in W$,*

$$(1) \qquad \left(\prod_{\gamma \in S(w^{-1}, v^{-1})} (1 - e^{\gamma}) \right) b_{w^{-1}, v^{-1}} \in R(T).$$

In particular,

$$(2) \qquad \left(\prod_{\gamma \in S(w^{-1}, v^{-1})} \gamma \right) c_{w^{-1}, v^{-1}} \in S(\mathfrak{h}^*).$$

Proof. We will freely use the notation (without explanation) from the proof of Proposition (5.2). Enumerate the elements of $S(w, v)$ as $\{\gamma_1, \dots, \gamma_m\}$ and define the $\mathbb{C}$-algebra homomorphism $\theta : R \to \mathbb{C}[Y]$ by $X_j \mapsto f^Y_{\gamma_j}$, where $R := \mathbb{C}[X_1, \dots, X_m]$ is the polynomial ring. Let $\mathfrak{m}_o$ (resp. $\mathfrak{m}_Y$) be the maximal ideal of R (resp. $\mathbb{C}[Y]$) consisting of polynomials with zero constant term (resp. regular functions on Y vanishing at 0). Then clearly $\theta(\mathfrak{m}_o) \subset \mathfrak{m}_Y$. By the proof of Proposition (5.2), since $Z_S = \{0\}$, the ideal I' of $\mathbb{C}[Y]$ generated by $\theta(\mathfrak{m}_o)$ satisfies

$$(3) \qquad \mathfrak{m}_Y^d \subset I' \subset \mathfrak{m}_Y,$$

for some $d > 0$. Consider $\mathbb{C}$ as an R-module under the evaluation at 0 and $\mathbb{C}[Y]$ as an R-module under the θ. Then $\mathrm{Tor}_p^R(\mathbb{C}[Y], \mathbb{C})$ is finite dimensional over $\mathbb{C}$ for any p: (I thank H. Bass for the following simple argument of this assertion.)

Considering a finitely generated R-free resolution of $\mathbb{C}$ and tensoring this resolution with $\mathbb{C}[Y]$ over R, we see that the R-module structure of $\mathrm{Tor}_p^R(\mathbb{C}[Y], \mathbb{C})$ comes from a $\mathbb{C}[Y]$-module structure and moreover (from the noetherian property of $\mathbb{C}[Y]$) $\mathrm{Tor}_p^R(\mathbb{C}[Y], \mathbb{C})$ is finitely generated $\mathbb{C}[Y]$-module. Further, since $\mathbb{C}$ is annihilated by $\mathfrak{m}_o$, $\mathrm{Tor}_p^R(\mathbb{C}[Y], \mathbb{C})$ (as a $\mathbb{C}[Y]$-module) is annihilated by the ideal I', i.e., $\mathrm{Tor}_p^R(\mathbb{C}[Y], \mathbb{C})$ is a finitely generated $\mathbb{C}[Y]/I'$ -module. But $\mathbb{C}[Y]/I'$ is finite dimensional over $\mathbb{C}$ (by (3)), in particular, $\mathrm{Tor}_p^R(\mathbb{C}[Y], \mathbb{C})$ is finite dimensional over $\mathbb{C}$.

Now by the same argument as in the proof of Corollary 3.2(a), we get that $(\prod_{\gamma \in S(w,v)} (1 - e^{\gamma}))\mathrm{ch}\,\mathbb{C}[Y] \in R(T)$. Hence $(\prod_{\gamma \in S(w,v)} (1 - e^{v\gamma}))\mathrm{ch}(\mathbb{C}[\mathscr{A}^v \cap X_w]) \in R(T)$, i.e., $(\prod_{\gamma \in S(w^{-1}, v^{-1})} (1 - e^{\gamma}))\,\mathrm{ch}(\mathrm{gr}\,\mathcal{O}_{\mathfrak{v}, X_w}) \in R(T)$. So (1) follows from Theorem (2.2) and (2) follows from (1) and Corollary 3.2(b). $\square$

The (b)-part of the following theorem is the main result of this paper.

(5.5) Theorem. *Fix $v \leqq w \in W$.*

(a) *The point $\mathfrak{v} \in X_w$ is rationally smooth $\Leftrightarrow$*
For all $v \leqq \theta \leqq w$, we have

$$(1) \qquad c_{w^{-1}, \theta^{-1}} = d_\theta (-1)^{\ell(w) - \ell(\theta)} \prod_{\beta \in S(w^{-1}, \theta^{-1})} \beta^{-1},$$

for some constants $d_\theta \in \mathbb{C}$.

(b) *The point* $\mathfrak{v} \in X_w$ *is smooth* $\Leftrightarrow$

$$c_{w^{-1},v^{-1}} = (-1)^{\ell(w)-\ell(v)} \prod_{\beta \in S(w^{-1},v^{-1})} \beta^{-1}. \tag{2}$$

Proof. (a) By [C1, Theorem E], $\mathfrak{v} \in X_w$ is rationally smooth if and only if for all $v \leqq \theta \leqq w$, $\#S(w^{-1},\theta^{-1}) = \ell(w)$. By Proposition (5.2), this is equivalent to the requirement that for all $v \leqq \theta \leqq w$,

$$[\mathrm{ch}\,(\mathrm{gr}\,(\mathcal{O}_{\theta,X_w}))] = d_\theta(-1)^{\ell(w)-\ell(\theta)} \prod_{\beta \in S(w^{-1},\theta^{-1})} \beta^{-1},$$

for some $d_\theta \in \mathbb{C}$. Now the (a)-part follows from Corollary 3.2(b).

(b) The point $\mathfrak{v} \in X_w$ is smooth if and only if the graded algebra $\mathrm{gr}(\mathcal{O}_{\mathfrak{v},X_w})$ is isomorphic with the symmetric algebra $S(\mathrm{gr}_1(\mathcal{O}_{\mathfrak{v},X_w}))$. We first prove the '$\Rightarrow$' implication: So assume that $\mathfrak{v} \in X_w$ is smooth. Then

$$\mathrm{ch}\,(\mathrm{gr}\,(\mathcal{O}_{\mathfrak{v},X_w})) = \mathrm{ch}\,(S(\mathrm{gr}_1(\mathcal{O}_{\mathfrak{v},X_w}))) = \prod_{\gamma \in S}(1-e^\gamma)^{-1},$$

if $\mathrm{ch}\,(\mathrm{gr}_1(\mathcal{O}_{\mathfrak{v},X_w})) = \sum_{\gamma \in S} e^\gamma$. It is easy to see that $S \subset v\Delta_+$ and moreover all the weight spaces of $\mathrm{gr}_1(\mathcal{O}_{\mathfrak{v},X_w})$ are one-dimensional. In particular,

$$c_{w^{-1},v^{-1}} = [\mathrm{ch}\,(\mathrm{gr}\,(\mathcal{O}_{\mathfrak{v},X_w}))] = \prod_{\gamma \in S \subset v\Delta_+} (-\gamma)^{-1}. \tag{3}$$

But since $\mathfrak{v} \in X_w$ is smooth, in particular, it is rationally smooth. So by the (a)-part of the theorem,

$$c_{w^{-1},v^{-1}} = (-1)^{\ell(w)-\ell(v)} d_v \prod_{\beta \in S(w^{-1},v^{-1})} \beta^{-1}, \tag{4}$$

for some positive integer d_v (see Remark 5.6(2)).

Equating (3) and (4), we get

$$d_v \prod_{\gamma \in S} \gamma = \pm \prod_{\beta \in S(w^{-1},v^{-1})} \beta. \tag{5}$$

Let $Q \subset \mathfrak{h}^*$ be the root lattice and let $Q_p := \mathbb{F}_p \otimes_{\mathbb{Z}} Q$ be the reduction mod p (for any prime p) of Q, where $\mathbb{F}_p$ is the prime field of order p. Reducing the equation (5) mod p (for any prime divisor p of d_v) and observing that no root mod p is 0 in Q_p, we get that $d_v = 1$. This proves the implication '$\Rightarrow$' of the (b)-part.

Conversely, assume that $c_{w^{-1},v^{-1}} = (-1)^{\ell(w)-\ell(v)} \prod_{\beta \in S(w^{-1},v^{-1})} \beta^{-1}$. By Corollary 3.2 (b), this gives

$$[\mathrm{ch}\,(\mathrm{gr}\,(\mathcal{O}_{\mathfrak{v},X_w}))] = (-1)^{\ell(w)-\ell(v)} \prod_{\beta \in S(w^{-1},v^{-1})} \beta^{-1}. \tag{6}$$

By (5) of the proof of Proposition (5.2), we get that

$$(7) \qquad [\mathrm{ch}\,(\mathrm{gr}\,(\mathcal{O}_{\mathfrak{v},X_w}))] = (-1)^{\ell(w)-\ell(v)}\, d \prod_{\beta\in S(w^{-1},v^{-1})} \beta^{-1},$$

where $d = \dim\left(\mathcal{O}_{0,Y}/I\right)$ (the notation is as in the proof of Proposition 5.2).

By comparing (6) and (7), we get that $d = 1$, i.e., I is the maximal ideal of $\mathcal{O}_{0,Y}$. In particular, by (1) of the proof of Proposition (5.2), $\mathrm{gr}(\mathcal{O}_{0,Y})$ is graded isomorphic with the polynomial ring $\mathbb{C}[X_1,\ldots,X_\ell]$. So we get that the point $0 \in Y$ is smooth, and hence the point $\mathfrak{v} \in X_w$ is smooth. This proves the theorem completely. □

(5.6) *Remarks.* (1) The (a)-part of the above theorem can also be proved immediately by combining a result of Dyer [Dy, Proposition, Sect. 3] with a result of Carrell–Peterson [C1, Theorem E], i.e., we can avoid the use of Proposition (5.2) and Corollary 3.2(b). But our geometric proof has the advantage that a similar argument (as seen above) gives our criterion for smoothness as in the (b)-part of the above theorem.

(2) In the case (a) as above (i.e. if $\mathfrak{v} \in X_w$ is rationally smooth), the constants d_θ are positive integers for any $v \leqq \theta \leqq w$ and in fact d_θ is the multiplicity of the point $\theta \in X_w$ if G does not contain any factors of type G_2 (cf. Remark 5.3).

(3) There are some examples of $\mathfrak{v} \in X_w$ (where X_w is even a codimension one Schubert variety in G/B) such that $c_{w^{-1},v^{-1}}$ satisfies condition (1) of the above theorem, but $\mathfrak{v}$ is not a rationally smooth point of X_w (cf. Remark 7.11(a)). In particular, to check the rational smoothness of a point $\mathfrak{v} \in X_w$, it is not sufficient (in general) to check the validity of condition (1) only for $\theta = v$.

(4) It is a result of V.V. Deodhar [D1] that any rationally smooth Schubert variety is in fact smooth for $G = SL(n)$. This result has recently been extended for any simply-laced G by D. Peterson. As is well known, this result is false in general for non simply-laced G.

6. Singular locus of Schubert varieties in rank-2 groups

As an immediate corollary of Theorem (5.5), we obtain the following result determining the singular locus of all the Schubert varieties in the case of any rank two group. I believe it should be well known, but I did not find it explicitly written down in the literature. We follow the indexing convention as in Bourbaki [B].

(6.1) Proposition. *The following is a complete description of the singular locus of the Schubert varieties in the case of rank two simple groups*:
Case I. G of type A_2 : *In this case all the six Schubert varieties are smooth.*
Case II. G of type C_2: *There are, in all, eight Schubert varieties. Out of these only* $X_{r_1r_2r_1}$ *is singular and it has singular locus* $= X_{r_1}$.

Case III. G of type G_2: *There are, in all, twelve Schubert varieties. Following is the complete list of singular ones and their singular loci:*

Singular locus

(1)	$X_{r_1r_2r_1}$	–	X_{r_1}
(2)	$X_{r_1r_2r_1r_2}$	–	$X_{r_1r_2}$
(3)	$X_{r_2r_1r_2r_1}$	–	$X_{r_2r_1}$
(4)	$X_{r_1r_2r_1r_2r_1}$	–	$X_{r_1r_2r_1}$
(5)	$X_{r_2r_1r_2r_1r_2}$	–	X_{r_2} .

Proof. As is well known, for any rank-2 group G, any $\mathfrak{v} \in X_w$ is rationally smooth. (This can also be obtained from Theorem 5.5(a) and the following Lemma 6.2.) In particular, $c_{w^{-1},v^{-1}}$ satisfies identity (1) of Theorem (5.5). Now the proposition follows immediately by combining Theorem (5.5)(b) and the following lemma. □

The following lemma can be easily proved by a straightforward calculation using the definition of the elements x_{r_i} in the nil Hecke ring Q_W (cf. Definition 3.1(b)).

(6.2) Lemma. *For any group G and any simple reflections $r_1, r_2 \in W$, we have the following (as elements of Q_W):*

(a) $$x_{r_1}x_{r_2} = \frac{1}{\alpha_1}\left(\frac{1}{\alpha_2}(\delta_e - \delta_{r_2}) - \frac{1}{r_1\alpha_2}(\delta_{r_1} - \delta_{r_1r_2})\right).$$

(b) $$x_{r_1}x_{r_2}x_{r_1} = \frac{1}{\alpha_1}\left(\frac{\alpha_2(\alpha_1^\vee)}{\alpha_2(r_1\alpha_2)}(\delta_e - \delta_{r_1}) + \frac{1}{\alpha_2(r_2\alpha_1)}(\delta_{r_2} - \delta_{r_2r_1}) - \frac{1}{(r_1\alpha_2)(r_1r_2\alpha_1)}(\delta_{r_1r_2} - \delta_{r_1r_2r_1})\right)$$

(c)
$$x_{r_1}x_{r_2}x_{r_1}x_{r_2} = \frac{1}{\alpha_1}\left(\frac{(m-1)}{\alpha_2(r_1\alpha_2)(r_2\alpha_1)}(\delta_e - \delta_{r_2}) - \frac{(m-1)}{\alpha_2(r_1\alpha_2)(r_1r_2\alpha_1)}(\delta_{r_1} - \delta_{r_1r_2}) + \frac{1}{\alpha_2(r_2\alpha_1)(r_2r_1\alpha_2)}(\delta_{r_2r_1} - \delta_{r_2r_1r_2}) - \frac{1}{(r_1\alpha_2)(r_1r_2\alpha_1)(r_1r_2r_1\alpha_2)}(\delta_{r_1r_2r_1} - \delta_{r_1r_2r_1r_2})\right)$$

(d)
$$x_{r_1}x_{r_2}x_{r_1}x_{r_2}x_{r_1} = \frac{1}{\alpha_1}\left(\frac{(m-1)(2-m)}{\alpha_2(r_1\alpha_2)(r_2\alpha_1)(r_1r_2\alpha_1)}(\delta_e - \delta_{r_1}) + \frac{(2-m)\alpha_2(\alpha_1^\vee)}{\alpha_2(r_1\alpha_2)(r_2\alpha_1)(r_2r_1\alpha_2)}(\delta_{r_2} - \delta_{r_2r_1})\right.$$

$$+\frac{(m-2)\alpha_2(\alpha_1^\vee)}{\alpha_2(r_1\alpha_2)(r_1r_2\alpha_1)(r_1r_2r_1\alpha_2)}(\delta_{r_1r_2}-\delta_{r_1r_2r_1})$$

$$+\frac{1}{\alpha_2(r_2\alpha_1)(r_2r_1\alpha_2)(r_2r_1r_2\alpha_1)}(\delta_{r_2r_1r_2}-\delta_{r_2r_1r_2r_1})$$

$$\left.-\frac{1}{(r_1\alpha_2)(r_1r_2\alpha_1)(r_1r_2r_1\alpha_2)(r_1r_2r_1r_2\alpha_1)}(\delta_{r_1r_2r_1r_2}-\delta_{r_1r_2r_1r_2r_1})\right),$$

where $m := \alpha_1(\alpha_2^\vee)\alpha_2(\alpha_1^\vee)$.

7. Singularity of codimension one Schubert varieties in G/B

Let w_0 be the longest element of the Weyl group W (of G). As is well known and easy to see, the codimension one Schubert varieties in G/B are precisely of the form X_w, where $w = w_0 r_i$ for a simple reflection r_i. In particular, the number of such Schubert varieties in G/B is equal to $n := \operatorname{rank} G$. We denote the Schubert variety $X_{w_0r_i}(1 \leqq i \leqq n)$ by X_i. Let $\chi_i \in \mathfrak{h}_{\mathbb{Z}}^*$ be the ith $(1 \leqq i \leqq n)$ fundamental weight, defined by $\chi_i(\alpha_j^\vee) = \delta_{i,j}$.

(7.1) Proposition. *Fix any* $1 \leqq i \leqq n$. *Then for any* $v \in W$ *such that* $v \leqq w_0 r_i$,

$$(1) \qquad c_{r_iw_0,v^{-1}} = [\operatorname{ch}(\operatorname{gr}\mathcal{O}_{v,X_i})] = (-1)^{\Delta_+-\ell(v)}\frac{1}{\prod_{\beta\in\Delta_+}\beta}(w_0\chi_i - v\chi_i),$$

where [ ] *is as in* Sect. 3.1(a).

Proof. Consider the ith fundamental representation $V(\chi_i)$ (with highest weight χ_i) and define the function

$$\varphi = \varphi_{i,v} : \mathfrak{u}^- \to \mathbb{C} \text{ by } \varphi(X) = \langle \exp X \cdot e_{\chi_i}, \bar{v}^{-1}e^*_{w_0\chi_i}\rangle, \quad \text{for } X \in \mathfrak{u}^-;$$

where $\bar{v}$ is a representative of v in $N(T)$, e_{χ_i} (resp. $e_{w_0\chi_i}$) is a non-zero vector in $V(\chi_i)$ of weight χ_i (resp. $w_0\chi_i$) and $e^*_{w_0\chi_i} \in V(\chi_i)^*$ is defined by $e^*_{w_0\chi_i}(e_{w_0\chi_i}) = 1$ and $e^*_{w_0\chi_i}(v_\mu) = 0$, for any weight vector $v_\mu \in V(\chi_i)$ of weight $\mu \neq w_0\chi_i$. Let Y be the closed subvariety of the affine space $\mathfrak{u}^-$ defined as $Y = \exp^{-1}(U^-\mathfrak{e} \cap v^{-1}X_i)$ (cf. proof of Proposition 5.2). It is easy to see that $Y \subset \mathfrak{u}^-$ is defined set-theoretically by the vanishing of the function $\varphi : \mathfrak{u}^- \to \mathbb{C}$ (use Lemma 7.2). Moreover, φ is obtained by restricting the section $\chi(\bar{v}^{-1}e^*_{w_0\chi_i}) \in H^0(G/B, \mathfrak{L}(\chi_i))$ to $U^-\mathfrak{e}$ (and using the identification exp: $\mathfrak{u}^- \to U^-\mathfrak{e} \subset G/B$), where χ is the Borel–Weil homomorphism (cf. Proof of Lemma 2.4). But the line bundle $\mathfrak{L}(\chi_i)$ on G/B corresponds to the irreducible divisor $X_i \subset G/B$ with multiplicity 1 (use, e.g., the Chern class calculation for the line bundle $\mathfrak{L}(\chi_i)$). This, in particular, implies that the ideal I of the irreducible hypersurface $Y \subset \mathfrak{u}^-$ (with the reduced structure) is generated by the function φ (cf. also [C2, Proposition 4.6]). This gives that (as graded T-algebras),

$$(2) \qquad \operatorname{gr}(\mathcal{O}_{\mathfrak{e},v^{-1}X_i}) \approx S(\mathfrak{u}^{-*})/\langle[\varphi]\rangle,$$

where (as earlier) $S(\mathfrak{u}^{-*})$ is the symmetric algebra of $\mathfrak{u}^{-*}$ and $\langle[\varphi]\rangle$ denotes the (homogeneous) ideal generated by the least degree non-zero homogeneous component $[\varphi]$ of φ. From the definition of φ, it is easy to see that $[\varphi]$ is a weight vector for the adjoint action of T on $\mathfrak{u}^-$ with weight $\chi_i - v^{-1}w_0\chi_i$. So by (2),

$$\mathrm{ch}(\mathrm{gr}\,\mathcal{O}_{e,v^{-1}X_i}) = (1 - e^{\chi_i - v^{-1}w_0\chi_i}) \prod_{\beta\in\Delta_+} (1-e^{\beta})^{-1} ,$$

and hence

$$(3) \qquad [\mathrm{ch}(\mathrm{gr}\,\mathcal{O}_{e,v^{-1}X_i})] = (-1)^{\#\Delta_+} \frac{(v^{-1}w_0\chi_i - \chi_i)}{\prod_{\beta\in\Delta_+}\beta} .$$

(Observe that by Lemma (7.2), $v^{-1}w_0\chi_i - \chi_i \neq 0$, since by assumption $v \leqq w_0 r_i$.) By applying v to (3) we get

$$[\mathrm{ch}(\mathrm{gr}\,\mathcal{O}_{v,X_i})] = (-1)^{\#\Delta_+ - \ell(v)} \frac{(w_0\chi_i - v\chi_i)}{\prod_{\beta\in\Delta_+}\beta} .$$

This proves the second equality of (1). First equality of (1) of course follows from Corollary 3.2(b). □

(7.2) Lemma. *For any simple reflection r_i and any $v \in W$, $v \leqq w_0 r_i$ if and only if $\chi_i \neq v^{-1}w_0\chi_i$.*

Proof. Let $Z \subset G$ be the zero set of the function $\hat{\varphi} : G \to \mathbb{C}$ given by $\hat{\varphi}(g) = \langle g e_{\chi_i}, e^*_{w_0\chi_i}\rangle$ (where e_{χ_i} and $e^*_{w_0\chi_i}$ are as in the proof of Proposition 7.1). Then clearly Z is B-stable under the left as well as right multiplication. In particular, $Z/B = \cup X_j$, where j runs over some subset $S \subset \{1,\dots,n\}$. Clearly $i \in S$, whereas for $j \neq i$, $j \notin S$, and hence $Z/B = X_i$. Hence $v \leqq w_0 r_i \Leftrightarrow \mathfrak{v} \in X_i = Z \Leftrightarrow v\chi_i \neq w_0\chi_i$. □

(7.3) *Remark* (due to Referee). A purely algebraic proof of Proposition (7.1) (assuming Corollary 3.2(b)) can be obtained by using the recurrence formula

$$\begin{aligned} \frac{r_i(c_{w,vr_i}) - c_{w,v}}{\alpha_i} &= c_{wr_i,v}, \quad \text{if } wr_i > w \\ &= 0, \ \text{otherwise} . \end{aligned}$$

Similarly, Lemma (7.2) can be derived on noting that $v \leqq w_0 r_i$ if and only if $r_i \leqq v^{-1}w_0$. But we have retained our more geometric proofs of these.

(7.4) Lemma. *Assume that $v \leqq w_0 r_i$. Then $\chi_i - v^{-1}w_0\chi_i$ is multiple of a root β if and only if $\pm v\beta \notin S(r_i w_0, v^{-1})$. In particular, $\chi_i - v^{-1}w_0\chi_i$ is multiple of a root if and only if $\#S(r_i w_0, v^{-1}) = N - 1$, where $N := \#\Delta_+$.*

Proof. If $\pm v\beta \notin S(r_i w_0, v^{-1})$, then by the above Lemma (7.2), $r_\beta v^{-1}w_0\chi_i = \chi_i$. In particular, $\chi_i - v^{-1}w_0\chi_i$ is a multiple of β.

Conversely, assume that

$$(1) \qquad \chi_i - v^{-1}w_0\chi_i = n\beta ,$$

for some number n and $\beta \in \Delta$. By Lemma (7.2), $n \neq 0$. To prove that $\pm v\beta \notin S(r_i w_0, v^{-1})$, it suffices to show (again by Lemma 7.2) that $r_\beta v^{-1} w_0 \chi_i = \chi_i$: By (1),

(2) $$\langle \chi_i - v^{-1} w_0 \chi_i, \beta^\vee \rangle = 2n, \quad \text{and}$$

(3) $$\langle \chi_i + v^{-1} w_0 \chi_i, \beta^\vee \rangle = \frac{2}{n\langle \beta, \beta \rangle} \langle \chi_i + v^{-1} w_0 \chi_i, \chi_i - v^{-1} w_0 \chi_i \rangle = 0 .$$

Combining (2) and (3) we get $\langle -v^{-1} w_0 \chi_i, \beta^\vee \rangle = n$; and hence $r_\beta v^{-1} w_0 \chi_i := v^{-1} w_0 \chi_i - \langle v^{-1} w_0 \chi_i, \beta^\vee \rangle \beta = v^{-1} w_0 \chi_i + n\beta = \chi_i$ (by (1)).

The 'in particular' statement of the lemma follows from Deodhar's conjecture (cf. Theorem 5.1). □

By virtue of Proposition (7.1), Lemma (7.4), and Theorem 5.5(b), we get the following characterization of the smooth points in the Schubert varieties X_i.

(7.5) Proposition. *Let X_i $(1 \leqq i \leqq n)$ be a codimension one Schubert variety. Then, for any $v \leqq w_0 r_i \in W$, the following are equivalent:*

(a$_1$) *$\mathfrak{v} \in X_i$ is smooth.*

(a$_2$) $c_{r_i w_0, v^{-1}} = (-1)^{N-1-\ell(v)} \frac{1}{\beta_1 \cdots \beta_{N-1}}$, *for some positive roots* $\{\beta_1, \ldots, \beta_{N-1}\}$ *(where $N = \dim G/B$).*

(a$_3$) *$\chi_i - v^{-1} w_0 \chi_i$ is a root.*

In particular, X_i is smooth if and only if $\chi_i - w_0 \chi_i$ is a root.

(7.6) *Remark.* If $\mathfrak{v} \in X_i$ is smooth, then the set $\{\beta_1, \ldots, \beta_{N-1}\}$, as in (a_2) above, coincides with the set $S(r_i w_0, v^{-1})$ (by Theorem 5.5 (b)).

(7.7) *Proof (of Proposition 7.5).* As follows from Theorem 5.5(b), (a$_1$)⇒(a$_2$). The implication (a$_2$)⇒(a$_3$) follows from Proposition (7.1). So we come to the proof of (a$_3$) ⇒ (a$_1$):

By Theorem 5.5(b), we need to show that

(1) $$c_{r_i w_0, v^{-1}} = (-1)^{N-1-\ell(v)} \prod_{\beta \in S(r_i w_0, v^{-1})} \beta^{-1} .$$

By (a$_3$), $\gamma := v\chi_i - w_0 \chi_i$ is a root (and in fact is positive since $v \leqq w_0$). In particular, by Proposition (7.1),

(2) $$c_{r_i w_0, v^{-1}} = (-1)^{N-1-\ell(v)} \frac{\gamma}{\prod_{\beta \in \Delta_+} \beta} .$$

But by Lemma (7.4), $S(r_i w_0, v^{-1}) = \Delta_+ \setminus \{\gamma\}$, and hence (1) follows from (2). This proves the implication (a$_3$) ⇒(a$_1$).

The 'in particular' statement of the proposition follows from the equivalence of (a$_1$) and (a$_3$) (since X_i is smooth if and only if $\mathfrak{e} \in X_i$ is smooth). □

By the same proof as above for the implication (a$_3$)⇒(a$_1$) (alternatively, by using Lemma (7.4) with [C1, Theorem E]) we obtain the following:

(7.8) Corollary. *With the notation as in Proposition* (7.5), $\mathfrak{v} \in X_i$ *is rationally smooth if and only if for all* $v \leqq \theta \leqq w_0 r_i$, $\chi_i - \theta^{-1} w_0 \chi_i$ *is multiple of a root* β_θ *(depending upon* θ*).*

We follow the indexing convention of simple roots as in [B, Planches I-IX]. The following lemma follows easily from the explicit knowledge of roots, coroots, fundamental weights etc. as given in loc. cit.. Recall that w_0 is the longest element of the Weyl group.

(7.9) Lemma. *Let* $\mathfrak{g}$ *be a simple Lie algebra. Then for any fundamental weight* χ_i $(1 \leqq i \leqq n)$,

(a) $\chi_i - w_0\chi_i$ *is a (positive) root precisely in the following cases* (A_n *etc. denotes the type of* $\mathfrak{g}$):

$$\begin{array}{llll} (\mathrm{a}_1) & A_n & (n \geqq 1) & : \ i = 1, n \\ (\mathrm{a}_2) & C_n & (n \geqq 2) & : \ i = 1\,. \end{array}$$

(b) $\chi_i - w_0\chi_i$ *is multiple of a root but not a root itself, precisely in the following cases*:

$$\begin{array}{llll} (\mathrm{b}_1) & B_n & (n \geqq 3) & : \ i = 1,2 \\ (\mathrm{b}_2) & C_n & (n \geqq 2) & : \ i = 2 \\ (\mathrm{b}_3) & D_n & (n \geqq 4) & : \ i = 2 \\ (\mathrm{b}_4) & E_6 & & : \ i = 2 \\ (\mathrm{b}_5) & E_7 & & : \ i = 1 \\ (\mathrm{b}_6) & E_8 & & : \ i = 8 \\ (\mathrm{b}_7) & F_4 & & : \ i = 1,4 \\ (\mathrm{b}_8) & G_2 & & : \ i = 1,2\,. \end{array}$$

As a consequence of the above lemma, we get the following complete list of codimension-1 Schubert varieties which are smooth or rationally smooth.

We assume that G is a simple group in the following proposition.

(7.10) Proposition. (c) *The following is a complete list of codimension one Schubert varieties* X_i *which are smooth*:

$$\begin{array}{llll} (\mathrm{c}_1) & A_n & (n \geqq 1) & : \ i = 1, n \\ (\mathrm{c}_2) & C_n & (n \geqq 2) & : \ i = 1\,. \end{array}$$

(d) *The following is a complete list of codimension one Schubert varieties* X_i *which are rationally smooth but not smooth*:

$$\begin{array}{llll} (\mathrm{d}_1) & C_2 & & : \ i = 2 \\ (\mathrm{d}_2) & G_2 & & : \ i = 1,2 \\ (\mathrm{d}_3) & B_n & (n \geqq 3) & : \ i = 1\,. \end{array}$$

Proof. The (c)-part follows immediately by combining Proposition (7.5) with Lemma (7.9).

To prove the (d)-part, in view of Corollary (7.8) and Lemma (7.9), it suffices to show that in all the cases covered by (b) of Lemma (7.9) but not in the list (d) above, there exists a $\theta \in W$ such that $\chi_i - \theta^{-1} w_0 \chi_i$ is not a multiple of any root (by Lemma 7.2, such a θ will automatically satisfy $\theta \leqq w_0 r_i$), whereas in the cases covered by (d), $\chi_i - \theta^{-1} w_0 \chi_i$ is indeed multiple of a root for any $\theta \in W$:

We freely use the notation without explanation from [B, Planches I-IX]. In the cases $(B_{n \geqq 3};\ i = 2)$, $(C_{n \geqq 3};\ i = 2)$, and $(D_{n \geqq 4};\ i = 2)$ take any $\theta \in W$ satisfying $\theta(\varepsilon_1) = \varepsilon_1$, $\theta(\varepsilon_3) = \varepsilon_2$. Then $\chi_i - \theta^{-1} w_0 \chi_i$ is not a multiple of any root.

In the cases $(E_6;\ i = 2)$, $(E_7;\ i = 1)$, and $(E_8;\ i = 8)$, χ_i is the highest root α_0. In these cases, take any $\theta \in W$ satisfying $\theta(\alpha_2) = \alpha_0$ (observe that $-w_0 \alpha_0 = \alpha_0$ and the W-orbit $W\alpha_0$ consists of all the roots). Then $\chi_i - \theta^{-1} w_0 \chi_i$ is not a multiple of any root.

In the case $(F_4;\ i = 1)$ (resp. $(F_4;\ i = 4)$), χ_1 (resp. χ_4) is the highest (resp. a short) root, in particular, $W\chi_1$ consists of all the long (resp. short) roots. Take any $\theta \in W$ satisfying $\theta(\varepsilon_2 + \varepsilon_3) = \chi_1$ (resp. $\theta\left(\frac{\varepsilon_1+\varepsilon_2+\varepsilon_3+\varepsilon_4}{2}\right) = \chi_4$), then $\chi_i - \theta^{-1} w_0 \chi_i$ is not a multiple of any root.

For $(C_2;\ i = 2)$ and $(G_2;\ i = 1, 2)$, it is easy to see that $\chi_i - \theta^{-1} w_0 \chi_i$ is multiple of a root for all $\theta \in W$.

So finally we come to $(B_{n \geqq 3};\ i = 1)$: In this case, $-w_0 = \mathrm{Id.}$, $\chi_1 = \varepsilon_1$ (a short root), and hence $W\chi_1 = \{\pm\varepsilon_i\}_{1 \leqq i \leqq n}$. In particular, $\chi_1 - \theta^{-1} w_0 \chi_1$ is multiple of a root for all $\theta \in W$. This finishes the proof of the (d)-part of the proposition. □

(7.11) *Remarks.* (a) In all the cases covered by Lemma 7.9(b) but not contained in Proposition 7.10(d), identity (1) of Theorem 5.5(a) is satisfied for $w = w_0 r_i$ and $\theta = e$ but is violated for some $e \leqq \theta \leqq w$ (use Proposition 7.1, Lemma 7.4 and Theorem 5.5(a)).

(b) I am informed that the (c)-part of the above proposition, as well as the equivalence of (a_1) and (a_3) in Proposition (7.5) for $v = e$ was contained in an earlier longer version of [C1] (cf. [C2, Sect. 4]). Of course (d_1), (d_2) are very well known, and example (d_3) was known to be rationally smooth due to Boe [Bo].

8. Extension of results to the Kac–Moody case

(8.1) *Notation.* We will follow the notation (often without explaining) from [Ku1, Sect. 1]. In particular, throughout this section $G = G(A)$ denotes the complex Kac–Moody group associated to an arbitrary $n \times n$ generalized Cartan matrix A (we do not put symmetrizability restriction on A), with the standard Borel subgroup B, and the standard maximal torus $T \subset B$. There is the Weyl group $W \approx N(T)/T$ associated to the pair (G, T) (where $N(T)$ is the normalizer of T in G). The Weyl group W is a Coxeter group with the simple reflections $\{r_i\}_{1 \leqq i \leqq n}$ as Coxeter generators (r_i is nothing but the reflection corresponding to the simple root α_i). Hence, for any $w \in W$, we can talk of its length $\ell(w)$ and also have Bruhat partial ordering $\leqq$ in W.

The Kac–Moody Lie algebra $\mathfrak{g} = \mathfrak{g}(A)$ admits the root space decomposition:

$$\mathfrak{g} = \mathfrak{h} \oplus \sum_{\alpha \in \Delta_+ \subset \mathfrak{h}^*} (\mathfrak{g}_\alpha \oplus \mathfrak{g}_{-\alpha}),$$

where $\mathfrak{g}_\alpha := \{X \in \mathfrak{g} : [h,X] = \alpha(h)X,$ for all $h \in \mathfrak{h}\}$ is the α-th root space, $\mathfrak{h} :=$ Lie T is the standard Cartan subalgebra of $\mathfrak{g}$, and $\Delta_+ := \{\alpha \neq 0 \in \sum_{i=1}^{n} \mathbb{Z}_+\alpha_i : \mathfrak{g}_\alpha \neq 0\}$ is the set of positive roots. We set $\Delta_- = -\Delta_+$ and $\Delta = \Delta_+ \cup \Delta_-$. The Weyl group W preserves Δ. The set of real roots $\Delta^{\mathrm{re}} \subset \Delta$ is defined to be $W\{\alpha_1, \ldots, \alpha_n\}$ and the set of imaginary roots $\Delta^{\mathrm{im}} := \Delta \backslash \Delta^{\mathrm{re}}$. We set $\Delta_+^{\mathrm{re}} = \Delta_+ \cap \Delta^{\mathrm{re}}$ (and $\Delta_-^{\mathrm{re}} = \Delta_- \cap \Delta^{\mathrm{re}}$); Δ_+^{im} and Δ_-^{im} have similar meanings. Recall that the real root spaces are of dimension one.

The group G (in particular, the torus T) acts on G/B by the left multiplication. For any $w \in W$, the Schubert variety X_w is by definition the closure of $B\bar{w}B/B$ in G/B, where $\bar{w}$ is a preimage of w in $N(T)$ and G/B is endowed with the Zariski topology as in [S]. Of course, X_w is T-stable. By the Bruhat decomposition, $X_w = \cup_{v \leqq w} B\bar{v}B/B$. In particular, for any $v \leqq w$, $\mathfrak{v} := \bar{v}B \in X_w$ and it is a T-fixed point. We will always endow X_w with the stable variety structure as given in [Ku1, Sect. 1]. With this structure X_w is an irreducible projective variety of dim $\ell(w)$.

For any real root β, there exists a unique additive one-parameter subgroup U_β and a homomorphism $u_\beta : \mathbb{C} \to G$ satisfying $u_\beta(\mathbb{C}) = U_\beta$ and such that

$$t u_\beta(z) t^{-1} = u_\beta(e^\beta(t) z) \,,$$

for any $z \in \mathbb{C}$, and $t \in T$. Furthermore, for any $w \in W$, $\bar{w} U_\beta \bar{w}^{-1} = U_{w\beta}$.

Now let U^- be the subgroup of G generated by the one-parameter subgroups $\{U_\beta\}_{\beta \in \Delta_-^{\mathrm{re}}}$. Then the map $U^- \to G/B$, taking $g \mapsto g\mathfrak{e}$, is injective and moreover $U^-\mathfrak{e} \subset G/B$ is an open subset.

For any $\lambda \in \mathfrak{h}_{\mathbb{Z}}^*$, recall the definition of the line bundle $\mathfrak{L}(\lambda) := G \times_B \mathbb{C}_{-\lambda} \to G/B$ from [Ku1, Sect.2.2] (where it is denoted by $\mathscr{L}(\lambda)$). For dominant $\lambda \in \mathfrak{h}_{\mathbb{Z}}^*$, let $V^{\max}(\lambda)$ be the maximal integrable highest weight $\mathfrak{g}$-module with highest weight λ (cf. [Ku1, Sect. 1.5], where it is denoted by $L^{\max}(\lambda)$). Define

$$H^0(G/B, \mathfrak{L}(\lambda)) = \operatorname{Inv} \lim_{w \in W} H^0(X_w, \mathfrak{L}(\lambda)_{|X_w}) \,.$$

The highest weight space $\mathbb{C}_\lambda := V^{\max}(\lambda)_{(\lambda)}$ of $V^{\max}(\lambda)$ is one dimensional. Define the map

$$\chi = \chi_\lambda : V^{\max}(\lambda)^* \longrightarrow H^0(G/B, \mathfrak{L}(\lambda))$$

by $\chi(f)(gB) = (g, (g^{-1}f)_{|\mathbb{C}_\lambda}) \bmod B$, for $f \in V^{\max}(\lambda)^*$, and $g \in G$.

The following result is due to Kumar [Ku1, Theorem 2.16] (and also Mathieu [Ma]).

(8.2) Theorem. *The map χ_λ as above is an isomorphism. Moreover, for any $v \leqq w \in W$, it induces an isomorphism*

$$\chi_\lambda(v, w) : (v^{-1} V_w^{\max}(\lambda))^* \xrightarrow{\sim} H^0(v^{-1} X_w, \mathfrak{L}(\lambda)_{|_{v^{-1}X_w}}) \,,$$

making the following diagram commutative:

$$\begin{array}{ccc} V^{\max}(\lambda)^* & \xrightarrow[\sim]{\chi_\lambda} & H^0(G/B, \mathfrak{L}(\lambda)) \\ \downarrow & & \downarrow \\ (v^{-1}V_w^{\max}(\lambda))^* & \xrightarrow[\sim]{\chi_\lambda(v,w)} & H^0(v^{-1}X_w, \mathfrak{L}(\lambda)_{|_{v^{-1}X_w}}) \end{array}$$

where $V_w^{\max}(\lambda) \subset V^{\max}(\lambda)$ is the B-submodule generated by the extremal weight space $V^{\max}(\lambda)_{(w\lambda)}$ of weight $w\lambda$, and the vertical maps are the canonical restriction maps.

For any non-zero $e_\lambda \in \mathbb{C}_\lambda$, define $e_\lambda^* \in V^{\max}(\lambda)^*$ as $e_\lambda^*(e_\lambda) = 1$ and $e_\lambda^*(y) = 0$, for any weight-vector y of weight $\mu \neq \lambda$. Now define the section $s_{e_\lambda} \in H^0(G/B, \ \mathfrak{L}(\lambda))$ by $s_{e_\lambda} = \chi_\lambda(e_\lambda^*)$.

The following lemma follows immediately from the Birkhoff decomposition [KP, §. 3].

(8.3) Lemma. *The zero set of $s_{e_\lambda}, Z(s_{e_\lambda}) = G/B \backslash (U^- \mathrm{e}\ )$, if $\lambda \in D^o$, where* (as in Sect. 1) *D^o is the set of dominant regular weights.*

The line bundle $\mathfrak{L}(\lambda)_{|_{v^{-1}X_w}}$ on the projective variety $v^{-1}X_w$ is ample for any $v \leqq w \in W$ and $\lambda \in D^o$. In particular, by Lemmas (2.3) and (8.3), $U^-\mathrm{e} \cap v^{-1}X_w$ is an affine open subset of $v^{-1}X_w$.

Define the T-equivariant map (cf. Sect. 2.6)

$$\varphi_\lambda(v,w) : (v^{-1}V_w^{\max}(\lambda))^* \otimes \mathbb{C}_\lambda \to \mathbb{C}[U^-\mathrm{e} \cap v^{-1}X_w] \text{ by}$$

$$(\varphi_\lambda(v,w)(f \otimes e_\lambda))(x) s_{e_\lambda}(x) = (\chi_\lambda(v,w)f)(x)\,,$$

for $f \in (v^{-1}V_w^{\max}(\lambda))^*$, $e_\lambda \neq 0 \in \mathbb{C}_\lambda$ and $x \in U^-\mathrm{e} \cap v^{-1}X_w$. (We set $\varphi_\lambda(v,w)$ $(f \otimes 0) = 0$.) By Lemma (8.3), the map $\varphi_\lambda(v,w)$ is well defined, and is injective by Theorem (8.2). Moreover, as in Sect. 2.7, for any $\lambda \in D^o$ and $\mu \in D$, the following diagram is commutative:

$$\begin{array}{ccc} (v^{-1}V_w^{\max}(\lambda))^* \otimes \mathbb{C}_\lambda & \overset{\delta_{\lambda,\mu}(v,w)}{\hookrightarrow} & (v^{-1}V_w^{\max}(\lambda+\mu))^* \otimes \mathbb{C}_{\lambda+\mu} \\ & {\scriptstyle\varphi_\lambda(v,w)}\searrow \quad \swarrow{\scriptstyle\varphi_{\lambda+\mu}(v,w)} & \\ & \mathbb{C}[U^-\mathrm{e} \cap v^{-1}X_w], & \end{array}$$

where the map $\delta_{\lambda,\mu}(v,w)$ is defined as in Lemma (2.7) (and is injective since $\varphi_\lambda(v,w)$ is injective). Taking the limit of the maps $\varphi_\lambda(v,w)$, we get the T-equivariant map

$$\varphi(v,w) : \underset{\lambda \in D^o}{\operatorname{limit}}{}_{\to} ((v^{-1}V_w^{\max}(\lambda))^* \otimes \mathbb{C}_\lambda) \to \mathbb{C}[U^-\mathrm{e} \cap v^{-1}X_w]\,.$$

The following proposition follows easily from Lemma (2.3) and Theorem (8.2).

(8.4) Proposition. *The above map $\varphi(v,w)$ is an isomorphism for any $v \leqq w \in W$.*

Define the Lie subalgebra $\mathfrak{u}^- = \bigoplus_{\alpha\in\Delta_-} \mathfrak{g}_\alpha$ of $\mathfrak{g}$ and (for any $m > 0$) the ideal $\mathfrak{u}_m^-$ of $\mathfrak{u}^-$ by

$$\mathfrak{u}_m^- = \bigoplus_{\substack{\alpha\in\Delta_- \\ |\alpha|\geqq m}} \mathfrak{g}_\alpha \,,$$

where for a root $\alpha = \sum m_i\alpha_i$, $|\alpha| := |\sum m_i|$.

The quotient algebra $F_m(\mathfrak{u}^-) := \mathfrak{u}^-/\mathfrak{u}_m^-$ is a finite dimensional nilpotent algebra. Let $F_m(U^-)$ be the associated unipotent complex algebraic group. Corresponding to the Lie algebra homomorphism $\mathfrak{u}^- \to F_m(\mathfrak{u}^-)$, there is associated a group homomorphism $\theta_m : U^- \to F_m(U^-)$. We state the following simple lemma without proof.

(8.5) Lemma. *Fix $v \leqq w \in W$. Then there exists a positive number $m_0(v,w)$ such that*

$$\theta_m(v,w) : U^-\mathrm{e} \cap v^{-1}X_w \to F_m(U^-)$$

(got by restricting the map θ_m) is a closed immersion for all $m \geqq m_0(v,w)$.

By an argument identical to the proof of Theorem (2.2) (as given in Sect. 2.12), Corollaries (3.2), and Lemma (5.4) (using Proposition 8.4, Lemma 8.5, and [Ku1, Theorem 3.4]), we get the following analog of Theorem (2.2), Corollaries (3.2), and Lemma (5.4) for an arbitrary Kac–Moody group G. For $v \leqq w \in W$, define $S(w,v) = \{\alpha \in \Delta_+^{\mathrm{re}} : vr_\alpha \leqq w\}$.

(8.6) Theorem. *Let G be an arbitrary Kac–Moody group.*

(a) *For any $v \leqq w \in W$, $\mathrm{gr}\,\mathcal{O}_{v,X_w}$ is an admissible T-module and moreover*

$$\mathrm{ch}(\mathrm{gr}\,\mathcal{O}_{v,X_w}) = {*b_{w^{-1},v^{-1}}} \,,$$

as elements of $\widetilde{Q(T)}$.

(b) *For any $v,w \in W$, $b_{w^{-1},v^{-1}} \neq 0$ if and only if $v \leqq w$, and in this case it has a pole of order exactly equal to $\ell(w)$. Further,*

$$\left(\prod_{\gamma\in S(w^{-1},v^{-1})} (1-e^\gamma)\right) b_{w^{-1},v^{-1}} \in R(T) \,.$$

(c) *$[*b_{w^{-1},v^{-1}}] = c_{w^{-1},v^{-1}}$; and hence for any $v \leqq w$, $[\mathrm{ch}\,(\mathrm{gr}\,\mathcal{O}_{v,X_w})] = c_{w^{-1},v^{-1}}$, as elements of $Q(\mathfrak{h})$.*

In particular, $c_{w^{-1},v^{-1}} \neq 0$ if and only if $v \leqq w$, and

$$\left(\prod_{\gamma\in S(w^{-1},v^{-1})} \gamma\right) c_{w^{-1},v^{-1}} \in S(\mathfrak{h}^*) \,.$$

We extend Proposition (5.2) to the Kac–Moody case.

(8.7) Proposition. *Let G be an arbitrary Kac–Moody group and let $v \leqq w \in W$. Then*

$$\# S(w^{-1}, v^{-1}) = \ell(w) \Leftrightarrow [\mathrm{ch}\,(\mathrm{gr}\,\mathcal{O}_{\mathfrak{v}, X_w})] = d(-1)^{\ell(w)-\ell(v)} \prod_{\beta \in S(w^{-1}, v^{-1})} \beta^{-1}\,,$$

for some $d \in \mathbb{C}$.

Proof. The proof is very similar to the proof of Proposition (5.2). But we need to make the following modifications:

Define $Y' = U^- \mathfrak{e} \cap v^{-1} X_w$. Fix any (regular) $\lambda \in D^o$ and a non-zero highest weight vector $e_\lambda \in V^{\max}(\lambda)$, and consider the element $e_\lambda^* \in V^{\max}(\lambda)^*$ as in Sect. 8.2. For any root $\alpha \in \Delta_+^{\mathrm{re}}$, choose a non-zero root vector $X_\alpha \in \mathfrak{g}_\alpha$ and define the map $\theta_\alpha : U^- \to \mathbb{C}$ by $\theta_\alpha(g) = e_\lambda^*(X_\alpha g e_\lambda)$, for $g \in U^-$. We claim that $\theta_\alpha(g) \neq 0$, for any $g \neq e \in U_{-\alpha}$:

Write $g = \exp(z X_{-\alpha})$, for some $z \neq 0 \in \mathbb{C}$, where $X_{-\alpha}$ is the root vector corresponding to the (real) root $-\alpha$ such that $[X_\alpha, X_{-\alpha}] = \alpha^\vee$ (cf. [K, Exercise 5.1]). Then

$$\begin{aligned}
\theta_\alpha(g) &= e_\lambda^*(X_\alpha \exp(z X_{-\alpha}) e_\lambda) \\
&= e_\lambda^*(z X_\alpha X_{-\alpha} e_\lambda) \\
&= e_\lambda^*(z[X_\alpha, X_{-\alpha}] e_\lambda) \\
&= z\lambda(\alpha^\vee) \\
&\neq 0, \qquad \text{since } \lambda \text{ is regular}\,.
\end{aligned}$$

Identifying $U^- \simeq U^- \mathfrak{e}$, we can (and do) consider θ_α as a function on Y'. Now define

$$Z'_S = \{x \in Y' : \theta_\alpha(x) = 0, \text{ for all } \alpha \in S := S(w, v)\}\,.$$

Rest of the argument to prove the proposition is similar to the proof of Proposition (5.2) provided we replace $\mathfrak{u}^-$ by U^- and use the following

(8.8) Lemma. *For any $v \leqq w \in W$, one dimensional T-orbits in $U^- \mathfrak{e} \cap v^{-1} X_w$ are precisely of the form $(U_{-\beta} \backslash e)\mathfrak{e}$, where β ranges over (positive real) roots $\in S(w, v)$.*

Proof. By the Bruhat decomposition

$$X_w = \bigcup_{\theta \leqq w} U\theta\mathfrak{e} \;= \bigcup_{\theta \leqq w} \theta(\theta^{-1} U \theta \cap U^-)\mathfrak{e}\;\;,$$

onedimensional T-orbits contained in $v^{-1} X_w$ are precisely of the form $I_{\theta,\beta} := v^{-1}\theta(U_{-\beta} \setminus e)\mathfrak{e}$, where $\theta \leqq w$ and $\beta \in \Delta_+ \cap \theta^{-1}\Delta_-$. (We are using the fact that any root in $\Delta_+ \cap \theta^{-1}\Delta_-$ is a real root and moreover for any real root β, $d\beta$ is not a root for any $d > 1$.) If $\theta = v$, clearly $I_{\theta,\beta} \subset U^- \mathfrak{e}$; and moreover $\beta \in \Delta_+ \cap v^{-1}\Delta_- \Leftrightarrow \beta \in \Delta_+^{\mathrm{re}}$ and $v r_\beta < v$ (by [BGG, Corollary 2.3]). So, assume that $\theta \neq v$. By the Bruhat decomposition for $SL(2)$, we get

$$(U_{-\beta}\mathrm{e})\bigcup\{r_\beta \mathrm{e}\} = \overline{Br_\beta B/B} \subset G/B ,$$

where the closure is taken with respect to the (inductive limit) Zariski topology on G/B. In particular,

$$\overline{I_{0,\beta}}\backslash I_{0,\beta} = \{v^{-1}\theta\mathrm{e} , v^{-1}\theta r_\beta \mathrm{e}\} .$$

By Lemma (8.5), it is easy to see that any closed T-stable subset of $U^{-}\mathrm{e}$ (under the induced subspace topology on $U^{-}\mathrm{e} \subset G/B$) contains e . Hence (if $\theta \neq v$)

$$I_{0,\beta} \subset U^{-}\mathrm{e} \Leftrightarrow \mathrm{e} \in \overline{I_{0,\beta}} \Leftrightarrow v^{-1}\theta r_\beta \mathrm{e} = \mathrm{e} ,$$

i.e., $v = \theta r_\beta$. Again, by the Bruhat decomposition for $SL(2)$, it is easy to see that in this case (i.e. $v = \theta r_\beta$) $v^{-1}\theta(U_{-\beta} \setminus e)\mathrm{e} = (U_{-\beta} \setminus e)\mathrm{e}$.

But by [BGG, Corollary 2.3],

$$\{\beta \in \Delta_+^{\mathrm{re}} : \beta \in (r_\beta v^{-1}\Delta_-) \text{ and } vr_\beta \leqq w\} = \{\beta \in S(w,v) : v < vr_\beta\} .$$

This proves the lemma. □

Now by an argument identical to the proof of Theorem (5.5), we obtain the following.

(8.9) Theorem. *Theorem* (5.5) *is true for an arbitrary Kac–Moody group.*

(8.10) *Remarks.* Even though we have taken the base field to be the field $\mathbb{C}$ of complex numbers, all the results of the paper carry over (with the same proofs) to an arbitrary algebraically closed field of char. 0.

Also, by a result of Polo [P, Sect. 4.1], the dimension of the Zariski tangent space $Z_\mathfrak{v}(X_w)$ is independent of the char. of the field. In particular, a point $\mathfrak{v} \in X_w$ is smooth in char. 0 if and only if it is smooth in any char. p. So our smoothness criterion (as in Theorem 5.5(b)) works in arbitrary char. p.

Note added in proof. I would like to ask the following questions:

(1) For any $\lambda \in D$ and $w \in W$, define the q-character $\mathrm{ch}_q(V_w(\lambda))$ of $V_w(\lambda)$ as follows:

$$\mathrm{ch}_q V_w(\lambda) = \sum_{p=0}^{\infty} \mathrm{ch}\left(\frac{V_w(\lambda) \cap \mathscr{F}_p(\lambda)}{V_w(\lambda) \cap \mathscr{F}_{p-1}(\lambda)}\right) q^p,$$

where the notation is as in §4.

It will be very interesting to determine the q-character $\mathrm{ch}_q(V_w(\lambda))$. (This will of course provide a q-version of Demazure character formula.)

2. Take $v \leqq w \in W$ and assume that there exist prositive roots $\{\gamma_1, \cdots, \gamma_\ell\}$ such that $c_{w^{-1},v^{-1}} = (-1)^{\ell(w)-\ell(v)} \prod \gamma_i^{-1}$. (By Lemma (5.4), γ_i's will automatically be distinct and moreover will lie in $S(w^{-1}, v^{-1})$.) Is it true that, in this case, the point $\mathfrak{v} \in X_w$ is smooth? (This is a weaker requirement than our Theorem 5.5(b).)

A similar question for rational smoothness.

3. Understand the local fundamental group of a rationally smooth point $\mathfrak{v} \in X_w$. In particular, is it true that it is always finite?

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