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Infinite Grassmannians and moduli spaces of G-bundles

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Introduction

Let C be a smooth projective irreducible algebraic curve over $\mathbb{C}$ of any genus and G a connected simply-connected simple affine algebraic group over $\mathbb{C}$. In this paper we elucidate the relationship between

(1) the space of vacua ("conformal blocks") defined in conformal field theory, using an integrable highest weight representation of the affine Kac-Moody algebra associated to G and

(2) the space of regular sections ("generalised theta functions") of a line bundle on the moduli space $\mathfrak{M}$ of semistable principal G-bundles on C.

Fix a point p in C and let $\hat{\mathscr{O}}_p$ (resp. $\hat{k}_p$) be the completion of the local ring $\mathscr{O}_p$ of C at p (resp. the quotient field of $\hat{\mathscr{O}}_p$). Let $\mathscr{G} := G(\hat{k}_p)$ (the $\hat{k}_p$-rational points of the algebraic group G) be the loop group of G and let $\mathscr{P} := G(\hat{\mathscr{O}}_p)$ be the standard maximal parahoric subgroup of $\mathscr{G}$. Then the generalised flag variety $X := \mathscr{G}/\mathscr{P}$ is an inductive limit of projective varieties, in fact of generalised Schubert varieties. One has a natural $\hat{\mathscr{G}}$-equivariant line bundle $\mathfrak{L}(\chi_o)$ on X (cf. Sect. 2.2), and the Picard group $\mathrm{Pic}(X)$ is isomorphic to $\mathbb{Z}$ which is generated by $\mathfrak{L}(\chi_o)$ (Proposition 2.3), where $\hat{\mathscr{G}}$ is the universal central extension of $\mathscr{G}$ by the multiplicative group $\mathbb{C}^*$ (cf. Sect. 2.2). By an analogue of the Borel-Weil theorem proved in the Kac-Moody setting by Kumar (and also by Mathieu), the space $H^0(\mathscr{G}/\mathscr{P}, \mathfrak{L}(d\chi_o))$ of the regular sections of the line bundle $\mathfrak{L}(d\chi_o) := \mathfrak{L}(\chi_o)^{\otimes d}$ (for any $d \geqq 0$) is canonically isomorphic with the full vector space dual $V(d\chi_o)^*$ of the (integrable) highest weight (irreducible) module $V(d\chi_o)$ of the affine Kac-Moody group $\hat{\mathscr{G}}$ (or of the associated affine Kac-Moody algebra, which is a certain one dimensional central extension of the loop algebra $(\mathrm{Lie}\, G) \otimes \hat{k}_p$) with highest weight $d\chi_o$ (cf. Sect. 6.1).

* A. Ramanathan passed away on March 12, 1993

Using the fact that any principal G-bundle on $C\backslash p$ is trivial (Proposition 1.3), one sees easily that the set of isomorphism classes of principal G-bundles on C is in bijective correspondence with the double coset space $\Gamma\backslash\mathscr{G}/\mathscr{P}$, where $\Gamma := \mathrm{Mor}(C\backslash p, G)$ is the subgroup of $\mathscr{G}$ consisting of all the algebraic morphisms of $C\backslash p \to G$. Moreover X parametrizes an algebraic family $\mathscr{U}$ of principal G-bundles on C (cf. Proposition 2.8). As an interesting byproduct of this parametrization, we obtain that the moduli space $\mathfrak{M}$ of semistable principal G-bundles on C is a unirational variety (cf. Corollary 6.3). Now, given a finite dimensional representation V of G, let $\mathscr{U}(V)$ be the family of associated vector bundles on C parametrized by X. We have then the determinant line bundle $\mathrm{Det}(\mathscr{U}(V))$ on X, defined as the dual of the determinant of the cohomology of the family $\mathscr{U}(V)$ of vector bundles on C (cf. Sect. 3.8). As we mentioned above, $\mathrm{Pic}(X)$ is freely generated by the homogeneous line bundle $\mathfrak{L}(\chi_o)$ on X, in particular, there exists a unique integer m_V (depending on the choice of the representation V) such that $\mathrm{Det}(\mathscr{U}(V)) \simeq \mathfrak{L}(m_V\chi_o)$. We determine this number explicitly in Theorem 5.4, the proof of which makes use of Riemann-Roch theorem. It may be mentioned that the number m_V is given explicitly in terms of the decomposition of V under $sl(2)$ "passing through the highest root space" (cf. Sect. 5.1), and coincides with the Dynkin index of the representation V. For example, if we take V to be the adjoint representation of G, then $m_V = 2\times$ dual Coxeter number of G (cf. Lemma 5.2 and Remark 5.3). The number m_V is also expressed in terms of the induced map at the third homotopy group level $\pi_3(G) \to \pi_3(\mathrm{SL}(V))$ (cf. Corollary 5.6).

The subgroup $\Gamma \subset \mathscr{G}$ can canonically be thought of as a subgroup of $\hat{\mathscr{G}}$ (cf. Lemma 2.7). Suggested by conformal field theory, we consider the space $H^0(\mathscr{G}/\mathscr{P}, \mathfrak{L}(d\chi_o))^\Gamma$ of Γ-invariant regular sections of the $\hat{\mathscr{G}}$-equivariant (in particular Γ-equivariant) line bundle $\mathfrak{L}(d\chi_o)$ (for any $d \geqq 0$). This space of invariants is called the space of vacua. More precisely, in conformal field theory, the space of vacua is defined to be the space of invariants of the Lie algebra $\mathfrak{g} \otimes R$ in $V(d\chi_o)^*$, where R is the ring of regular functions on the affine curve $C\backslash p$ and $\mathfrak{g}$ is the Lie algebra of the group G. We have (by Proposition 6.7) $[V(d\chi_o)^*]^\Gamma = [V(d\chi_o)^*]^{\mathfrak{g}\otimes R}$ and, as already mentioned above, $H^0(\mathscr{G}/\mathscr{P}, \mathfrak{L}(d\chi_o)) \simeq V(d\chi_o)^*$. Thus, by Theorem 6.6, we see that (for any $d \geqq 0$) the space $H^0(\mathfrak{M}, \Theta(V)^{\otimes d})$ of the regular sections of the d-th power of the Θ-bundle $\Theta(V)$ (cf. Sect. 3.8) on the moduli space $\mathfrak{M}$ is isomorphic with the space of vacua $[V(dm_V\chi_o)^*]^{\mathfrak{g}\otimes R}$. This is the connection, alluded to in the beginning of the introduction, between the space of vacua and the space of generalised theta functions. (In the case $G = \mathrm{SL}(n, \mathbb{C})$, this result has also independently been obtained recently by A. Beauville and Y. Laszlo by different methods.)

The proof of our Theorem 6.6 uses geometric invariant theory; in particular, we make crucial use of the following extension lemma (cf. Proposition 7.2):

Let H be a reductive group and Q be a projective scheme with a H-linearised ample line bundle $\mathfrak{L}$ on Q, and let Q^s denote the (open) subset of semistable points of Q. Then, for any irreducible normal open H-invariant subscheme

$U \supseteqq Q^s$ of Q, the canonical restriction map $H^0(U, \mathfrak{L}^N)^H \to H^0(Q^s, \mathfrak{L}^N)^H$ is an isomorphism, for any $N \geqq 1$.

We also make crucial use of a "descent" lemma (cf. Proposition 4.1), in the proof of Theorem 6.6.

Our Theorem 6.6 can be generalised to the situation where the curve C has n marked points $\{p_1, \ldots, p_n\}$ together with finite dimensional G-modules $\{V_1, \ldots, V_n\}$ attached to them respectively, by bringing in moduli space of parabolic G-bundles on C.

It should be mentioned that Tsuchiya–Ueno–Yamada [TUY] have obtained a factorization theorem for the space of vacua, from which one gets the validity of the Verlinde's formula for the dimension of the space of vacua. In view of our identification of the space of generalised theta functions with the space of vacua, one gets the same formula for the dimension of the space of generalised theta functions (for general G). Recently G. Faltings has also announced a proof of the Verlinde's formula. A purely algebro-geometric study (which does not use loop groups) of generalised theta functions on the moduli space of (parabolic) rank two torsion-free sheaves on a nodal curve is made by Narasimhan and Ramadas [NRa]. A factorization theorem and a vanishing theorem for the theta line bundle are proved there. In addition, several other mathematicians (A. Bertram, S. Bradlow, S. Chang, G. Daskalopoulos, B. van Geemen, E. Previato, A. Szenes, M. Thaddeus, R. Wentworth, D. Zagier, ...) and physicists have studied the space of generalized theta functions (from different view points) in the case when $G = \mathrm{SL}(2)$, in the last few years.

The organization of the paper is as follows:

Apart from introducing some notation in Sect. 1, we realize the affine flag variety X as a parameter set for G-bundles. Section 2 is devoted to recalling some basic facts (we need) about the affine Kac-Moody groups and their flag varieties. In this section we prove that the affine flag variety is the parameter space for an algebraic family of G-bundles on the curve C (cf. Proposition 2.8). Section 3 is devoted to recalling some basic definitions and results on the moduli space of semistable G-bundles, including the definitions of the determinant line bundle and the Θ-bundle on the moduli space. We prove a result (cf. Proposition 4.1) on algebraic descent in Sect. 4. Section 5 is devoted to identifying the determinant line bundle on the affine flag variety with a suitable power of the basic homogeneous line bundle. Section 6 contains the statement and the proof of the main result (Theorem 6.6). Finally in Sect. 7 we prove the basic extension result (Proposition 6.5), using Geometric Invariant Theory.

1 Affine flag variety as parameter set for G-bundles

(1.1) **Notation.** Throughout the paper k denotes an algebraically closed field of char. 0. By a scheme we will mean a scheme over k. Let us fix a projective curve C over k, and a smooth point $p \in C$. Let C^* denote the open set $C \backslash p$. We also fix an affine algebraic connected reductive group G over k.

For any k-algebra A, by $G(A)$ we mean the A-rational points of the algebraic group G. We fix the following notation to be used throughout the paper:

$$\mathscr{G} = \mathscr{G}_{\mathfrak{T}} = G(\hat{k}_p)\,,$$
$$\mathscr{P} = \mathscr{P}_{\mathfrak{T}} = G(\hat{\mathscr{O}}_p)\,, \quad \text{and}$$
$$\Gamma = \Gamma_{\mathfrak{T}} = G(k[C^*])\,,$$

where $\hat{\mathscr{O}}_p$ is the completion of the local ring $\mathscr{O}_p$ of C at p, $\hat{k}_p$ is the quotient field of $\hat{\mathscr{O}}_p$, $k[C^*]$ is the ring of regular functions on the affine curve C^* (which can canonically be viewed as a subring of $\hat{k}_p$), and $\mathfrak{T}$ is the triple (G, C, p).

We recall the following

(1.2) **Definition.** *Let G be any (not necessarily reductive) affine algebraic group over k. By a* **principal G-bundle** *(for short* **G-bundle***) on an algebraic variety X, we mean an algebraic variety E on which G acts algebraically from the right and a G-equivariant morphism $\pi: E \to X$ (where G acts trivially on X), such that π is locally isotrivial (i.e. locally trivial in the étale topology).*

Let G act algebraically on a quasi-projective variety F from the left. We can then form the **associated bundle with fiber** F, denoted by $E(F)$. Recall that $E(F)$ is the quotient of $E \times F$ under the G-action given by $g(e, f) = (eg^{-1}, gf)$, for $g \in G$, $e \in E$ and $f \in F$.

Reduction of structure group of E to a closed algebraic subgroup $H \subset G$ is, by definition, an H-bundle E_H such that $E_H(G) \approx E$, where H acts on G by left multiplication. Reduction of structure group to H can canonically be thought of as a section of the associated bundle $E(G/H) \to X$.

Let $\mathscr{X} = \mathscr{X}(G, C)$ denote the set of isomorphism classes of G-bundles on the base C, and $\mathscr{X}_0 = \mathscr{X}_0(\mathfrak{T}) \subset \mathscr{X}$ denote the subset consisting of those G-bundles on C which are algebraically trivial restricted to C^*.

(1.3) **Proposition.** *Let G be a connected reductive algebraic group over k. Then the structure group of a G-bundle on a smooth affine curve Y can be reduced to the connected component $Z^0(G)$ of the centre $Z(G)$ of G.*

In particular, if G as above is semi-simple, then any G-bundle on Y is trivial.

Proof. This proposition is essentially proved in Harder's paper [H1, Satz 3.3 and the remarks following it], (as pointed out by the referee). We also need to use the following facts:

(a) An affine group scheme over Y is rationally quasi-trivial (a result due to Springer and Steinberg, cf. [Se2, Chap. III, Sect. 2.3]).

(b) $\operatorname{Pic} Y$ is a divisible group. □

The following map is of basic importance for us in this paper. This provides a bridge between the moduli space of G-bundles and the affine (Kac-Moody) flag variety.

(1.4) **Definition** (of the map $\varphi: \mathscr{G} \to \mathscr{X}_0$). *Let G be a connected reductive algebraic group over k. Consider the canonical morphisms i_1: $\operatorname{Spec}(\hat{\mathscr{O}}_p) \to C$*

and $i_2 : C^* \hookrightarrow C$. The morphisms i_1 and i_2 together provide a flat cover of C. Let us take the trivial G-bundles on both the schemes $\mathrm{Spec}(\hat{\mathscr{O}}_p)$ *and C^*. The fiber product*

$$F := \mathrm{Spec}(\hat{\mathscr{O}}_p) \underset{C}{\times} C^*$$

of i_1 and i_2 can canonically be identified with $\mathrm{Spec}(\hat{k}_p)$. *This identification* $\mathrm{Spec}(\hat{k}_p) \simeq F$ *is induced from the natural morphisms*

$$\begin{array}{ccccc} & & \mathrm{Spec}(\hat{k}_p) & & \\ & \swarrow & & \searrow & \\ \mathrm{Spec}(\hat{\mathscr{O}}_p) & & \downarrow \wr & & C^* \, . \\ & \nwarrow & & \nearrow & \\ & & F & & \end{array}$$

By a "glueing" lemma of Grothendieck [Mi, Part I, Theorem 2.23, p. 19], to give a G-bundle on C, it suffices to give an automorphism of the trivial G-bundle on $\mathrm{Spec}(\hat{k}_p)$, i.e., to give an element of $\mathscr{G} := G(\hat{k}_p)$. (Observe that since we have a flat cover of C by only two schemes, the cocycle condition is vacuously satisfied.) This is, by definition, the map $\varphi : \mathscr{G} \to \mathscr{X}_0$.

(1.5) **Proposition.** *The map φ (defined above) factors through the double coset space to give a bijective map (denoted by)*

$$\bar{\varphi} : \Gamma \backslash \mathscr{G} / \mathscr{P} \to \mathscr{X}_0 \, .$$

(Observe that, by Proposition 1.3, $\mathscr{X}_0 = \mathscr{X}$ if G is assumed to be connected and semi-simple and C is smooth and irreducible.)

Proof. From the above construction, it is clear that for $g, g' \in \mathscr{G}$, $\varphi(g)$ is isomorphic with $\varphi(g')$ (written $\varphi(g) \approx \varphi(g')$) if and only if there exist two G-bundle isomorphisms:

$$\begin{array}{ccccc} \mathrm{Spec}(\hat{\mathscr{O}}_p) \times G & & \xrightarrow[\sim]{\theta_1} & & \mathrm{Spec}(\hat{\mathscr{O}}_p) \times G \\ & \searrow & & \swarrow & \\ & & \mathrm{Spec}(\hat{\mathscr{O}}_p) & & \end{array}$$

and

$$\begin{array}{ccccc} C^* \times G & & \xrightarrow[\sim]{\theta_2} & & C^* \times G \\ & \searrow & & \swarrow & \\ & & C^* & & \end{array}$$

such that the following diagram is commutative:

$$(*)\qquad \begin{array}{ccc} \mathrm{Spec}(\hat{k}_p) \times G & \xrightarrow{\theta_1|_{\mathrm{Spec}(\hat{k}_p)}} & \mathrm{Spec}(\hat{k}_p) \times G \\ \downarrow g' & & \downarrow g \\ \mathrm{Spec}(\hat{k}_p) \times G & \xrightarrow{\theta_2|_{\mathrm{Spec}(\hat{k}_p)}} & \mathrm{Spec}(\hat{k}_p) \times G\,. \end{array}$$

Any G-bundle isomorphism θ_1 (resp. θ_2) as above is given by an element $h \in \mathscr{P}$ (resp. $\gamma \in \Gamma$). In particular, from the commutativity of the above diagram $(*)$, $\varphi(g) \approx \varphi(g')$ if and only if there exists $h \in \mathscr{P}$ and $\gamma \in \Gamma$ such that $gh = \gamma g'$, i.e., $\gamma^{-1} gh = g'$. This shows that the map φ factors through $\Gamma \backslash \mathscr{G} / \mathscr{P}$ to give an injective map $\bar{\varphi}$. The surjectivity of $\bar{\varphi}$ follows immediately from the definition of $\mathscr{X}_0$, and the fact that any G-bundle on $\mathrm{Spec}(\hat{\mathscr{O}}_p)$ is trivial. $\square$

(1.6) *Remarks.* (a) We will show (cf. Proposition 2.8) that $\mathscr{G}/\mathscr{P}$ in fact is a parameter space for an algebraic family of G-bundles.

(b) The correspondence given in the above proposition is parallel to the correspondence from the Adele group to bundles on a curve (cf. [H1, H2], also see [PS, Sect. 8.11]). Some other analogous constructions are given by Beilinson-Schechtman, Mulase [Mu], ...

(c) $\mathscr{G}/\mathscr{P}$ should be thought of as a the parameter space for G-bundles E on C together with a trivialization of $E_{|C^*}$ (cf. Proposition 2.8).

(1.7) *An alternative description of the map φ for vector bundles.* We give an alternative description of the map φ in the case when $G = \mathrm{GL}_n$. In this case $\mathscr{X}_0$ can also be thought of as the set of isomorphism classes of locally free $\mathscr{O}_C$-modules of rank n (where $\mathscr{O}_C$ is the structure sheaf of C) which are free as $\mathscr{O}_{C^*}$-modules.

Let us denote by $E = E_n$ the n-dimensional standard representation of GL_n. Then the group $\mathscr{G}$ has a canonical representation in $E(\hat{k}_p)$ and $\mathscr{P}$ is precisely the stabilizer of $E(\hat{\mathscr{O}}_p)$. Let $\mathfrak{E} := C \times E(k) \to C$ be the trivial rank-n vector bundle over C. Fix any $g \in \mathscr{G}$, and define the presheaf $\tilde{\varphi}(g)$ of $\mathscr{O}_C$-modules on C as follows: For any Zariski open $U \subset C$, set

$$\tilde{\varphi}(g)(U) = H^0(U, \mathfrak{E})\,, \quad \text{if} \quad p \notin U \quad \text{and}$$
$$\tilde{\varphi}(g)(U) = \{\sigma \in H^0(U \backslash p, \mathfrak{E}) : (\sigma)_p \in g(E(\hat{\mathscr{O}}_p))\}\,, \quad \text{if} \quad p \in U\,,$$

where $(\sigma)_p$ denotes the germ of the rational section σ at p viewed canonically as an element of $E(\hat{k}_p)$.

Now let $\varphi(g)$ be the associated sheaf of $\mathscr{O}_C$-modules on C. Since the representation of $\mathscr{G}$ in $E(\hat{k}_p)$ is $\hat{k}_p$-linear (in particular $\hat{\mathscr{O}}_p$-linear), it is easy to see that the sheaf $\varphi(g)$ is a locally free sheaf of $\mathscr{O}_C$-modules of rank n and of course (by construction) $\varphi(g)_{|C^*}$ is trivial. It can be easily seen that the map $\varphi : \mathscr{G} \to \mathscr{X}_0$ thus obtained is the same as the map φ defined in Sect. 1.4.

2 Affine Kac-Moody groups and their flag varieties

Let $\mathfrak{T} = (G, C, p)$ be as in Sect. 1.1. *In this section we will assume that the base field k is $\mathbb{C}$ and further assume that G is a connected simply-connected simple affine algebraic group over* $\mathbb{C}$. We fix a Borel subgroup $B \subset G$ and a maximal torus $T \subset B$, and define the **standard Borel subgroup** $\mathscr{B}$ of $\mathscr{G}$ as $ev_p^{-1}(B)$, where $ev_p : \mathscr{P} = G(\hat{\mathscr{O}}_p) \to G$ is the group homomorphism induced from the $\mathbb{C}$-algebra homomorphism $\hat{\mathscr{O}}_p \to \mathbb{C}$, which takes $f \mapsto f(p)$.

(2.1) *Generalized Schubert varieties.* The generalised flag variety $X := \mathscr{G}/\mathscr{P}$ (where $\mathscr{G}, \mathscr{P}$ are as in Sect. 1.1) has the following *Bruhat decomposition*:

$$(1) \qquad X = \bigcup_{\mathfrak{w} \in \tilde{W}/W} \mathscr{B}\mathfrak{w}\mathscr{P}/\mathscr{P},$$

where $W := N_G(T)/T$ is the (finite) Weyl group of G, $N_G(T)$ is the normalizer of T in G, and $\tilde{W}$ is the affine Weyl group of G (cf. [K, Sect. 6.6]). Moreover the union in (1) is disjoint.

The affine Weyl group $\tilde{W}$ is a Coxeter group and hence has a Bruhat partial order $\leqq$. This induces a partial order (again denoted by) $\leqq$ in $\tilde{W}/W$ defined by

$$\mathfrak{u} := u \bmod W \leqq \mathfrak{v} \quad (\text{for } u, v \in \tilde{W})$$

if and only if there exists a $w \in W$ such that

$$u \leqq vw .$$

We define the **generalized Schubert variety** $X_{\mathfrak{w}}$ (for any $\mathfrak{w} \in \tilde{W}/W$) by

$$(2) \qquad X_{\mathfrak{w}} = \bigcup_{\mathfrak{v} \leqq \mathfrak{w}} \mathscr{B}\mathfrak{v}\mathscr{P}/\mathscr{P}.$$

Then clearly $X_{\mathfrak{v}} \subseteqq X_{\mathfrak{w}}$ if and only if $\mathfrak{v} \leqq \mathfrak{w}$. The set $X_{\mathfrak{w}}$ has the structure of a (not necessarily smooth) finite dimensional projective variety over $\mathbb{C}$. Moreover, the inclusion $X_{\mathfrak{v}} \subseteqq X_{\mathfrak{w}}$ (for $\mathfrak{v} \leqq \mathfrak{w}$) is a closed immersion.

We put the inductive limit Hausdorff (resp. Zariski) topology on $\mathscr{G}/\mathscr{P}$, i.e., a set $U \subset \mathscr{G}/\mathscr{P}$ is open if and only if $U \cap X_{\mathfrak{w}}$ is open in $X_{\mathfrak{w}}$ in the Hausdorff (resp. Zariski) topology for all $\mathfrak{w} \in \tilde{W}/W$. The decomposition (1) provides a cellular decomposition of $\mathscr{G}/\mathscr{P}$, where $\mathscr{B}\mathfrak{w}\mathscr{P}/\mathscr{P}$ is biregular isomorphic with $\mathbb{C}^{\ell(\mathfrak{w})}$ and $\ell(\mathfrak{w})$ is the length of the smallest element in the coset $\mathfrak{w} := wW$.

(2.2) *Line bundles on* $\mathscr{G}/\mathscr{P}$. We define

$$(1) \qquad \mathrm{Pic}(\mathscr{G}/\mathscr{P}) = \varprojlim_{\mathfrak{w} \in \tilde{W}/W} \mathrm{Pic}(X_{\mathfrak{w}}),$$

where $\mathrm{Pic}(X_{\mathfrak{w}})$ is of course the set of isomorphism classes of (algebraic) line bundles on $X_{\mathfrak{w}}$. Clearly an element $\mathscr{L} \in \mathrm{Pic}(\mathscr{G}/\mathscr{P})$ is given by a collection of algebraic line bundles $\mathscr{L}_{\mathfrak{w}}$ on $X_{\mathfrak{w}}$ (for every $\mathfrak{w} \in \tilde{W}/W$) together with morphisms $i_{\mathfrak{w},\mathfrak{v}}$ (for all $\mathfrak{v} \leqq \mathfrak{w}$)

$$\begin{array}{ccc} \mathscr{L}_{\mathfrak{v}} & \overset{i_{\mathfrak{w},\mathfrak{v}}}{\hookrightarrow} & \mathscr{L}_{\mathfrak{w}} \\ \downarrow & & \downarrow \\ X_{\mathfrak{v}} & \hookrightarrow & X_{\mathfrak{w}}\,, \end{array}$$

satisfying $i_{\mathfrak{w},\mathfrak{v}} \circ i_{\mathfrak{v},\mathfrak{u}} = i_{\mathfrak{w},\mathfrak{u}}$, for all $\mathfrak{u} \leqq \mathfrak{v} \leqq \mathfrak{w}$.

One can similarly define the notion of vector bundles or principal bundles on $\mathscr{G}/\mathscr{P}$.

Let us recall that the group $\mathscr{G}$ admits a "canonical" one-dimensional central extension:

$$(2) \qquad 1 \to \mathbb{C}^* \underset{i}{\to} \tilde{\mathscr{G}} \underset{\beta}{\to} \mathscr{G} \to 1\,.$$

The "Lie algebra" Lie$(\tilde{\mathscr{G}})$ of $\tilde{\mathscr{G}}$ is described explicitly in [K, Chap. 7, Identity 7.2.1] and is denoted by $\tilde{L}(\mathfrak{g})$.

The composite map $\mathbb{C}^* \underset{i}{\to} \tilde{\mathscr{P}} \underset{q}{\to} \tilde{\mathscr{P}}/[\tilde{\mathscr{P}},\tilde{\mathscr{P}}]$ is an isomorphism, where $\tilde{\mathscr{P}} := \beta^{-1}(\mathscr{P})$ and q is the canonical projection. In particular, identifying $\tilde{\mathscr{P}}/[\tilde{\mathscr{P}},\tilde{\mathscr{P}}]$ with $\mathbb{C}^*$ (under $q \circ i$), we get the character denoted $e^{\chi_o} : \tilde{\mathscr{P}} \to \mathbb{C}^*$. Alternatively, this is the unique character which is identically 1 restricted to the commutator $[\tilde{\mathscr{P}},\tilde{\mathscr{P}}]$, and restricted to the *standard maximal torus* $\tilde{T} := \beta^{-1}(T)$ it is got by exponentiating the "integral" weight χ_o: Lie$(\tilde{T}) \to \mathbb{C}$, where χ_o is defined by

$$(3) \qquad \begin{array}{c} \chi_o(\alpha_0^\vee) = 1\,, \quad \text{and} \\ \chi_o(\alpha_i^\vee) = 0\,, \quad \text{for all} \quad 1 \leqq i \leqq \ell\,, \end{array}$$

where $\{\alpha_0^\vee, \alpha_1^\vee, \ldots, \alpha_\ell^\vee\}$ (resp. $\{\alpha_1^\vee, \ldots, \alpha_\ell^\vee\}$) are the simple coroots for $\tilde{L}(\mathfrak{g})$ (resp. $\mathfrak{g} :=$ Lie G) (cf. [K, Sect. 7.4]).

For any $d \in \mathbb{Z}$, let $\mathscr{L}(d\chi_o)$ be the homogeneous line bundle on the base $\tilde{\mathscr{G}}/\tilde{\mathscr{P}} \approx \mathscr{G}/\mathscr{P}$, which is associated to the principal $\tilde{\mathscr{P}}$-bundle $\tilde{\mathscr{G}} \to \tilde{\mathscr{G}}/\tilde{\mathscr{P}}$ by the character $(e^{\chi_o})^{-d}$. We denote its restriction to $X_{\mathfrak{w}}$ by $\mathscr{L}_{\mathfrak{w}}(d\chi_o)$. Then $\mathscr{L}_{\mathfrak{w}}(d\chi_o)$ has a canonical structure of an algebraic line bundle, which is compatible with respect to the inclusions, i.e., $\mathscr{L}_{\mathfrak{w}}(d\chi_o)_{|X_{\mathfrak{v}}} = \mathscr{L}_{\mathfrak{v}}(d\chi_o)$ for any $\mathfrak{v} \leqq \mathfrak{w}$ (cf. [S1, Sect. 2.7]). In particular, we get an element (again denoted by) $\mathscr{L}(d\chi_o) \in$ Pic$(\mathscr{G}/\mathscr{P})$.

We have the following proposition determining Pic$(\mathscr{G}/\mathscr{P})$.

(2.3) **Proposition.** *The map* $\mathbb{Z} \to$ Pic$(\mathscr{G}/\mathscr{P})$ *given by*

$$d \mapsto \mathscr{L}(d\chi_o)$$

is an isomorphism.

Proof. Since $X_{\mathfrak{w}}$ is a projective variety, by GAGA, the natural map

$$(1) \qquad \text{Pic}(X_{\mathfrak{w}}) \xrightarrow{\sim} \text{Pic}_{\text{an}}(X_{\mathfrak{w}})$$

is an isomorphism, where $\mathrm{Pic}_{\mathrm{an}}(X_{\mathfrak{w}})$ is the set of isomorphism classes of analytic line bundles on $X_{\mathfrak{w}}$.

We have the sheaf exact sequence:

(2) $$0 \to \mathbb{Z} \to \mathcal{O}_{\mathrm{an}} \to \mathcal{O}_{\mathrm{an}}^* \to 0,$$

where $\mathcal{O}_{\mathrm{an}}$ (resp. $\mathcal{O}_{\mathrm{an}}^*$) denotes the sheaf of analytic functions (resp. the sheaf of invertible analytic functions) on $X_{\mathfrak{w}}$. Taking the associated long exact cohomology sequence, we get

(3) $$\dots \to H^1(X_{\mathfrak{w}}, \mathcal{O}_{\mathrm{an}}) \to H^1(X_{\mathfrak{w}}, \mathcal{O}_{\mathrm{an}}^*) \xrightarrow{c_1} H^2(X_{\mathfrak{w}}, \mathbb{Z}) \to H^2(X_{\mathfrak{w}}, \mathcal{O}_{\mathrm{an}}) \to \dots,$$

where the map c_1 associates to any line bundle its first Chern class. Now

(4) $$H^i(X_{\mathfrak{w}}, \mathcal{O}) = 0, \quad \text{for all} \quad i > 0,$$

by [Ku, Theorem 2.16(3)] (also proved in [M]), and by GAGA

(5) $$H^i(X_{\mathfrak{w}}, \mathcal{O}) \approx H^i(X_{\mathfrak{w}}, \mathcal{O}_{\mathrm{an}}),$$

and hence the map c_1 is an isomorphism. But

(6) $$\mathrm{Pic}_{\mathrm{an}}(X_{\mathfrak{w}}) \approx H^1(X_{\mathfrak{w}}, \mathcal{O}_{\mathrm{an}}^*).$$

Hence, by combining (1) and (3)–(6), we get the isomorphism (again denoted by)

(7) $$c_1 : \mathrm{Pic}(X_{\mathfrak{w}}) \xrightarrow{\approx} H^2(X_{\mathfrak{w}}, \mathbb{Z}).$$

Further the following diagram is commutative (whenever $X_{\mathfrak{v}} \subseteqq X_{\mathfrak{w}}$):

($\mathcal{D}$) $$\begin{array}{ccc} \mathrm{Pic}(X_{\mathfrak{w}}) & \xrightarrow[\sim]{c_1} & H^2(X_{\mathfrak{w}}, \mathbb{Z}) \\ \downarrow & & \downarrow \\ \mathrm{Pic}(X_{\mathfrak{v}}) & \xrightarrow[\sim]{c_1} & H^2(X_{\mathfrak{v}}, \mathbb{Z}), \end{array}$$

where the vertical maps are the canonical restriction maps. But from the Bruhat decomposition (1) of Sect. 2.1, for any $\mathfrak{w} \geqq \mathfrak{s}_o$, the restriction map

(8) $$H^2(X_{\mathfrak{w}}, \mathbb{Z}) \to H^2(X_{\mathfrak{s}_o}, \mathbb{Z})$$

is an isomorphism, where $\mathfrak{s}_o$ is the (simple) reflection corresponding to the simple coroot $\alpha_0^\vee$, and $\mathfrak{s}_o := s_o \bmod W$. Moreover, $X_{\mathfrak{s}_o}$ being isomorphic with the complex projective line $\mathbb{P}^1$, $H^2(X_{\mathfrak{s}_o}, \mathbb{Z})$ is a free $\mathbb{Z}$-module of rank 1, which is generated by the first Chern class -1 of the line bundle $\mathcal{L}_{\mathfrak{s}_o}(\chi_o)$.

Since any element $\mathfrak{w} \neq \mathfrak{e} \in \tilde{W}/W$ satisfies $\mathfrak{w} \geqq \mathfrak{s}_o$ (in particular the elements $\mathfrak{w} \geqq \mathfrak{s}_o$ are cofinal in $\tilde{W}/W$), taking the inverse limit of diagram ($\mathcal{D}$), we get the proposition. □

(2.4) *Topology on Γ*. We fix an embedding $G \hookrightarrow \mathrm{GL}_m(\mathbb{C})$ (for some large m), and define a filtration of Γ as follows:

$$G = \Gamma_0 \subset \Gamma_1 \subset \dots ,$$

where $\Gamma_i := \{f : C^* \to G \subset \mathrm{GL}_m(\mathbb{C})$ such that all the matrix coefficients of f have poles of order $\leqq i$ at $p\}$.

It is easy to see that Γ_i's admit canonically a compatible structure of finite dimensional affine varieties. In particular, we have Hausdorff as well as Zariski topology on Γ_i's. Now we define the corresponding (Hausdorff or Zariski) topology on Γ as the inductive limit topology from Γ_i's. It is easy to see that neither topology on Γ depends upon the particular embedding of $G \hookrightarrow \mathrm{GL}_m(\mathbb{C})$.

We prove the following lemma.

(2.5) **Lemma.** *Let X be a connected variety over $\mathbb{C}$. Then any regular map $X \to \mathbb{C}^*$, which is null-homotopic in the topological category, is a constant.*

(Observe that if the singular cohomology $H^1(X, \mathbb{Z}) = 0$, then any continuous map $X \to \mathbb{C}^*$ is null-homotopic.)

Proof. Assume, if possible, that there exists a null-homotopic non-constant regular map $\lambda : X \to \mathbb{C}^*$. Since λ is algebraic, there exists a number $N > 0$ such that the number of irreducible components of $\lambda^{-1}(z) \leqq N$, for all $z \in \mathbb{C}^*$. Now we consider the N'-sheeted covering $\pi_{N'} : \mathbb{C}^* \to \mathbb{C}^*(z \mapsto z^{N'})$, for any $N' > N$. Since λ is null-homotopic, there exists a (regular) lift $\tilde{\lambda} : X \to \mathbb{C}^*$, making the following diagram commutative:

$$\begin{array}{ccc} & & \mathbb{C}^* \\ & {}^{\tilde{\lambda}}\nearrow & \downarrow \pi_{N'} \\ X & \xrightarrow[\lambda]{} & \mathbb{C}^* . \end{array}$$

Since $\tilde{\lambda}$ is regular and non-constant, by Chevalley's theorem, $\mathrm{Im}\,\tilde{\lambda}$ (being a constructible set) misses only finitely many points of $\mathbb{C}^*$. In particular, there exists a $z_o \in \mathbb{C}^*$ (in fact a Zariski open set of points) such that $\pi_{N'}^{-1}(z_o) \subset \mathrm{Im}\,\tilde{\lambda}$. But then the number of irreducible components of $\lambda^{-1}(z_o) = \tilde{\lambda}^{-1}(\pi_{N'}^{-1} z_o) \geqq N' > N$, a contradiction to the choice of N. This proves the lemma. $\square$

(2.6) **Corollary.** *There does not exist any non-constant regular map $\lambda : \Gamma \to \mathbb{C}^*$.*

(A regular map $\lambda : \Gamma \to \mathbb{C}^*$ is, by definition, a map such that $\lambda|_{\Gamma_n}$ is regular for each n, cf. Sect. 2.4.)

Proof. By Segal [S] (see also [PS, Proposition 8.11.6(i), p. 157]), Γ is connected and simply-connected, in particular, $H^1(\Gamma, \mathbb{Z}) = 0$ (where $H^*(\Gamma, \mathbb{Z})$ denotes the singular cohomology of the topological space Γ). This gives that the map λ is null-homotopic. By using the above Lemma 2.5, λ is constant on each connected component of Γ_n (for any $n \geqq 0$) and hence λ itself is constant. $\square$

Restrict the central extension (2) of Sect. 2.2 to get a central extension

(1) $$1 \to \mathbb{C}^* \xrightarrow[i]{} \tilde{\Gamma} \xrightarrow[\beta]{} \Gamma \to 1 ,$$

where $\tilde{\Gamma}$ is by definition $\beta^{-1}(\Gamma)$. The group $\tilde{\Gamma}$ admits a canonical structure of an inductive limit of affine algebraic varieties.

(2.7) **Lemma.** *There exists a unique regular group homomorphism $\Gamma \to \tilde{\Gamma}$, which splits the above central extension.*

In particular, we can canonically view Γ as a subgroup of $\tilde{\mathcal{G}}$.

Proof. The existence of a regular splitting on Γ is well known (cf., e.g., [W, Sect. 4]). The uniqueness follows immediately from the above corollary. □

We have the following proposition.

(2.8) **Proposition.** (a) *There is an algebraic G-bundle $\mathcal{U} \to C \times \mathcal{G}/\mathcal{P}$ (i.e. $\mathcal{U}_{|C\times X_{\mathfrak{w}}}$ is algebraic for any $\mathfrak{w} \in \tilde{W}/W$) such that, for any $x \in \mathcal{G}/\mathcal{P}$ the G-bundle $\mathcal{U}_x := \mathcal{U}_{|C\times x}$ is isomorphic with $\varphi(x)$ (where φ is the map of Sect. 1.4). Moreover the bundle $\mathcal{U}_{|C^*\times\mathcal{G}/\mathcal{P}}$ comes equipped with a trivialization $\alpha: \varepsilon \xrightarrow{\sim} \mathcal{U}_{|C^*\times\mathcal{G}/\mathcal{P}}$, where ε is the trivial G-bundle on $C^* \times \mathcal{G}/\mathcal{P}$.*

(b) *Let $\mathcal{E} \to C \times T$ be an algebraic family of G-bundles (parametrized by an algebraic variety T), such that $\mathcal{E}$ is trivial over $C^* \times T$ and also over* (Spec $\hat{\mathcal{O}}_p$) $\times T$. *Then, if we choose a trivialization $\beta: \varepsilon' \xrightarrow{\sim} \mathcal{E}_{|C^*\times T}$, we get a Schubert variety $X_{\mathfrak{w}}$ and a unique morphism $f: T \to X_{\mathfrak{w}}$ together with a G-bundle morphism $\hat{f}: \mathcal{E} \to \mathcal{U}_{|C\times X_{\mathfrak{w}}}$ inducing the map* Id $\times f$ *at the base such that $\hat{f} \circ \beta = \alpha \circ \theta$, where ε' is the trivial bundle on $C^* \times T$ and θ is the canonical G-bundle morphism $\varepsilon' \to \varepsilon$ inducing the map* Id $\times f$ *at the base.*

Proof. Let R be a $\mathbb{C}$-algebra and let $T :=$ Spec R be the corresponding scheme. Suppose $E \to C \times T$ is a G-bundle with trivializations α of E over $C^* \times T$ and β of E over (Spec $\hat{\mathcal{O}}_p$) $\times T$. Note that the fiber product $(C^* \times T) \times_{C\times T}$ (Spec $\hat{\mathcal{O}}_p \times T$) is canonically isomorphic with (Spec $\hat{k}_p$) $\times T$ (cf. Sect. 1.4). Therefore the trivializations α and β give rise to an element $\alpha\beta^{-1} \in G(\hat{k}_p \otimes_{\mathbb{C}} R)$. Conversely, given an element $g \in G(\hat{k}_p \otimes R)$, we can construct the family $E \to C \times$ Spec R by taking the trivial bundles on $C^* \times T$ and (Spec $\hat{\mathcal{O}}_p$) $\times T$ and glueing them via the element g. Moreover, if g_1 and g_2 are two elements of $G(\hat{k}_p \otimes R)$ such that $g_2 = g_1 h$ with $h \in G(\hat{\mathcal{O}}_p \otimes R)$, then h induces a canonical isomorphism of the bundles corresponding to g_1 and g_2. All these assertions are easily verified.

To construct the family parametrized by $\mathcal{G}/\mathcal{P}$, we note that it is enough to construct the families $\mathcal{U}_{\mathfrak{w}} \to C \times X_{\mathfrak{w}}$ parametrized by the Schubert varieties $X_{\mathfrak{w}}$ together with certain isomorphisms $\phi_{\mathfrak{w},\mathfrak{v}}$ of $\mathcal{U}_{\mathfrak{w}}|_{C\times X_{\mathfrak{v}}}$ with $\mathcal{U}_{\mathfrak{v}}$, for any $X_{\mathfrak{v}} \subset X_{\mathfrak{w}}$, such that the isomorphisms $\phi_{\mathfrak{w},\mathfrak{v}}$ satisfy the cocycle condition $\phi_{\mathfrak{w},\mathfrak{v}}\phi_{\mathfrak{v},\mathfrak{u}} = \phi_{\mathfrak{w},\mathfrak{u}}$, for all $\mathfrak{w} \geqq \mathfrak{v} \geqq \mathfrak{u}$.

Choose a local parameter t for C at p and set $\mathcal{N}^- = G(\mathbb{C}[t^{-1}])$. Then $\mathcal{N}^-$ can canonically be thought of as a subgroup of $\mathcal{G}$. Further the $\mathcal{N}^-$-orbit U^- through the base point $\mathrm{e} \in X := \mathcal{G}/\mathcal{P}$ is open in the Zariski topology on X. In particular, by the Bruhat decomposition $\{\mathfrak{w}U^-\}_{\mathfrak{w}}$ provides an open cover of X. The map $U^- \to \mathcal{G}$, defined by $x.\mathrm{e} \mapsto x$ for $x \in \mathcal{N}^-$, provides a section σ of the principal $\mathcal{P}$-bundle $\mathcal{G} \to \mathcal{G}/\mathcal{P}$ over the open set U^-, and by translating this section we also get sections $\sigma_{\mathfrak{w}}$ over any $\mathfrak{w}U^-$ ($\sigma_{\mathfrak{w}}$ does depend upon the coset representative w of $\mathfrak{w} = wW$, but for each, $\mathfrak{w} \in \tilde{W}/W$ we fix one coset representative w and then $\sigma_{\mathfrak{w}}$ really means σ_w). Now fix any Schubert variety $X_{\mathfrak{v}}$ and cover this by the affine open sets $\{(\mathfrak{w}U^-) \cap X_{\mathfrak{v}}\}$ and take the sections $\sigma_{\mathfrak{w}}$ over them. In view of the discussion above, this canonically gives rise to G-bundles $\mathscr{U}_{\mathfrak{w}} = \mathscr{U}^{\mathfrak{v}}_{\mathfrak{w}}$ on $C \times (\mathfrak{w}U^- \cap X_{\mathfrak{v}})$. Further, for any x in the intersection $U_{\mathfrak{w}_1} \cap U_{\mathfrak{w}_2}$, where $U_{\mathfrak{w}_1} := (\mathfrak{w}_1 U^-) \cap X_{\mathfrak{v}}$, we have $\sigma_{\mathfrak{w}_2}(x) = \sigma_{\mathfrak{w}_1}(x) h_{\mathfrak{w}_1,\mathfrak{w}_2}(x)$ with $h_{\mathfrak{w}_1,\mathfrak{w}_2}(x) \in G(\hat{\mathscr{O}}_p)$. These $h_{\mathfrak{w}_1,\mathfrak{w}_2}$ canonically give rise to the isomorphisms $\mathscr{U}_{\mathfrak{w}_1} \to \mathscr{U}_{\mathfrak{w}_2}$ over the intersection $C \times (\mathscr{U}_{\mathfrak{w}_1} \cap \mathscr{U}_{\mathfrak{w}_2})$, which obviously satisfy the cocycle condition. Thus the bundles $\{\mathscr{U}^{\mathfrak{v}}_{\mathfrak{w}}\}_{\mathfrak{w}}$ patch-up to give the G-bundle $\mathscr{U} = \mathscr{U}^{\mathfrak{v}}$ on $C \times X_{\mathfrak{v}}$. Since the sections $\sigma_{\mathfrak{w}}$ are defined on the whole of $\mathfrak{w}U^-$, it is easy to see that $\mathscr{U}^{\mathfrak{v}_1}$ canonically restricts to $\mathscr{U}^{\mathfrak{v}_2}$ whenever $\mathfrak{v}_1 \geqq \mathfrak{v}_2$. This completes the (a)-part, i.e., the construction of the family $\mathscr{U}$ parametrized by $\mathscr{G}/\mathscr{P}$.

To prove the (b) part, let us choose a trivialization τ of the bundle $\mathscr{E}$ restricted to $(\operatorname{Spec} \hat{\mathscr{O}}_p) \times T$. As above, this (together with the trivialization β) gives rise to a map $f_\tau : T \to \mathscr{G}$ and hence a map $f : T \to \mathscr{G}/\mathscr{P}$. (It is easy to see that the map f does not depend upon the choice of the trivialization τ.) We claim that there exists a large enough $X_{\mathfrak{w}}$ such that $\operatorname{Im} f \subset X_{\mathfrak{w}}$ and moreover $f : T \to X_{\mathfrak{w}}$ is a morphism:

For both of these assertions, we can assume that T is an affine variety $T = \operatorname{Spec} R$, for some $\mathbb{C}$-algebra R. Then the map f_τ can be thought of as an element (again denoted by) $f_\tau \in G(\hat{k}_p \otimes R)$. Choose an embedding $G \hookrightarrow \mathrm{GL}(N)$, and also choose a local parameter t around $p \in C$. Then we can write $f_\tau = (f_\tau^{i,j})_{1 \leqq i,j \leqq N}$, with $f_\tau^{i,j} \in \hat{k}_p \otimes R$. In particular, there exists a large enough $l \geqq 0$ such that (for any $1 \leqq i, j \leqq N$) $f_\tau^{i,j} \in t^{-l}\mathbb{C}[[t]] \otimes R$. From this one can see that $\operatorname{Im} f$ is contained in a Schubert variety $X_{\mathfrak{w}}$. Now the assertion that $f : T \to X_{\mathfrak{w}}$ is a morphism follows from the description of the map f_τ as an element of $G(\hat{k}_p \otimes R)$ together with the explicit description of the variety structure on $\mathscr{G}/\mathscr{P}$, as given, e.g., in [KL, Sect. 5.2]. The remaining assertions of (b) are easy to verify, thereby completing the proof of (b). □

Let $X_0 \subset X_1 \subset X_2 \subset \ldots$ be a sequence of algebraic varieties such that $X_i \subset X_{i+1}$ is a closed immersion, for all i. Let $X := \cup X_i$ be the corresponding ind-variety. For any $x \in X$, we define the **Zariski tangent space** $T_x(X) := \lim T_x(X_i)$, where $T_x(X_i)$ is the Zariski tangent space of X_i at x. If X as above is an algebraic ind-group, then $T_e(X)$ has a canonical structure of a Lie algebra (see [Sa, Sect. 1]). Endowed with this Lie algebra structure, $T_e(X)$ is denoted by $\operatorname{Lie} X$.

We define the map $\operatorname{Lie} \Gamma \to (\operatorname{Lie} G) \otimes_k k[C^*]$, by considering the differential of the evaluation map at each point of C^*, where (as in Sect. 1.1) $k[C^*]$ is the ring of regular functions on the affine curve $C^* := C \backslash p$. The following lemma determines the Lie algebra of the algebraic ind-group Γ.

(2.9) **Lemma.** *Under the above map,* $\operatorname{Lie} \Gamma$ *is isomorphic with* $\mathfrak{g} \otimes_k k[C^*]$ *as Lie algebras, where* $\mathfrak{g} := \operatorname{Lie} G$ *and the bracket in* $\mathfrak{g} \otimes k[C^*]$ *is defined as* $[X \otimes p, Y \otimes q] = [X, Y] \otimes pq$, *for* $X, Y \in \mathfrak{g}$ *and* $p, q \in k[C^*]$.

Proof. Embed G as a (closed) algebraic subgroup $i: G \hookrightarrow \mathrm{SL}_N(k)\,(\subset M_N(k))$, for some $N > 0$. This gives rise to a closed immersion $\tilde{i}: G(R) = \Gamma \hookrightarrow M_N(R)$ (where $M_N(R)$ is the space of $N \times N$ matrices over the ring $R := k[C^*]$). In particular, it induces an injective map $d\tilde{i}: T_e(\Gamma) = \mathrm{Lie}\,\Gamma \hookrightarrow T_I(M_N(R)) = M_N(R)$, at the Zariski tangent space level (where I is the identity matrix). We claim that $d\tilde{i}$ is a Lie algebra homomorphism, if we endow $M_N(R)$ with the standard Lie algebra structure. To prove this, consider the following commutative diagram (for any fixed $x \in C^*$):

$$\begin{array}{ccc} T_e(\Gamma) & \overset{d\tilde{i}}{\hookrightarrow} & M_N(R) \\ \downarrow & & \downarrow \\ T_e(G) & \overset{di}{\hookrightarrow} & M_N(k)\,, \end{array}$$

where the vertical maps are induced by the evaluation map $e_x: R \to k$ given by $p \mapsto p(x)$. Since di is a Lie algebra homomorphism, and so are the vertical maps, we obtain that $d\tilde{i}$ itself is a Lie algebra homomorphism. It is further clear, from the above commutative diagram, that the image of $d\tilde{i}$ is contained in $\mathfrak{g} \otimes R$, where $\mathfrak{g}$ is identified with its image in $M_N(k)$ via the Lie algebra homomorphism di.

Next, we prove that the image of $d\tilde{i}$ contains at least the set $\mathfrak{g} \otimes R$: Fix any ad-nilpotent vector $X \in \mathfrak{g}$ and $p \in R$, and define a morphism $\mathbb{A}^1 \to \Gamma$ by $z \mapsto \exp(zX \otimes p)$. (Since X is ad-nilpotent, the image is indeed contained in Γ.) It is easy to see that the image of the induced map (at the tangent space level at 0) is precisely the space $k(X \otimes p)$. But since the image of $d\tilde{i}$ is a Lie subalgebra, and ad-nilpotent vectors $X \in \mathfrak{g}$ generate $\mathfrak{g}$ (as a Lie algebra), the assertion follows. This completes the proof of the lemma. □

3 Preliminaries on moduli space of G-bundles and the determinant bundle

Throughout this section, G denotes a connected reductive group over $\mathbb{C}$ and C a smooth projective irreducible curve over $\mathbb{C}$.

We recall some basic concepts and results on semistable G-bundles on C. The references are [NS, R1, R2, RR]. Recall the definition of G-bundles and reduction of structure group from Sect. 1.2.

(3.1) **Definition.** *Let $E \to C$ be a G-bundle. Then E is said to be* **semistable** *(resp.* **stable***), if for any reduction E_P of structure group of E to any parabolic subgroup $P \subset G$ and any non-trivial character $\chi: P \to G_m$ which is dominant with respect to some Borel subgroup contained in P, the degree of the associated line bundle $E_P(\chi)$ is $\leqq 0$ (resp. < 0). (Note that, by definition, a dominant character is taken to be trivial on the connected component of the centre of G.)*

(3.2) *Remark.* When $G = \mathrm{GL}_n$, this definition coincides with the usual definition of semistability (resp. stability) due to Mumford (cf. [NS]) viz. a vector bundle $V \to C$ is semistable (resp. stable) if for every subbundle $W \subsetneqq V$, we have $\mu(W) \leqq \mu(V)$ (resp. $\mu(W) < \mu(V)$), where $\mu(V) := \deg V / \operatorname{rank} V$.

Let $V \to C$ be a semistable vector bundle. Then there exists a filtration by subbundles

$$V_0 = 0 \subsetneqq V_1 \subsetneqq V_2 \subsetneqq \dots \subsetneqq V,$$

such that $\mu(V_i) = \mu(V)$ and V_i/V_{i-1} are stable. Though such a filtration in general is not unique, the associated graded

$$\mathrm{gr}(V) := \bigoplus_{i \geqq 1} V_i/V_{i-1}$$

is uniquely determined by V (upto an isomorphism).

We will now describe the corresponding notion of $\mathrm{gr}(E)$ for a semistable G-bundle E.

(3.3) **Definition.** *A reduction of structure group of a G-bundle $E \to C$ to a parabolic subgroup P is called* **admissible** *if for any character of P, which is trivial on the connected component of the centre of G, the associated line bundle of the reduced P-bundle has degree 0.*

It is easy to see that if E_P is an admissible reduction of structure group of E to a parabolic subgroup P, then E is semistable if and only if the P/U-bundle $E_P(P/U)$ is semistable, where U is the unipotent radical of P. Moreover, a semistable G-bundle E admits an admissible reduction to some parabolic subgroup P such that $E_P(P/U)$ is, in fact, a stable P/U-bundle. Let M be a Levi component of P. Then $M \approx P/U$ (as algebraic groups) and thus we get a stable M-bundle $E_P(M)$. Extend the structure group of this M-bundle to G to get a semistable G-bundle denoted by $\mathrm{gr}(E)$. Then $\mathrm{gr}(E)$ is uniquely determined by E (up to an isomorphism) (see [R1]).

Two semistable G-bundles E_1 and E_2 are said to be ***S*-equivalent** if $\mathrm{gr}(E_1) \approx \mathrm{gr}(E_2)$. We call a semistable G-bundle E **quasistable** if $E \approx \mathrm{gr}(E)$. (It can be seen that a semistable vector bundle is quasistable if and only if it is a direct sum of stable vector bundles with the same μ.)

Two G-bundles E_1 and E_2 on C are said to be of the **same topological type** if they are isomorphic as G-bundles in the topological category. The topological types of all the algebraic G-bundles on C are bijectively parametrized by the first fundamental group $\pi_1(G)$ (cf. [R2, Sect. 5]).

(3.4) **Theorem.** *The set $\mathfrak{M}$ of S-equivalence classes of all the semistable G-bundles on C of a fixed topological type admits the structure of a normal, irreducible, projective variety over k, making it into a coarse moduli.*

In particular, for any algebraic family $\mathscr{E} \to C \times T$ of semistable G-bundles of the same topological type (parametrized by a variety T), the set map $\beta : T \to \mathfrak{M}$, which takes $t \in T$ to the S-equivalence class of $\mathscr{E}_t$ in $\mathfrak{M}$ is a morphism.

The details can be found in [NS, R1, R2, Ses, ...].

(3.5) *Remarks.* (a) In general $\mathfrak{M}$ is not a *fine moduli*, i.e., there may not exist any family $\mathscr{F} \to C \times \mathfrak{M}$ (parametrized by $\mathfrak{M}$) such that $\mathscr{F}_m$ belongs to the S-equivalence class $m \in \mathfrak{M}$.

(b) For $G = \mathrm{GL}_n$, i.e., for the case of rank-n vector bundles, the topological type is nothing but its degree. When the degree is coprime to the rank, the coarse moduli is in fact a fine moduli. (When the degree is not coprime to the rank, the coarse moduli is *not* a fine moduli.)

We prove a result on $\mathrm{gr}(E)$ which we will need in Sect. 6. We first prove the following:

(3.6) **Lemma.** *Let H be a connected affine algebraic group and C a smooth projective curve over k. Then any principal H-bundle on C is locally trivial in the Zariski topology.*

Proof. Let E be a principal H-bundle on C and U the unipotent radical of H. Since the group $M = H/U$ is connected and reductive, the M-bundle $E(M)$, obtained from E by extension of structure group to M, is locally trivial in the Zariski topology [R3, Proposition 4.3].

Let W be a non-empty affine open subset of C such that the restriction of $E(M)$ to W is trivial. We shall show that $E_{|W}$ is trivial (which will prove the lemma): Observe that a trivialization of $E(M)$ on W gives a reduction of the structure group H of $E_{|W}$ to the subgroup U. So, it suffices to show that any (principal) U-bundle on W is trivial:

We may assume $U \neq e$. Then there exists a (finite) filtration of U by closed normal subgroups such that the successive quotients are isomorphic to the additive group G_a. Now the assertion follows since any principal G_a-bundle on W is trivial, W being affine (see [Se1, Sect. 5.1]). □

Let P be a parabolic subgroup of G and $P = MU$ a Levi decomposition, where U is the unipotent radical of P and M a Levi component. The next proposition will be used in Sect. 6 in the case of an admissible reduction of a semistable bundle E.

(3.7) **Proposition.** *Let E be a G-bundle on C and E_P a reduction of the structure group of E to P. Denote by $\mathrm{gr}(E_P)$ the G-bundle on C obtained from the P-bundle E_P by extension of the structure group via the homomorphism*

$$P \to P/U \approx M \hookrightarrow G\,.$$

Assume that G is semisimple and connected.

Then there exists a G-bundle $\mathscr{E}$ on $C \times \mathbb{A}^1$, where $\mathbb{A}^1$ is the affine line, such that we have

(a) $\mathscr{E}_{|C\times(\mathbb{A}^1\setminus 0)} \approx p_C^*(E)$, $\mathscr{E}_{|C\times\{0\}} \approx \mathrm{gr}(E_P)$ and

(b) *$\mathscr{E}_{|C^*\times\mathbb{A}^1}$ is trivial and also the pull-back of $\mathscr{E}$ to* $(\mathrm{Spec}\,\hat{\mathscr{O}}_p) \times \mathbb{A}^1$ *is trivial, where p_C is the projection on the C-factor.*

Proof. By [R1, Lemma 2.5.12], there exists a one-parameter group $\lambda: G_m (:= \mathbb{A}^1\setminus 0) \to M$, such that the regular map

$$G_m \times P \to P\,, \quad \text{given by} \quad (t,p) \mapsto \lambda(t)p\lambda(t)^{-1} \quad \text{for} \quad t \in G_m,\, p \in P\,,$$

extends to a regular map $\phi: \mathbb{A}^1 \times P \to P$ satisfying $\phi(0, mu) = m$, for $m \in M$, $u \in U$. By Lemma (3.6), the P-bundle E_P is locally trivial in the Zariski topology. Let $\{U_i\}$ be an affine open covering of C in which the bundle E_P

is given by the transition functions $p_{ij}: U_i \cap U_j \to P$. Let $\mathcal{F}$ be the (Zariski locally trivial) P-bundle on $C \times \mathbb{A}^1$ defined by the covering $\{U_i \times \mathbb{A}^1\}$ and the transition functions

$$h_{ij}: (U_i \cap U_j) \times \mathbb{A}^1 \to P,$$

where $h_{ij}(z,t) = \phi(t, p_{ij}(z))$, for $t \in \mathbb{A}^1$, $z \in U_i \cap U_j$. Now let $\mathcal{E}$ be the G-bundle obtained from the P-bundle $\mathcal{F}$ by extension of the structure group to G. Then clearly $\mathcal{E}$ satisfies condition (a).

We next show that for any non-empty affine open subset W of C, the restriction of $\mathcal{E}$ to $W \times \mathbb{A}^1$ is trivial (this will, in particular, imply that condition (b) is satisfied): Note that, by our construction, there exists a (finite) open covering W_i of W such that $\mathcal{E}_{|W_i \times \mathbb{A}^1}$ is trivial, for every i. Now by an analogue of a result of Quillen (cf. [Ra, Theorem 2]) $\mathcal{E}_{|W \times \mathbb{A}^1}$ is the pull-back of a G-bundle on W. But, by Proposition (1.3), any G-bundle on W is trivial. □

(3.8) *Determinant bundle and Θ-bundle.* We now briefly recall a few definitions and facts on the determinant bundles and Θ-bundles associated to families of bundles on C. We follow [DN, NRa].

In the case of the moduli J_d of line bundles of fixed degree d on C, i.e., the Jacobian, there is a natural divisor (on the Jacobian) called the Θ-divisor. It is defined only up to an algebraic equivalence in general, but on the Jacobian J_{g-1} it is canonically defined (where g is the genus of C). Since we have chosen a base point p on C, the Θ-divisor on any J_d is canonically defined.

To generalise this notion to the moduli of higher rank vector bundles, one makes use of the determinant bundle associated to any family of vector bundles.

Let $\mathcal{V} \to C \times T$ be a vector bundle. Then there exists a complex of vector bundles $\mathcal{V}_i^{\cdot}$ on T (with $\mathcal{V}_i^{\cdot} = 0$, for all $i \geqq 2$):

$$\mathcal{V}_0^{\cdot} \to \mathcal{V}_1^{\cdot} \to 0 \to 0 \to \dots,$$

such that for any base change $f: Z \to T$, the i^{th} direct image on Z (under the projection $C \times Z \to Z$) of the pull back $(\mathrm{id} \times f)^* \mathcal{V}$ is given by the i^{th} cohomology of the pull back of the above complex to Z. We define the **determinant line bundle** $\operatorname{Det} \mathcal{V}^{\cdot}$ on T to be the product $\overset{\text{top}}{\bigwedge}(\mathcal{V}_1^{\cdot}) \otimes \left(\overset{\text{top}}{\bigwedge}(\mathcal{V}_0^{\cdot})^*\right)$. (Notice that our $\operatorname{Det} \mathcal{V}^{\cdot}$ is dual to the determinant line bundle as defined, e.g., in [L, Chap. 6, Sect. 1].)

The above base change property gives rise to the base change property for $\operatorname{Det} \mathcal{V}^{\cdot}$, i.e., if $f: Z \to T$ is a morphism then $\operatorname{Det}((\mathrm{id} \times f)^* \mathcal{V}^{\cdot}) \cong f^*(\operatorname{Det} \mathcal{V}^{\cdot})$.

Let $\mathfrak{L}$ be a line bundle on T, and let $p_2: C \times T \to T$ be the projection on the second factor. Then for the family $\mathcal{V}^{\cdot} \otimes p_2^* \mathfrak{L} \to C \times T$, we have $\operatorname{Det}(\mathcal{V}^{\cdot} \otimes p_2^* \mathfrak{L}) = (\operatorname{Det} \mathcal{V}^{\cdot}) \otimes \mathfrak{L}^{-\chi(\mathcal{V}^{\cdot})}$, where $\chi(\mathcal{V}^{\cdot}) := h^0(\mathcal{V}_t) - h^1(\mathcal{V}_t)$ is the Euler characteristic and $\mathcal{V}_t^{\cdot} := \mathcal{V}^{\cdot}|_{C \times t}$. (Observe that $h^0(\mathcal{V}_t) - h^1(\mathcal{V}_t)$ remains constant on any connected component of T.)

We now define the **Θ-bundle** $\Theta(\mathcal{V}^{\cdot})$ of a family of rank r and degree 0 bundles $\mathcal{V}^{\cdot} \to C \times T$ to be the modified determinant bundle given by $(\operatorname{Det} \mathcal{V}^{\cdot}) \otimes \det(\mathcal{V}_p^{\cdot})^{\chi(\mathcal{V}^{\cdot})/r}$, where $\mathcal{V}_p^{\cdot}$ is the bundle $\mathcal{V}^{\cdot}|_{p \times T}$ on T, and $\det \mathcal{V}_p^{\cdot}$ is its usual determinant line bundle. It follows then that $\Theta(\mathcal{V}^{\cdot}) = \Theta(\mathcal{V}^{\cdot} \otimes p_2^* \mathfrak{L})$, for any line bundle $\mathfrak{L}$ on T. Moreover $\Theta(\mathscr{V})$ also has the functorial property $\Theta((\mathrm{id} \times f)^* \mathscr{V}) \cong f^*(\Theta(\mathscr{V}))$.

If $\mathscr{E} \to C \times T$ is a family of G-bundles and V is a G-module, then $\mathrm{Det}(\mathscr{E}(V))$ and $\Theta(\mathscr{E}(V))$ are defined to be the corresponding line bundles of the associated family of vector bundles, via the representation V of G, for G semisimple.

For the family $\mathscr{U} \to C \times \mathscr{G}/\mathscr{P}$ (cf. Proposition 2.8), the line bundles $\Theta(\mathscr{U}(V))$ and $\mathrm{Det}(\mathscr{U}(V))$ coincide, since $\mathscr{U}_{|p \times \mathscr{G}/\mathscr{P}}$ is trivial.

It is known ([DN, NRa]; see also Remark 7.6) that there exists a line bundle Θ on the moduli space $\mathfrak{M}_o$ of rank r and degree 0 (semistable) bundles, such that for any family $\mathscr{V}$ of rank r and degree 0 semistable bundles parametrized by T we have $f^*(\Theta) \cong \Theta(\mathscr{V})$, where $f : T \to \mathfrak{M}_o$ is the morphism given by the coarse moduli property of $\mathfrak{M}_o$ (cf. Theorem 3.4).

Assume that G is semisimple (and connected) and let V be a finite dimensional representation of G of dimension r. Then for any semistable G-bundle on C, the associated vector bundle (via the representation V) is semistable (cf. [RR, Theorem 3.18]). Thus, given a family of semistable G-bundles on C parametrized by T, we have a canonical morphism (induced from the representation V) $T \to \mathfrak{M}_o$ (where $\mathfrak{M}_o$ as above is the moduli space of semistable bundles of rank r and degree 0). Let $\mathfrak{M}$ be the moduli space of semistable G-bundles on C. By the coarse moduli property of $\mathfrak{M}$, we see that we have a canonical morphism $\phi_V : \mathfrak{M} \to \mathfrak{M}_o$. We define the **theta bundle $\Theta(V)$ on $\mathfrak{M}$ associated to V** to be the pull-back of the line bundle Θ on $\mathfrak{M}_o$ via the morphism ϕ_V. It can be easily seen that for any family $\mathscr{V} \to C \times T$ of semistable G-bundles, $f^*(\Theta(V)) \simeq \Theta(\mathscr{V}(V))$, where $f : T \to \mathfrak{M}$ is the morphism (induced from the family $\mathscr{V}$) given by the coarse moduli property of $\mathfrak{M}$.

4 A result on algebraic descent

We prove the following technical result, which will crucially be used in the paper. Even though we believe that it should be known, we did not find a precise reference.

(4.1) **Proposition.** *Let $f : X \to Y$ be a surjective morphism between irreducible algebraic varieties X and Y over an algebraically closed field k of* char 0. *Assume that Y is normal and let $\mathscr{E} \to Y$ be an algebraic vector bundle on Y.*

Then any set theoretic section σ of the vector bundle $\mathscr{E}$ is regular if and only if the induced section $f^*(\sigma)$ of the induced bundle $f^*(\mathscr{E})$ is regular.

Proof. The "only if" part is of course trivially true. So we come to the "if" part.

Since the question is local (in Y), we can assume that Y is affine and moreover the vector bundle $\mathscr{E}$ is trivial, i.e., it suffices to show that any (set theoretic) map $\sigma : Y \to k$ is regular, provided $\bar{\sigma} := \sigma \circ f : X \to k$ is regular (under the assumption that $Y = \mathrm{Spec}\, R$ is irreducible normal and affine):

Since the map f is surjective (in particular dominant), the ring R is canonically embedded in $\Gamma(X) := H^0(X, \mathscr{O}_X)$. Let $R[\bar{\sigma}]$ denote the subring of $\Gamma(X)$ generated by R and $\bar{\sigma} \in \Gamma(X)$. Then $R[\bar{\sigma}]$ is a (finitely generated) domain (as X

is irreducible by assumption), and we get a dominant morphism $\hat{f}: Z \to \operatorname{Spec} R$, where $Z := \operatorname{Spec}(R[\bar{\sigma}])$. Consider the commutative diagram:

$$\begin{array}{ccc} & X & \\ {}^{\theta}\swarrow & & \searrow^{f} \\ Z & \xrightarrow[\hat{f}]{} & Y \end{array}$$

where θ is the dominant morphism induced from the inclusion $R[\bar{\sigma}] \hookrightarrow \Gamma(X)$. In particular, $\operatorname{Im}\theta$ contains a non-empty Zariski open subset U of Z. Let $x_1, x_2 \in X$ be closed points such that $f(x_1) = f(x_2)$. Then $r(x_1) = r(x_2)$, for all $r \in R$ and also $\bar{\sigma}(x_1) = \bar{\sigma}(x_2)$. This forces $\theta(x_1) = \theta(x_2)$, in particular, $\hat{f}_{|U}$ is injective on the closed points of U.

Since $\hat{f}$ is dominant, by cutting down U if necessary, we can assume that $\hat{f}_{|U}: U \to V$ is a bijection, for some open subset $V \subset Y$. Now since Y is (by assumption) normal and Z is irreducible, by Zariski's main theorem (cf. [Mum, p. 288, I. Original form]), $\hat{f}_{|U}: U \to V$ is an isomorphism, and hence σ is regular on V. Now we give two different proofs for the remaining part:

First proof. Assume, if possible, that $\sigma_{|V}$ does not extend to a regular function on the whole of Y. Then, by [B, Lemma 18.3, AG], there exists a point $y_0 \in Y$ and a regular function h on a Zariski neighborhood W of y_0 such that $h(y_0) = 0$ and $h\sigma = 1$ on $W \cap V$. But then $\bar{h}\bar{\sigma} = 1$ on $f^{-1}(W \cap V)$ (where $\bar{h} := h \circ f$) and hence, $\bar{\sigma}$ being regular on the whole of X, $\bar{h}\bar{\sigma} = 1$ on $f^{-1}(W)$. Taking $\bar{y}_0 \in f^{-1}(y_0)$ (f is, by assumption, surjective), we get $\bar{h}(\bar{y}_0)\bar{\sigma}(\bar{y}_0) = 0$. This contradiction shows that $\sigma_{|V}$ does extend to some regular function (say σ') on the whole of Y. Hence $\bar{\sigma} = \bar{\sigma}'$, in particular, by the surjectivity of f, $\sigma = \sigma'$. This proves the proposition.

Second proof. Let us define a subset $U_0 \subset Z$ by

$$U_0 = \{x \in Z : \dim e(x) = 0\},$$

where $e(x)$ is the union of all the irreducible components of $\hat{f}^{-1}(\hat{f}(x))$ containing x. Then, by Chevalley's theorem, U_0 is open (possibly empty) in Z and the map $\hat{f}_{|U_0}: U_0 \to Y$ has all its fibers finite. But since $\hat{f}$ is birational, U_0 is non-empty. Further, by Zariski's main theorem, $V_0 := \hat{f}(U_0)$ is open in Y and the map $\hat{f}_{|U_0}: U_0 \to V_0$ is an isomorphism. This gives that $\sigma_{|V_0}$ is a regular function.

Consider the surjective map

$$\hat{f}: Z \backslash \hat{f}^{-1}(V_0) \to Y \backslash V_0 .$$

Then, by the definition of V_0, every fiber of the above map has at least one irreducible component of dim $\geqq 1$ (actually of dim exactly 1). Hence

$$\dim(Y \backslash V_0) \leqq \dim(Z \backslash \hat{f}^{-1}(V_0)) - 1 \leqq \dim Y - 2$$

(since $\hat{f}: Z \to Y$ is birational and Z is irreducible), i.e., $\operatorname{codim}_Y(Y \backslash V_0) \geqq 2$. But since Y is assumed to be normal, the regular function $\sigma_{|V_0}$ admits a regular extension σ' to the whole of Y. Now by the same argument as in the first proof, we get that $\sigma = \sigma'$ on the whole of Y. This completes the second proof as well. $\square$

(4.2) *Remark.* Even though we do not need it, the same result as above is true in the analytic category if the underlying field k is taken to be $\mathbb{C}$.

5 Identification of the determinant bundle

In this section we consider the triple $\mathfrak{T} = (G, C, p)$, where G is a connected, simply-connected, simple algebraic group over $\mathbb{C}$, C is a smooth projective irreducible curve over $\mathbb{C}$, *and p is any point of C. We follow the notation as in Sect. 1.1.*

(5.1) Recall from Proposition 2.8 that $\mathcal{G}/\mathcal{P}$ is a parameter space for an algebraic family $\mathcal{U}$ of G-bundles on C. Let us fix a (finite dimensional) representation V of G. In particular, we can talk of the determinant line bundle $\mathrm{Det}(\mathcal{U}(V))$ (cf. Sect. 3.8). Also recall the definition of the (fundamental) homogeneous line bundle $\mathfrak{L}(\chi_o)$ on $\mathcal{G}/\mathcal{P}$ from Sect. 2.2. Our aim in this section is to determine the line bundle $\mathrm{Det}(\mathcal{U}(V))$ in terms of $\mathfrak{L}(\chi_o)$. We begin with the following preparation.

Let θ be the highest root of $\mathfrak{g}$. Define the following Lie subalgebra $sl_2(\theta)$ of the Lie algebra $\mathfrak{g}$ of G:

(1) $$sl_2(\theta) := \mathfrak{g}_{-\theta} \oplus \mathbb{C}\theta^\vee \oplus \mathfrak{g}_\theta \, ,$$

where $\mathfrak{g}_\theta$ is the θ-th root space, and $\theta^\vee$ is the corresponding coroot. Clearly $sl_2(\theta) \approx sl_2$ as Lie algebras. Decompose

(2) $$V = \oplus_i V_i \, ,$$

as a direct sum of irreducible $sl_2(\theta)$-submodules V_i of $\dim m_i$. Now we define

(3) $$m_V = \sum_i \binom{m_i + 1}{3} , \quad \text{where we set} \quad \binom{2}{3} = 0 \, .$$

The number m_V coincides with the *Dynkin index* of the representation V (cf. [D, Sect. 2]). We give an expression for m_V in the following lemma.

Write the formal character

(4) $$\mathrm{ch}\, V = \sum n_\lambda e^\lambda \, .$$

(5.2) **Lemma.**

(1) $$m_V = \frac{1}{2} \sum_\lambda n_\lambda \langle \lambda, \theta^\vee \rangle^2 \, .$$

In particular, for the adjoint representation ad *of* $\mathfrak{g}$ *we have*

(2) $$m_{\mathrm{ad}} = 2(1 + \langle \varrho, \theta^\vee \rangle) \, ,$$

where ϱ as usual is the half sum of the positive roots of $\mathfrak{g}$.

Similarly, for the standard n-dim. representation E_n of sl_n, $m_{E_n} = 1$.

Proof. It suffices to show that, for the irreducible representation $W(m)$ (of $\dim m+1$) of sl_2

$$\frac{1}{2}\sum_{n=0}^{m}\langle m\varrho_1 - n\alpha, H\rangle^2 = \binom{m+2}{3}, \tag{3}$$

where α is the (unique) positive root of sl_2, H the corresponding coroot and $\varrho_1 := \frac{1}{2}\alpha$.

Now the left side of (3) is equal to

$$\begin{aligned}
2\sum_{n=0}^{m}\left(\frac{m}{2}-n\right)^2 &= 4\sum_{k=1}^{k_o} k^2 = \frac{m(m+1)(m+2)}{6}, \quad \text{if } m = 2k_o \text{ is even, and}\\
&= 2\sum_{n=0}^{m}\left(k_o - \frac{1}{2} - n\right)^2, \quad \text{if } m = 2k_o - 1 \text{ is odd}\\
&= 4\sum_{k=1}^{k_o}\left(k-\frac{1}{2}\right)^2 = \left(4\sum_{k=1}^{k_o} k^2\right) + k_o - 4\sum_{k=1}^{k_o} k\\
&= \frac{m(m+1)(m+2)}{6}.
\end{aligned}$$

So in either case the left side of (3) $= \dfrac{m(m+1)(m+2)}{6} = \dbinom{m+2}{3}$. This proves the first part of the lemma.

For the assertion regarding the adjoint representation, we have

$$\mathrm{ch}(\mathrm{ad}) = (\dim \mathfrak{h}).e^0 + \sum_{\beta\in\Delta_+}(e^{\beta} + e^{-\beta}).$$

So

$$\begin{aligned}
m_{\mathrm{ad}} &= \sum_{\beta\in\Delta_+}\langle\beta,\theta^\vee\rangle^2\\
&= 4 + \sum_{\beta\in\Delta_+\backslash\theta}\langle\beta,\theta^\vee\rangle, \quad \text{since } \langle\beta,\theta^\vee\rangle = 0 \text{ or } 1, \text{ for any } \beta\in\Delta_+\backslash\theta\\
&= 4 + \langle 2\varrho - \theta, \theta^\vee\rangle\\
&= 2(1 + \langle\varrho,\theta^\vee\rangle).
\end{aligned}$$

The assertion about m_{E_n} is easy to verify. □

(5.3) *Remark.* The number $(1 + \langle\varrho,\theta^\vee\rangle)$ is called the **dual Coxeter number**(cf. [K, Sect. 6.1 and Exercise 6.2]) of $\mathfrak{g}$ (rather of the corresponding affine Kac-Moody algebra). Its value is given as below.

Type of $\mathfrak{g}$	Dual Coxeter number
A_l	$l+1$
B_l	$2l-1$
C_l	$l+1$
D_l	$2l-2$
E_6	12
E_7	18
E_8	30
G_2	4
F_4	9

Now we can state the main theorem of this section.

(5.4) **Theorem.** *With the notation as in Sect. 5.1 (as elements of* Pic $\mathscr{G}/\mathscr{P}$*)*

$$\mathrm{Det}(\mathscr{U}(V)) = \mathfrak{L}(m_V\chi_o)\,,$$

for any finite dimensional representation V of G, where the number m_V is defined by (3) *of Sect. 5.1.*

Proof. By Proposition 2.3, there exists an integer m such that

$$\mathrm{Det}(\mathscr{U}(V)) = \mathfrak{L}(m\chi_o) \in \mathrm{Pic}(\mathscr{G}/\mathscr{P})\,.$$

We want to prove that $m = m_V$:

Set $\mathscr{U}_o := \mathscr{U}(V)_{|C\times X_o}$ as the family restricted to the Schubert variety $X_o := X_{\mathfrak{s}_o}$ (cf. proof of Proposition 2.3). Denote by α (resp. β) the canonical generator of $H^2(X_o,\mathbb{Z})$ (resp. $H^2(C,\mathbb{Z})$). Then it suffices to show that $\mathrm{Det}\,\mathscr{U}_o \simeq \mathfrak{L}_{\mathfrak{s}_o}(m_V\chi_o)$, which is equivalent to showing that the first Chern class

$$c_1(\mathrm{Det}\,\mathscr{U}_o) = m_V\alpha\,: \tag{1}$$

From the definition of the determinant bundle we have

$$c_1(\mathrm{Det}\,\mathscr{U}_o) = -c_1(\pi_{2*}\mathscr{U}_o)\,, \tag{2}$$

where π_2 is the projection $C\times X_o \to X_o$, and the notation π_{2*} is as in [F, Chap. 9].

Since $\mathscr{U}_{o|C^*\times X_o}$ as well as $\mathscr{U}_{o|C\times \mathrm{e}}$ is trivial (where e is the base point of $\mathscr{G}/\mathscr{P}$), we get

$$c_1(\mathscr{U}_o) = 0\,. \tag{3}$$

Let $\tilde{\alpha}$ (resp. $\tilde{\beta}$) be the pull back of α (resp. β) under π_2 (resp. π_1). Now write

$$c_2(\mathscr{U}_o) = l\tilde{\alpha}\tilde{\beta}\,,\quad \text{for some (unique) } l\in\mathbb{Z}\,. \tag{4}$$

Let T_{π_2} be the relative tangent bundle along the fibers of π_2. Let us denote by c_1 (resp. c_2) the first (resp. second) Chern class of $\mathscr{U}_o$. By the Grothendieck's Riemann-Roch theorem [F, Sect. 9.1] applied to the (proper) map π_2, we get

$$\begin{aligned}\mathrm{ch}(\pi_{2*}\mathscr{U}_o) &= \pi_{2*}(\mathrm{ch}(\mathscr{U}_o).\,\mathrm{td}(T_{\pi_2}))\\ &= \pi_{2*}\left[\left(\mathrm{rk}\,\mathscr{U}_o + c_1 + \tfrac{1}{2}(c_1^2 - 2c_2)\right)\left(1 + \tfrac{1}{2}c_1(T_{\pi_2})\right)\right]\\ &= \pi_{2*}\left[(\mathrm{rk}\,\mathscr{U}_o - c_2)\left(1 + \tfrac{1}{2}c_1(T_{\pi_2})\right)\right], \quad \text{by (3)},\end{aligned}$$

where ch denotes the Chern character and td denotes the Todd class. Hence

$$\begin{aligned}c_1(\pi_{2*}\mathscr{U}_o) &= \pi_{2*}(-c_2(\mathscr{U}_o))\\ &= \pi_{2*}(-l\tilde{\alpha}\tilde{\beta}), \quad \text{by (4)}\\ &= -l\alpha, \quad \text{since } \pi_{2*}(\tilde{\alpha}\tilde{\beta}) = \alpha.\end{aligned} \tag{5}$$

So to prove the theorem, by (1), (2) and (5), we need to show that $l = m_V$, where l is given by (4):

It is easy to see (from its definition) that topologically the bundle $\mathscr{U}_o$ is pull back of the bundle $\mathscr{U}_o'$ on $\mathbb{P}^1 \times X_o$ (where $\mathscr{U}_o'$ is the same as $\mathscr{U}_o$ for $C = \mathbb{P}^1$) via the map

$$C \times X_o \xrightarrow{\delta\times\mathrm{Id}} \mathbb{P}^1 \times X_o,$$

where $\delta : C \to \mathbb{P}^1$ pinches all the points outside a small open disc around p to a point. Of course the map δ is of degree 1, so the cohomology generator α pulls back to the generator β (observe that $X_o \approx \mathbb{P}'$ as shown below). Hence it suffices to compute the second Chern class of the bundle $\mathscr{U}_o'$ on $\mathbb{P}^1 \times X_o$:

Choose $X_\theta \in \mathfrak{g}_\theta$ (where θ is the highest root of $\mathfrak{g}$) such that $\langle X_\theta, -\omega X_\theta\rangle = 1$, where ω is the Cartan involution of $\mathfrak{g}$ and $\langle\,,\rangle$ is the Killing form on $\mathfrak{g}$, normalized so that $\langle\theta,\theta\rangle = 2$. Set $Y_\theta = -\omega(X_\theta) \in \mathfrak{g}_{-\theta}$. Define a Lie algebra homomorphism $sl_2 \to \mathbb{C}[t, t^{-1}] \otimes \mathfrak{g}$, by

$$\begin{aligned}X &\mapsto t \otimes Y_\theta\\ Y &\mapsto t^{-1} \otimes X_\theta\\ H &\mapsto -1 \otimes \theta^\vee,\end{aligned}$$

where $\{X, Y, H\}$ is the standard basis of sl_2. The corresponding group homomorphism (choosing a local parameter t around p) $\eta : \mathrm{SL}_2(\mathbb{C}) \to \mathscr{G}$ induces a biregular isomorphism $\bar{\eta} : \mathbb{P}^1 \approx \mathrm{SL}_2(\mathbb{C})/B_1 \xrightarrow{\sim} X_o$, where B_1 is the standard Borel subgroup of $\mathrm{SL}_2(\mathbb{C})$ consisting of upper triangular matrices. In what follows we will identify X_o with $\mathbb{P}^1$ under $\bar{\eta}$. The representation V of G on restriction, under the decomposition (2) of Sect. 5.1, gives rise to a continuous group homomorphism

$$\psi : \mathrm{SU}_2(\theta) \to \prod_i (\mathrm{Aut}\, V_i),$$

where $\mathrm{SU}_2(\theta)$ is the (standard) compact form (induced from the involution ω) of the group $\mathrm{SL}_2(\theta)$ (with Lie algebra $sl_2(\theta)$).

There is a principal SU_2-bundle $\mathscr{W}$ on S^4 (in the topological category) got by the clutching construction from the identity map $S^3 \approx \mathrm{SU}_2 \to \mathrm{SU}_2$. In particular, we obtain the vector bundle $\mathscr{W}(\psi) \to S^4$ associated to the principal bundle $\mathscr{W}$ via the representation ψ, which breaks up as a direct sum of subbundles $\mathscr{W}_i(\psi)$ (got from the representations V_i).

We further choose a degree 1 continuous map $\nu : \mathbb{P}^1 \times \mathbb{P}^1 \to S^4$. We claim that the vector bundle $\mathscr{U}_o'$ on $\mathbb{P}^1 \times \mathbb{P}^1$ is isomorphic (in the topological category) with the pull back $\nu^*(\mathscr{W}(\psi))$:

Define a map $\Phi : (\mathrm{SU}_2/D) \times S^1 \to \mathrm{SU}_2$ by

$$\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix} \bmod D, t \right) \mapsto \begin{pmatrix} d & ct^{-1} \\ bt & a \end{pmatrix} \begin{pmatrix} d & c \\ b & a \end{pmatrix}^{-1},$$

for $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SU}_2$ and $t \in S^1$, where D is the diagonal subgroup of SU_2. It is easy to see that the principal SU_2-bundle $\nu^*(\mathscr{W})$ on $\mathbb{P}^1 \times \mathbb{P}^1$ is isomorphic with the principal SU_2-bundle obtained by the clutching construction from the map Φ (by covering $\mathbb{P}^1 \times \mathbb{P}^1 = S^2 \times S^2 = S^2 \times H^+ \cup S^2 \times H^-$, where H^+ and H^- are resp. the upper and lower closed hemispheres). By composing Φ with the isomorphism $\mathrm{SU}_2 \to \mathrm{SU}_2(\theta)$ (induced from the Lie algebra isomorphism $sl_2 \to sl_2(\theta)$ taking $X \mapsto X_\theta$, $Y \mapsto Y_\theta$, and $H \mapsto \theta^\vee$), and using the isomorphism $\bar{\eta}$ together with the definition of the vector bundle $\mathscr{U}_o$ we get the assertion that $\mathscr{U}_o' \approx \nu^*(\mathscr{W}(\psi))$.

So

$$c_2(\mathscr{U}_o') = \nu^*(c_2(\mathscr{W}(\psi))) = \nu^* \sum_i c_2(\mathscr{W}_i(\psi))$$

$$= \sum_i \binom{m_i + 1}{3} \tilde{\alpha}\tilde{\beta}, \text{ by the following lemma}$$

(since ν is a map of degree 1).

Hence $l = \sum_i \binom{m_i+1}{3} = m_V$, proving the theorem modulo the next lemma. □

(5.5) **Lemma.** *Let $W(m)$ be the $(m+1)$-dimensional irreducible representation of SU_2 and let $\mathscr{W}(m)$ be the vector bundle on S^4 associated to the principal SU_2-bundle $\mathscr{W}$ on S^4 (defined in the proof of Theorem 5.4) by the representation $W(m)$ of SU_2. Then*

(1)
$$c_2(\mathscr{W}(m)) = \binom{m+2}{3} \Omega,$$

where Ω is the fundamental cohomology generator of $H^4(S^4, \mathbb{Z})$.

Proof. By the Clebsch-Gordan theorem (cf. [Hu, p. 126]), we have the following decomposition as SU_2-modules:

$$W(m) \otimes W(1) = W(m+1) \oplus W(m-1), \quad \text{for any } m \geqq 1.$$

In particular, the Chern character

$$\text{ch}\,\mathscr{W}(m).\,\text{ch}\,\mathscr{W}(1) = \text{ch}\,\mathscr{W}(m+1) + \text{ch}\,\mathscr{W}(m-1)\,. \tag{2}$$

Assume, by induction, that (1) is true for all $l \leqq m$. (The validity of (1) for $m = 1$ is trivial to see.) Then by (2) we get

$$\begin{aligned}\text{ch}\,\mathscr{W}(m+1) &= \text{ch}\,\mathscr{W}(m).\,\text{ch}\,\mathscr{W}(1) - \text{ch}\,\mathscr{W}(m-1)\\ &= ((m+1).1 - c_2\mathscr{W}(m))\,(2.1 - c_2\mathscr{W}(1))\\ &\quad - (m.1 - c_2\mathscr{W}(m-1))\,,\\ &\quad \text{since } c_1\mathscr{W}(l) = 0 \text{ as it is a SU}_2\text{-bundle}.\end{aligned}$$

Hence (by induction)

$$\begin{aligned}\text{ch}\,\mathscr{W}(m+1) &= \left((m+1).1 - \binom{m+2}{3}\Omega\right)(2.1 - \Omega)\\ &\quad - \left(m.1 - \binom{m+1}{3}\Omega\right).\end{aligned} \tag{3}$$

Writing $\text{ch}\,\mathscr{W}(m+1) = (m+2).1 - c_2\mathscr{W}(m+1)$, and equating the coefficients of (3), we get

$$\begin{aligned}c_2\mathscr{W}(m+1) &= \left(2\binom{m+2}{3} + m + 1 - \binom{m+1}{3}\right)\Omega\\ &= \binom{m+3}{3}\Omega\,.\end{aligned}$$

This completes the induction and hence proves the lemma. □

Recall that for any connected complex simple group G, the third homotopy group $\pi_3(G)$ is canonically isomorphic with $\mathbb{Z}$.

(5.6) **Corollary.** *For any representation ϱ of G in a finite dimensional (complex) vector space V, the induced map $\pi_3(G) \to \pi_3(\text{SL}(V))$ is multiplication by the number m_V.*

Proof. We can clearly assume that G is simply connected. The representation $\varrho: G \to \text{SL}(V)$ gives rise to a morphism $\tilde{\varrho}: \mathscr{G}/\mathscr{P} \to \mathscr{G}_o/\mathscr{P}_o$, where $\mathscr{G}_o := \text{SL}(V)(\hat{k}_p)$ and $\mathscr{P}_o := \text{SL}(V)(\hat{\mathscr{O}}_p)$. Moreover, the family $\mathscr{U}_o(V)$ parametrized by $\mathscr{G}_o/\mathscr{P}_o$ (got from the standard representation of $\text{SL}(V)$ in V) pulls back to the family $\mathscr{U}(V)$ (parametrized by $\mathscr{G}/\mathscr{P}$). In particular, from the functoriality of the determinant bundle (cf. Sect. 3.8), Theorem 5.4, and Lemma 5.2, we see that the induced map $\tilde{\varrho}^*: H^2(\mathscr{G}_o/\mathscr{P}_o, \mathbb{Z}) \to H^2(\mathscr{G}/\mathscr{P}, \mathbb{Z})$ is multiplication by the number m_V (under the canonical identifications $H^2(\mathscr{G}_o/\mathscr{P}_o, \mathbb{Z}) \simeq \mathbb{Z} \simeq H^2(\mathscr{G}/\mathscr{P}, \mathbb{Z})$). But the flag variety $\mathscr{G}/\mathscr{P}$ is homotopic to the based loop group $\Omega_e(K)$ (where K is a compact form of G), and similarly $\mathscr{G}_o/\mathscr{P}_o$ is homotopic to $\Omega_e(\text{SU}(V))$. In particular, by the Hurewicz's theorem and the long exact homotopy sequence corresponding to the fibration $\Omega_e(K) \to P(K) \to K$ (where $P(K)$ is the path space of K consisting of the paths starting at the base point e), the corollary follows. □

(5.7) *Remark.* J.-L. Brylinski has observed a direct proof of the above corollary using Lemma 5.2.

6 Statement of the main theorem and its proof

Let the triple $\mathfrak{T} = (G, C, p)$ be as in the beginning of Sect. 5.

(6.1) **Definition.** *Recall the definition of the homogeneous line bundle $\mathfrak{L}(m\chi_o)$ on $X := \mathscr{G}/\mathscr{P} \approx \tilde{\mathscr{G}}/\tilde{\mathscr{P}}$ (for any $m \in \mathbb{Z}$) from Sect. 2.2. Define, for any $p \in \mathbb{Z}$, (cf.* [Ku, *Sect. 3.8*]*)*

$$H^p(X, \mathfrak{L}(m\chi_o)) = \varprojlim_{\mathfrak{w} \in \tilde{W}/W} H^p(X_{\mathfrak{w}}, \mathfrak{L}_{\mathfrak{w}}(m\chi_o)) . \tag{1}$$

Since $\mathfrak{L}(m\chi_o)$ is a $\tilde{\mathscr{G}}$-equivariant line bundle, $H^p(X, \mathfrak{L}(m\chi_o))$ is canonically a $\tilde{\mathscr{G}}$-module. This module is determined in [Ku, Corollary 3.11] (and also in [M]). We summarize the results:

$$H^p(X, \mathfrak{L}(m\chi_o)) = 0 , \quad \text{if} \quad p > 0 \quad \text{and} \quad m \geqq 0 , \tag{2}$$

$$H^0(X, \mathfrak{L}(m\chi_o)) = 0 , \quad \text{if} \quad m < 0 , \quad \text{and} \tag{3}$$

$$H^0(X, \mathfrak{L}(m\chi_o)) \simeq V(m\chi_o)^* \quad \text{for} \quad m \geqq 0 , \tag{4}$$

where $V(m\chi_0)$ is the irreducible highest weight $\tilde{\mathscr{G}}$-module with highest weight $m\chi_o$, and $V(m\chi_o)^*$ denotes its full vector space dual. Recall from Lemma 2.7 that Γ is canonically embedded in $\tilde{\mathscr{G}}$. By $H^p(X, \mathfrak{L}(m\chi_o))^\Gamma$ we mean the Γ-invariants in $H^p(X, \mathfrak{L}(m\chi_o))$.

Recall the definition of the map $\varphi: \mathscr{G} \to \mathscr{X}_o$ from Sect. 1.4, and the family $\mathscr{U}$ parametrized by X from Proposition 2.8. Now define

$$\begin{aligned} X^s &= \{g\mathscr{P} \in \mathscr{G}/\mathscr{P} : \ \varphi(g) \text{ is semistable}\} \\ &= \{x \in X : \mathscr{U}_x := \ \mathscr{U}_{|C \times x} \text{ is semistable}\} , \end{aligned}$$

and set (for any $\mathfrak{w} \in \tilde{W}/W$)

$$X^s_{\mathfrak{w}} = X^s \cap X_{\mathfrak{w}} .$$

Then by the same proof as [R2, Proposition 4.1], $X^s_{\mathfrak{w}}$ is a Zariski open (and non-empty, since $\mathsf{e} \in X^s_{\mathfrak{w}}$) subset of $X_{\mathfrak{w}}$, in particular, X^s is a Zariski open subset of X. Now define

$$H^p(X^s, \mathfrak{L}(m\chi_o)) = \varprojlim_{\mathfrak{w} \in \tilde{W}/W} H^p(X^s_{\mathfrak{w}}, \mathfrak{L}_{\mathfrak{w}}(m\chi_o)) . \tag{5}$$

Clearly Γ keeps X^s stable, in particular, Γ acts on the cohomology

$$H^p(X^s, \mathfrak{L}(m\chi_o)) ,$$

and we can talk of the Γ-invariants $H^p(X^s, \mathfrak{L}(m\chi_o))^\Gamma$.

The family $\mathscr{U}_{|X^s}$ yields a morphism $\psi: X^s \to \mathfrak{M}$, which maps any $x \in X^s$ to the S-equivalence class of the semistable bundle $\mathscr{U}_x$, where $\mathfrak{M}$ is the moduli space of semistable G-bundles on C (cf. Theorem 3.4). (By a morphism $X^s \to \mathfrak{M}$ we mean a map which is a morphism restricted to any $X^s_{\mathfrak{w}}$.)

(6.2) **Lemma.** *There exists a* $\mathfrak{v}_o \in \tilde{W}/W$ *such that*

$$\psi(X^s_{\mathfrak{v}_o}) = \mathfrak{M}\,.$$

Proof. Since $\bigcup_{\mathfrak{w}} X^s_{\mathfrak{w}} = \mathscr{G}^s/\mathscr{P}$ and $\psi(\mathscr{G}^s/\mathscr{P}) = \mathfrak{M}$, we get $\mathfrak{M} = \bigcup_{\mathfrak{w}} \psi(X^s_{\mathfrak{w}})$. But by a result of Chevalley (cf. [B, Chap. AG, Corollary 10.2]), $\psi(X^s_{\mathfrak{w}})$ is a finite union of locally closed subvarieties $\{\mathfrak{M}^i_{\mathfrak{w}}\}$ of $\mathfrak{M}$, hence $\mathfrak{M}$ is a countable union $\bigcup \mathfrak{M}^i_{\mathfrak{w}}$ of locally closed subvarieties. But then, by a Baire category argument, $\mathfrak{M}$ is a certain finite union of (locally closed) subvarieties $\{\mathfrak{M}^{i_1}_{\mathfrak{w}_1}, \ldots, \mathfrak{M}^{i_n}_{\mathfrak{w}_n}\}$. Now choosing a $\mathfrak{v}_o \in \tilde{W}/W$ such that $\mathfrak{v}_o \geqq \mathfrak{w}_i$, for all $1 \leqq i \leqq n$, we get that $\mathfrak{M} = \psi(X^s_{\mathfrak{v}_o})$. This proves the lemma. □

(6.3) **Corollary.** *The moduli space* $\mathfrak{M}$ *is a unirational variety.*

Proof. Since $X^s_{\mathfrak{w}}$ is an open subset of $X_{\mathfrak{w}}$ and $X_{\mathfrak{w}}$ is a rational variety (by the Bruhat decomposition, cf. Sect. 2.1), the corollary follows from the above Lemma 6.2. □

(6.4) **Proposition.** *For any* $d \geqq 0$ *and any finite dimensional representation* V *of* G*, the canonical map*

$$\psi^* : H^0(\mathfrak{M}, \Theta(V)^{\otimes d}) \to H^0(X^s, \psi^*(\Theta(V))^{\otimes d})^{\Gamma}$$

is an isomorphism, where $\Theta(V)$ *is the theta bundle on the moduli space* $\mathfrak{M}$ *associated to the representation* V *(cf. Sect. 3.8), and the vector space on the right denotes the space of* Γ*-invariants under its natural action on the line bundle* $\psi^*(\Theta(V))$*. (Since the map* $\psi : X^s \to \mathfrak{M}$ *is* Γ*-equivariant, with trivial action of* Γ *on* $\mathfrak{M}$*, the pull back bundle* $\psi^*(\Theta(V))$ *has a natural* Γ*-action.)*

Proof. Using Lemma 6.2, we see that the map ψ^* is injective. Now part b) of Proposition 2.8, and Proposition 3.7 show that if x and y are two points in X^s with $\mathscr{U}_y \simeq \mathrm{gr}(\mathscr{U}_x)$, then y belongs to the Zariski closure of the Γ-orbit of x. In particular, two points in X^s are in the same fiber of ψ if and only if the closures of their Γ-orbits intersect. This, in turn, shows that if σ is a Γ-invariant regular section of $\psi^*(\Theta(V))^{\otimes d}$ on X^s, it is induced from a set theoretic section $\underline{\sigma}$ of $\Theta(V)^{\otimes d}$ on $\mathfrak{M}$. That $\underline{\sigma}$ is regular, is seen by taking all those Schubert varieties $X_{\mathfrak{w}}$ such that $\psi(X^s_{\mathfrak{w}}) = \mathfrak{M}$ (cf. Lemma 6.2) and applying Proposition 4.1 to the morphism $\psi_{|X^s_{\mathfrak{w}}} : X^s_{\mathfrak{w}} \to \mathfrak{M}$. □

By the functorial property of the theta bundle, $\Theta(\mathscr{U}(V))_{|X^s}$ is canonically isomorphic to $\psi^*(\Theta(V))$, since ψ is defined using the restriction of the family $\mathscr{U}(V)$ to X^s (cf. Sect. 3.8). Moreover, as observed in Sect. 3.8, the line bundles $\Theta(\mathscr{U}(V))$ and $\mathrm{Det}(\mathscr{U}(V))$ coincide on the whole of X.

(6.5) **Proposition.** *Any* Γ*-invariant regular section of* $\psi^*(\Theta(V))^{\otimes d}$ *on* X^s *extends uniquely to a regular section of* $(\mathrm{Det}\,\mathscr{U}(V))^{\otimes d}$ *on* X.

This proposition will be proved in the next section.

We now state and prove our main theorem, assuming the validity of Proposition 6.5.

(6.6) **Theorem.** *Let the triple $\mathfrak{T} = (G, C, p)$ be as in the beginning of Sect. 5 and let V be a finite dimensional representation of G. Then, for any $d \geqq 0$,*

$$H^0(\mathfrak{M}, \Theta(V)^{\otimes d}) \simeq H^0(\mathcal{G}/\mathcal{P}, \mathfrak{L}(dm_V\chi_o))^\Gamma ,$$

where the latter space of Γ-invariants is defined in Sect. 6.1, the integer m_V is the same as in Theorem 5.4, and the moduli space $\mathfrak{M}$ and the theta bundle $\Theta(V)$ are as in Proposition 6.4.

In particular, $H^0(\mathcal{G}/\mathcal{P}, \mathfrak{L}(dm_V\chi_o))^\Gamma$ is finite dimensional.

(Observe that, by (4) of Sect. 6.1, $H^0(\mathcal{G}/\mathcal{P}, \mathfrak{L}(dm_V\chi_o))^\Gamma$ is isomorphic with the space of Γ-invariants in the dual space $V(dm_V\chi_o)^*$.)

Proof. We first begin with some simple observations:

(a) *For any line bundle $\mathfrak{L}$ on X, the canonical restriction map $H^0(X, \mathfrak{L}) \to H^0(X^s, \mathfrak{L}_{|X^s})$ is injective:* This is seen by restricting any section to each Schubert variety $X_{\mathfrak{w}}$, and observing that (cf. Sect. 6.1) $X^s_{\mathfrak{w}}$ is non-empty (since the base point e corresponds to the trivial bundle, which is semistable), and open (and hence dense) in the irreducible variety $X_{\mathfrak{w}}$.

(b) *If $\mathfrak{L}$ is a Γ-equivariant line bundle on X (with respect to the standard action of Γ on X) and σ is a regular section of $\mathfrak{L}$ such that its restriction to X^s is Γ-invariant, then σ itself is Γ-invariant:* By assumption, for $\gamma \in \Gamma$, the section $\gamma^*(\sigma) - \sigma$ vanishes on X^s and hence on the whole of X.

(c) *Suppose that $\mathfrak{L}'$ and $\mathfrak{L}''$ are two Γ-equivariant line bundles on X^s. Then any algebraic isomorphism of line bundles $\xi: \mathfrak{L}' \to \mathfrak{L}''$ (inducing the identity on the base) in fact is Γ-equivariant. In particular, ξ induces an isomorphism of the corresponding spaces of Γ-invariant sections:*

Define a map $\varepsilon: \Gamma \times X^s \to \mathbb{C}^*$ by

$$\varepsilon(\gamma, x) = L_{\gamma^{-1}}\xi_{\gamma x}L_\gamma(\xi_x)^{-1} \in \mathrm{Aut}(\mathfrak{L}''_x) = \mathbb{C}^* ,$$

for $\gamma \in \Gamma$ and $x \in X^s$, where L_γ is the action of γ on the appropriate line bundles, and ξ_x denotes the restriction of ξ to the fiber over $x \in X^s$. It is easy to see that ε is a regular map and of course $\varepsilon(1, x) = 1$, for all $x \in X^s$. In particular, by Corollary 2.6, $\varepsilon(\gamma, x) = 1$, for all $\gamma \in \Gamma$. This proves the assertion (c).

We now consider $(\mathrm{Det}\, \mathcal{U}(V))^{\otimes d}_{|X^s}$ as a Γ-equivariant line bundle by transporting the natural Γ-action on $\psi^*(\Theta(V))^{\otimes d}$ (cf. Proposition 6.4), via the canonical identification

$$\mathrm{Det}\, \mathcal{U}(V)_{|X^s} \simeq \psi^*(\Theta(V)) . \tag{1}$$

Choose an isomorphism of line bundles on X

$$\xi : (\mathrm{Det}\, \mathcal{U}(V))^{\otimes d} \to \mathfrak{L}(\chi_o)^{\otimes dm_V} ,$$

which exists by Theorem 5.4. Recall from Sect. 2.2 that $\mathfrak{L}(\chi_o)^{\otimes dm_V}$ is a $\tilde{\mathcal{G}}$-equivariant line bundle, in particular, by Lemma 2.7, it is a Γ-equivariant line bundle on X. Hence by (c) above, the map $\xi_o := \xi_{|X^s}$ is automatically

Γ-equivariant. We have the following commutative diagram:

$$\begin{array}{ccc} H^0(X, (\operatorname{Det} \mathscr{U}(V))^{\otimes d}) & \xrightarrow[\sim]{\bar{\xi}} & H^0(X, \mathfrak{L}(\chi_o)^{\otimes d m_V}) \\ \downarrow & & \downarrow \\ H^0(X^s, (\operatorname{Det} \mathscr{U}(V))^{\otimes d}) & \xrightarrow[\sim]{\overline{\xi_o}} & H^0(X^s, \mathfrak{L}(\chi_o)^{\otimes d m_V}) \end{array}$$

where $\bar{\xi}$ (resp. $\overline{\xi_o}$) is induced from ξ (resp. ξ_o), and the vertical maps are the canonical restriction maps. Observe that $\overline{\xi_o}$ is Γ-equivariant (since ξ_o is so).

Further, we have

$$\begin{aligned} H^0(\mathfrak{M}, \Theta(V)^{\otimes d}) &\simeq H^0(X^s, (\operatorname{Det} \mathscr{U}(V))^{\otimes d})^\Gamma \quad \text{(by (1) and Proposition 6.4)} \\ &\simeq H^0(X^s, \mathfrak{L}(\chi_o)^{\otimes d m_V})^\Gamma \quad (\text{under } \overline{\xi_o}). \end{aligned}$$

We complete the proof of the theorem by showing that the restriction map

$$H^0(X, \mathfrak{L}(\chi_o)^{\otimes d m_V})^\Gamma \to H^0(X^s, \mathfrak{L}(\chi_o)^{\otimes d m_V})^\Gamma$$

is an isomorphism:

It suffices to show that any Γ-invariant section σ of $\mathfrak{L}(\chi_o)^{\otimes d m_V}$ over X^s extends to a section over X, for then the extension will automatically be Γ-invariant by (b) and unique by (a). By the above commutative diagram, this is equivalent to showing that any Γ-invariant section σ_o of $(\operatorname{Det} \mathscr{U}(V))^{\otimes d}$ over X^s extends to the whole of X. But this is the content of Proposition 6.5, thereby completing the proof of the theorem. □

Recall the definition of the $\tilde{\mathscr{G}}$-module $V(m\chi_o)$ from Sect. 6.1, and the definition of $\operatorname{Lie} \Gamma$ from Sect. 2.9.

(6.7) **Proposition.** *For any $m \geqq 0$, we have*

$$[V(m\chi_o)^*]^\Gamma = [V(m\chi_o)^*]^{\operatorname{Lie} \Gamma} = [V(m\chi_o)^*]^{\mathfrak{g} \otimes \mathbb{C}[C^*]},$$

where $\mathfrak{g}$ is the Lie algebra of the group G and (as in Sect. 1.1) $\mathbb{C}(C^*]$ is the ring of regular functions on the affine curve C^*.

Proof. Abbreviate $V(m\chi_o)$ by V. Fix $v \in V$ and consider the morphism $\pi_v : \Gamma \to V$ given by $\pi_v(\gamma) = \gamma.v$ for $\gamma \in \Gamma$ (where Γ is considered as a subgroup of $\tilde{\mathscr{G}}$, by Lemma 2.7). Recall that, by definition, the action of the Lie algebra $\operatorname{Lie} \Gamma$ on $v \in V$ is given by the induced map $(d\pi_v)_e : T_e(\Gamma) = \operatorname{Lie} \Gamma \to T_v(V) = V$.

Fix $\theta \in V^*$. For any $v \in V$, define the map $\theta_v : \Gamma \to \mathbb{A}^1$ by $\theta_v(\gamma) = \theta(\gamma.v)$. The induced map $(d\theta_v)_e : T_e(\Gamma) = \operatorname{Lie} \Gamma \to T_{\theta(v)}(\mathbb{A}^1) = \mathbb{A}^1$ is given by

(1) $$(d\theta_v)_e(a) = \theta(a.v), \quad \text{for} \quad a \in \operatorname{Lie} \Gamma.$$

For any $\gamma_o \in \Gamma$, we now determine the map $(d\theta_v)_{\gamma_o}$: Consider the right translation map $R_{\gamma_o} : \Gamma \to \Gamma$, given by $R_{\gamma_o}(\gamma) = \gamma\gamma_o$. Then we have

(2) $$(d\theta_v)_{\gamma_o} \circ (dR_{\gamma_o})_e = (d\theta_{\gamma_o.v})_e.$$

If $\theta \in [V^*]^\Gamma$, then θ_v (for any fixed $v \in V$) is the constant map $\gamma \mapsto \theta(v)$. In particular, $(d\theta_v)_e \equiv 0$, proving (by 1) that $\theta \in [V^*]^{\mathrm{Lie}\,\Gamma}$. Conversely, take $\theta \in [V^*]^{\mathrm{Lie}\,\Gamma}$. Then by (1) and (2), for any fixed $v \in V$, $(d\theta_v)_{\gamma_o} \equiv 0$ for any $\gamma_o \in \Gamma$. In particular, for any fixed $v \in V$ and $i \geqq 0$, the map $\theta_{v_{|\Gamma_i}} : \Gamma_i \to \mathbb{A}^1$ (Γ_i is as in Sect. 2.4) is constant on the irreducible components of Γ_i (as the base field is of char. 0). But since Γ is connected, θ_v itself is forced to be a constant. Thus, we have $(\gamma\theta - \theta)v = 0$, for every $v \in V$ and $\gamma \in \Gamma$; proving that $\theta \in [V^*]^\Gamma$. Moreover, by Lemma 2.9, we have $\mathrm{Lie}\,\Gamma = \mathfrak{g} \otimes \mathbb{C}[C^*]$. This proves the proposition. □

(6.8) *Remarks.* (a) From the proof it is clear that the above proposition is true with $V(m\chi_o)$ replaced by any algebraic representation of the algebraic group Γ.

(b) In conformal field theory, the space of vacua is defined to be the space of invariants $[V(m\chi_o)^*]^{\mathfrak{g}\otimes\mathbb{C}[C^*]}$ of the Lie algebra $\mathfrak{g} \otimes \mathbb{C}[C^*]$ in the affine Kac-Moody algebra module $V(m\chi_o)^*$ (cf. [TUY, Definition 2.2.2]). We see, by Theorem 6.6 and Proposition 6.7, that the space of vacua is isomorphic to the space of generalised theta functions.

As an immediate consequence of the above remark (b), we obtain the following.

(6.9) **Corollary.** *Let the notation and assumptions be as in Theorem 6.6. Then the space of coinvariants $V(dm_V\chi_o)/((\mathfrak{g} \otimes_{\mathbb{C}} \mathbb{C}[C^*]).V(dm_V\chi_o))$ is finite dimensional. (Cf.* [K, *Exercise 11.10, p. 209*] *for a purely algebraic proof of this Corollary.).*

7 Geometric invariant theory – Proof of Proposition 6.5

In this section C is a smooth projective irreducible curve over $\mathbb{C}$ with a fixed base point p.

(7.1) **Lemma.** *Let X be an irreducible normal variety, $U \subset X$ a non-empty open subset and $\mathfrak{L}$ a line bundle on X. Then any element of $\bigoplus_{n\in\mathbb{Z}} H^0(U, \mathfrak{L}^n)$ which is integral over $\bigoplus_n H^0(X, \mathfrak{L}^n)$ belongs to $\bigoplus_n H^0(X, \mathfrak{L}^n)$.*

Proof. Since the rings in question are graded, it suffices to prove the lemma only for homogeneous elements. Let $b \in H^0(U, \mathfrak{L}^{n_o})$ be integral over $\oplus H^0(X, \mathfrak{L}^n)$, i.e., b satisfies a relation $b^m + a_1 b^{m-1} + \ldots + a_m = 0$ with $a_i \in \oplus H^0(X, \mathfrak{L}^n)$. Let D be a prime divisor in $X \backslash U$ and let b have a pole of order $\ell \geqq 0$ along D. Then the order of the pole of b^m along D is of course ℓm and that of $a_i b^{m-i}$ is $\leqq \ell(m-1)$ for every $i \geqq 1$. But since $b^m + a_1 b^{m-1} + \ldots + a_{m-1} b$ is by assumption regular along D, we are forced to have $\ell = 0$, i.e., b is regular along D. Hence $b \in H^0(X, \mathfrak{L}^{n_o})$. □

We state now a general result on the extendability of invariant sections for actions of reductive groups.

Let a reductive group H operate on a projective scheme Q along with a linearization with respect to an ample line bundle $\mathfrak{L}$ on Q. Let Q^s denote the open subset of Q of semistable points (with respect to the H-equivariant ample line bundle $\mathfrak{L}$) for the action of H. Recall that $Q^s := \{x \in Q : \exists \sigma \in H^0(Q, \mathfrak{L}^N)^H$ for some $N \geqq 1$ such that $\sigma(x) \neq 0\}$. We then have the following proposition (cf. [NRa, Se1]).

(7.2) **Proposition.** *Let $U \supset Q^s$ be a H-invariant open subset of Q, which (i.e. U) is normal and irreducible. Then, for $N \geqq 1$, any H-invariant section of $\mathfrak{L}^N$ on Q^s can be extended to a H-invariant section of $\mathfrak{L}^N$ on U.*

Proof. We indicate the proof when Q is normal and $U = Q$. (The general case can be reduced to this case by the arguments as in [NRa, Lemma 4.15].) Let $\sigma \in H^0(Q^s, \mathfrak{L}^N)^H$. If D is an irreducible divisor in $Q \backslash Q^s$ along which σ has a pole, then we can find a $\tau \in H^0(Q, \mathfrak{L}^{N'})^H$ (for some $N' \gg 0$) such that $\sigma^{N_1}\tau^{N_2}$ will not vanish identically on D for suitable $N_1, N_2 > 0$. This is a contradiction since $D \subset Q \backslash Q^s$, in particular, any invariant section vanishes on D. □

(7.3) G.I.T. *and moduli of vector bundles.* We recall the construction of the moduli spaces of vector bundles on C using G.I.T.. Let $r \geqq 1$ and δ be integers. For the fixed point $p \in C$ and for a coherent sheaf $\mathbf{F}$ on C, put $\mathbf{F}(m) = \mathbf{F} \otimes_{\mathscr{O}} \mathscr{O}(mp)$, for any $m \in \mathbb{Z}$, where $\mathscr{O} = \mathscr{O}_C$ is the structure sheaf of C. We can choose an integer $m_o = m_o(r, \delta)$ such that for any $m \geqq m_o$ and any semistable vector bundle $\mathbf{E}$ of rank r and degree δ on C, we have $H^1(\mathbf{E}(m)) = 0$ and $\mathbf{E}(m)$ is generated by its global sections. Let $q = \dim H^0(\mathbf{E}(m)) = \delta + r(m + 1 - g)$ and consider the *Grothendieck quot scheme* Q consisting of coherent sheaves on C which are quotients of $\mathbb{C}^q \otimes \mathscr{O}$ with Hilbert polynomial (in the indeterminate v) $rv + q$ (where g is the genus of C). The group $\mathrm{GL}(q, \mathbb{C})$ operates canonically on Q and the action on $C \times Q$ (with the trivial action on C) lifts to an action of the tautological sheaf $\mathscr{E}$ on $C \times Q$.

We denote by R_o the $\mathrm{GL}(q)$-invariant open subset of Q consisting of those $x \in Q$ such that $\mathscr{E}_x = \mathscr{E}_{|C \times x}$ is locally free and such that the following canonical map is an isomorphism:

$$\mathbb{C}^q = H^0(\mathbb{C}^q \otimes \mathscr{O}) \xrightarrow{\sim} H^0(\mathscr{E}_x).$$

Then R_o is smooth and irreducible. We still denote by $\mathscr{E}$ the restriction of the family to R_o.

We obtain a $\mathrm{GL}(q)$-linearized ample line bundle $\mathfrak{L}$ on Q by embedding Q in a suitable Grassmannian as follows: We choose an integer $k_o = k_o(m)$ such that for $k \geqq k_o$ the composite map

$$\mathbb{C}^q \otimes H^0(\mathscr{O}(k)) \to H^0(\mathscr{E}_x) \otimes H^0(\mathscr{O}(k)) \to H^0(\mathscr{E}_x(k))$$

is surjective for all $x \in Q$, and such that the morphism $Q \to$ Grass (taking $x \mapsto H^0(\mathscr{E}_x(k))$) is a closed embedding, where $\mathscr{O}(k) := \mathscr{O}(kp)$ and Grass denotes the Grassmannian of $\delta + 1 - g + r(m + k)$ dimensional quotient spaces of $\mathbb{C}^q \otimes H^0(\mathscr{O}(k))$. We define the ample line bundle $\mathfrak{L}$ on Q to be the pull back of the natural ample line bundle on Grass, namely the determinant of the universal quotient bundle on Grass. The action of $\mathrm{GL}(q)$ clearly lifts to $\mathfrak{L}$.

There exists a positive integer m'_o with $m'_o \geqq m_o$ such that for any integer $m \geqq m'_o$ there is a positive integer $k'_o = k'_o(m) \geqq k_o(m)$ with the property that the following conditions are equivalent (for any $k \geqq k'_o$):

(1) A point $x \in Q$ is semistable in the sense of G.I.T. for the $\mathrm{SL}(q)$-linearized bundle $\mathfrak{L}$.

(2) $x \in R_o$ (in particular, the sheaf $\mathscr{E}_x$ is locally free) and the bundle $\mathscr{E}_x$ is a semistable vector bundle on C.

We denote by R_o^s, by abuse of notation, the set of semistable points (in the sense of G.I.T.) in Q. By the above equivalent conditions, we have $R_o^s \subset R_o$. Now the G.I.T. quotient $R_o^s // \mathrm{GL}(q)$ yields the *moduli space* $\mathfrak{M}_o$ *of vector bundles* of rank r and degree δ.

(For all this, see [NRa, Appendix A] or [Le].)

(7.4) We note that we can arrange the above construction in such a way that any fixed bounded family of vector bundles of rank r and degree δ occurs in R_o. (This observation will be crucial for us.) More precisely, let $\mathscr{V}_o \to C \times T_o$ be a family of vector bundles of rank r and degree δ (parametrized by a variety T_o). We can find an integer m_{T_o} such that for $m \geqq m_{T_o}$, we have:

(1) $R^1 p_{T_o *}(\mathscr{V}_o(m)) = 0$,

(2) $p_{T_o *}(\mathscr{V}_o(m))$ is a vector bundle on T_o (say of rank q).

(3) The canonical map $p^*_{T_o} p_{T_o *}(\mathscr{V}_o(m)) \to \mathscr{V}_o(m)$ is surjective, where $p_{T_o}: C \times T_o \to T_o$ is the projection on the second factor,

$$\mathscr{V}_o(m) := \mathscr{V}_o \otimes_{\mathscr{O}_{C \times T_o}} p_C^* \mathscr{O}(m),$$

and $p_C: C \times T_o \to C$ is the projection on the first factor.

Choose $m > \max(m_{T_o}, m'_o)$, where m'_o is as in Sect. 7.3. Let $\mathbf{P}_o$ be the frame bundle of $p_{T_o *}(\mathscr{V}_o(m))$ with the projection $\pi_o: \mathbf{P}_o \to T_o$. Then there exists a canonical $\mathrm{GL}(q)$-equivariant morphism $\varphi_o: \mathbf{P}_o \to R_o$ such that the families $\pi_o^*(\mathscr{V}_o)$ and $\varphi_o^*(\mathscr{E}(-m))$ are isomorphic, where the family $\mathscr{E}$ on R_o is as in Sect. 7.3, $\mathscr{E}(-m) := \mathscr{E} \otimes_{\mathscr{O}_{C \times R_o}} \bar{p}_C^*(\mathscr{O}(-m))$ and $\bar{p}_C: C \times R_o \to C$ is the projection on the first factor.

(7.5) **Lemma.** *Suppose that $\delta = 0$. Let $\Theta(\mathscr{F})$ be the theta bundle (on R_o) of the family $\mathscr{F} := \mathscr{E}(-m)$ (cf. Sect. 3.8). Then there exist positive integers e and f such that, as* $\mathrm{GL}(q)$ *equivariant line bundles,*

$$\Theta(\mathscr{F})^{\otimes e} \cong (\mathfrak{L}_{|R_o})^{\otimes f},$$

where $\mathfrak{L}$ is the ample line bundle on Q defined in Sect. 7.3.

Proof. For any integer $\ell \geqq 1$, we have

$$\mathrm{Det}(\mathscr{F}(\ell)) = (\mathrm{Det}\,\mathscr{F}) \otimes (\det(\mathscr{F}_{|p \times R_o}))^{-\ell},$$

as is seen from the exact sequence

$$0 \to \mathscr{F} \to \mathscr{F}(\ell) \to \mathscr{F} \otimes_{\mathscr{O}_{C \times R_o}} \bar{p}_C^*(\mathscr{O}/\mathfrak{m}_p^\ell) \to 0,$$

where $\mathfrak{m}_p \subset \mathscr{O}$ is the sheaf of functions on C vanishing at p. Observing that $\mathfrak{L}^{-1}_{|R_o} \simeq \mathrm{Det}(\mathscr{F}(k+m))$ (where m and k are as in Sect. 7.3) and $\mathrm{Det}\,\mathscr{F}(m)$ is trivial, we see that $\mathfrak{L}_{|R_o} \simeq (\det(\mathscr{F}_{|p\times R_o}))^k$ and $\Theta(\mathscr{F}) \simeq (\det(\mathscr{F}_{|p\times R_o}))^{m+1-g}$. (By choosing m large enough in Sect. 7.3, we may assume that $m+1-g > 0$.) This proves the lemma. (Compare [NRa, Proof of Theorem 1(B)].) □

(7.6) *Remark.* One knows that $\Theta(\mathscr{F})_{|R_o^s}$ descends to a line bundle Θ on $\mathfrak{M}_o$ [DN], [NRa, Proof of Theorem 1(A)]. By G.I.T., some power of $\mathfrak{L}_{|R_o^s}$ descends as an ample line bundle on $\mathfrak{M}_o$. Using Lemma 7.5, we see that Θ is an ample line bundle on $\mathfrak{M}_o$.

(7.7) **Proposition.** *Let* $f_o: R_o^s \to \mathfrak{M}_o = R_o^s // \mathrm{GL}(q)$ *be the canonical map. Let* σ *be a section of* $\Theta^{\otimes \ell}$ *over* $\mathfrak{M}_o$ *(for any* $\ell \geqq 1$*). Then the section* $f_o^*(\sigma)$ *over* R_o^s *of the line bundle* $f_o^*(\Theta^{\otimes \ell}) \simeq (\Theta\mathscr{F})^{\otimes \ell}$ *extends uniquely as a* $\mathrm{GL}(q)$*-invariant section of* $(\Theta\mathscr{F})^{\otimes \ell}$ *over* R_o*, where, as in Lemma 7.5,* $\mathscr{F} = \mathscr{E}(-m)$.

Proof. By Proposition 7.2, any $\mathrm{GL}(q)$-invariant section of any positive power of $\mathfrak{L}$ over R_o^s extends to R_o, as R_o is smooth. Thus, by Lemma 7.5, some power of $f_o^*(\sigma)$ extends to R_o. Hence, by Lemma 7.1, $f_o^*(\sigma)$ itself extends. Observe that $R_o^s \neq \emptyset$, as the trivial bundle is semistable. Since R_o is irreducible, the extension is unique and invariant. □

(7.8) *Moduli of principal G-bundles.* Assume that G is a connected semisimple algebraic group. Let T be a variety parametrizing a family $\mathscr{V}$ of G-bundles on C. Then there exists a smooth quasi-projective irreducible variety R with an action of $\mathrm{GL}(N)$ (for some N), a family $\mathscr{W}$ of G-bundles on C parametrized by R and a lift of the $\mathrm{GL}(N)$-action to $\mathscr{W}$ (as bundle automorphisms), such that the following holds:

(I) Let $R^s := \{x \in R: \mathscr{W}_x := \mathscr{W}_{|C\times x}$ is a semistable G-bundle$\}$ be the $\mathrm{GL}(N)$-invariant open subset of R. Then the canonical map $R^s \to \mathfrak{M}$ is surjective, where $\mathfrak{M}$ is the moduli space of semistable G-bundles.

(II) Moreover, there exists a principal $\mathrm{GL}(N)$-bundle $\pi: \mathbf{P} \to T$ and a $\mathrm{GL}(N)$-equivariant morphism $\varphi: \mathbf{P} \to R$ such that the families $\varphi^*(\mathscr{W})$ and $\pi^*(\mathscr{V})$ are isomorphic. (See [R1].)

Now if V is a finite dimensional representation of G, we denote by $\Theta(\mathscr{W}(V))$ the theta bundle on R of the family $\mathscr{W}(V)$, of vector bundles of rank r ($r = \dim V$) and degree 0 parametrized by R, obtained from the family $\mathscr{W}$ of (principal) G-bundles via the representation V. Note that $\mathrm{GL}(N)$ operates on $\Theta(\mathscr{W}(V))$. Let $\Theta(V)$ be the theta bundle on the moduli space $\mathfrak{M}$ associated to the representation V of G (cf. Sect. 3.8). If $f: R^s \to \mathfrak{M}$ is the canonical map, we have

$$f^*(\Theta(V)) \simeq \Theta(\mathscr{W}(V))_{|R^s}\,.$$

(7.9) **Proposition.** *Any section of* $\Theta(V)^{\otimes \ell}$ *over* $\mathfrak{M}$ *(for* $\ell \geqq 1$*), considered as a* $\mathrm{GL}(N)$*-invariant section of* $(\Theta(\mathscr{W}(V)))^{\otimes \ell}$ *over* R^s*, extends uniquely as an invariant section of* $(\Theta(\mathscr{W}(V)))^{\otimes \ell}$ *over* R.

Proof. We will prove the proposition by showing that such a section of $\Theta(\mathscr{W}(V))^{\otimes \ell}$ over R^s is integral over $\bigoplus_n H^0(R, \Theta(\mathscr{W}(V))^n)$, and then applying Lemma 7.1:

We will apply the results of Sects. 7.3 and 7.4. With the notation of Sect. 7.4, choose for T_o the variety R and for $\mathscr{V}_o$ the vector bundle $\mathscr{W}(V)$ on $C \times R$ defined above in Sect. 7.8. Let $h = h_V : \mathfrak{M} \to \mathfrak{M}_o$ be the morphism defined by V, where (as in Sect. 7.3) $\mathfrak{M}_o$ is the moduli space of rank r and degree 0 vector bundles on C. We have $h^*(\Theta) \simeq \Theta(V)$, where Θ is the theta bundle on $\mathfrak{M}_o$ (see Remark 7.6 and Sect. 3.8).

Since Θ is ample and h is a projective morphism, we see that

$$\bigoplus_n H^0(\mathfrak{M}, \Theta(V)^{\otimes n})$$

is a module of finite type over $\bigoplus_n H^0(\mathfrak{M}_o, \Theta^{\otimes n})$. In particular, the former ring is integral over the latter. Let σ be a section of $\Theta(V)^{\otimes \ell}$ over $\mathfrak{M}$. Then σ satisfies an equation

$$\sigma^d + a_{d-1}\sigma^{d-1} + \ldots + a_1\sigma + a_0 = 0\,,$$

where $a_j \in \bigoplus_n H^0(\mathfrak{M}_o, \Theta^{\otimes n})$. Let $f_o : R_o^s \to R_o^s // \mathrm{GL}(q) = \mathfrak{M}_o$ be the canonical map (as in Proposition 7.7). If $\{a_{ij}\}_i$ are the homogeneous components of a_j, using Proposition 7.7, we can extend $f_o^*(a_{ij})$ to an invariant section (say) σ_{ij} of some appropriate power of $\Theta(\mathscr{F})$ over R_o, where $\mathscr{F} = \mathscr{E}(-m)$ (as in Lemma 7.5). Pulling back σ_{ij} via $\varphi_o : \mathbf{P}_o \to R_o$ (cf. Sect. 7.4) and descending them via the projection $\pi_o : \mathbf{P}_o \to R$ (cf. Sect. 7.4) to sections of some appropriate powers of $\Theta(\mathscr{W}(V))$ over R), we see that $f^*(\sigma)$ is integral over $\bigoplus_n H^0(R, \Theta(\mathscr{W}(V))^{\otimes n})$, where $f : R^s \to \mathfrak{M}$ is the canonical map as in Sect. 7.8. (Observe that φ_o maps $\pi_o^{-1}(R^s)$ into R_o^s.) $\square$

Finally we prove Proposition 6.5 and thus complete the proof of Theorem 6.6.

(7.10) *Proof of Proposition 6.5.* Let $\tilde{\sigma}$ be a Γ-invariant section of $\psi^*(\Theta(V))^{\otimes d}$ on X^s. By Proposition 6.4, there is a section σ of $\Theta(V)^{\otimes d}$ over $\mathfrak{M}$ such that $\psi^*(\sigma) = \tilde{\sigma}$. Let $X_{\mathfrak{w}}$ be a Schubert variety. With the notation of Sect. 7.8 we construct R, where we take for T the variety $X_{\mathfrak{w}}$ and for $\mathscr{V}$ the restriction of the family $\mathscr{U}$ (Proposition 2.8) to $X_{\mathfrak{w}}$. Now σ can be viewed as an invariant section of $\Theta(\mathscr{W}(V))^{\otimes d}$ over R^s and hence (by Proposition 7.9) extends to an invariant section σ' of $\Theta(\mathscr{W}(V))^{\otimes d}$ over R. Pulling back σ' via $\varphi : \mathbf{P} \to R$ (cf. Sect. 7.8) and descending via $\pi : \mathbf{P} \to T = X_{\mathfrak{w}}$, we obtain a section of $(\Theta(\mathscr{U}(V)_{|X_{\mathfrak{w}}}))^{\otimes d}$ which extends the section $\tilde{\sigma}_{|X_{\mathfrak{w}}^s}$. Moreover, this extension is unique as $X_{\mathfrak{w}}^s \neq \emptyset$ (cf. Sect. 6.1). Varying $X_{\mathfrak{w}}$ we see that $\tilde{\sigma}$ extends to a section of $\Theta(\mathscr{U}(V))^{\otimes d}$ over X. This completes the proof of the proposition. $\square$

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