Representations of Quantum Groups at Roots of Unity

SHRAWAN KUMAR
Department of Mathematics
University of North Carolina
Chapel Hill
N.C. 27599-3250
USA

0. Introduction

Let $\mathfrak{g}$ be a (finite dimensional) complex semi-simple Lie algebra, and let $U(\mathfrak{g})$ be its universal enveloping algebra. Fix a prime ℓ . To $\mathfrak{g}$, there is associated (via Chevalley basis) a (semi-simple) simply-connected algebaic group G over the field K (where K is the algebraic closure of the prime field F_ℓ). Also there is an integral form $U_{\mathbb{Z}}(\mathfrak{g})$ of $U(\mathfrak{g})$ (which is a Hopf subalgebra), in particular, we get the Hopf algebra (called the Hyper algebra) $U_K(\mathfrak{g}) := K \otimes_{\mathbb{Z}} U_{\mathbb{Z}}(\mathfrak{g})$. As is known, the category of rational G-modules over K is the same as the category of locally finite $U_K(\mathfrak{g})$ -modules. Further, there is a finite dimensional Hopf subalgebra $u_K(\mathfrak{g})$ (known as the restricted enveloping algebra) of $U_K(\mathfrak{g})$, such that the study of irreducible $U_K(\mathfrak{g})$ -modules reduces (by Steinberg's tensor product theorem) to that of irreducible $u_K(\mathfrak{g})$-modules. We refer to this as the modular case.

Drinfeld and Jimbo have independently defined a Hopf algebra, the quantized universal enveloping algebra U over the field of rational functions in one variable $\mathbb{C}(v)$, which is a certain Hopf algebra deformation of $U(\mathfrak{g})$. Let $\mathcal{A} = \mathbb{Z}[v, v^{-1}] \subset \mathbb{C}(v)$ be the subring of Laurent polynomials. Analogous to the integral form $U_{\mathbb{Z}}(\mathfrak{g})$ of $U(\mathfrak{g})$, Lusztig has defined an $\mathcal{A}$-form $U_{\mathcal{A}}$ of U (which is a Hopf subalgebra). Thus, for any commutative $\mathcal{A}$-algebra $\mathcal{B}$, one gets the Hopf algebra $U_{\mathcal{B}} := \mathcal{B} \otimes_{\mathcal{A}} U_{\mathcal{A}}$. Now fix an odd integer $\ell > 1$ (assume, in addition, $(3, \ell) = 1$, if G_2 is a factor of $\mathfrak{g}$), and a primitive ℓ-th root of unity ξ. If we take for $\mathcal{B}$ the cyclotomic field $\mathbb{Q}_\xi$, viewed as an $\mathcal{A}$-algebra (where v acts via the multiplication with ξ), we get the Hopf algebra $U_\xi := U_{\mathbb{Q}_\xi}$.

As shown by Lusztig, U_ξ admits a certain finite dimensional Hopf subalgebra u_ξ and a certain quotient u_ξ of u_ξ (see §1.3). The algebra u_ξ is a 'quantization' of the restricted enveloping algebra $u_K(\mathfrak{g})$ (in the case of ℓ prime), and is called the quantized restricted enveloping algebra. By an analogue of Steinberg tensor product theorem (proved by Lusztig), to understand the irreducible representations of U_ξ , it suffices to understand the irreducible representations of u_ξ.

The aim of this paper is to study some aspects of the representation theory of $\mathrm{u} := \mathrm{u}_\xi$, in particular, we extend some of the results known for the restricted enveloping algebra $u_K(\mathfrak{g})$ to that for the quantized restricted enveloping algebra u_ξ.

In more detail; we prove that the algebra u is symmetric (Theorem 2.2) in the sense of Nesbitt [N], in particular, the Cartan matrix C associated to the algebra u (cf. §3.2) is symmetric. The notion of Verma modules easily carries over to u, and moreover (as in the modular case) any projective u-module admits a Verma filtration (Proposition 5.16). Let B (resp. D) be the matrix, obtained from the multiplicity of Verma (resp. irreducible) modules in projective (resp. Verma) modules. We show that $C = BD$, D is a block matrix with blocks parametrized by the linkage classes and having same row vectors, and moreover we prove the reciprocity : $B = D^t$ (where D^t is the transpose of D) (Theorem 3.9 and Corollary 5.17). As in the modular case, the Steinberg module St_ξ for u is projective (Proposition 4.1); in particular, for any u-module V, $St_\xi \otimes V$ is projective. As an immediate consequence of this we get Corollary (4.2). We give certain expressions for the decomposition of $St_\xi \otimes V$ in Propositions (4.5) and (4.10). The concept of "(u, T)-modules" (introduced by Jantzen) can be easily extended to the quantum case; by considering the representations of u, which are also modules (in a compatible manner) for U_ξ^0 (where U_ξ^0 is as defined in §5.1). We show that the irreducible as well as projective modules of u have such a structure (Lemma 5.7 and Proposition 5.16). This concept is used, as in [J], to prove that any projective u-module admits a Verma filtration (Proposition 5.16), and also to prove the reciprocity $B = D^t$ mentioned earlier (Corollary 5.17).

An informed reader will readily see that many ideas from $[H_1]$, $[H_2]$, and $[J]$ have been used in the paper. Also various results due to Lusztig $[L_1 - L_4]$, a commutation relation due to Levendorskii-Soibelman (cf. Proposition 2.4), and the structure of $Gr\ U$ due to DeConcini-Kac (cf. Proposition 2.6) have repeatedly been used in the proofs.

This is a shortened version of an earlier preprint (with the same title) distributed in March, 1991. Subsequently I received an Aarhus preprint " Injective modules for quantum algebras", by Andersen-Polo-Wen, which has some overlap with §5 of our paper (which we retain for the sake of completeness). I also have received the preprint "Representations of finite dimensional Hopf algebras arising from quantum groups", by Xi Nanhua. But this preprint has several gaps, e.g., his proof of Lusztig's conjecture as well as his proof of symmetry of u is not complete. He subsequently sent me (in response to my letter dated 15th May, 1992 to him) a revised version of his paper fixing the gap in his proof of symmetry of u. This paper of Nanhua also has some overlap with our §5.

Acknowledgements. I sincerely thank D.N. Verma for teaching me whatever modular representation theory I know, an area in which his ideas have led to several basic results. My thanks are also due to J.E. Humphreys for some references and clarifications, and the Referee for some suggestions on the exposition. This work was done while the author was a member of Tata Institute of Fundamental Research, Bombay.

1. Quantized Enveloping Algebras - Preliminaries and Notation

The general reference for this introductory section is Lusztig's papers $[L_3]$, $[L_4]$.

(1.1) Quantized enveloping algebra: Let us fix a Cartan matrix of finite type $A = (a_{ij})_{1 \le i,j \le n}$. Then there is a unique diagonal matrix D with positive integral diagonal entries $(d_1, \cdots, d_n)$ such that g.c.d. $(d_1, \cdots, d_n) = 1$ and DA is symmetric and positive definite.

It is a well known result that, in fact, $d_i \in \{1,2,3\}$.

Let $\mathfrak{g} = \mathfrak{g}(A)$ be the semi-simple (finite dimensional) Lie algebra over $\mathbb{C}$, associated to the Cartan matrix A. Recall that $\mathfrak{g}$ is defined by generators and relations. We denote the standard Cartan subalgebra of $\mathfrak{g}$ by $\mathfrak{h}$, and the (standard) simple roots by $\{\alpha_1, \cdots, \alpha_n\}$. The set (resp. number) of positive roots is denoted by Δ_+ (resp. N). The associated Weyl group $W \subset \mathrm{Aut}_{\mathbb{C}}(\mathfrak{h}^*)$ (where $\mathfrak{h}^* := \mathrm{Hom}_{\mathbb{C}}(\mathfrak{h}, \mathbb{C})$) is a Coxeter group generated by the simple reflections $\{s_1, \cdots, s_n\}$ (where the reflection s_i corresponds to the simple root α_i). There is a (unique) W-invariant non-degenerate symmetric bilinear form, denoted $<,>$, on $\mathfrak{h}^*$ satisfying

(1) ⋯ $$< \alpha_i, \alpha_j > = d_i a_{ij}, \text{ for all } 1 \le i,j \le n.$$

Denote by $\mathcal{A} = \mathbb{Z}[v, v^{-1}]$ the ring of Laurent polynomials over $\mathbb{Z}$, and $\mathbb{Q}(v)$ its quotient field. Following Drinfeld and Jimbo one associates an associative $\mathbb{Q}(v)$-algebra $U = U(A)$ to A (called the *quantized universal enveloping algebra*), defined by the generators $E_i, F_i, K_i, K_i^{-1} (1 \le i \le n)$, subject to the relations $(2-6)$:

(2) ⋯ $$K_iK_j = K_jK_i, \quad K_iK_i^{-1} = K_i^{-1}K_i = 1, \quad \text{for all } i,j$$

(3) ⋯ $$K_iE_jK_i^{-1} = v^{<\alpha_i,\alpha_j>}E_j, \quad K_iF_jK_i^{-1} = v^{-<\alpha_i,\alpha_j>}F_j, \quad \text{for all } i,j$$

(4) ⋯ $$E_iF_j - F_jE_i = \delta_{ij}\frac{K_i - K_i^{-1}}{v^{d_i} - v^{-d_i}}, \quad \text{for all } i,j$$

(5) ⋯ $$\sum_{m=0}^{1-a_{ij}} (-1)^m E_i^{(1-a_{ij}-m)} E_j E_i^{(m)} = 0, \quad \text{if } i \ne j \text{ and}$$

(6) ⋯ $$\sum_{m=0}^{1-a_{ij}} (-1)^m F_i^{(1-a_{ij}-m)} F_j F_i^{(m)} = 0, \quad \text{if } i \ne j; \text{where}$$

(7) ⋯ $$E_i^{(m)} := E_i^m/[m]!_{d_i}, \quad F_i^{(m)} := F_i^m/[m]!_{d_i}, \text{for any } m \ge 0,$$

(8) ⋯ $$[m]_d := (v^{dm} - v^{-dm})/(v^d - v^{-d}), \quad \text{and}$$

(9) ⋯ $$[m]!_d := [m]_d[m-1]_d \cdots [1]_d \qquad (\text{we set } [0]!_d = 1).$$

Then U is a Hopf algebra with *comultiplication* Δ, *antipode* S and *counit* ϵ defined by

(10) ⋯
$$\Delta(E_i) = E_i \otimes 1 + K_i \otimes E_i, \Delta(F_i) = F_i \otimes K_i^{-1} + 1 \otimes F_i, \Delta(K_i) = K_i \otimes K_i$$

(11) ⋯ $$S(E_i) = -K_i^{-1}E_i, \quad S(F_i) = -F_iK_i, \quad S(K_i) = K_i^{-1}$$

(12) ⋯ $$\epsilon(E_i) = \epsilon(F_i) = 0, \quad \epsilon(K_i) = 1.$$

The $\mathbb{Q}$-algebra U admits the following *antiautomorphisms* Ω and Ψ (of order 2) defined by

(13) ⋯ $$\Omega(E_i) = F_i, \Omega(F_i) = E_i, \Omega(K_i) = K_i^{-1}, \Omega(v) = v^{-1}$$

(14) ⋯ $$\Psi(E_i) = E_i, \Psi(F_i) = F_i, \Psi(K_i) = K_i^{-1}, \Psi(v) = v.$$

(1.2) The algebra $U_{\mathcal{A}}$. Following Lusztig [L_4; §1.3], define $U_{\mathcal{A}}$ to be the $\mathcal{A}$-subalgebra of U generated by the elements

$$\{E_i^{(m)}, F_i^{(m)}, K_i, K_i^{-1}\}_{1 \le i \le n \text{ and } 0 \le m}.$$

Then, by loc. cit., $U_{\mathcal{A}}$ is a Hopf subalgebra of U.

Given any commutative $\mathcal{A}$-algebra $\mathcal{B}$, by change of base, we get the Hopf algebra

(1) ⋯ $$U_{\mathcal{B}} := \mathcal{B} \otimes_{\mathcal{A}} \mathcal{U}_{\mathcal{A}}.$$

In particular, fix an integer $\ell \ge 1$ and ξ a primitive ℓ-th root of unity. Let $\mathbb{Q}_\xi$ denote the corresponding cyclotomic field. Then $\mathbb{Q}_\xi$ is an $\mathcal{A}$-algebra via $\mathcal{A} \to \mathbb{Q}_\xi, v \mapsto \xi$. So we get the Hopf algebra $U_\xi := U_{\mathbb{Q}_\xi}$.

(1.3) Quantized restricted enveloping algebra. We put the following restriction on ℓ :

(*) *$\ell > 1$ is an odd integer and is coprime to 3 if G_2 is a component of* $\mathfrak{g}(A)$.

Let u_ξ be the $\mathbb{Q}_\xi$ -subalgebra of U_ξ generated by $\{E_i, F_i, K_i, K_i^{-1}\}$ $(1 \le i \le n)$. Then

$$(1)\cdots \qquad E_i^\ell = [\ell]!_{d_i} E_i^{(\ell)} = 0 \quad \text{in} \quad u_\xi.$$

Similarly, $F_i^\ell = 0$. Moreover, by (3) of §1.1, it follows that $K_i^\ell - 1$ is central in u_ξ (in fact in U_ξ). Let u_ξ be the quotient algebra of u_ξ divided by the (two-sided) ideal generated by $\{K_i^\ell - 1\}_{1\le i \le n}$. The algebra u_ξ is called the *Quantized Restricted Enveloping Algebra.* It is easy to see that the comultiplication, defined in (10) of §1.1, induces a Hopf algebra structure on u_ξ.

Let us recall from [L_4; Theorem 3.2] and [L_3; Proposition 1.7] that, for any $w \in W$, there is an automorphism T_w of U which keeps the subalgebra $U_{\mathcal{A}}$ stable, and commutes with Ω.

From now on, we fix one reduced decomposition of the longest element w_o of the Weyl group W :

$$(2)\cdots \qquad w_o = s_{i_1} \cdots s_{i_N} \ , \quad \text{where (as in §1.1) } N := \mid \Delta_+ \mid .$$

This gives rise to an enumeration of the set of positive roots Δ_+:

$$(3)\cdots \qquad \Delta_+ = \{\beta_1, \beta_2, \cdots, \beta_N\},$$

where $\beta_j := (s_{i_1} \cdots s_{i_{j-1}})\alpha_{i_j}$, for any $1 \le j \le N$.

For any $\psi = (\psi_1, \cdots, \psi_N) \in \mathbb{Z}_+^N$ (where $\mathbb{Z}_+$ is the set of non-negative integers), define (cf. [L_4; § 4, and the Appendix] and [L_3; §§ 1, 6])

$$(4)\cdots \qquad E^\psi = \prod_{j=1}^{N} E_{\beta_j}^{\psi_j}, \text{ where}$$

$$(5)\cdots \qquad E_{\beta_j} := T_{w_{j-1}}(E_{\alpha_{i_j}}), \quad (w_{j-1} := s_{i_1} \cdots s_{i_{j-1}}).$$

Similarly we define

$$(6)\cdots \qquad F_{\beta_j} = T_{w_{j-1}}(F_{\alpha_{i_j}}) = \Omega(E_{\beta_j})$$

$$(7)\cdots \qquad F^\psi = \Omega(E^\psi) \ ,$$

where Ω is given by (13) of §1.1.

We also define, for any $\mathbf{t} = (t_1, \cdots, t_n) \in \mathbb{Z}_+^n$,

$$(8)\cdots \qquad K^{\mathbf{t}} = \prod_{i=1}^{n} K_i^{t_i} \ .$$

The elements E^ψ, F^ψ, and $K^{\mathbf{t}}$ may be viewed as elements of any of $U, U_{\mathcal{A}}$, or more generally $U_{\mathcal{B}}$ for any $\mathcal{A}$ algebra $\mathcal{B}$. The algebra to which they belong will be clear from the context.

Now we recall the following result due to Lusztig:

(1.4) Theorem [L_3; §6.5]. *The algebra u_ξ defined above has the following as a basis over $\mathbb{Q}_\xi$:*

$$\{F^\varphi K^{\mathbf{t}} E^\psi\}_{\varphi,\psi \in \mathbb{Z}_\ell^N, \mathbf{t} \in \mathbb{Z}_\ell^n},$$

where $\mathbb{Z}_\ell := \{0, 1, \cdots, \ell - 1\} \subset \mathbb{Z}_+$.

2. Quantized Restricted Enveloping Algebra is Symmetric

(2.1) Definition [N]. Let $\mathfrak{A}$ be a finite dimensional associative algebra over a field k. Then $\mathfrak{A}$ is called *Frobenius*, if there exists a non-degenerate bilinear form $f : \mathfrak{A} \times \mathfrak{A} \to k$, which is associative in the sense that

$$f(ab, c) = f(a, bc) \ , \text{ for } a, b, c \in \mathfrak{A}.$$

A Frobenius algebra $\mathfrak{A}$ is called *Symmetric* if f is a symmetric form. Equivalently, a finite dimensional associative algebra $\mathfrak{A}$ over k is symmetric if there exists a non-zero linear form $\lambda : \mathfrak{A} \to k$, such that

(1) $\lambda(ab) = \lambda(ba)$, for $a, b \in \mathfrak{A}$, and

(2) $\ker \lambda$ does not contain any non-zero right ideal of $\mathfrak{A}$.

The main result of this section is the following:

(2.2) Theorem. *Let ξ be a primitive ℓ-th root of unity, where ℓ satisfies the condition (*) of §1.3. Then the quantized restricted enveloping algebra $\mathfrak{u}_\xi$ (defined in §1.3) is symmetric.*

In particular, $\mathfrak{u}_\xi$ is unimodular (cf. [H_3; Theorem 2]).

We begin with some preparation towards its proof.

(2.3) Definition. Let U^o (resp. $\overline{U}^o$) denote the $\mathbb{Q}(v)$-subalgebra (resp. $\mathbb{Q}[v,v^{-1}]$-subalgebra) of U, generated by $\{K_i^{\pm 1}\}_{i\le i\le n}$ (resp. $\{K_i, \binom{K_i;0}{1}; 1 \le i \le n\}$), where $\binom{K_i;0}{1} := (K_i - K_i^{-1})/(v^{d_i} - v^{-d_i})$.

For any $\varphi,\ \psi \in \mathbb{Z}_+^N$, following [$DK$; §1.7], define

(1) ⋯

$$deg(\varphi,\psi) := (\varphi_N, \varphi_{N-1}, \cdots, \varphi_1, \psi_1, \psi_2, \cdots, \psi_N, \| \varphi \| + \| \psi \|) \in \mathbb{Z}_+^{2N+1},$$

where $\varphi = (\varphi_1, \cdots, \varphi_N)$, $\psi = (\psi_1, \cdots, \psi_N)$, $\| \varphi \| := \sum \varphi_i \| \beta_i \|$, and for any root $\alpha = \sum_{i=1}^{n} m_i\alpha_i$, $\| \alpha \| := \sum_{i=1}^{n} | m_i |$.

Further we define the *degree*, denoted $d(x) \in \mathbb{Z}_+^{2N+1}$, of any non-zero element $x \in F^\varphi U^o E^\psi \subset U$ by

(2) ⋯
$$d(x) := deg(\varphi,\psi).$$

Since, by [L_4; Proposition 4.2],

(3) ⋯
$$U = \oplus_{\varphi,\psi \in \mathbb{Z}_+^N} F^\varphi U^o E^\psi,$$

$d(x)$ is well defined.

For any $x \in U$ and pair (φ,ψ), by (φ,ψ)-*th component* $x_{\varphi,\psi}$ of x, we mean the component of x in the space $F^\varphi U^o E^\psi$ under the decomposition (3).

Following [DK], $\mathbb{Z}_+^{2N+1}$ is viewed as a totally ordered semigroup (under addition) with the lexicographic order $\le$.

Define an increasing filtration $\{U_{\mathbf{r}}\}_{\mathbf{r}\in \mathbb{Z}_+^{2N+1}}$ of U by

(4) ⋯
$$U_{\mathbf{r}} = \sum_{deg(\varphi,\psi)\le \mathbf{r}} F^\varphi U^o E^\psi.$$

We recall the following result due to Levendorskii and Soibelman [LS].

(2.4) Proposition. *For any $\varphi', \varphi'', \psi', \psi'' \in \mathbb{Z}_+^N$*

$$(F^{\varphi'}\overline{U}^o E^{\psi'})(F^{\varphi''}\overline{U}^o E^{\psi''}) \subset \sum_{deg(\varphi,\psi)\le\ deg(\varphi'+\varphi'',\psi'+\psi'')} F^\varphi \overline{U}^o E^\psi.$$

In particular, for any $\mathbf{r}_1, \mathbf{r}_2 \in \mathbb{Z}_+^{2N+1}$, $U_{\mathbf{r}_1} U_{\mathbf{r}_2} \subset U_{\mathbf{r}_1+\mathbf{r}_2}$, i.e., U is a filtered algebra over $\mathbb{Q}(v)$ with respect to the filtration $\{U_{\mathbf{r}}\}_{\mathbf{r}\in \mathbb{Z}_+^{2N+1}}$.

(2.5) Corollary. *Define $\overline{U} = \oplus_{\varphi,\psi\in \mathbb{Z}_+^N} F^\varphi \overline{U}^o E^\psi$. Then $\overline{U}$ is a $\mathbb{Q}[v,v^{-1}]$-subalgebra of U.*

In view of the above proposition, we can define the corresponding graded algebra

(1) ⋯
$$GrU := \sum_{\mathbf{r}\in \mathbb{Z}_+^{2N+1}} (U_{\mathbf{r}} / \sum_{\mathbf{r}'<\mathbf{r}} U_{\mathbf{r}'}).$$

Recall the definition of E_{β_j}, F_{β_j} from (5) and (6) of §1.3. The following result due to DeConcini-Kac gives the structure of GrU.

(2.6) Proposition [DK; Proposition 1.7]. *GrU is an associative algebra over $\mathbb{Q}(v)$ generated by $E_{\beta_j}, F_{\beta_j} (1 \le j \le N)$, and $K_i^{\pm 1} (1 \le i \le n)$; subject to the following relations:*

(R_1) ⋯
$$K_iK_j = K_jK_i,\ K_iK_i^{-1} = 1\ ,\ \textit{for}\ \ 1 \le i,j \le n$$

(R_2) ⋯
$$E_{\beta_j}F_{\beta_k} = F_{\beta_k}E_{\beta_j}\ ,\ \textit{for}\ 1 \le j,k \le N$$

(R_3) ⋯
$$K_iE_{\beta_j}K_i^{-1} = v^{<\beta_j,\alpha_i>}E_{\beta_j}, K_iF_{\beta_j}K_i^{-1} = v^{-<\beta_j,\alpha_i>}F_{\beta_j},$$
$$\textit{for}\ 1 \le i \le n\ ,\ 1 \le j \le N,\ \textit{and}$$

(R_4) ⋯
$$E_{\beta_i}E_{\beta_j} = v^{<\beta_i,\beta_j>}E_{\beta_j}E_{\beta_i},\quad F_{\beta_i}F_{\beta_j} = v^{<\beta_i,\beta_j>}F_{\beta_j}F_{\beta_i}\ ,\ \textit{for}\ i > j.$$

(2.7) Definition. Let $u_\xi^o \subset u_\xi$ be the $\mathbb{Q}_\xi$-subalgebra generated by $\{K_i\}_{1\le i\le n}$. Then u_ξ^o has $\{K^{\mathbf{t}}\}_{\mathbf{t}\in \mathbb{Z}_\ell^n}$ as a $\mathbb{Q}_\xi$-basis, where $K^{\mathbf{t}}$ is defined by (8) of §1.3.

By the $\mathbf{t}$-th *component* $\mathbf{x}_{\mathbf{t}}$ (for $\mathbf{t} \in \mathbb{Z}_\ell^n$) of any $\mathbf{x} \in u_\xi^o$, we mean the coefficient of $K^{\mathbf{t}}$ in the expansion of $\mathbf{x}$ in the above basis.

Clearly, for any non-zero $\mathbf{x} \in u_\xi^o$, there exists a $\mathbf{x}' \in u_\xi^o$ such that the 0_n-th component of $\mathbf{x}\mathbf{x}'$ is non-zero, where $0_n = 0_{n,\ell} := (0,0,\cdots,0) \in \mathbb{Z}_\ell^n$.

(2.8) Definition. An element $x \in U$ is said to be of *weight* $\alpha \in \sum_{i=1}^{n}\mathbb{Z}\alpha_i$, if

$$(1)\cdots \qquad K_i x K_i^{-1} = v^{<\alpha,\alpha_i>}x\ , \quad \text{for all } 1\le i\le n.$$

By (3) of §1.1, E_j (resp. F_j) is of weight α_j (resp. $-\alpha_j$), for any $1\le j\le n$. From [L_4; Theorem 3.1], it is clear that E_{β_j} (resp. F_{β_j}), for any $1\le j\le N$, is of weight β_j (resp. $-\beta_j$). In particular, E^φ (resp. F^φ), for any $\varphi = (\varphi_1,\cdots,\varphi_N) \in \mathbb{Z}_+^N$, is of weight $\sum_{j=1}^{N}\varphi_j\beta_j$ (resp. $-\sum_{j=1}^{N}\varphi_j\beta_j$).

(2.9) Lemma. *For any $1\le i\le n$, $E_iE^{\mathbb{1}_N} = E^{\mathbb{1}_N}E_i = 0$, as elements of u_ξ, where $\mathbb{1}_N = \mathbb{1}_{N,\ell} := (\ell-1,\ell-1,\cdots,\ell-1) \in \mathbb{Z}_+^N$.*

Proof. Recall the definition of $\overline{U}$ from Corollary (2.5). By virtue of Proposition (2.4), we can write

$$(1)\cdots \qquad E_iE^{\mathbb{1}_N} = \sum c^\psi E^\psi,$$

as elements of $\overline{U}$, where the sum ranges over $deg(0,\psi) \le deg(0,\mathbb{1}_N+\overline{\delta}_{n(i)})$, $n(i)$ is the index such that $\beta_{n(i)} = \alpha_i$, and

$$\overline{\delta}_{n(i)} := (0,\cdots 0,1,0,\cdots,0) \in \mathbb{Z}_+^N,$$

where 1 is put in the $n(i)$-th place. Considering the weights of both the sides in (1), we see that any ψ such that $c^\psi \ne 0$ satisfies $\psi_j \ge \ell$ for some $1\le j\le N$. In particular, taking the image of (1) under the $\mathbb{Q}$-algebra homomorphism $\theta : \overline{U} \to u_\xi$ (taking $E_{\beta_j} \mapsto E_{\beta_j}, F_{\beta_j} \mapsto F_{\beta_j}$, for any

$1\le j\le N; K_i \mapsto K_i$, $\binom{K_i;0}{1} \mapsto \binom{K_i;0}{1}$, for any $1\le i\le n$; and $v\mapsto\xi$), and using (1) of §1.3, we get the lemma. □

With these preparations we come to the proof of Theorem (2.2).

(2.10) *Proof of Theorem (2.2).* By Theorem (1.4), the elements

$$\{F^\varphi K^{\mathbf{t}}E^\psi\}(\varphi,\psi \in \mathbb{Z}_\ell^N \text{ and } \mathbf{t} \in \mathbb{Z}_\ell^n)$$

form a $\mathbb{Q}_\xi$-basis of u_ξ. We define a $\mathbb{Q}_\xi$ -linear form $\lambda : u_\xi \to \mathbb{Q}_\xi$, by setting $\lambda(x)$ to be the coefficient of $\phi_o := F^{\mathbb{1}_N}E^{\mathbb{1}_N}$ in the decomposition of x with respect to the above basis.

Assertion I. ker λ *does not contain any non-zero right ideal:*

Let us fix any non-zero element $x \in \ker\lambda$ and write

$$x = \sum_{\varphi,\mathbf{t},\psi} c_{\varphi,\mathbf{t},\psi}(x)F^\varphi K^{\mathbf{t}}E^\psi, \quad \text{for some (unique) } c_{\varphi,\mathbf{t},\psi} \in \mathbb{Q}_\xi.$$

We set for any φ,ψ

$$x_{\varphi,\psi} := \sum_{\mathbf{t}\in \mathbb{Z}_\ell^n} c_{\varphi,\mathbf{t},\psi}(x)F^\varphi K^{\mathbf{t}}E^\psi.$$

Let us choose any lifts $\overline{c}_{\varphi,\mathbf{t},\psi}(x) \in \mathbb{Q}[v,v^{-1}]$ of $c_{\varphi,\mathbf{t},\psi}$ under the $\mathbb{Q}$-algebra homomorphism $\mathbb{Q}[v,v^{-1}] \to \mathbb{Q}_\xi$ (taking $v\mapsto\xi$). (If $c_{\varphi,\mathbf{t},\psi} = 0$, we choose $\overline{c}_{\varphi,\mathbf{t},\psi} = 0$.) Set

$$\overline{x} = \sum_{\varphi,\psi} \overline{x}_{\varphi,\psi} \in \overline{U}\ , \text{ where}$$
$$\overline{x}_{\varphi,\psi} := \sum_{\mathbf{t}} \overline{c}_{\varphi,\mathbf{t},\psi}(x)F^\varphi K^{\mathbf{t}}E^\psi.$$

Let (φ_o,ψ_o) be the (unique) pair such that $\overline{x}_{\varphi_o,\psi_o} \ne 0$ and $d(\overline{x}_{\varphi_o,\psi_o})$ is largest with this property. We now consider the element

$$\overline{y} := \overline{x}F^{\mathbb{1}_N-\varphi_o}E^{\mathbb{1}_N-\psi_o} \in \overline{U}.$$

Then, by Propositions (2.4) and (2.6), $(\mathbb{1}_N,\mathbb{1}_N)$–th component $\overline{y}_o$ of $\overline{y}$ (under the decomposition of Corollary 2.5) is non-zero. As in §2.3, for any $\varphi,\psi \in \mathbb{Z}_+^N$, we denote the (φ,ψ)-th component of $\overline{y}$ by $\overline{y}_{\varphi,\psi}$. Put

$$y = \theta(\overline{y}),\ y_o = \theta(\overline{y}_o) \text{ and } y_{\varphi,\psi} = \theta(\overline{y}_{\varphi,\psi}),$$

where θ is the $\mathbb{Q}$-algebra homomorphism $\overline{U} \to u_\xi$ defined in the proof of Lemma (2.9). Then, by (1) of §1.3, if φ_j or $\psi_j \geq \ell$ for some $1 \leq j \leq N$, then $y_{\varphi,\psi} = 0$. In particular,

$$y = y_o + \sum y_{\varphi,\psi},$$

where the sum ranges over $\varphi, \psi \in \mathbb{Z}_\ell^N$ and $(\varphi,\psi) \neq (\mathbb{1}_N, \mathbb{1}_N)$.

By Proposition (2.6), it is easy to see that $y_o \neq 0$. Clearly $F^\varphi u_\xi^o E^\psi$ (cf. Definition 2.7) is stable, for any fixed $\varphi, \psi \in \mathbb{Z}_\ell^N$, under the left (as well as right) multiplication by u_ξ^o. Write $y_o = F^{\mathbb{1}_N} k E^{\mathbb{1}_N}$, for some $k \neq 0 \in u_\xi^o$. There exists an element $k' \in u_\xi^o$ such that kk' has 0_n-th component non-zero (cf. Definition 2.7). But $F^{\mathbb{1}_N} kk' E^{\mathbb{1}_N} = y_o \widetilde{k}'$, for some $\widetilde{k}' \in u_\xi^o$. Hence $\lambda(y\widetilde{k}') \neq 0$. This proves Assertion I. □

Assertion II. $\lambda(ab) = \lambda(ba)$, *for all* $a, b \in u_\xi$:

Clearly, it suffices to prove the above for the algebra generators $b = E_i, F_i, K_i$ $(1 \leq i \leq n)$ of u_ξ and the $\mathbb{Q}_\xi$-basis elements $a = F^\varphi K^{\mathbf{t}} E^\psi$ $(\varphi, \psi \in \mathbb{Z}_\ell^N$, and $\mathbf{t} \in \mathbb{Z}_\ell^n)$:

Case I. $b = K_i$:

$$\text{Clearly } K_i F^\varphi u_\xi^o E^\psi K_i^{-1} = F^\varphi u_\xi^o E^\psi \text{ (for any } \varphi, \psi),$$

and moreover $K_i F^{\mathbb{1}_N} K^{\mathbf{t}} E^{\mathbb{1}_N} K_i^{-1} = F^{\mathbb{1}_N} K^{\mathbf{t}} E^{\mathbb{1}_N}$ (cf. Definition 2.8). This proves this case, i.e., $\lambda(aK_i) = \lambda(K_i a)$, for any $a \in u_\xi$.

Case II. $b = E_i$:

We first claim that unless $\varphi = \mathbb{1}_N$ and $\psi = \mathbb{1}_N - \overline{\delta}_{n(i)}$ (where $\overline{\delta}_{n(i)}$ is as defined in the proof of Lemma 2.9), we have

$$(1)\cdots \qquad \lambda(E_i F^\varphi K^{\mathbf{t}} E^\psi) = 0, \quad \text{and also}$$

$$(2)\cdots \qquad \lambda(F^\varphi K^{\mathbf{t}} E^\psi E_i) = 0, \text{ for any } \mathbf{t} \in \mathbb{Z}_\ell^n :$$

By Proposition (2.4), considered as elements of $\overline{U}$,

$$(3)\cdots \qquad E^\psi E_i = cE^{\psi + \overline{\delta}_{n(i)}} + \sum_{deg(0,\eta) < deg(0,\psi+\overline{\delta}_{n(i)})} c^\eta E^\eta,$$

for some $c, c^\eta \in \mathbb{Q}[v, v^{-1}]$. If $c^{\mathbb{1}_N} \neq 0$ we get, from the weight considerations (cf. Definition 2.8), that $\psi = \mathbb{1}_N - \overline{\delta}_{n(i)}$. This proves (2).

Similarly, by Proposition (2.4), again as elements of $\overline{U}$, we get

$$E_i F^\varphi \overline{U}^o E^\psi \subset \sum_{deg(\varphi',\psi') \leq deg(\varphi, \psi + \overline{\delta}_{n(i)})} F^{\varphi'} \overline{U}^o E^{\psi'}.$$

So, if $\lambda(E_i F^\varphi u_\xi^o E^\psi) \neq 0$, $deg(\mathbb{1}_N, \mathbb{1}_N) \leq deg(\varphi, \psi + \overline{\delta}_{n(i)})$. In particular, only two possibilities can occur:

(p_1) $\varphi = \mathbb{1}_N - \overline{\delta}_{n(j)}$, for some simple root α_j, and $\psi = \mathbb{1}_N$, or

(p_2) $\varphi = \mathbb{1}_N$, and $\psi = \mathbb{1}_N - \overline{\delta}_{n(k)}$, for some simple root α_k.

Comparing the weights again, we deduce that the possibility (p_1) can not occur and, moreover in the possibility (p_2), $\alpha_k = \alpha_i$. This proves (1).

So, to prove the result in case II, it suffices to show that (abbreviating $E^{\mathbb{1}_N - \overline{\delta}_{n(i)}}$ by E)

$$(4)\cdots \qquad \lambda(F^{\mathbb{1}_N} K^{\mathbf{t}} E E_i) = \lambda(E_i F^{\mathbb{1}_N} K^{\mathbf{t}} E) :$$

In (3) if we take $\psi = \mathbb{1}_N - \overline{\delta}_{n(i)}$, then by Proposition (2.6),

$$(5)\cdots \qquad \lambda(F^{\mathbb{1}_N} K^{\mathbf{t}} E E_i) = 1, \text{ if } \mathbf{t} = 0_n$$
$$= 0, \text{ otherwise.}$$

Now, as elements of U,

$$E_i K^{\mathbf{t}} E = v^{-<\alpha_{\mathbf{t}}, \alpha_i>} K^{\mathbf{t}} E_i E,$$

where $\alpha_{\mathbf{t}} := \sum_{i=1}^n t_i \alpha_i$. So

$$(6)\cdots \qquad \lambda(E_i F^{\mathbb{1}_N} K^{\mathbf{t}} E) = \lambda(F^{\mathbb{1}_N} E_i K^{\mathbf{t}} E), \text{ by Proposition (2.6)}$$
$$= \begin{cases} 0, & \text{if } \mathbf{t} \neq 0_n \\ 1, & \text{if } \mathbf{t} = 0_n. \end{cases}$$

Now the validity of Assertion II in case II (i.e. $b = E_i$) follows by combining (5) and (6). The case III (i.e. $b = F_i$) is exactly parallel to the case II, and hence is left to the reader. So Assertion II is established. □

Combining Assertions I and II, Theorem (2.2) follows from the definition (2.1). □

(2.11) Remarks. (a) A similar proof gives Theorem (2.2) with $\mathfrak{u}_\xi$ replaced by u_ξ (defined in §1.3).

(b) Theorem (2.2) was motivated by the corresponding result for the restricted enveloping algebras (associated to semi-simple algebraic groups over algebraically closed field of prime char. p) due to Schue [S] (see also [H_3]).

3. The Cartan matrix associated to Quantized Restricted Enveloping Algebras

Let $\ell, \xi, \mathfrak{u}_\xi$ be as in §§1.2, 1.3. In particular, ℓ satisfies the restriction (*) of §1.3. From now on we shall abbreviate $\mathfrak{u}_\xi$ by $\mathfrak{u}$. Also recall the definition of the subalgebra $\mathfrak{u}^o := \mathfrak{u}^o_\xi \subset \mathfrak{u}_\xi$ from §2.7. We define $\mathfrak{b}$ (resp. $\mathfrak{b}^-$) as the $\mathbb{Q}_\xi$ -subalgebra of $\mathfrak{u}$ generated by $\{E_i, K_i\}_{1\leq i\leq n}$ (resp. $\{F_i, K_i\}_{1\leq i\leq n}$). Similarly, let $\mathfrak{u}^+$ (resp. $\mathfrak{u}^-$) be the $\mathbb{Q}_\xi$- subalgebra of $\mathfrak{u}$ generated by $\{E_i\}_{1\leq i\leq n}$ (resp. $\{F_i\}_{1\leq i\leq n}$). *In the sequel, by an* $\mathfrak{u}$*-module we will always mean a left representation of* $\mathfrak{u}$ *in a finite dimensional vector space over* $\mathbb{Q}_\xi$.

(3.0) Definition. (Verma, Projective and Irreducible modules of $\mathfrak{u}$). Recall from § 1.1 that $\mathfrak{h} \subset \mathfrak{g} = \mathfrak{g}(A)$ is the Cartan subalgebra of the complex semi-simple Lie algebra $\mathfrak{g}$. Define

(1) ⋯ $$\mathfrak{h}^*_{\mathbb{Z}} = \{\lambda \in \mathfrak{h}^* : \lambda(\alpha_i^\vee) \in \mathbb{Z}, \text{for all simple coroots } \alpha_i^\vee\},$$

where $\mathfrak{h}^* := \mathrm{Hom}_{\mathbb{C}}(\mathfrak{h}, \mathbb{C})$. Put

(2) ⋯ $$\mathfrak{h}^*_\ell := (\mathbb{Z}/(\ell)) \otimes_{\mathbb{Z}} \mathfrak{h}^*_{\mathbb{Z}}.$$

For any $\lambda \in \mathfrak{h}^*_\ell$, we define the *Verma module* $Z(\lambda)$ for the algebra $\mathfrak{u}$ by

(3) ⋯ $$Z(\lambda) = \mathfrak{u} \otimes_{\mathfrak{b}} \mathbb{Q}_\xi(\lambda),$$

where $\mathbb{Q}_\xi(\lambda)$ is the one dimensional module of $\mathfrak{b}$ over $\mathbb{Q}_\xi$, such that E_i' s($1 \leq i \leq n$) act trivially and $K_i.x = \xi^{<\widetilde{\lambda},\alpha_i>}x$, for $x \in \mathbb{Q}_\xi(\lambda)$; where $\widetilde{\lambda}$ is any lift in $\mathfrak{h}^*_{\mathbb{Z}}$ of the element $\lambda \in \mathfrak{h}^*_\ell$, under the canonical map $\mathfrak{h}^*_{\mathbb{Z}} \to \mathfrak{h}^*_\ell$. It is easy to see that $\{\mathbb{Q}_\xi(\lambda)\}_{\lambda\in\mathfrak{h}^*_\ell}$ bijectively parametrizes the set of isomorphism classes of one dimensional modules of $\mathfrak{b}$. (We are using the condition (*) of § 1.3.)

If we take $\lambda = (\ell - 1)\rho$ (where $\rho \in \mathfrak{h}^*_\ell$ is defined by $\rho(\alpha_i^\vee) = 1$, for all the simple coroots $\alpha_i^\vee$), then the Verma module $Z(\lambda)$ will be called the *Steinberg module* and denoted St_ξ.

It can be seen (cf. [L_3; Proof of Proposition 5.11]) that any $Z(\lambda)$ has a unique proper maximal $\mathfrak{u}$-submodule and hence a unique *irreducible quotient* (denoted by) $M(\lambda)$. Moreover, by loc. cit., any irreducible $\mathfrak{u}$-module is isomorphic with precisely one $M(\lambda)$.

From [CR; Corollary 54.14], the set of isomorphism classes of principal indecomposable modules (for short PIM) of $\mathfrak{u}$ is bijectively parametrized by $\mathfrak{h}^*_\ell$. We choose the parametrization so that $Q(\lambda)$ is the PIM with the unique irreducible quotient $M(\lambda)$ (cf. [CR; Theorem 54.11]). Observe that, by [CR; Theorem 56.6 and the Remark at the end of its proof], $\{Q(\lambda)\}_{\lambda\in\mathfrak{h}^*_\ell}$ are precisely the isomorphism classes of indecomposable projective modules of $\mathfrak{u}$.

We record the following lemma for its subsequent use.

(3.1) Lemma. *The irreducible* $\mathfrak{u}$*−module* $M(\lambda)$ *(for any* $\lambda \in \mathfrak{h}^*_\ell$*) remains irreducible under field extensions, i.e., for any field* $K \supset \mathbb{Q}_\xi$, $M_K(\lambda) := K \otimes_{\mathbb{Q}_\xi} M(\lambda)$ *is irreducible as a module for* $\mathfrak{u}_K := K \otimes_{\mathbb{Q}_\xi} \mathfrak{u}$.

Proof. It is easy to see, by the same argument as in [L_3; Proof of Proposition 5.11], that $M_K(\lambda)$ has a unique proper maximal $\mathfrak{u}_K$ submodule $S_K(\lambda)$. By going to a further extension, we assume w. l. o. g. that K is algebraically closed. Let $\mathcal{G} = \mathcal{G}_K$ be the group of field automorphisms of K over $\mathbb{Q}_\xi$. Clearly $\mathcal{G}$ acts on $M_K(\lambda)$ by automorphisms (over the field $\mathbb{Q}_\xi$), commuting with the action of $\mathfrak{u}$. In particular, $S_K(\lambda)$ is stable under $\mathcal{G}$ and hence (by Hilbert's theorem 90), $S_K(\lambda)$ is defined over $\mathbb{Q}_\xi$, i.e., $K \otimes_{\mathbb{Q}_\xi} (S_K(\lambda) \cap M(\lambda)) \approx S_K(\lambda)$. But $S_K(\lambda) \cap M(\lambda)$, being $\mathfrak{u}$-stable, is 0. □

(3.2) Cartan Matrix. For any $\mathfrak{u}$-module M and irreducible $\mathfrak{u}$-module

$M(\mu)$, we denote the multiplicity of $M(\mu)$ in M by $(M : M(\mu))$. In particular, taking $M = Q(\lambda)$ (for $\lambda \in \mathfrak{h}_\ell^*$), we get the matrix

(1) ⋯ $$C = (c_{\lambda,\mu})_{\lambda,\mu\in\mathfrak{h}_\ell^*}, \text{ where } c_{\lambda,\mu} = c^{\xi}_{\lambda,\mu} := (Q(\lambda) : M(\mu)).$$

The matrix C is called the *Cartan matrix* associated to the algebra u. Similarly, define the matrix

(2) ⋯ $$D = (d_{\lambda,\mu})_{\lambda,\mu\in\mathfrak{h}_\ell^*}, \text{ where } d_{\lambda,\mu} = d^{\xi}_{\lambda,\mu} := (Z(\lambda) : M(\mu)).$$

So, in the Grothendieck group of finite dimensional u-modules, we get

(3) ⋯ $$Q(\lambda) = \sum c_{\lambda,\mu} M(\mu) \quad \text{and}$$

(4) ⋯ $$Z(\lambda) = \sum d_{\lambda,\mu} M(\mu).$$

(3.3) Lemma. *Any projective* u^-*-module is free. The same statement is of course true for* u^+ *as well.*

Proof. Let $I^- \subset u^-$ be the ideal spanned as a vector space over $\mathbb{Q}_\xi$ by the elements $\{F^\varphi\}$, where φ ranges over all the non-zero elements of $\mathbb{Z}_\ell^N$. Then, by [L_3; Lemma 5.10], I^- is a nilpotent ideal, i.e., there exists an integer m such that $(I^-)^m = 0$. In particular, by [CR; Lemma 54.8], u^- is a completely primary ring. Now the lemma follows by [CR; Exercise 2 p. 383]. □

(3.4) Lemma. *The left regular representation of* $\mathfrak{b}^-$ *(onto itself) decomposes as* $\oplus_{\lambda\in\mathfrak{h}_\ell^*} Z(\lambda)$, *where the* u*-module* $Z(\lambda)$ *is considered as a* $\mathfrak{b}^-$*-module by restriction. In particular, any* $Z(\lambda)$ *is a projective* $\mathfrak{b}^-$*-module.*

Moreover, $\{Z(\lambda)\}_{\lambda\in\mathfrak{h}_\ell^*}$ *is a complete set of non-isomorphic indecomposable projective* $\mathfrak{b}^-$*-modules. Further,* $\{\overline{\mathbb{Q}}_\xi(\lambda)\}_{\lambda\in\mathfrak{h}_\ell^*}$ *is a complete set of non-isomorphic irreducible* $\mathfrak{b}^-$*-modules, where* $\overline{\mathbb{Q}}_\xi(\lambda)$ *is the one dimensional* $\mathfrak{b}^-$*-module such that* F_i $(1 \le i \le n)$ *acts trivially and* K_i *acts by the multiplication with* $\xi^{<\lambda,\alpha_i>}$.

(3.5) Remark. Since u is $\mathfrak{b}^-$-free (under left multiplication), any projective u-module is also projective considered as a $\mathfrak{b}^-$-module.

(3.6) *Proof of Lemma 3.4.* For any $1 \le i \le n$ and $j \in \mathbb{Z}/(\ell)$, define

(1) ⋯ $$k_{i,j} = \sum_{m=0}^{\ell-1} (\xi^j K_i)^m \in u^o.$$

Then

(2) ⋯ $$K_i k_{i,j} = \xi^{-j} k_{i,j}.$$

In particular, for fixed i, $\{k_{i,j}\}_{j\in\mathbb{Z}/(\ell)}$ is linearly independent over $\mathbb{Q}_\xi$. By dimension counting, $\{\mathbf{k_j} = k_{1,j_1}\cdots k_{n,j_n}\}_{\mathbf{j}=(j_1,\dots,j_n)\in(\mathbb{Z}/(\ell))^n}$ is a $\mathbb{Q}_\xi$-basis for u^o. This gives that

(3) ⋯ $$\mathfrak{b}^- = \oplus_{\mathbf{j}\in(\mathbb{Z}/(\ell))^n}(u^-\mathbf{k_j}).$$

It is clear from (2) that, for any $\mathbf{j}$, $u^-\mathbf{k_j}$ is $\mathfrak{b}^-$-stable. Define a bijection

$$\theta : \mathfrak{h}_\ell^* \to (\mathbb{Z}/(\ell))^n,$$

by $\theta(\lambda) = (-<\lambda,\alpha_1>,\cdots,-<\lambda,\alpha_n>)$. (Observe that the assumption (*) of § 1.3 on ℓ is being used here to show that θ is bijective .) From (2), it is clear that $u^-\mathbf{k}_{\theta(\lambda)}$ is isomorphic with the $\mathfrak{b}^-$-module $Z(\lambda)$. This proves the first part of the lemma.

We next show that $Z(\lambda)$ is $\mathfrak{b}^-$-indecomposable : If M is a non-trivial direct summand of $Z(\lambda)$, then M is $\mathfrak{b}^-$-projective and hence u^--projective (note : $\mathfrak{b}^-$ is u^-–free). But then by Lemma (3.3), M is free as u^-–module. So, by dimension counting, $M = Z(\lambda)$. This proves the assertion that $Z(\lambda)$ is indecomposable.

Since the number of non-isomorphic irreducible representations of $\mathfrak{b}^-$ is $\ge |\mathfrak{h}_\ell^*| = |(\mathbb{Z}/(\ell))^n|$ (as $\overline{\mathbb{Q}}_\xi(\lambda)$ are all non-isomorphic), and the number of indecomposable components of $\mathfrak{b}^-$ is equal to $|(\mathbb{Z}/(\ell))^n|$ (by (3)), we obtain by [CR; Corollary 54.14] that $u^-\mathbf{k_j}$ are all non-isomorphic $\mathfrak{b}^-$-modules and the irreducible representations of $\mathfrak{b}^-$ are precisely $\overline{\mathbb{Q}}_\xi(\lambda)$ $(\lambda \in \mathfrak{h}_\ell^*)$. This, together with [CR; Theorem 56.6], completes the proof of the Lemma. □

(3.7) Definitions. (a) *The matrix B*: The u- projective module $Q(\lambda)$ $(\lambda \in$

$\mathfrak{h}_\ell^*$) is projective considered as a $\mathfrak{b}^-$-module (cf. Remark 3.5). In particular, by Lemma (3.4) and [CR; Theorem 56.6], we obtain the decomposition (as $\mathfrak{b}^-$-modules)

$$(1)\cdots \qquad Q(\lambda) \approx \oplus_{\mu\in\mathfrak{h}_\ell^*}\ b_{\lambda,\mu}\ Z(\mu), \text{ for some (unique) } b_{\lambda,\mu} \in \mathbb{Z}_+.$$

We denote $b_{\lambda,\mu}$ by $(Q(\lambda) : Z(\mu))$, and define the matrix $B = (b_{\lambda,\mu})_{\lambda,\mu\in\mathfrak{h}_\ell^*}$.

(b) The linkage relation $\sim$: For $\lambda, \mu \in \mathfrak{h}_\ell^*$ call λ *linked to* μ, written $\lambda \sim \mu$, if there exist indecomposable u-modules $V_1, \cdots, V_k$ (for some $k \geq 1$) and weights $\lambda_o = \lambda, \lambda_1, \cdots, \lambda_k = \mu$ belonging to $\mathfrak{h}_\ell^*$ such that $M(\lambda_{i-1})$ and $M(\lambda_i)$ are both subquotients of V_i , for all $1 \leq i \leq k$. Clearly $\sim$ is an equivalence relation. We denote the equivalence class of λ by $e(\lambda)$.

We recall the following result due to Andersen - Polo - Wen.

(3.8) Theorem [*APW*; Corollary 8.2]. *Let $\lambda, \mu \in \mathfrak{h}_\ell^*$. Then $\lambda \sim \mu$ if and only if there exists a $w \in W$ such that $\mu = w * \lambda$ as elements of $\mathfrak{h}_\ell^*$, where $w * \lambda := w(\lambda + \rho) - \rho$ and ρ is as defined in § 3.0.*

We come to the following main result of this section. Recall the definition of the matrices B, C, and D from (1) of §3.7, (1) of §3.2 and (2) of §3.2 respectively; and denote by D^t the transpose of D.

(3.9) Theorem. *The matrix C is a symmetric matrix, which admits the following decompositions:*

$$(1)\cdots \qquad C = BD = D^tD .$$

Further, for $\lambda \sim \lambda'$,

$$(2)\cdots \qquad d_{\lambda,\mu} = d_{\lambda',\mu} \ \ (\textit{for any } \mu \in \mathfrak{h}_\ell^*), \textit{ and}$$

$$(3)\cdots \qquad d_{\lambda,\mu} = 0 \ \ \textit{unless } \lambda \sim \mu .$$

Moreover, for any $\lambda \in \mathfrak{h}_\ell^*$,

$$(4)\cdots \qquad \dim_{\mathbb{Q}_\xi}(Q(\lambda)) = \mid e(\lambda) \mid d_\lambda \ell^N,$$

where $N = \mid \Delta_+ \mid$ *and* $d_\lambda := d_{\lambda,\lambda}$.

Remark. We prove in Section 5 that $B = D^t$ (cf. Corollary 5.17). In particular, this gives a different proof of (4) and proves the symmetry of C without using Theorem (2.2).

As a preparation for the proof of the above theorem, we give the following.

(3.10) Definition. A u-module V is said to admit a *Verma filtration*, if there exists a filtration of V by u-submodules :

$$V_o = 0 \subset V_1 \subset \cdots \subset V_m = V ,$$

such that, for any $1 \leq j \leq m, V_j/V_{j-1}$ is u-isomorphic with $Z(\lambda_j)$ (for some $\lambda_j \in \mathfrak{h}_\ell^*$). Such a filtration is called a *Verma filtration.*

By virtue of Proposition (5.16), any projective u–module admits a Verma filtration.

(3.11) Lemma. *Let Q be any projective* u*-module. Then (as* u*-modules),*

$$(1)\cdots \qquad Q \approx \oplus_{\lambda\in\mathfrak{h}_\ell^*}\ n(\lambda)Q(\lambda) ,$$

where $n(\lambda) := \dim_{\mathbb{Q}_\xi}(\mathrm{Hom}_u(Q, M(\lambda))) = \dim_{\mathbb{Q}_\xi}(\mathrm{Hom}_u(M(\lambda), Q))$.

We call $n(\lambda)$ as the *multiplicity* of $Q(\lambda)$ in Q and denote it by $\mathrm{mult}_{Q(\lambda)}\ Q$.

Proof. By § 3.0

$$Q \approx \oplus_{\mu\in\mathfrak{h}_\ell^*}\ m(\mu)Q(\mu) \text{ , for some } m(\mu) \in \mathbb{Z}_+.$$

For any $\lambda \in \mathfrak{h}_\ell^*$

$$\begin{aligned}\mathrm{Hom}_u(Q, M(\lambda)) &\approx \oplus_{\mu\in\mathfrak{h}_\ell^*}\ m(\mu)\ \mathrm{Hom}_u\ (Q(\mu), M(\lambda)) \\ &\approx m(\lambda)\mathrm{Hom}_u(Q(\lambda), M(\lambda)),\end{aligned}$$

since $Q(\mu)$ has unique irreducible quotient $M(\mu)$ (cf. §3.0). But, by Schur's lemma and Lemma (3.1), $\mathrm{Hom}_u(Q(\lambda), M(\lambda))$ is one dimensional over $\mathbb{Q}_\xi$. In particular, $m(\lambda) = \dim(\mathrm{Hom}_u(Q, M(\lambda)))$.

The other statement follows since socle $Q(\lambda) = M(\lambda)$, as u is a symmetric algebra by Theorem (2.2) (cf. [N] or [CR; Page 401 and Exercise 83.1]). □

With these preparations we come to the proof of Theorem (3.9).

(3.12) *Proof of Theorem* (3.9). **Assertion I.** *The matrix C is symmetric :*

This follows from Theorem (2.2) and [N; Theorem 8].

Assertion II. $C = BD$:

By Proposition (5.16), $Q(\lambda)$ admits a (Verma) filtration

$$0 = V_o \subset V_1 \subset \cdots \subset V_m = Q(\lambda) \ ,$$

such that, for any $1 \leq j \leq m, V_j/V_{j-1}$ is u-isomorphic with $Z(\lambda_j)$ (for some $\lambda_j \in \mathfrak{h}_\ell^*$). But since $Z(\lambda_j)$ are projective $\mathfrak{b}^-$-modules (by Lemma 3.4), we get the decomposition as $\mathfrak{b}^-$-modules :

(1) ⋯
$$Q(\lambda) \approx \oplus_{j=1}^m Z(\lambda_j).$$

But clearly

(2) ⋯
$$(Q(\lambda) : M(\mu)) = \sum_{j=1}^m \ (Z(\lambda_j) : M(\mu)).$$

Considering (1) and (2) the assertion II follows. □

Assertion III. $d_{\lambda,\mu} = d_{\lambda',\mu}$ *for* $\lambda \sim \lambda'$ *:*

(From the definition of $\sim$, it is clear that $d_{\lambda,\mu} = 0$ unless $\lambda \sim \mu$.)

In view of Theorem (3.8), we can assume that $\lambda' = s_i * \lambda$, for some simple reflection s_i. Define u-module maps $\theta_1 : Z(\lambda') \to Z(\lambda)$ and $\theta_2 : Z(\lambda) \to Z(\lambda')$, determined by $\theta_1(1 \otimes 1_{\lambda'}) = F_i^{\widetilde{\lambda}(\alpha_i^\vee)+1} \otimes 1_\lambda$ and $\theta_2(1 \otimes 1_\lambda) = F_i^{\ell-1-\widetilde{\lambda}(\alpha_i^\vee)} \otimes 1_{\lambda'}$, where 1_λ is any fixed non-zero vector of $\mathcal{Q}_\xi(\lambda)$ and $\widetilde{\lambda} \in \mathfrak{h}^*_{\mathbb{Z}}$ is the unique lift of λ under the canonical map $\mathfrak{h}^*_{\mathbb{Z}} \to \mathfrak{h}_\ell^*$ with the requirement that $0 \leq \widetilde{\lambda}(\alpha_j^\vee) < \ell$ for all the simple coroots $\alpha_j^\vee$. (It is easy to see that $\theta_1(1 \otimes 1_{\lambda'})$ and $\theta_2(1 \otimes 1_\lambda)$ have restricted weights λ' and λ respectively, and moreover both are annihilated by E_j for all $1 \leq j \leq n$ by [L_4; §6.5], and hence they extend uniquely to u-module maps; cf. §4.3 for the definition of restricted weight.) Clearly $\theta_1\theta_2 = \theta_2\theta_1 = 0$. So im $\theta_1 \subset$ ker θ_2 (and im $\theta_2 \subset$ ker θ_1), where im denotes the image. We claim, in fact, that im θ_1 = ker θ_2 :

Now im $\theta_1 = (\mathrm{u}^- \ F_i^{\widetilde{\lambda}(\alpha_i^\vee)+1}) \otimes 1_\lambda$. By a suitable choice of the reduced decomposition of the longest element w_o, we can assume that $i = 1$. This gives that im θ_1 has basis $\{F^\varphi \otimes 1_\lambda\}$, where $\varphi = (\varphi_1, \cdots, \varphi_N)$ ranges over $\mathbb{Z}_\ell^N$ with $\varphi_1 \geq \widetilde{\lambda}(\alpha_i^\vee) + 1$. But ker θ_2 has the same basis, and thus

(3) ⋯
$$\text{im } \theta_1 = \text{ker } \theta_2.$$

Exactly similar argument also gives that

(4) ⋯
$$\text{im } \theta_2 = \text{ker } \theta_1.$$

For any $\mu \in \mathfrak{h}_\ell^*$

(5) ⋯
$$(Z(\lambda') : M(\mu)) = (\text{ker } \theta_1 : M(\mu)) + (\text{im } \theta_1 : M(\mu)).$$

Similarly,

(6) ⋯
$$(Z(\lambda) : M(\mu)) = (\text{ker } \theta_2 : M(\mu)) + (\text{im } \theta_2 : M(\mu)) \ .$$

Now putting (3)- (6) together, we get

$$(Z(\lambda) : M(\mu)) = (Z(\lambda') : M(\mu)) \text{ ,i.e., } \ d_{\lambda,\mu} = d_{\lambda',\mu} \text{ if } \lambda \sim \lambda'. \ \square$$

Assertion IV. $\dim Q(\lambda) = | e(\lambda) | d_\lambda \ell^N$:

For any $\lambda \in \mathfrak{h}_\ell^*$, denote by $\mathcal{B}_{e(\lambda)}$ the sum of all the PIM's of u isomorphic to $Q(\mu)$ ($\mu \in e(\lambda)$). Then, from Lemma (3.4) and Definition (3.10), it is clear that in the direct sum decomposition (as $\mathfrak{b}^-$-module) of the projective module $\mathcal{B}_{e(\lambda)}$, $Z(\mu)$ occurs exactly ℓ^N times, for any $\mu \sim \lambda$. But $\text{mult}_{Q(\mu)} \mathcal{B}_{e(\lambda)} = \text{mult}_{Q(\mu)}$ u $= \dim_{\mathcal{Q}_\xi} M(\mu)$, by Lemma (3.11). These give rise to two different expressions of $\dim_{\mathcal{Q}_\xi} \mathcal{B}_{e(\lambda)}$:

(7) ⋯
$$\dim \mathcal{B}_{e(\lambda)} = \sum_{\mu \in e(\lambda)} \ \dim M(\mu) \dim Q(\mu) \text{ ,and}$$

(8) $\cdots$ $$\dim \mathcal{B}_{e(\lambda)} = | e(\lambda) | \ell^{2N}.$$

Further, by Assertion III,

(9) $\cdots$ $$\dim Z(\lambda) = \sum_{\mu \in e(\lambda)} d_{\lambda,\mu} \dim M(\mu) = \sum_{\mu \in e(\lambda)} d_\mu \dim M(\mu) .$$

By (1) of §3.7

(10) $\cdots$ $$\dim Q(\lambda) = b_\lambda \ell^N, \text{where } b_\lambda := \sum_{\theta \in e(\lambda)} b_{\lambda,\theta}.$$

But, by Assertion II, if $\lambda \sim \mu$

$$\begin{aligned} c_{\lambda,\mu} &= \sum_{\theta \in e(\lambda)} b_{\lambda,\theta}\, d_{\theta,\mu} \\ &= (\sum_{\theta \in e(\lambda)} b_{\lambda,\theta}) d_\mu \text{ ,by Assertion III ,i.e.,} \end{aligned}$$

(11) $\cdots$ $$c_{\lambda,\mu} = b_\lambda d_\mu.$$

So, by Assertion I, $b_\lambda d_\mu = b_\mu d_\lambda$,i.e.,

(12) $\cdots$ $$\frac{b_\lambda}{d_\lambda} = \frac{b_\mu}{d_\mu}, \quad \text{if } \lambda \sim \mu .$$

By (7) and (10), we get

$$\begin{aligned} \dim \mathcal{B}_{e(\lambda)} &= (\sum_{\mu \in e(\lambda)} \tfrac{b_\mu}{d_\mu} d_\mu \dim M(\mu))\ell^N \\ &= \tfrac{b_\lambda}{d_\lambda}(\dim Z(\lambda))\ell^N \text{ , by (9), i.e.,} \end{aligned}$$

(13) $\cdots$ $$\dim \mathcal{B}_{e(\lambda)} = \frac{b_\lambda}{d_\lambda} \ell^{2N}.$$

Combining (8) and (13), we get

(14) $\cdots$ $$| e(\lambda) | = \frac{b_\lambda}{d_\lambda} ,$$

and hence from (10), $\dim Q(\lambda) = | e(\lambda) | d_\lambda \ell^N$. This proves Assertion IV. □

Assertion V. $C = D^t.D$:
By (11), if $\lambda \sim \mu$,

$$\begin{aligned} c_{\lambda,\mu} &= b_\lambda d\mu \\ &= | e(\lambda) | d_\lambda d_\mu , \quad \text{by (14)} \\ &= \sum_{\theta \in e(\lambda)} d_{\theta,\lambda} d_{\theta,\mu} , \quad \text{by Assertion III.} \end{aligned}$$

This proves Assertion V. □

Combining Assertions I - V, we get Theorem (3.9). □

(3.13) Remark. The analogues of most of the results (in particular, Theorem 3.9) in this section for the modular case are due to Humphreys [H_1].

4. Some consequences of Theorem (3.9)

We follow the notation and conventions as in the beginning of § 3.

The following result (at least in the case when ℓ is a power of a prime number) is obtained by Andersen-Polo-Wen [APW; Corollary 7.6 and Theorem 9.8], as a consequence of their linkage principle. Even though we do not give the details, a simple and direct proof (for the assertion that St_ξ is irreducible and projective u-module) can be given along the lines of the proof of the corresponding classical result as in [H_2; §5.5]. Now $St_\xi \otimes V$ is a projective u-module, follows from a general fact about Hopf algebras (cf., e.g., [GL; Proposition 1.7 ff.]). Since u is a Frobenius algebra, any projective u-module is injective (cf. [CR; Theorem 58.14]).

(4.1) Proposition. *St_ξ is an irreducible* u*-module, which is projective (and hence injective), where St_ξ is as defined in Definition (3.0).*

In particular, for any u*-module $V, St_\xi \otimes V$ is a projective (and injective)* u*-module.*

As an immediate consequence of Theorem (3.9), we obtain the following:

(4.2) Corollary. *For $\lambda \sim \lambda' \in \mathfrak{h}_\ell^*$ (cf. Definition 3.7 (b)),*

$$St_\xi \otimes Z(\lambda) \approx St_\xi \otimes Z(\lambda') \quad \text{as u-modules.}$$

Proof. Since, by the previous proposition, St_ξ is a projective module,

(1) $\cdots$ $St_\xi \otimes Z(\lambda) \approx \sum_\mu d_{\lambda,\mu} St_\xi \otimes M(\mu)$ (as u – modules), by (4) of §3.2.

But then the corollary follows from (2) of Theorem (3.9). □

(4.3) Definition. Recall the definition of the algebra $\mathfrak{u}^o \subset \mathfrak{u}$ from §2.7 and the beginning of §3. It can be easily seen that the map

$$\chi : \mathfrak{h}_\ell^* \to \mathrm{Spec}(\mathfrak{u}^o),$$

given by $\chi(\lambda)(K_i) = \xi^{<\lambda,\alpha_i>}$, for $\lambda \in \mathfrak{h}_\ell^*$ and $1 \le i \le n$, is a bijection, where $\mathrm{Spec}(\mathfrak{u}^o)$ denotes the set of all the algebra homomorphisms : $\mathfrak{u}^o \to \mathcal{Q}_\xi$.

For any $\mathfrak{u}^o$-module V and $\lambda \in \mathfrak{h}_\ell^*$, we define the λ-th *restricted weight space*

(1) $\cdots$ $$V_{(\lambda)} := \{v \in V : K_i v = \chi(\lambda)(K_i).v\}.$$

Since $\mathfrak{u}^o$ is a semi-simple algebra, being the tensor product

$$\bigotimes_{i=1}^{n} (\mathcal{Q}_\xi[X_i]/ < X_i^\ell - 1 >),$$

and $\mathfrak{u}^o$ is a direct sum of certain one dimensional ideals of $\mathfrak{u}^o$ (cf. §3.6), any $\mathfrak{u}^o$-module V decomposes as the (direct) sum of its restricted weight spaces (cf. [CR; Theorem 25.10]) :

$$V = \sum_{\lambda \in \mathfrak{h}_\ell^*} V_{(\lambda)} .$$

We define its *formal restricted character*, denoted $ch_\ell(V)$, by

(2) $\cdots$ $$ch_\ell(V) = \sum_{\lambda \in \mathfrak{h}_\ell^*} \dim\,(V_{(\lambda)})\, e^\lambda.$$

Let $\mathfrak{h}_{\ell,\mathrm{root}}^* \subset \mathfrak{h}_\ell^*$ be the subgroup generated by the image of the simple roots $\{\alpha_1, \cdots, \alpha_n\}$ in $\mathfrak{h}_\ell^*$.

(4.4) Lemma. *For any $\lambda \in \mathfrak{h}_\ell^*$*

$$ch_\ell(Z(\lambda)) = (\ell^N / \#(\mathfrak{h}_{\ell,\mathrm{root}}^*))\, e^\lambda \sum_{\beta \in \mathfrak{h}_{\ell,\mathrm{root}}^*} e^\beta .$$

Proof. From the identity (2) of Theorem (3.9) (for any simple root α_i)

$$\begin{aligned} ch_\ell(Z(0)) &= ch_\ell(Z(-\alpha_i))\,, \quad \text{since } -\alpha_i \sim 0 \\ &= e^{-\alpha_i}\, ch_\ell(Z(0)). \end{aligned}$$

This gives $ch_\ell(Z(0)) = e^\beta ch_\ell(Z(0))$, for any $\beta \in \mathfrak{h}_{\ell,\mathrm{root}}^*$. In particular,

$$ch_\ell(Z(0)) = (\ell^N / \#(\mathfrak{h}_{\ell,\mathrm{root}}^*)) \sum_{\beta \in \mathfrak{h}_{\ell,\mathrm{root}}^*} e^\beta .$$

This proves the lemma. □

(4.5) Proposition. *For any* u*-module V, we have (as* u*-modules)*

(1) $\cdots$ $$St_\xi \otimes V \approx \oplus_{\mu \in \mathfrak{h}_\ell^*} \dim(\mathrm{Hom}_{\mathfrak{b}}(\mathcal{Q}_\xi((\ell-1)\rho) \otimes V, M(\mu)))\; Q(\mu)\,,$$

and also

(2) $\cdots$ $$St_\xi \otimes V \approx \oplus_{\mu \in \mathfrak{h}_\ell^*} \dim(\mathrm{Hom}_{\mathfrak{b}^-}(\overline{\mathcal{Q}}_\xi(\rho) \otimes V, M(\mu)))\; Q(\mu)\,,$$

where $\mathcal{Q}_\xi(\lambda)$ is as in §3.0 and $\overline{\mathcal{Q}}_\xi(\lambda)$ is as in Lemma (3.4).

In particular, (for any $\lambda \in \mathfrak{h}_\ell^*$)

(3) $\cdots$ $$St_\xi \otimes Z(\lambda) \approx \sum_{\mu \in \mathfrak{h}_\ell^*} \dim\,(M(\mu)_{(\lambda+\rho)})\; Q(\mu)\,.$$

Proof. By Proposition (4.1), Lemma (3.11), and [GL; Proposition 1.7]

$$\begin{aligned} St_\xi \otimes V &\approx \oplus_\mu \dim(\mathrm{Hom}_{\mathfrak{u}}(St_\xi \otimes V, M(\mu)))\; Q(\mu) \\ &\approx \oplus \dim(\mathrm{Hom}_{\mathfrak{u}}(\mathfrak{u} \otimes_{\mathfrak{b}} (\mathcal{Q}_\xi((\ell-1)\rho) \otimes V), M(\mu)))\; Q(\mu) \\ &\approx \oplus \dim(\mathrm{Hom}_{\mathfrak{b}}(\mathcal{Q}_\xi((\ell-1)\rho) \otimes V, M(\mu)))\; Q(\mu), \end{aligned}$$ by [CE; Chap. II, §

The proof of (2) is exactly the same.

To prove the "In particular" part, observe first that (using the definition of Δ given in (10) of §1.1)

(4) $\cdots$ $$\overline{Q}_\xi(\rho) \otimes Z(\lambda) \approx Z(\lambda+\rho), \quad \text{as } \mathfrak{b}^- - \text{modules}.$$

Further, since $Z(\lambda+\rho)$ is freely generated as an $\mathfrak{u}^-$-module by the element $1 \otimes 1_{\lambda+\rho}$, it is easy to see that the restriction map $\mathrm{Hom}_{\mathfrak{b}^-}(Z(\lambda+\rho), M(\mu)) \to M(\mu)_{(\lambda+\rho)}$, given by $f \mapsto f(1\otimes 1_{\lambda+\rho})$, is an isomorphism. So (3) follows by combining (2) and (4). □

(4.6) Definition. Following Bernshtein-Gel'fand-Gel'fand [BGG], define for any $\lambda \in \mathfrak{h}^*_\ell$,

$$I(\lambda) = \mathfrak{u} \otimes_{\mathfrak{u}^o} Q_\xi(\lambda),$$

where $Q_\xi(\lambda)$, a $\mathfrak{b}$-module, is considered as an $\mathfrak{u}^o$-module by restriction.

(4.7) Proposition. *For any $\lambda \in \mathfrak{h}^*_\ell$, $I(\lambda)$ is a projective* u*-module. Moreover*

(1) $\cdots$ $$I(\lambda) \approx \oplus_{\mu \in \mathfrak{h}^*_\ell} \dim(M(\mu)_{(\lambda)}) \; Q(\mu), \quad \textit{as } \mathfrak{u} - \textit{modules}.$$

Proof. For any u-module V, the restriction map $\varphi : \mathrm{Hom}_\mathfrak{u}(I(\lambda), V) \to V_{(\lambda)}$ is clearly an isomorphism, where $\varphi(f) = f(1 \otimes 1_\lambda)$, for $f \in \mathrm{Hom}_\mathfrak{u}(I(\lambda), V)$. This, in particular, implies (using the restricted weight space decomposition, cf. §4.3) that for any surjective u-module map $f : V \to W$ between two u-modules, the induced map $\tilde{f}$: $\mathrm{Hom}_\mathfrak{u}(I(\lambda), V) \to \mathrm{Hom}_\mathfrak{u}(I(\lambda), W)$ is surjective. This proves that $I(\lambda)$ is u-projective.

Now (1) follows from Lemma (3.11), using the isomorphism φ. □

(4.8) Remark. Combining (3) of Proposition (4.5) and (1) of Proposition (4.7), it follows that for any $\lambda \in \mathfrak{h}^*_\ell$, $St_\xi \otimes Z(\lambda) \approx I(\lambda+\rho)$ as u-modules.

(4.9) Let V be any u-module. For any $\lambda \in \mathfrak{h}^*_\ell$, decompose (as u-modules)

(1) $\cdots$ $$Q(\lambda) \otimes V \approx \oplus_{\mu \in \mathfrak{h}^*_\ell} n_{\lambda,\mu}(V) \; Q(\mu),$$

for some (unique) $n_{\lambda,\mu}(V) \in \mathbb{Z}_+$.

By Lemma (3.11) and [APW; Proposition 1.18 (iii), and Remark (7.6)], it is easy to see that

(2) $\cdots$ $$n_{(\ell-1)\rho,\mu}(M(\theta)) = n_{(\ell-1)\rho,\theta}(M(\mu)), \text{ for any } \mu, \theta \in \mathfrak{h}^*_\ell.$$

(4.10) Proposition. *For any $\lambda, \mu \in \mathfrak{h}^*_\ell$ and any* u*-module V*

(1) $\cdots$ $$d_\lambda \sum_{\lambda' \sim \lambda} \dim V_{(\mu-\lambda')} = \sum_{\theta\sim\mu} d_\theta \; n_{\lambda,\theta}(V),$$

where d_λ is as defined in Theorem (3.9).

In particular,

(2) $\cdots$ $$\dim V_{(\mu+\rho)} = \sum_{\theta\sim\mu} d_\theta \; n_{-\rho,\theta}(V), \; \textit{and}$$

(3) $\cdots$ $$\sum_{\theta\sim\mu} d_\theta \dim(M(\theta)_{(\lambda+\rho)}) \begin{cases} = \ell^N/\#\mathfrak{h}^*_{\ell,\mathrm{root}}\,, & \textit{if } \lambda-\mu-\rho \in \mathfrak{h}^*_{\ell,\mathrm{root}} \\ = 0\,, & \textit{otherwise.} \end{cases}$$

Proof. As $\mathfrak{b}^-$-modules

$$Q(\lambda) \otimes V \approx \bigoplus_{\lambda' \in \mathfrak{h}^*_\ell} b_{\lambda,\lambda'}(Z(\lambda') \otimes V), \quad \text{by (1) of §3.7.}$$

This gives, by Corollary (5.17) and Theorem (3.9),

(4) $\cdots$ $$Q(\lambda) \otimes V \approx d_\lambda \bigoplus_{\lambda'\sim\lambda} (Z(\lambda') \otimes V).$$

Similarly (as $\mathfrak{b}^-$-modules),

(5) $\cdots$ $$\bigoplus_\theta n_{\lambda,\theta}(V) \; Q(\theta) \approx \bigoplus_\theta n_{\lambda,\theta}(V) \; d_\theta \left(\bigoplus_{\mu\sim\theta} Z(\mu) \right).$$

Putting (4) and (5) in (1) of §4.9, and equating coefficients of $Z(\mu)$ we get

$$\sum_{\theta\sim\mu} d_\theta \; n_{\lambda,\theta}(V) = d_\lambda \sum_{\lambda'\sim\lambda} \dim V_{(\mu-\lambda')} \,.$$

This proves (1). The identity (2) follows by specializing (1) to $\lambda = -\rho$, since $d_{-\rho} = 1$ and $\lambda' \sim -\rho$ if and only if $\lambda' = -\rho$ (by Theorem 3.8). The identity (3) follows from (2) by taking $V = Z(\lambda)$ and using (3) of Proposition (4.5) and Lemma (4.4). □

5. Modules for the pair $(\mathfrak{u}, \mathfrak{U}_\xi^0)$

We continue to use the same notation and conventions as in the beginning of § 3.

(5.1) Definition. Let $U_{\mathcal{A}}^0$ be the $\mathcal{A}$-subalgebra of $U_{\mathcal{A}}$ (cf. §1.2) generated by $\left\{K_i^{\pm 1}, \binom{K_i;c}{t}\right\}$ $(1 \le i \le n, c \in \mathbb{Z}, t \in \mathbb{Z}_+)$, and set

$$(1) \cdots \qquad U_\xi^0 := \mathbb{Q}_\xi \otimes_{\mathcal{A}} U_{\mathcal{A}}^0,$$

where $\mathbb{Q}_\xi$ is an $\mathcal{A}$-algebra as in §1.2, and

$$(2) \cdots \qquad \binom{K_i;c}{t} := \prod_{s=1}^{t} \frac{K_i v^{d_i(c-s+1)} - K_i^{-1} v^{-d_i(c-s+1)}}{v^{d_i s} - v^{-d_i s}}.$$

By $[L_4$; Theorem 6.7], the canonical map : $U_\xi^0 \to U_\xi$ is injective, where $U_\xi := U_{\mathbb{Q}_\xi}$ (cf. (1) of §1.2). Let $\widetilde{u}$ be the $\mathbb{Q}_\xi$-subalgebra of U_ξ generated by U_ξ^0 and u_ξ (cf. §1.3). By $[L_3; §6.5]$, $K_i^\ell - 1$ is central in U_ξ (in particular in $\widetilde{u}$). Let $\widetilde{\mathfrak{u}}$ (resp. $\mathfrak{U}_\xi^0$) be the quotient algebra of $\widetilde{u}$ (resp. U_ξ^0) divided by the (two sided) ideal generated by $\{K_i^\ell - 1\}_{1 \le i \le n}$. It is easy to see that the comultiplication Δ (cf. (10) of §1.1) defines a Hopf algebra structure on $\widetilde{u}, \widetilde{\mathfrak{u}}, U_\xi^0$, and $\mathfrak{U}_\xi^0$. We denote by $\widetilde{\mathfrak{b}}$ the subalgebra of $\widetilde{\mathfrak{u}}$ generated by $E_i (1 \le i \le n)$ and $\mathfrak{U}_\xi^0$. By $[L_4$;§ 6.5], the multiplication map : $u_\xi \otimes U_\xi^0 \to \widetilde{u}$ is surjective.

Let $\mathrm{Spec}(\mathfrak{U}_\xi^0)$ be the set of all the $\mathbb{Q}_\xi$-algebra homomorphisms $\mathfrak{U}_\xi^0 \to \mathbb{Q}_\xi$, and let $\mathrm{Spec}_{\mathbb{Z}}(\mathfrak{U}_\xi^0) \subset \mathrm{Spec}(\mathfrak{U}_\xi^0)$ be the subset consisting of those $f \in \mathrm{Spec}(\mathfrak{U}_\xi^0)$ such that $f\binom{K_i;0}{\ell} \in \mathbb{Z}$, for all $1 \le i \le n$. Define a map

$$\widetilde{\chi} : \mathfrak{h}^*_{\mathbb{Z}} \to \mathrm{Spec}_{\mathbb{Z}}(\mathfrak{U}_\xi^0) \text{ by}$$

$$(3) \cdots \qquad \widetilde{\chi}(\lambda)(K_i^{\pm 1}) = \xi^{\pm <\lambda, \alpha_i>} \text{ and}$$

$$(4) \cdots \qquad \widetilde{\chi}(\lambda)\binom{K_i;c}{t} = \binom{<\lambda, \alpha_i^\vee> + c}{t}_{d_i}^{\xi},$$

for any $\lambda \in \mathfrak{h}^*_{\mathbb{Z}}, 1 \le i \le n, c \in \mathbb{Z}$, and $t \in \mathbb{Z}_+$; where for $m \in \mathbb{Z}$ and $n \in \mathbb{Z}_+$, $\binom{m}{n}_{d_i}^{\xi}$ is by definition $\prod_{s=1}^{n} \frac{v^{d_i(m-s+1)} - v^{-d_i(m-s+1)}}{v^{d_i s} - v^{-d_i s}}$, evaluated at $v = \xi$.

It is easy to see, from the relations $[L_4$; § 6.4] and $[L_2$;§ 4.1(e)], that $\widetilde{\chi}(\lambda)$ indeed extends (uniquely) to an algebra homomorphism, and moreover by $[L_2$; Corollary 3.3(a)], $\widetilde{\chi}(\lambda)\binom{K_i;0}{\ell} \in \mathbb{Z}$.

(5.2) Lemma. *The map $\widetilde{\chi} : \mathfrak{h}^*_{\mathbb{Z}} \to \mathrm{Spec}_{\mathbb{Z}}(\mathfrak{U}_\xi^0)$, defined above, is a bijection.*

(We denote by $\widetilde{\mathbb{Q}}_\xi(\lambda)$ the one dimensional $\mathfrak{U}_\xi^0$-module given by the character $\widetilde{\chi}(\lambda)$.)

Proof. Injectivity of $\widetilde{\chi}$ follows from evaluating $\widetilde{\chi}(\lambda)$ at K_i and $\binom{K_i;0}{\ell}$ (for all $1 \le i \le n$), and using $[L_2$; Proposition 3.2(a)]. (Observe that we are using the condition (∗) of § 1.3.)

To prove surjectivity; given $f \in \mathrm{Spec}_{\mathbb{Z}}(\mathfrak{U}_\xi^0)$, define $\lambda = \lambda(f) \in \mathfrak{h}^*_{\mathbb{Z}}$ by

$$(1) \cdots \qquad <\lambda, \alpha_i^\vee> = m_i + \ell f\binom{K_i;0}{\ell}, \text{ for any } 1 \le i \le n,$$

where $0 \le m_i < \ell$ is the unique integer such that $f(K_i) = (\xi^{d_i})^{m_i}$. (Since $K_i^\ell = 1, f(K_i)$ is a ℓ-th root of unity, and d_i is coprime to ℓ by (∗) of § 1.3.)

The elements $\left\{K_i, \binom{K_i;0}{\ell}\right\}_{1 \le i \le n}$ generate the algebra $\mathfrak{U}_\xi^0$ over $\mathbb{Q}_\xi$, as follows from $[L_3$; §2.3, Identity g.9 and g.10] and $[L_4$; §6.4, Identity (b3)]. In particular, using $[L_2$; Proposition 3.2(a)], it is easy to see that $\widetilde{\chi}(\lambda) = f$. This proves the lemma. □

(5.3) Remark. There is an algebra isomorphism

$$\mathbb{Q}_\xi[X_1, \cdots, X_n, Y_1, \cdots, Y_n] / < X_1^\ell - 1, \cdots, X_n^\ell - 1 > \xrightarrow{\sim} \mathfrak{U}_\xi^0,$$

given by $X_i \mapsto K_i$, and $Y_i \mapsto \binom{K_i;0}{\ell}$. To prove this, use [$L_3$; § 2.3, relations (g5)-(g10)].

In particular, Spec $_{\mathbb{Z}}\left(\mathfrak{U}_\xi^0\right)$ is a proper subset of Spec $\left(\mathfrak{U}_\xi^0\right)$.

(5.4) Definitions. (a) Let V be any representation (over $\mathbb{Q}_\xi$) of the algebra $\mathfrak{U}_\xi^0$. Then, for any $\lambda \in \mathfrak{h}^*_{\mathbb{Z}}$, the λ-th *weight space* V_λ of V is defined by

(1) $\cdots$ $$V_\lambda = \{v \in V : xv = \widetilde{\chi}(\lambda)x.v, \text{ for all } x \in \mathfrak{U}_\xi^0\},$$

where $\widetilde{\chi}(\lambda)$ is as defined in §5.1.

Clearly the sum $\sum_{\lambda\in\mathfrak{h}^*_{\mathbb{Z}}} V_\lambda$ is direct. We call the representation V a *weight module* if

$$V = \oplus_{\lambda\in\mathfrak{h}^*_{\mathbb{Z}}} V_\lambda\ .$$

We define the *formal character ch* of a finite dimensional weight module V as

(2) $\cdots$ $$ch\ V = \sum_{\lambda\in\mathfrak{h}^*_{\mathbb{Z}}} (\dim V_\lambda)\ e^\lambda.$$

It is easy to see that any submodule (as well as a quotient module) of a weight module is a weight module. It follows by [L_4; § 6.5] that for any $\widetilde{\mathfrak{u}}$-module V, any $\psi \in \mathbb{Z}_\ell^N$, and $\lambda \in \mathfrak{h}^*_{\mathbb{Z}}$

(3) $\cdots$ $$E^\psi \cdot V_\lambda \subset V_{\lambda'} \ \ (\text{where } \lambda' := \lambda + \sum_{k=1}^N \psi_k\beta_k\ ), \text{ and}$$

(4) $\cdots$ $$F^\psi \cdot V_\lambda \subset V_{\lambda''} \ \ (\text{where } \lambda'' := \lambda - \sum_{k=1}^N \psi_k\beta_k\ ),$$

where E^ψ, F^ψ are as defined by (4) and (7) of §1.3.

(b) Let $\mathcal{F}$ be the full category of $\widetilde{\mathfrak{u}}$-modules V in finite dimensional vector spaces over $\mathbb{Q}_\xi$, such that V is a weight module for $\mathfrak{U}_\xi^0$.

The following lemma is "essentially" [APW; Lemma 1.1].

(5.5) Lemma. *For any $\lambda, \mu \in \mathfrak{h}^*_{\mathbb{Z}}$, the tensor product $\mathfrak{U}_\xi^0$-module $\widetilde{\mathbb{Q}}_\xi(\lambda) \otimes \widetilde{\mathbb{Q}}_\xi(\mu)$ (cf. Lemma 5.2) has weight $\lambda + \mu$.*

(5.6) Definition. Just as in §3.0, we define the *Verma module* $\widetilde{Z}(\lambda)$ ($\lambda \in \mathfrak{h}^*_{\mathbb{Z}}$) for $\widetilde{\mathfrak{u}}$ by

$$\widetilde{Z}(\lambda) = \widetilde{\mathfrak{u}} \otimes_{\widetilde{\mathfrak{b}}} \widetilde{\mathbb{Q}}_\xi(\lambda),$$

where $\mathfrak{U}_\xi^0$-module structure on $\widetilde{\mathbb{Q}}_\xi(\lambda)$ is extended to a $\widetilde{\mathfrak{b}}$-module structure by demanding E_i's ($1 \le i \le n$) to act by 0.

It is easy to see that $\widetilde{Z}(\lambda) \in \mathcal{F}$. Further, by (4) of §5.4, all the weights μ of $\widetilde{Z}(\lambda)$ satisfy $\mu \le \lambda$ (where the notation $\mu \le \lambda$ means that $\lambda - \mu \in \sum_{i=1}^n \mathbb{Z}_+\alpha_i$). In particular, $\widetilde{Z}(\lambda)$ has a unique proper maximal submodule, and hence a unique *irreducible quotient* $\widetilde{M}(\lambda)$.

It can be easily seen that the map $\lambda \mapsto \widetilde{M}(\lambda)$ defines a bijection between $\mathfrak{h}^*_{\mathbb{Z}}$ and the set of isomorphism classes of simple $\widetilde{\mathfrak{u}}$-modules in the category $\mathcal{F}$. As in § 3.2, for any $V \in \mathcal{F}$, by $(V : \widetilde{M}(\lambda))$ we mean the *multiplicity* of $\widetilde{M}(\lambda)$ in V.

It follows from [L_2; Proposition 7.2(a)] that for any $\lambda \in \mathfrak{h}^*_{\mathbb{Z}}$, $\widetilde{M}(\ell\lambda)$ is a one dimensional module, such that the augmentation ideal of $\mathfrak{u}$ acts trivially. Further, for any $\lambda, \mu \in \mathfrak{h}^*_{\mathbb{Z}}$, as $\widetilde{\mathfrak{u}}$-modules (by Lemma 5.5)

(1) $\cdots$ $$\widetilde{Z}(\lambda + \ell\mu) \approx \widetilde{Z}(\lambda) \otimes \widetilde{M}(\ell\mu), \text{ and}$$

(2) $\cdots$ $$\widetilde{M}(\lambda + \ell\mu) \approx \widetilde{M}(\lambda) \otimes \widetilde{M}(\ell\mu)\ .$$

(5.7) Lemma. *For $\lambda \in \mathfrak{h}^*_{\mathbb{Z}}$, $\widetilde{M}(\lambda)$ considered as a $\mathfrak{u}$-module (via restriction) is isomorphic with $M(\overline{\lambda})$ (cf. § 3.0), where $\overline{\lambda}$ is the image of λ under the canonical map $\mathfrak{h}^*_{\mathbb{Z}} \to \mathfrak{h}^*_\ell$.*

Proof. Clearly $\widetilde{Z}(\lambda) \approx Z(\overline{\lambda})$, as $\mathfrak{u}$-modules. So, it suffices to show that the maximal proper $\mathfrak{u}$-submodule $K(\lambda)$ of $\widetilde{Z}(\lambda)$ is stable under $\mathfrak{U}_\xi^0$ as well :

By [L_4; § 6.5], $\mathfrak{U}_\xi^0 \cdot K(\lambda) \subset \widetilde{Z}(\lambda)$ is stable under $\mathfrak{u}$. Further, since $K(\lambda) \subset \widehat{\mathcal{M}}$ (cf. [L_3; Proof of Proposition 5.11]; the notation $\widehat{\mathcal{M}}$ is as in loc. cit.) and $\mathfrak{U}_\xi^0 \cdot \widehat{\mathcal{M}} \subset \widehat{\mathcal{M}}$, we see that $\mathfrak{U}_\xi^0 \cdot K(\lambda)$ is properly contained in $\widetilde{Z}(\lambda)$. In particular, by maximality of $K(\lambda)$, $K(\lambda) = \mathfrak{U}_\xi^0 \cdot K(\lambda)$, i.e., $K(\lambda)$ is $\mathfrak{U}_\xi^0$-stable. □

Analogous to Definition (4.6), we define (for any $\lambda \in \mathfrak{h}^*_{\mathbb{Z}}$) the $\tilde{\mathfrak{u}}$-module

(1) $\cdots$ $$\widetilde{I}(\lambda) := \tilde{\mathfrak{u}} \otimes_{\mathfrak{U}_\xi^0} \widetilde{\mathcal{Q}}_\xi(\lambda)\ .$$

It is easy to see that $\widetilde{I}(\lambda) \in \mathcal{F}$, and moreover

(2) $\cdots$ $$\widetilde{I}(\lambda) \approx I(\overline{\lambda}), \text{considered as } \mathfrak{u}-\text{modules}.$$

The following holds by the same argument as Proposition (4.7).

(5.8) Lemma. *For any $\tilde{\mathfrak{u}}$-module V and $\lambda \in \mathfrak{h}^*_{\mathbb{Z}}$, the restriction map $\widetilde{\varphi} : \mathrm{Hom}_{\tilde{\mathfrak{u}}}(\widetilde{I}(\lambda), V) \to V_\lambda$, given by $\widetilde{\varphi}(f) = f(1 \otimes 1_\lambda)$, is an isomorphism.*

In particular, $\widetilde{I}(\lambda)$ is a projective module in the category $\mathcal{F}$.

(5.9) Corollary. *For any $\lambda \in \mathfrak{h}^*_{\mathbb{Z}}$, there exists a unique (upto an isomorphism) indecomposable projective module $\widetilde{Q}(\lambda)$ in the category $\mathcal{F}$, such that $\widetilde{Q}(\lambda)$ has $\widetilde{M}(\lambda)$ as quotient. Moreover, $\widetilde{Q}(\lambda)$ has a unique proper maximal submodule, and any indecomposable projective module $\in \mathcal{F}$ is isomorphic with exactly one $\widetilde{Q}(\lambda)$.*

Proof. Since $\widetilde{I}(\lambda)$ is projective, and the canonical map $\widetilde{I}(\lambda) \to \widetilde{M}(\lambda)$ is surjective, there exists an indecomposable summand of $\widetilde{I}(\lambda)$ which maps surjectivily onto the irreducible module $\widetilde{M}(\lambda)$.

We now come to the uniqueness statements [1] :

Let $\widetilde{Q}$ be any indecomposable projective module $\in \mathcal{F}$ with a surjective $\tilde{\mathfrak{u}}$-module map $f : \widetilde{Q} \to \widetilde{M}(\lambda)$. We first claim that any proper submodule of $\widetilde{Q}$ is contained in ker f (in particular, $\widetilde{Q}$ has a unique proper maximal submodule) : Let V be any submodule of $\widetilde{Q}$ such that $f(V) \neq 0$, and choose a minimal such. (This is possible since $\widetilde{Q}$ is finite dimensional.) Since $\widetilde{Q}$ is projective, there exists a map $\widetilde{f} : \widetilde{Q} \to V$ such that $f\,|_V \circ \widetilde{f} = f$. Since $f_{|V}$ is surjective and V is minimal with this property, $\widetilde{f}(V) = V$. But then, by dimension counting, $\widetilde{f}_{|V} : V \to V$ is an isomorphism. Now the map $(\widetilde{f}_{|V})^{-1}$ provides a splitting for the exact sequence

$$0 \to \ker \widetilde{f} \to \widetilde{Q} \xrightarrow{\widetilde{f}} V \to 0.$$

But $\widetilde{Q}$ being indecomposable, this is possible only if $V = \widetilde{Q}$. This proves the assertion that any proper submodule of $\widetilde{Q}$ is contained in ker f.

Now let $\widetilde{Q}_1 \xrightarrow{f_1} \widetilde{M}(\lambda)$ and $\widetilde{Q}_2 \xrightarrow{f_2} \widetilde{M}(\lambda)$ be two indecomposable projective covers $\in \mathcal{F}$. This gives rise to a commutative diagram :

$$\begin{array}{ccccc} \widetilde{Q}_1 & \xrightarrow{\varphi_1} & \widetilde{Q}_2 & \xrightarrow{\varphi_2} & \widetilde{Q}_1 \\ & f_1 \searrow & \downarrow f_2 & \swarrow f_1 & \\ & & \widetilde{M}(\lambda) & & \end{array}$$

But, from the previous assertion, φ_1 and φ_2 are both surjective. This proves that φ_1 is an isomorphism. Now the corollary follows, since $\{\widetilde{M}(\lambda)\}_\lambda$ are precisely the simple objects $\in \mathcal{F}$. □

(5.10) Lemma. (a) *For any $\lambda, \mu \in \mathfrak{h}^*_{\mathbb{Z}}$ (as $\tilde{\mathfrak{u}}$-modules)*

$$\widetilde{Q}(\lambda + \ell\mu) \approx \widetilde{Q}(\lambda) \otimes \widetilde{M}(\ell\mu).$$

(b) *For any module $V \in \mathcal{F}$, and $\lambda \in \mathfrak{h}^*_{\mathbb{Z}}$*

$$(V : \widetilde{M}(\lambda)) = \dim_{\mathcal{Q}_\xi}(\mathrm{Hom}_{\tilde{\mathfrak{u}}}(\widetilde{Q}(\lambda), V)).$$

Proof. By [GL; Proposition 1.7 ff.], $\widetilde{Q}(\lambda) \otimes \widetilde{M}(\ell\mu)$ is a projective module. Further $\widetilde{Q}(\lambda) \approx \widetilde{Q}(\lambda) \otimes \widetilde{M}(\ell\mu) \otimes \widetilde{M}(-\ell\mu)$ (cf. Lemma 5.5 and §5.6). Hence, $\widetilde{Q}(\lambda)$ being indecomposable, $\widetilde{Q}(\lambda) \otimes \widetilde{M}(\ell\mu)$ is indecomposable. In particular, (a) follows from (2) of §5.6 and Corollary (5.9).

(b) For any exact sequence in $\mathcal{F}$:

$$0 \to V_1 \to V \to V_2 \to 0\ ,$$

we clearly have $(V : \widetilde{M}(\lambda)) = (V_1 : \widetilde{M}(\lambda)) + (V_2 : \widetilde{M}(\lambda))$. Further, $\widetilde{Q}(\lambda)$ being projective,

(1) $\cdots$

$$\dim(\mathrm{Hom}_{\tilde{\mathfrak{u}}}(\widetilde{Q}(\lambda), V)) = \dim(\mathrm{Hom}_{\tilde{\mathfrak{u}}}(\widetilde{Q}(\lambda), V_1)) + \dim(\mathrm{Hom}_{\tilde{\mathfrak{u}}}(\widetilde{Q}(\lambda), V_2)).$$

[1] I thank R. Parthasarathy for this argument.

So, to prove the (b) part, it suffices to prove it (by induction on the length of V) for $V = \widetilde{M}(\mu)$ (for any $\mu \in \mathfrak{h}^*_{\mathbb{Z}}$). But this follows from the above Corollary (5.9). □

(5.11) Definition. Exactly parallel to the notion of Verma filtration for a u-module (Definition 3.10), one has the notion of *Verma filtration* for $\widetilde{\mathfrak{u}}$-modules (where we demand the successive quotients be isomorphic to $\widetilde{Z}(\lambda_j)$).

For a $\widetilde{\mathfrak{u}}$-module $V \in \mathcal{F}$ admitting a Verma filtration

$$V_0 = 0 \subset V_1 \subset \cdots \subset V_m = V,$$

and $\lambda \in \mathfrak{h}^*_{\mathbb{Z}}$, we define

(1) $\cdots$ $$(V : \widetilde{Z}(\lambda)) = \#\{1 \leq j \leq m : V_j/V_{j-1} \approx \widetilde{Z}(\lambda)\}.$$

From $ch\ V$, it is easy to see that $(V : \widetilde{Z}(\lambda))$ is well defined, i.e., it does not depend upon the choice of Verma filtration.

(5.12) Lemma. *For any $\lambda \in \mathfrak{h}^*_{\mathbb{Z}}$, the $\widetilde{\mathfrak{u}}$-module $\widetilde{I}(\lambda)$ (cf.(1) of §5.7) admits a Verma filtration. Moreover,*

(1) $\cdots$ $$(\widetilde{I}(\lambda) : \widetilde{Z}(\mu)) = \dim\ \widetilde{Z}(\mu)_\lambda.$$

Proof. By the definition, $\widetilde{I}(\lambda) = \widetilde{\mathfrak{u}} \otimes_{\mathfrak{u}^0_\xi} \widetilde{\mathcal{Q}}_\xi(\lambda) \approx \widetilde{\mathfrak{u}} \otimes_{\widetilde{\mathfrak{b}}} \left(\widetilde{\mathfrak{b}} \otimes_{\mathfrak{u}^0_\xi} \widetilde{\mathcal{Q}}_\xi(\lambda)\right)$. It is easy to see, using Proposition (2.4) and the weight consideration (cf. Definition 2.8), that $\widetilde{\mathfrak{b}} \otimes_{\mathfrak{u}^0_\xi} \widetilde{\mathcal{Q}}_\xi(\lambda)$ has a filtration by $\widetilde{\mathfrak{b}}$-stable submodules:

$$F_0 = 0 \subset F_1 \subset \cdots \subset F_k = \widetilde{\mathfrak{b}} \otimes_{\mathfrak{u}^0_\xi} \widetilde{\mathcal{Q}}_\xi(\lambda),$$

such that F_j/F_{j-1} (for any j) is isomorphic with $\widetilde{\mathcal{Q}}_\xi(\lambda_j)$ (for some $\lambda_j \in \mathfrak{h}^*_{\mathbb{Z}}$). Further, for any $\mu \in \mathfrak{h}^*_{\mathbb{Z}}$,

$$\#\{1 \leq j \leq k : \lambda_j = \mu\} = \dim\ \widetilde{Z}(\mu)_\lambda.$$

Since $\widetilde{\mathfrak{u}}$ is $\widetilde{\mathfrak{b}}$-free (under right multiplication), the above filtration gives rise to a Verma filtration of $\widetilde{I}(\lambda)$. □

The following lemma can be obtained by the same argument as in [BGG; §6].

(5.13) Lemma. *For two $\widetilde{\mathfrak{u}}$-modules $V_1, V_2 \in \mathcal{F}$, $V_1 \oplus V_2$ admits a Verma filtration if and only if both of V_1 and V_2 admit Verma filtrations.*

(5.14) Proposition. *Any projective $\widetilde{\mathfrak{u}}$-module $\in \mathcal{F}$ admits a Verma filtration.*

Further, for any $\lambda, \mu \in \mathfrak{h}^*_{\mathbb{Z}}$, we have the BGG- duality

(1) $\cdots$ $$\left(\widetilde{Q}(\lambda) : \widetilde{Z}(\mu)\right) = \left(\widetilde{Z}(\mu) : \widetilde{M}(\lambda)\right).$$

Proof. The first part follows immediately from Lemmas (5.12), (5.13) and (5.8).

The identity (1) follows by an argument same as [J; Proof of Satz 3.8], using Lemmas (5.10)(b), (5.12), and (the following) Proposition (5.15). □

By an argument same as proof of Lemma (3.11) (together with Lemma 5.8 and Corollary 5.9), we get the following.

(5.15) Proposition. *For any projective module $\widetilde{Q} \in \mathcal{F}$,*

(1) $\cdots$ $$\widetilde{Q} \approx \bigoplus_{\lambda \in \mathfrak{h}^*_{\mathbb{Z}}} n(\lambda)\ \widetilde{Q}(\lambda) \quad (as\ \widetilde{\mathfrak{u}}-modules),$$

where $n(\lambda) := \dim_{\mathcal{Q}_\xi}(\mathrm{Hom}_{\widetilde{\mathfrak{u}}}(\widetilde{Q}, \widetilde{M}(\lambda)))$.

In particular, for any $\mu \in \mathfrak{h}^*_{\mathbb{Z}}$,

(2) $\cdots$ $$\widetilde{I}(\mu) \approx \bigoplus_\lambda \dim(\widetilde{M}(\lambda)_\mu)\ \widetilde{Q}(\lambda).$$

(5.16) Proposition. *For any $\lambda \in \mathfrak{h}^*_{\mathbb{Z}}$,*

$$\widetilde{Q}(\lambda) \approx Q(\overline{\lambda}), \quad as\ \mathfrak{u}-modules,$$

where $\overline{\lambda}$ is as in Lemma 5.7.

In particular, any projective u*-module admits a Verma filtration.*

Proof. Since $\widetilde{I}(\mu) \approx I(\overline{\mu})$ as u-modules (by (2) of §5.7), $\widetilde{Q}(\mu)$ is a projective u-module which surjects onto $\widetilde{M}(\mu) \approx M(\overline{\mu})$ (cf. Lemma 5.7). In particular, we can decompose as u-modules (cf. Lemma 5.10(a)) :

$$\widetilde{Q}(\mu) \approx \bigoplus_{\overline{\nu} \in \mathfrak{h}^*_\ell} n_{\overline{\mu},\overline{\nu}} Q(\overline{\nu})\ , \text{ for some } n_{\overline{\mu},\overline{\nu}} \geq 0 \text{ and } n_{\overline{\mu},\overline{\mu}} > 0.$$

Substituting this in (2) of Proposition (5.15), we get (as u-modules)

$$I(\overline{\lambda}) \approx \bigoplus_{\mu\in\mathfrak{h}^*_{\mathbb{Z}}} \dim(\widetilde{M}(\mu)_\lambda)\ (\bigoplus_{\overline{\nu}\in\mathfrak{h}^*_\ell} n_{\overline{\mu},\overline{\nu}} Q(\overline{\nu}))$$
$$\approx \bigoplus_{\overline{\nu}\in\mathfrak{h}^*_\ell} \bigoplus_{\substack{\mu_0\in\mathfrak{h}^*_\ell \text{ and} \\ \mu_1\in\mathfrak{h}^*_{\mathbb{Z}}}} \dim(\widetilde{M}(\mu_0+\ell\mu_1)_\lambda)\ n_{\overline{\mu_0},\overline{\nu}}\ Q(\overline{\nu})$$
$$\approx \bigoplus_{\overline{\nu}\in\mathfrak{h}^*_\ell} \bigoplus_{\substack{\mu_0\in\mathfrak{h}^*_\ell \text{ and} \\ \mu_1\in\mathfrak{h}^*_{\mathbb{Z}}}} \dim(\widetilde{M}(\mu_0)_{\lambda-\ell\mu_1})\ n_{\overline{\mu}_0,\overline{\nu}}\ Q(\overline{\nu}), \text{ by (2) of §5.6}$$
$$(1) \quad \cdots \approx \bigoplus_{\overline{\nu}\in\mathfrak{h}^*_\ell} \bigoplus_{\overline{\mu}_0\in\mathfrak{h}^*_\ell} \dim(M(\overline{\mu}_0)_{(\overline{\lambda})})\ n_{\overline{\mu}_0,\overline{\nu}}\ Q(\overline{\nu}), \text{ by Lemma (5.7).}$$

But, by Proposition (4.7),

$$I(\overline{\lambda}) \approx \bigoplus_{\overline{\nu}\in\mathfrak{h}^*_\ell} \dim(M(\overline{\nu})_{(\overline{\lambda})})\ Q(\overline{\nu})\ .$$

This, together with (1), forces $n_{\overline{\mu}_0,\overline{\nu}} = \delta_{\overline{\mu}_0,\overline{\nu}}$, proving the proposition. □

(5.17) Corollary. *For any* $\overline{\lambda},\overline{\mu}\in\mathfrak{h}^*_\ell$

$$(Q(\overline{\lambda}):Z(\overline{\mu})) = (Z(\overline{\mu}):M(\overline{\lambda})),\ i.e.,$$

the matrix $D = B^t$ *(cf. Theorem 3.9).*

Proof. Fix any preimage $\lambda\in\mathfrak{h}^*_{\mathbb{Z}}$ (resp. μ) of $\overline{\lambda}$ (resp. $\overline{\mu}$) under the canonical map $\mathfrak{h}^*_{\mathbb{Z}}\to\mathfrak{h}^*_\ell$. Then, by Proposition (5.16) and (1) of §5.6,

$$(Q(\overline{\lambda}):Z(\overline{\mu})) = \sum_{\theta\in\mathfrak{h}^*_{\mathbb{Z}}} (\widetilde{Q}(\lambda):\widetilde{Z}(\mu+\ell\theta))$$
$$(1) \quad \cdots \quad = \sum_{\theta\in\mathfrak{h}^*_{\mathbb{Z}}} (\widetilde{Q}(\lambda-\ell\theta):\widetilde{Z}(\mu)), \text{ by Lemma 5.10(a).}$$

Similarly,

$$(2) \quad \cdots \quad (Z(\overline{\mu}):M(\overline{\lambda})) = \sum_{\theta\in\mathfrak{h}^*_{\mathbb{Z}}} (\widetilde{Z}(\mu):\widetilde{M}(\lambda+\ell\theta)).$$

Now the corollary follows by combining (1) and (2) with Proposition (5.14).

References

[APW] Andersen, H. H.; Polo, P.; and Wen, K.: Representations of quantum algebras. Invent. Math. 104, 1-59 (1991).

[B] Ballard, J. W.: Injective modules for restricted enveloping algebras. Math. Z. 163, 57-63 (1978).

[Bo] Borel, A.: Linear representations of semi-simple algebraic groups. Proc. of Symp. in Pure Math. vol. 29, 421-440 (1975).

[BGG] Bernshtein, I. N.; Gel'fand, I. M.; and Gel'fand, S. I.: Category of $\mathfrak{g}$-modules. Functional Anal. Appl. 10, 87-92 (1976).

[CE] Cartan, H. and Eilenberg, S.: "Homological Algebra". Princeton Univ. Press, Princeton, N. J., (1956).

[CR] Curtis C. W. and Reiner, I.: "Representation theory of finite groups and associative algebras". Interscience Publishers (1962).

[DK] DeConcini, C. and Kac, V. G.: Representations of quantum groups at roots of 1. In:" Operator Algebras, Unitary Representations, Enveloping Algebras, and Invariant Theory" (ed. by A. Connes et. al.), Progr. Math. 92, 471-506 (1990).

[GL] Garland, H. and Lepowsky, J.: Lie-algebra homology and the Macdonald-Kac formulas. Invent. Math. 34, 37-76 (1976).

[H_1] Humphreys, J. E.: Modular representations of classical Lie algebras and semi-simple groups. J. of Algebra 19, 51-79 (1971).

[H_2] Humphreys, J. E.: "Ordinary and modular representations of Chevalley groups ". Lecture Notes in Mathematics vol. 528, Springer-Verlag, (1976).

[H_3] Humphreys, J.E.: Symmetry for finite dimensional Hopf algebras. Proc. of AMS 68, 143-146 (1978).

[J] Jantzen, J. C.: Über darstellungen höherer Frobenius-kerne halbeinfacher algebraischer gruppen. Math. Z. 164, 271-292 (1979).

[L_1] Lusztig, G.: Quantum deformations of certain simple modules over enveloping algebras. Adv. in Maths. 70, 237-249 (1988).

[L_2] Lusztig, G.: Modular representations and quantum groups. In: "Classical groups and related topics", Contemp. Math. 82, 59-77 (1989).

[L_3] Lusztig, G.: Finite dimensional Hopf algebras arising from quantized universal enveloping algebras. J. Amer. Math. Soc. 3, 257-296 (1990).

[L_4] Lusztig, G.: Quantum groups at roots of 1. Geometriae Dedicata 35, 89-113 (1990).

[LS] Levendorskii, S. Z. and Soibelman, Ya. S. : Some applications of quantum Weyl group I. J. Geom. and Phys. 7(4), 1-14 (1991).

[LSw] Larson, R. G. and Sweedler, M. E.: An associative orthogonal bilinear form for Hopf algebras. Amer. J. of Maths. 91, 75-94 (1969).

[N] Nesbitt, C.: On the regular representations of algebras. Annals of Maths. 39, 634-658 (1938).

[S] Schue, J. R.: Symmetry for the enveloping algebra of a restricted Lie algebra. Proc. of AMS 16, 1123-1124 (1965).

[Sw] Sweedler, M. E.: "Hopf algebras". W. A. Benjamin, Inc., New York (1969).

[V] Verma, D. N.: The role of affine Weyl groups in the representation theory of algebraic Chevalley groups and their Lie algebras. In:"Lie groups and their representations" (ed. by I. M. Gel'fand), Summer School of the Bolyai János Math. Soc., Halsted Press, New York, pp. 653-705 (1975).