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Symmetric and exterior powers of homogeneous vector bundles

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Introduction

Throughout this note, the base field K is assumed to be algebraically closed of char 0.

The following question was asked by Y. Laszlo:

Question. Let G be a connected semi-simple algebraic group over K and let P be a parabolic subgroup. Then for any irreducible P-module V, such that V is a submodule of a G-module, is it true that the canonical map

$$S^n(H^0(G/P, \mathcal{L}(V^*))) \to H^0(G/P, \mathcal{L}(S^n(V^*)))$$

is surjective (for all $n \geqq 1$)?

(Here S^n denotes the n-th symmetric power, V^* denotes the dual of V, and $\mathcal{L}(V^*)$ denotes the vector bundle on the base space G/P associated to the principal P-bundle $G \to G/P$ via the representation V^* of P.)

The aim of this note is to answer the above question affirmatively. In fact, we prove the following more general theorem:

Theorem. *Let V_1 and V_2 be two completely reducible P-modules, such that both of them are P-submodules of certain G-modules. Then the canonical map, induced from the diagonal embedding $G/P \hookrightarrow G/P \times G/P$,*

$$\beta: H^0(G/P \times G/P, \mathcal{L}(V_1^*) \boxtimes \mathcal{L}(V_2^*)) \to H^0(G/P, \mathcal{L}(V_1^* \otimes V_2^*))$$

is surjective, where $\mathcal{L}(V_1^*) \boxtimes \mathcal{L}(V_2^*)$ denotes the external tensor product of the vector bundles $\mathcal{L}(V_1^*)$ and $\mathcal{L}(V_2^*)$.

The main theorem and its proof

Let G be a connected semi-simple algebraic group over K, and let $P \subset G$ be any parabolic subgroup with unipotent radical U and with a fixed Levi component $L \subset P$. We also fix a Borel subgroup $B \subset P$ of G. For any algebraic group H over K, by an H-module we shall always mean an algebraic H-module in a finite dimensional K-vector space.

(1) Lemma. *For any irreducible P-module V, U acts trivially on V.*

In particular, for any two completely reducible modules V_1 and V_2 of P, $V_1 \otimes V_2$ is again completely reducible.

Proof. The Borel subgroup B being solvable, B fixes a line in V, say Kv_0 (cf. [B, Theorem (10.4), Chap. III]). Clearly $Uv_0 = v_0$ (since U is unipotent and $U \subset B$). But U being a normal subgroup of P, for any $p \in P$ and $u \in U$

$$p^{-1}upv_0 = v_0, \quad \text{i.e.,}$$

$$u(pv_0) = pv_0 .$$

But V being irreducible, the linear span of $\{pv_0\}_{p \in P}$ is the whole of V. This proves the first part of the lemma.

To prove the 'in particular' statement, we can assume, without loss of generality, that V_1 and V_2 are irreducible P-modules. Now, L being reductive, $V_1 \otimes V_2$ can be decomposed as a direct sum of irreducible L-modules, on which U acts trivially (by the first part of the lemma). □

(2) Corollary. *Irreducible P-modules are canonically in one-to-one correspondence with irreducible L-modules.*

In particular, any irreducible P-module has a unique B-fixed line, and the character of B obtained by its action on any non-zero vector in this line (called the highest weight of V) completely determines the P-module V.

Proof. Let $B(L) \subset B$ be the Borel subgroup of L. Then L being reductive, from the general theory of representations of reductive groups over char 0, there is a unique $B(L)$-fixed line in V, and the weight of any non-zero vector in this line completely determines the L-module V. From this the corollary follows easily. □

(3) Definition. *We denote the irreducible P-module with highest weight λ by $E_P(\lambda)$.*

Then $E_P(\lambda)$ are precisely the irreducible P-modules, where λ varies over those characters of B which are dominant with respect to the Borel subgroup $B \cap [L,L]$ of the "semi-simple part" $[L,L]$ of L.

(4) Lemma. *Let $E(\lambda)$ be the irreducible G-module with highest weight vector v_λ (of weight λ) and let E be the P-submodule of $E(\lambda)$ generated by v_λ. Then E is an irreducible P-module (isomorphic to $E_P(\lambda)$).*

In particular, any irreducible P-module $E_P(\lambda)$, with λ any character of B which is dominant with respect to B, arises this way.

Proof. Since U acts trivially on the vector v_λ and U is normal in P, it is easy to see (cf. proof of Lemma 1) that U acts trivially on E. We next claim that E is already irreducible as an L-module (in particular will be irreducible as a P-module):

Let $B(L) \subset B$ be the Borel subgroup of L. To prove the claim, it suffices to show that $B(L)$ has a unique fixed line in E. Since U acts trivially on E and the Borel subgroup B is generated by $B(L)$ and U, any $B(L)$-fixed line in E will automatically be fixed by B. But $E(\lambda)$ being G-irreducible, $E(\lambda)$ (in particular E) has a unique B-fixed line. This proves the claim, thereby proves the lemma. □

(5) Lemma. *Let V be a completely reducible P-module such that $V \subset W$ for some G-module W. Write*

$$V = \bigoplus_{i=1}^{n} V_i \,,$$

where any V_i is an irreducible P-module. Then there exist irreducible G-modules $W_i \subset W$ and a G-submodule $W' \subset W$ such that

(a) *$V_i \subset W_i$, for all i, and*

(b) *$W = (\oplus_i W_i) \oplus W'$.*

Proof. We prove the lemma by induction on n: If $n = 1$, take the unique B-fixed line (say ℓ) in V. (In fact, there is a unique $B(L)$-fixed line in V, by Lemma 1 and Corollary 2.) The G-span W_1 of this line inside W is an irreducible G-module: This follows since, as is well known (and easy to prove), for any G-module W and any B-fixed line in W, the G-submodule of W generated by this line is irreducible. Hence the lemma follows in this case (i.e. the case when $n = 1$).

Now let $V = V_1 \oplus \ldots \oplus V_n \oplus V_{n+1} \hookrightarrow W$. By the induction hypothesis, we have $W = W_1 \oplus \ldots \oplus W_n \oplus W'$, with $V_i \subset W_i$ for all $1 \leqq i \leqq n$. Also, by the case $n = 1$, there exists an irreducible G-submodule $V_{n+1} \subseteq W_{n+1} \subseteq W$. We next claim that $(W_1 \oplus \ldots \oplus W_n) \cap W_{n+1} = 0$: For otherwise ($W_{n+1}$ being irreducible), $W_1 \oplus \ldots \oplus W_n \supseteq W_{n+1}$, and hence $V_1 \oplus \ldots \oplus V_{n+1} \subseteq W_1 \oplus \ldots \oplus W_n$. But the dimension of N-fixed vectors in $V_1 \oplus \ldots \oplus V_{n+1}$ is $n+1$, whereas dimension of N-fixed vectors in $W_1 \oplus \ldots \oplus W_n$ is n (where N is the unipotent radical of B). A contradiction! Hence the sum $(W_1 \oplus \ldots \oplus W_n) + W_{n+1}$ is direct. This finishes the proof of the lemma. □

(6) Definition. *Let us call a P-module to be an* ***extendable*** *P-module if it is a submodule of a G-module. An extendable P-module is said to be* ***irreducible*** *if it is irreducible as a P-module.*

(7) *Remark.* By Lemmas 4 and 5, extendable irreducible P-modules are precisely those of the form $E_P(\lambda)$, for λ varying over those characters of B which are dominant with respect to B, and in this case $E_P(\lambda)$ is a P-submodule of the irreducible G-module $E(\lambda)$. In particular, with the notation as in Lemma 5, if $V_i \approx E_P(\lambda_i)$, then $W_i \approx E(\lambda_i)$.

Let V be any P-module. This gives rise to a vector bundle $\mathcal{L}(V) := G \times_P V$ on G/P, i.e., $\mathcal{L}(V)$ is the vector bundle associated to the principal P-bundle $G \to G/P$ via the P-module V.

(8) Lemma. *Let $E_P(\lambda)$ be an extendable irreducible P-module. Then the canonical map $r: H^0(G/P, \mathcal{L}(E(\lambda)^*)) \to H^0(G/P, \mathcal{L}(E_P(\lambda)^*))$ (got from the restriction map $E(\lambda)^* \to E_P(\lambda)^*$) is an isomorphism.*

Proof. Since $E(\lambda)$ is a G-module, the vector bundle $\mathcal{L}(E(\lambda)^*)$ on G/P is a trivial vector bundle. In particular,

$$H^0(G/P, \mathcal{L}(E(\lambda)^*)) \approx E(\lambda)^* .$$

Further, since the character λ (of B) is dominant with respect to B, by the Borel-Weil theorem,

$$H^0(G/B, \mathcal{L}(K_\lambda^*)) \approx E(\lambda)^* ,$$

and

$$H^0(P/B, \mathcal{L}(K_\lambda^*)) \approx E_P(\lambda)^* ,$$

where K_λ denotes the one dimensional representation of B, given by the character λ. (Observe that even though P is not reductive, the Borel-Weil theorem is still available for P/B, as can be seen by passing to the Levi component L of P.) Now, considering the fibration $\pi: G/B \to G/P$ (with fiber P/B) and taking the line bundle $\mathcal{L}(K_\lambda^*)$ on G/B, we get

$$H^0(G/B, \mathcal{L}(K_\lambda^*)) \simeq H^0(G/P, \pi_*(\mathcal{L}(K_\lambda^*))) .$$

But, by the homogeneity, the sheaf $\pi_*(\mathcal{L}(K_\lambda^*))$ is locally free and moreover

$$\pi_*(\mathcal{L}(K_\lambda^*)) \simeq \mathcal{L}(H^0(P/B, \mathcal{L}(K_\lambda^*))) \simeq \mathcal{L}(E_P(\lambda)^*) .$$

Combining these, we get

$$H^0(G/P, \mathcal{L}(E_P(\lambda)^*)) \approx E(\lambda)^* .$$

So, both of the domain and the range of the G-module map r are irreducible G-modules. In particular, if non-zero, the map r will be an isomorphism. It is easy to see that r is indeed non-zero. □

(9) Theorem. *For any two extendable completely reducible P-modules V_1 and V_2, the canonical restriction map (induced from the diagonal embedding $G/P \hookrightarrow G/P \times G/P$)*

$$\beta: H^0(G/P \times G/P, \mathcal{L}(V_1^*) \boxtimes \mathcal{L}(V_2^*)) \to H^0(G/P, \mathcal{L}(V_1^* \otimes V_2^*))$$

is surjective, where $\mathcal{L}(V_1^*) \boxtimes \mathcal{L}(V_2^*)$ denotes the external tensor product of the vector bundles $\mathcal{L}(V_1^*)$ and $\mathcal{L}(V_2^*)$.

Proof. We can clearly assume that V_1 and V_2 are extendable irreducible P-modules (say) $E_P(\lambda)$ and $E_P(\mu)$ respectively. We have the following commutative diagram:

$$\begin{array}{ccccc} E(\lambda)^* \otimes E(\mu)^* \approx & H^0(\widetilde{G/P}, \mathcal{L}(E(\lambda)^*)\boxtimes\mathcal{L}(E(\mu)^*)) & \xrightarrow[\sim]{\theta_1} & H^0(\widetilde{G/P}, \mathcal{L}(E_P(\lambda)^*)\boxtimes\mathcal{L}(E_P(\mu)^*)) \\ \Big\| \text{Id} & \Big\downarrow \beta_1 & & \Big\downarrow \beta \\ E(\lambda)^* \otimes E(\mu)^* \approx & H^0(G/P, \mathcal{L}(E(\lambda)^*\otimes E(\mu)^*)) & \xrightarrow[\theta_2]{} & H^0(G/P, \mathcal{L}(E_P(\lambda)^*\otimes E_P(\mu)^*)), \end{array}$$

where $\widetilde{G/P} := G/P \times G/P$, the maps θ_1 and θ_2 are induced by the canonical restriction map $E(\lambda)^* \otimes E(\mu)^* \to E_P(\lambda)^* \otimes E_P(\mu)^*$, and the vertical maps are induced by the diagonal inclusion. By Lemma 8, the map θ_1 is an isomorphism. So, to prove the surjectivity of β, it suffices to prove the surjectivity of θ_2: Now to prove the surjectivity of θ_2, by virtue of Lemmas 1, 4, and 5 (see also Remark 7), we only need to prove that for any character λ of B which is dominant with respect to B,

$$H^0(G/P, \mathcal{L}(E(\lambda)^*)) \to H^0(G/P, \mathcal{L}(E_P(\lambda)^*))$$

is surjective. But this follows from Lemma 8. This proves the theorem. □

(10) Corollary. *Let V be an extendable completely reducible P-module. Then, for any $n \geqq 1$, the symmetric power*

$$S^n(H^0(G/P, \mathcal{L}(V^*))) \to H^0(G/P, \mathcal{L}(S^n(V^*)))$$

is surjective.

Similarly, the exterior power

$$\Lambda^n(H^0(G/P, \mathcal{L}(V^*))) \to H^0(G/P, \mathcal{L}(\Lambda^n(V^*)))$$

is surjective, for all $n \geqq 1$.

Proof. As an easy consequence of the above theorem, the canonical map

$$\bigotimes_{i=1}^{n} H^0(G/P, \mathcal{L}(V_i^*)) \to H^0\left(G/P, \mathcal{L}\left(\bigotimes_{i=1}^{n} V_i^*\right)\right)$$

is surjective, for any extendable completely reducible P-modules V_i. (Observe that the tensor product of any two extendable completely reducible P-modules is again extendable completely reducible, by Lemma 1.)

Now the corollary follows from the above by the standard symmetrization (resp. antisymmetrization) argument. □

(11) *Remarks.* (a) It will be interesting to know if Theorem 9 and Corollary 10 remain true in arbitrary characteristic.

(b) The restriction in Theorem 9, that the P-modules V_1 and V_2 are extendable, can not be removed. This can be easily seen by taking, e.g., V_1 to be $E_P(\lambda)$ with λ

'very dominant' along B, and V_2 to be $E_P(\mu)$ with μ dominant along $B \cap [L, L]$ but non-dominant along the whole of B.
(c) It can be easily seen that Theorem 9 is equivalent to the assertion (under the hypothesis and the notation of Theorem 9) that for all $i \geqq 1$,

$$H^i(G/P \times G/P, \mathcal{I}_D \otimes (\mathcal{L}(V_1^*) \boxtimes \mathcal{L}(V_2^*))) = 0,$$

where $\mathcal{I}_D$ denotes the ideal sheaf of the diagonal $D \simeq G/P \subset G/P \times G/P$.

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Reference

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