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Finiteness of local fundamental groups for quotients of affine varieties under reductive groups

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0. Introduction

Let us recall the following conjecture due to C. T. C. Wall:

(C_1) CONJECTURE [W; §1]. *Let G be a reductive linear algebraic group* $/\mathbb{C}$ *acting linearly on an affine space* $\mathbb{C}^n$. *Assume that* $\dim \mathbb{C}^n//G = 2$ (cf. §1). *Then the variety* $\mathbb{C}^n//G$ *is biregular isomorphic with the variety* $\mathbb{C}^2/\Gamma$, *where* Γ *is some finite group acting linearly on* $\mathbb{C}^2$.

In our attempt to prove the above conjecture, we (together with R. V. Gurjar) were led to the following question (or conjecture) vastly generalizing the above conjecture:

(C_2) CONJECTURE. *Let G be as above, and assume that G acts on an irreducible normal affine variety X over* $\mathbb{C}$. *If the local fundamental groups* (cf. §1.2) *of X at all the points of X are finite, then the same is true for the quotient variety* $X//G$, *provided* $\dim X//G \geq 2$.

Recently Gurjar obtained a proof of the above conjecture (C_2) in the case when X is smooth; in particular he proved Wall's conjecture (C_1). But his proof relies heavily on the assumption that X is smooth.

The aim of this note is to prove the conjecture (C_2); but we need to assume that all the local rings of X have fully-torsion divisor class groups. (In fact a more general result is proved; see our theorem 2.1, and remark 2.2.)

The 'Kempf–Ness theory', as developed by Neeman, is the main ingredient in our proof. We also make use of the Luna slice theorem.

I thank R. V. Gurjar for explaining to me his proof of Wall's conjecture, in particular I make use of his crucial proposition from [G]. I also thank J. N. Damon and J. Wahl for some references, and the Referee for his (her) suggestions to improve the exposition.

1. Notation and preliminaries

By a variety X we shall always mean an algebraic variety $/\mathbb{C}$, and its ring of regular functions is denoted by $\mathbb{C}[X]$. We denote the singular locus of X by Σ_X. Let X be an affine variety on which a reductive linear algebraic group $G/\mathbb{C}$ acts, then by $X//G$ we mean the affine variety $\operatorname{Spec}(\mathbb{C}[X]^G)$, where $\mathbb{C}[X]^G$ denotes the ring of G-invariants in $\mathbb{C}[X]$.

Let us recall the following well known fact about CW complexes (see, e.g., arguments in [LW; Chapter II, Sec. 6]):

(1.1) LEMMA. *Let X be a CW complex, and $Y \subsetneq X$ a (closed) subcomplex. For any $x \in X$, there exists a fundamental system $\{U\}_{U \in \mathscr{U}}$ of (open) neighborhoods of x in X satisfying the following condition*:

Given any $U, V \in \mathscr{U}$, $V \subset U$, the inclusion $V \setminus Y_x \hookrightarrow U \setminus Y_x$ is a homotopy equivalence, where $Y_x := \{x\} \cup Y$. $(\mathscr{A})$

Now for any neighborhood $W \subset U$ of x ($U \in \mathscr{U}$, but W not necessarily in $\mathscr{U}$), there of course exists a V in $\mathscr{U}$ such that $V \subset W$. From the condition $(\mathscr{A})$, we easily see that, for any $* \in V \setminus Y_x$, the canonical map

$$\pi_1(W \setminus Y_x, *) \to \pi_1(U \setminus Y_x, *) \quad \text{is surjective.} \qquad (\mathscr{I})$$

(1.2) DEFINITION. With the notation as in the above lemma, let us further assume that $U \setminus Y_x$ (for some, and hence any $U \in \mathscr{U}$) is connected and non-empty. If this is satisfied, we say that *Y does not disconnect X locally at x*. In this case, we define the *local fundamental group of X at x with respect to Y*, denoted $\pi_1^{x,Y}(X)$, as the fundamental group $\pi_1(U \setminus Y_x, *)$, for any base point $* \in U \setminus Y_x$ and any $U \in \mathscr{U}$.

Observe that, by the condition $(\mathscr{A})$, for any $V \in \mathscr{U}$ and $*' \in V \setminus Y_x$, $\pi_1(U \setminus Y_x, *)$ is isomorphic with $\pi_1(V \setminus Y_x, *')$, and moreover the isomorphism is unique up to an inner automorphism of $\pi_1(U \setminus Y_x, *)$. *In particular, the group $\pi_1^{x,Y}(X)$ is defined only up to an inner automorphism.* It is easy to see from $(\mathscr{I})$ that $\pi_1^{x,Y}(X)$ does not depend upon the choice of the fundamental system of neighborhods $\mathscr{U}$ satisfying $(\mathscr{A})$.

As is well known, for any variety X and a closed subvariety Y, X is a CW complex such that $Y \subset X$ is a subcomplex (see [Gi; §5, Satz 4] or [Lo]). Moreover if X is an irreducible normal variety, then for any closed subvariety $Y \subsetneq X$, Y does not disconnect X locally at any $x \in X$. (This can easily be deduced from [M; page 288, Topological form].) In particular $\pi_1^{x,Y}(X)$ is well defined.

If X is an irreducible normal variety, we will often abbreviate $\pi_1^{x,\Sigma_X}(X)$ as $\pi_1^x(X)$; and call it the *local fundamental group of X at x*.

2. The main theorem and its proof

Following is our main theorem:

(2.1) THEOREM. *Let X be an irreducible normal affine variety, on which a (not necessarily connected) reductive linear algebraic group* $G/\mathbb{C}$ *acts with quotient* $q: X \to X//G$, *such that* $\dim X//G \geq 2$. *We assume that the following condition* ($\mathscr{C}$) *is satisfied*:

The union of the codimension-one irreducible components of $q^{-1}(\Sigma_{X//G})$ *is locally (in the Zariski topology) set theoretically defined by a single equation.* ($\mathscr{C}$)

Assume, in addition, that the local fundamental groups of X at all the points in X are finite. Then the same is true for X//G (i.e. the local fundamental groups of X//G at all the points are finite).

(2.2) REMARKS. (a) If all the irreducible components of $q^{-1}(\Sigma_{X//G})$ have codim ≥ 2, then of course the condition ($\mathscr{C}$) is vacuously satisfied.

(b) As pointed out by Gurjar; if all the local rings of the variety X (at the closed points) have fully-torsion divisor class groups, then the condition ($\mathscr{C}$) is automatically satisfied for any G action on X.

If X (as in the above theorem) is assumed to be smooth, then all the hypotheses are clearly satisfied. In particular, as a special case of the above theorem, we recover the following main result of [G]:

(2.3) COROLLARY. *Let X be an irreducible smooth affine variety, on which a reductive linear algebraic group G acts, such that* $\dim X//G \geq 2$. *Then* $X//G$ *has all its local fundamental groups finite.*

(2.4) *Proof of Theorem* (2.1). Set $Y = X//G$, and write $q^{-1}(\Sigma_Y) = D \cup E$; where D (resp. E) is the union of all the irreducible components of $q^{-1}(\Sigma_Y)$ of codim 1 (resp. codim > 1). Then, by the condition ($\mathscr{C}$), $X \backslash D$ is again an affine variety (cf. [N; Corollary 1 on page 52, Chapter V]), and clearly (D being G-stable) $X \backslash D$ is G-stable. Now, by a proposition of Gurjar [G], $(X \backslash D)//G$ is biregular isomorphic with $X//G$. (To prove this, use the fact that the canonical morphism:

$(X\backslash D)//G \to X//G$ is an isomorphism outside the singular locus and, by assumption X being normal, $(X\backslash D)//G$ as well as $X//G$ are normal.) In particular, we can (and will) replace X by $X\backslash D$ throughout the proof of the theorem; and hence assume that all the irreducible components of $q^{-1}(\Sigma_Y)$ have codim ≥ 2.

If $\bar{x} \in Y\backslash\Sigma_Y$, $\pi_1^{\bar{x}}(Y)$ is clearly trivial (since dim $Y \geq 2$, by assumption). Hence, in what follows, we can assume that $\bar{x} \in \Sigma_Y$.

We first take a G-fixed point $x \in X$ (such that $\bar{x} := q(x) \in \Sigma_Y$), and prove that $\pi_1^{\bar{x}}(Y)$ is finite by crucially using the Kempf–Ness theory:

We fix a maximal compact subgroup $K \subset G$. Then there is a real algebraic K-stable closed subvariety X_c of X and, by Neeman's deformation theorem [Ne] (also given in [S; §5]), a continuous deformation $\varphi_t : X \to X$ $(0 \leq t \leq 1)$ satisfying the following properties (P_1)–(P_6):

(P_1) X_c is contained in the union of all the closed G-orbits of X, and moreover any closed G-orbit intersects X_c in precisely one K-orbit.

(P_2) The canonical map: $X_c/K \to X//G$ is a homeomorphism in the Hausdorff topology, where X_c/K denotes the orbit space with the quotient topology coming from the Hausdorff topology on X_c.

(P_3) φ_0 is the identity map Id.

(P_4) $\varphi_{t|X_c} = Id$, for all $0 \leq t \leq 1$.

(P_5) Image $\varphi_1 \subset X_c$.

(P_6) $\{\varphi_t(x)\}_{0 \leq t < 1} \subset G \cdot x$, for any $x \in X$. In particular $\varphi_1(x) \in \overline{G \cdot x} \cap X_c$, where $\overline{G \cdot x}$ is the closure in the Hausdorff topology.

Continuing with the proof of our theorem (2.1); from the property (P_6), it is easy to see that $\varphi_t(X\backslash\Sigma) \subset X\backslash\Sigma$, for any $0 \leq t \leq 1$, where we set $\Sigma := q^{-1}(\Sigma_Y)$. (Even though we do not need, the same is true for any subset $A \subset Y$ instead of Σ_Y.) Further, by the property (P_1), (x being G-fixed) $x \in X_c$, and by assumption $x \in \Sigma$.

Let W be a (small enough) neighborhood of x in X_c, such that $\pi_1^{x, X_c \cap \Sigma}(X_c) \approx \pi_1(W\backslash\Sigma)$. (It is easy to see, from the above deformation, that $X_c \cap \Sigma$ does not disconnect X_c locally at x.) Since $\varphi_1(x) = x$ (cf. P_4), there exists a (small enough) neighborhood U of x in X such that $\varphi_1(U) \subset W$ (in particular $\varphi_1(U\backslash\Sigma) \subset W\backslash\Sigma$), and moreover $\pi_1^{x,\Sigma}(X) \approx \pi_1(U\backslash\Sigma)$. Since $W \cap U$ is a neighborhood of x in X_c and $\varphi_{1|W \cap U} = Id$ (cf. P_4), it is easy to see, from $(\mathscr{I})$ of §1.1, that the induced map

$$\varphi_{1*} : \pi_1^{x,\Sigma}(X) \to \pi_1^{x, X_c \cap \Sigma}(X_c)$$

is surjective (in fact an isomorphism).

Let q_0 denote the canonical map: $X_c \to X_c/K$. By virtue of (P_2), we identify X_c/K with Y. Let us take a (small enough) neighborhood N of $\bar{x}$ in Y (resp. W of x in

X_c), such that $\pi_1^{\bar{x}}(Y) \approx \pi_1(N \backslash \Sigma_Y)$ (resp. $\pi_1^{x, X_c \cap \Sigma}(X_c) \approx \pi_1(W \backslash \Sigma)$). We can assume that $q_0(W) \subset N$, and hence $q_0(W \backslash \Sigma) \subset N \backslash \Sigma_Y$. Since x is a G-fixed (in particular K-fixed) point and K is compact, there exists a fundamental system of neighborhoods of x in X_c, which are all K-stable. We take such a $W' \subset W$. (We can choose W' such that $W' \backslash \Sigma$ is connected.) Then by [B; Chap. II, Theorem 6.2], the induced map $\pi_1(W' \backslash \Sigma) \to \pi_1((W'/K) \backslash \Sigma_Y)$ (got by the restriction of q_0) has finite cokernel (bounded by the order of K/K^0, where K^0 is the identity component of K). But q_0 being an open map, W'/K is again a neighborhood of $\bar{x}$ in Y. Hence, by ($\mathscr{I}$) of §1.1, the canonical map $\pi_1((W'/K) \backslash \Sigma_Y) \to \pi_1(N \backslash \Sigma_Y)$ is surjective. In particular, the induced map

$$q_{0*} : \pi_1^{x, X_c \cap \Sigma}(X_c) \to \pi_1^{\bar{x}}(Y)$$

has finite cokernel. On composition, we get the map

$$q_{0*} \varphi_{1*} : \pi_1^{x, \Sigma}(X) \to \pi_1^{\bar{x}}(Y),$$

which has finite cokernel. So, to prove the finiteness of $\pi_1^{\bar{x}}(Y)$, it suffices to show that $\pi_1^{x, \Sigma}(X)$ is finite:

Consider the canonical maps α and β as follows:

$$\pi_1^{x, \Sigma}(X) \xleftarrow{\alpha} \pi_1^{x, \Sigma \cup \Sigma_X}(X) \xrightarrow{\beta} \pi_1^{x}(X).$$

Since $X \backslash \Sigma_X$ is smooth and all the irreducible components of Σ are of codim ≥ 2 (by assumption), the map β is an isomorphism. As is well known, the map α is surjective; but we give an argument (told to me by R. R. Simha) for completeness:

Let U be a non-empty connected open subset (in the Hausdorff topology) of an irreducible normal variety X. Since any subvariety $Y \subsetneqq X$ does not disconnect X locally at any point (cf. §1.2), $U \backslash Y$ is connected. Let $p : \tilde{U} \to U$ be the simply connected cover of U, viewed canonically as a complex analytic variety. Since $\tilde{U}$ is locally homeomorphic to U, $Z := p^{-1}(U \cap Y)$ does not disconnect $\tilde{U}$ locally at any point of $\tilde{U}$. But then, by a straightforward pointset topological argument, $\tilde{U} \backslash Z$ itself is connected. From this the surjectivity of $\pi_1(U \backslash Y) \to \pi_1(U)$ follows immediately. This gives the surjectivity of α.

This proves the finiteness of $\pi_1^{x, \Sigma}(X)$ (since, by assumption, $\pi_1^{x}(X)$ is finite); thereby proving the finiteness of $\pi_1^{\bar{x}}(Y)$, in the case when $G \cdot x = x$.

Now we come to an arbitrary point $\bar{x} \in \Sigma_Y$, and let $G \cdot x$ be the (unique) closed G-orbit lying inside $q^{-1}(\bar{x})$.

By Luna's slice theorem [L; §III], there exists an irreducible affine locally closed subvariety $x \in S \subset X$, which is stable under the reductive subgroup G_x (where

$G_x \subset G$ is the isotropy subgroup at x), and an affine open subset $N \subset Y$, such that the canonical map $\psi : G \times_{G_x} S \to X$ is étale onto the open subset $q^{-1}(N)$ of X, and moreover the induced map $\bar{\psi} : S//G_x \to X//G$ is étale onto N. So to prove the finiteness of $\pi_1^x(X//G) \approx \pi_1^x(S//G_x)$, since any descending chain of algebraic subgroups of G becomes stationary, it suffices to show that the G_x-variety S satisfies:

(F_1) S is normal,
(F_2) The local fundamental groups of S at all the points of S are finite, and
(F_3) $q_S : S \to S//G_x$ satisfies the condition ($\mathscr{C}$) of Theorem (2.1).

(F_1) follows trivially, since the map ψ is étale and $G \times_{G_x} S$ fibres over the smooth variety G/G_x with fibre S. Since $\Sigma_{G \times_{G_x} S} = G \times_{G_x} \Sigma_S$ and, by assumption, all the local fundamental groups of X are finite, (F_2) follows.

Observe that $(\bar{\psi})^{-1}(\Sigma_{X//G}) = \Sigma_{S//G_x}$ (since $\bar{\psi}$ is étale). So $q_S^{-1}(\Sigma_{S//G_x}) = q^{-1}(\Sigma_{X//G}) \cap S$, which gives

$$G \times_{G_x} (q_S^{-1}(\Sigma_{S//G_x})) = \psi^{-1}(q^{-1}(\Sigma_{X//G})). \qquad (*)$$

The equality (*) clearly shows the validity of (F_3) (since the same is true, by assumption, for the map $q : X \to X//G$).

This completes the proof of the theorem. □

(2.5) REMARK (due to R. V. Gurjar). The condition ($\mathscr{C}$) in Theorem (2.1) is not always satisfied. Consider, e.g.,

$$X = \mathrm{Spec}\,(\mathbb{C}[x_1, x_2, x_3, x_4]/\langle x_1x_2 - x_3x_4\rangle),$$

and $G = \mathbb{C}^*$ acting on X by $t \cdot x_1 = tx_1$, $t \cdot x_2 = t^{-1}x_2$, $t \cdot x_3 = tx_3$, $t \cdot x_4 = t^{-1}x_4$ (for any $t \in \mathbb{C}^*$). Then $\Sigma_{X//\mathbb{C}^*} = \{0\}$, and $q^{-1}(\Sigma_{X//\mathbb{C}^*})$ is the union of two irreducible components (each isomorphic with $\mathbb{C}^2$); and this does not satisfy the condition ($\mathscr{C}$). Observe however that in this example, $X//\mathbb{C}^*$ has all its local fundamental groups finite.

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