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PROOF OF WAHL'S CONJECTURE ON SURJECTIVITY OF THE GAUSSIAN MAP FOR FLAG VARIETIES

By SHRAWAN KUMAR

Introduction. Let X be a smooth projective variety over $\mathbf{C}$, with ample line bundles $\mathcal{L}_1$ and $\mathcal{L}_2$ on X. We can form their external tensor product to get a line bundle $\mathcal{L}_1 \boxtimes \mathcal{L}_2$ on $X \times X$. Let $\mathcal{I}_D$ be the ideal sheaf of the diagonal $D \subset X \times X$. Wahl defines a map (which he calls as the Gaussian map) $\Phi_{\mathcal{L}_1,\mathcal{L}_2} : H^0(X \times X, \mathcal{I}_D \otimes (\mathcal{L}_1 \boxtimes \mathcal{L}_2)) \to H^0(X, \Omega^1_X \otimes \mathcal{L}_1 \otimes \mathcal{L}_2)$; which is induced from the canonical projection: $\mathcal{I}_D \to \mathcal{I}_D/\mathcal{I}_D^2$ by identifying the $\mathcal{O}_{X\times X}/\mathcal{I}_D \approx \mathcal{O}_D$-module $\mathcal{I}_D/\mathcal{I}_D^2$ (supported in D) with the sheaf of 1-forms Ω^1_X on $D \approx X$. Wahl has extensively studied this map in [W_1], [W_2], [W_3], and [W_4]; and he made the following conjecture in [W_4].

CONJECTURE (Wahl). *The Gaussian map $\Phi_{\mathcal{L}_1,\mathcal{L}_2}$, defined above, is surjective for any (generalized) flag variety $X = G/P$ (where G is any complex semi-simple group and $P \subset G$ a parabolic subgroup) and any ample homogeneous line bundles $\mathcal{L}_1$ and $\mathcal{L}_2$ on X.*

Wahl proved the above conjecture in the case when $X = SL(n, \mathbf{C})/B$, and also in the case when $X = G/P$ but P is a maximal parabolic subgroup corresponding to a miniscule weight (cf. [W_4]).

One of the principal aims of this paper is to prove the above conjecture in full generality. In fact we prove the following:

(2.5) THEOREM. *With the notation and assumptions as in the above conjecture; $H^p(G/P \times G/P, \mathcal{I}_D^2 \otimes (\mathcal{L}_1 \boxtimes \mathcal{L}_2)) = 0$, for all $p > 0$.*

Considering the long exact cohomology sequence associated to the sheaf exact sequence:

$$0 \to \mathcal{I}_D^2 \otimes (\mathcal{L}_1 \boxtimes \mathcal{L}_2) \to \mathcal{I}_D \otimes (\mathcal{L}_1 \boxtimes \mathcal{L}_2) \to (\mathcal{I}_D/\mathcal{I}_D^2) \otimes (\mathcal{L}_1 \boxtimes \mathcal{L}_2) \to 0,$$

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we immediately see that the above Theorem (2.5) implies the validity of Wahl's conjecture. (Actually, as we have shown in Remark 2.10, the two are equivalent.)

We prove the above theorem by first proving the following result on the existence of certain components in the tensor product of two representations.

Let $\mathfrak{g}$ be a complex semi-simple Lie algebra with a fixed Borel subalgebra $\mathfrak{b}$ and a Cartan subalgebra $\mathfrak{h} \subset \mathfrak{b}$. We denote the set of roots for the pair $(\mathfrak{b}, \mathfrak{h})$ by Δ_+, and call them as the positive roots. Let $\{\alpha_1, \ldots, \alpha_\ell\}$ be the set of simple roots in Δ_+ and let $\{\alpha_1^\vee, \ldots, \alpha_\ell^\vee\}$ be the corresponding co-roots. For any $\lambda \in \mathfrak{h}^*$, define $S_\lambda = \{1 \le i \le \ell : \lambda(\alpha_i^\vee) = 0\}$; and for any $\beta \in \Delta_+$, define $F_\beta = \{1 \le i \le \ell : \beta - \alpha_i \notin \Delta_+ \cup \{0\}\}$. With this notation, we have the following:

(1.1) THEOREM. *Let $V(\lambda)$ and $V(\mu)$ be two (finite dimensional) irreducible $\mathfrak{g}$-modules (with highest weights λ and μ resp.). Take any $\beta \in \Delta_+$ satisfying:*

$$(P_1) \qquad \lambda + \mu - \beta \text{ is a dominant weight, and}$$

$$(P_2) \qquad S_\lambda \cup S_\mu \subset F_\beta .$$

Then the $\mathfrak{g}$-module $V(\lambda + \mu - \beta)$ occurs as a component in the tensor product $\mathfrak{g}$-module $V(\lambda) \otimes V(\mu)$.

(To understand the condition P_2, it is instructive to keep the case $\mu = 0$ in mind.)

We next show that the validity of Theorem (2.5) is 'essentially' equivalent to the validity of the above Theorem (1.1) (see Proposition 2.6 for a more precise statement). It may be mentioned that Theorem (2.5) easily implies (1.1) (see [W_4; Proposition 3.9]). But we have reverted the roles.

Theorem (1.1) should be contrasted with the PRV (Parthasarathy-Ranga Rao-Varadarajan) conjecture (proved in [Ku_1]) which asserts:

THEOREM [Ku_1; Theorem 2.10]. *Let $\mathfrak{g}$, $V(\lambda)$, and $V(\mu)$ be as in Theorem (1.1). Then for any $w \in W$ the irreducible $\mathfrak{g}$-module $V(\overline{\lambda + w\mu})$ occurs in the tensor product $V(\lambda) \otimes V(\mu)$ with multiplicity $m_w(\lambda, \mu) \ge 1$; where W is the Weyl group associated to $(\mathfrak{g}, \mathfrak{h})$, and $\overline{\lambda + w\mu}$ is the unique dominant element in the W-orbit of $\lambda + w\mu$.*

In fact in [Ku_2; *Theorem* 1.2] (*following a conjecture due to D. N. Verma*), *we have shown that* $m_w(\lambda, \mu) \geq \#\{v \in W_\lambda \backslash W/W_\mu : \overline{\lambda + v\mu} = \overline{\lambda + w\mu}\}$, *where* W_λ *is the stabilizer of* λ *in* W.

It should be mentioned that our Theorem (1.1) detects certain components in the tensor product $V(\lambda) \otimes V(\mu)$ not covered by the above theorem (PRV conjecture) and conversely. (Assuming λ and μ to be regular the above theorem detects $|W|$ many components whereas Theorem 1.1 only gives $\leq |\Delta_+|$ components.)

Section (0) is devoted to establishing notation. Section (1) contains the statement of our Theorem (1.1) and its proof, whereas Section (2) contains the proof of Wahl's conjecture.

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0. Notation. We will adhere (often without explaining again) to the following notational conventions throughout the paper:

$\mathfrak{g}$ = a complex finite dimensional semisimple Lie algebra
$\mathfrak{b}$ = a fixed Borel subalgebra of $\mathfrak{g}$
$\mathfrak{n}$ = the nil-radical $[\mathfrak{b}, \mathfrak{b}]$ of $\mathfrak{b}$
$\mathfrak{h}$ = a fixed Cartan subalgebra $\subset \mathfrak{b}$
$U(\mathfrak{a})$ = the universal enveloping algebra of $\mathfrak{a}$ (for any Lie algebra $\mathfrak{a}$)
G = Simply-connected complex (semi-simple) group with Lie algebra $\mathfrak{g}$
B = the Borel subgroup of G with Lie algebra $\mathfrak{b}$
N = the unipotent radical $[B, B]$ of B
T = the (complex) maximal torus of G with Lie algebra $\mathfrak{h}$
Δ = the set of roots for the pair $(\mathfrak{g}, \mathfrak{h})$
$\mathfrak{g}_\alpha$ = the root space corresponding to the root α (for any $\alpha \in \Delta$)
Δ_+ = the set of roots for the pair $(\mathfrak{b}, \mathfrak{h})$
W = the Weyl group associated to $(\mathfrak{g}, \mathfrak{h})$
$\langle\ ,\ \rangle$ = a fixed killing form on $\mathfrak{h}^*$
$\{\alpha_1, \ldots, \alpha_\ell\}$ denotes the set of simple roots $\subset \Delta_+$

$\{\alpha_1^\vee, \ldots, \alpha_\ell^\vee\}$ denotes the corresponding (simple) co-roots: $\alpha_i^\vee := 2\alpha_i/\langle\alpha_i, \alpha_i\rangle$

$\{r_1, \ldots, r_\ell\}$ denotes the (simple) reflections $\in W$ corresponding to the simple roots $\{\alpha_1, \ldots, \alpha_\ell\}$ respectively

$\mathfrak{h}_{\mathbf{Z}}^*$ = the set of integral weights, i.e., $\{\lambda \in \mathfrak{h}^* : \langle\lambda, \alpha_i^\vee\rangle \in \mathbf{Z}$, for all $i\}$

D = the set of dominant integral weights, i.e., $\{\lambda \in \mathfrak{h}_{\mathbf{Z}}^* : \langle\lambda, \alpha_i^\vee\rangle \geq 0$, for all $i\}$

$D_S^0 = \{\lambda \in D : \langle\lambda, \alpha_i^\vee\rangle = 0$ if and only if $i \in S\}$, where S is any subset of $\{1, \ldots, \ell\}$

$D^0 = D_\phi^0$

For $\lambda \in \mathfrak{h}_{\mathbf{Z}}^*$, $\mathbf{C}_\lambda$ denotes the one dimensional B-module, such that the torus T acts by the character e^λ and (of course) the unipotent radical N acts trivially

$V(\lambda)$ = the (finite dimensional) irreducible $\mathfrak{g}$-module with highest weight $\lambda \in D$

ρ = the (unique) element in D, such that $\langle\rho, \alpha_i^\vee\rangle = 1$, for all the simple co-roots $\alpha_i^\vee$.

Unless otherwise stated, vector spaces and tensor products are over $\mathbf{C}$. For a vector space V, V^* denotes its dual $\mathrm{Hom}_{\mathbf{C}}(V, \mathbf{C})$. For any $\lambda \in \mathfrak{h}_{\mathbf{Z}}^*$, $\bar{\lambda}$ denotes the unique dominant element in the W-orbit of λ. For any $1 \leq i \leq \ell$, we fix a root vector $e_i \in \mathfrak{g}_{\alpha_i}$ and $f_i \in \mathfrak{g}_{-\alpha_i}$ such that $\langle e_i, f_i\rangle = 2/\langle\alpha_i, \alpha_i\rangle$, where $\langle\ ,\ \rangle$ denotes the killing form on $\mathfrak{g}$ induced from $\langle\ ,\ \rangle$ on $\mathfrak{h}^*$. In the sequel, we shall often denote $\langle\lambda, \alpha_i^\vee\rangle$ by $\lambda(\alpha_i^\vee)$.

1. Existence of certain components in the tensor product of two representations. For any $\lambda \in \mathfrak{h}^*$, define $S_\lambda = \{1 \leq i \leq \ell : \lambda(\alpha_i^\vee) = 0\}$; where $\{\alpha_i^\vee\}_{1\leq i\leq \ell}$ are the simple co-roots. Also for any $\beta \in \Delta_+$, define $F_\beta = \{1 \leq i \leq \ell : \beta - \alpha_i$ does not belong to $\Delta_+ \cup \{0\}\}$.

With this notation we have the following main result of this section:

(1.1) THEOREM. *Let $\mathfrak{g}$ be any complex semi-simple Lie algebra. Take any $\lambda, \mu \in D$ and $\beta \in \Delta_+$ satisfying*

$$(P_1) \quad \lambda + \mu - \beta \in D \quad \textit{and}$$

$$(P_2) \quad S_\lambda \cup S_\mu \subset F_\beta.$$

Then the irreducible $\mathfrak{g}$*-module* $V(\lambda + \mu - \beta)$ *occurs as a component in the tensor product* $\mathfrak{g}$*-module* $V(\lambda) \otimes V(\mu)$ *with multiplicity* (*denoted* $m_\beta(\lambda, \mu)$) ≥ 1.

Before we come to the proof of the above theorem, we record a particular case of this in the following:

(1.2) COROLLARY. *Let* $\mathfrak{g}$ *be any complex semi-simple Lie algebra such that the simple Lie algebra of type* G_2 *does not occur as a component of* $\mathfrak{g}$. *Then, for any* $\lambda, \mu \in D^0$ (*cf. Section* 0) *and any* $\beta \in \Delta_+$, $m_\beta(\lambda, \mu) \geq 1$.

(As seen below $\lambda + \mu - \beta \in D$ under the assumptions of the corollary.)

Proof. For any $\mathfrak{g}$ as in the corollary and any $\beta \in \Delta_+$, $\langle \beta, \alpha_i^\vee \rangle \leq 2$ for all the simple co-roots $\alpha_i^\vee$ (cf. [B; page 278]). This, in particular, gives that $\lambda + \mu - \beta \in D$. Now the corollary follows as a particular case of the above theorem. □

(1.3) *Remark.* The above corollary in the special case when β is a sum of simple roots with coefficients ≤ 1, is earlier obtained by Wahl [W_4; Section 5].

As a preparation for the proof of Theorem (1.1), we observe the following crucial

(1.4) PROPOSITION. *For any* $\lambda, \mu, \theta \in D$, *let* $m = m(\lambda, \mu; \theta)$ *denote the multiplicity of* $V(\theta)$ *in the tensor product* $V(\lambda) \otimes V(\mu)$. *Then* $m = \dim V$; *where* $V = V(\lambda, \mu; \theta) := \{v \in [V(\mu)]_{\theta-\lambda} : e_i^{\lambda(\alpha_i^\vee)+1} v = 0$, *for all* $1 \leq i \leq \ell\}$, $[V(\mu)]_{\theta-\lambda}$ *denotes the* $(\theta - \lambda)$*-th weight space of* $V(\mu)$, *and* e_i *is as in Section* 0.

Proof. Clearly $m = \dim[V(\lambda) \otimes V(\mu)]_\theta^{\mathfrak{n}}$, where $[V(\lambda) \otimes V(\mu)]_\theta^{\mathfrak{n}}$ denotes the space of $\mathfrak{n}$-invariants of weight θ in $V(\lambda) \otimes V(\mu)$. We can rewrite $[V(\lambda) \otimes V(\mu)]_\theta^{\mathfrak{n}} \approx [\mathrm{Hom}_{\mathfrak{n}}(V_\lambda^*, V(\mu))]_\theta$.

Now, by a result of Harish-Chandra, the map $\varphi : U(\mathfrak{n}) \to V(\lambda)^*$ given by $X \mapsto X v_\lambda^*$, for $X \in U(\mathfrak{n})$ (where $v_\lambda^* \neq 0 \in [V(\lambda)^*]_{-\lambda}$), is surjective and moreover

$$\ker \varphi = \sum_{1 \leq i \leq \ell} U(\mathfrak{n}) \cdot e_i^{\lambda(\alpha_i^\vee)+1}.$$

(This also follows from the BGG-resolution.)

In particular, the map $\psi : V \to [\mathrm{Hom}_{\mathfrak{n}}(V(\lambda)^*, V(\mu))]_\theta$, given by $\psi(v)(Xv_\lambda^*) = Xv$ (for $v \in V$ and $X \in U(\mathfrak{n})$), is well defined and is an isomorphism. □

The following result is well known.

(1.5) Corollary. *Let* $\lambda, \mu, \theta, \lambda', \mu' \in D$. *Then*

$$m(\lambda, \mu; \theta) \leq m(\lambda + \lambda', \mu + \mu'; \theta + \lambda' + \mu').$$

Proof. Define the map $\xi : V(\mu) \to V(\mu + \mu')$ by $\xi(v) = \pi(v \otimes v_{\mu'})$, for $v \in V(\mu)$; where $0 \neq v_{\mu'} \in [V(\mu')]_{\mu'}$ and π is the unique $\mathfrak{g}$-module projection: $V(\mu) \otimes V(\mu') \to V(\mu + \mu')$.

Since for any $0 \neq v \in V(\mu)$, $v_\mu \otimes v_{\mu'} \in U(\mathfrak{g})(v \otimes v_{\mu'})$, it is easy to see that the map ξ is injective. Further ξ takes the set $V(\lambda, \mu; \theta)$ into the set $\{w \in [V(\mu + \mu')]_{\theta+\mu'-\lambda} : e_i^{\lambda(\alpha_i^\vee)+1} w = 0$, for all $1 \leq i \leq \ell\}$. In particular, by the above proposition, the corollary follows. □

We also record the following lemma on the property of finite reduced root systems.

(1.6) Lemma. *Let* $\mathfrak{g}$ *be any simple Lie algebra. Then*

(a) *If* $\beta \in \Delta_+$ *is such that* $\beta - 2\alpha_j$ *is a root* (*for some simple root* α_j), *then none of the* $\beta - 2\alpha_i$ *and* $\beta - 2\alpha_i - \alpha_j$ *belong to* $\Delta \cup \{0\}$, *for any simple root* $\alpha_i \neq \alpha_j$.

We denote this (unique) α_j (whenever it exists) by $\alpha_j(\beta)$.

(b) *Assume* $\mathfrak{g}$ *to be simply laced. Then, for any* $\beta \in \Delta_+$ *and simple root* α_j, $\beta - 2\alpha_j$ *is a root if and only if* $\beta = \alpha_j$.

Proof. The (b)-part follows from the fact that for any $\beta \in \Delta_+$ and any simple co-root $\alpha_i^\vee$ such that $\beta \neq \alpha_i$, $-1 \leq \langle \beta, \alpha_i^\vee \rangle \leq 1$ (see [B; Page 278]).

To prove the (a)-part; (in view of the b-part and the classification of simple Lie algebras) we can assume that the type of $\mathfrak{g}$ is one of (following the notation of [B]) $B_\ell (\ell \geq 2)$, $C_\ell (\ell \geq 2)$, F_4, or G_2. Now, from the explicit knowledge of Δ_+ (see [B]), one easily finds the validity of (a).

Alternatively (as B. Kostant has suggested), by looking at the span of β, α_i, and α_j (for any $i \neq j$), we can reduce the problem to the simple Lie algebras of rank ≤ 3; where we can explicitly check. □

(1.7) *Proof of Theorem* (1.1). By Proposition (1.4), to prove the theorem, it suffices to show that $V = V(\lambda, \mu; \lambda + \mu - \beta) := \{v \in [V(\mu)]_{\mu-\beta} : e_i^{\lambda(\alpha_i^\vee)+1}v = 0$, for all $1 \leq i \leq \ell\} \neq 0$. Next observe that (to prove the theorem), it suffices to assume that $\mathfrak{g}$ in fact is simple.

We need to deal with the following three cases separately:

Case (a). $\beta - 2\alpha_i$ *is not a positive root for any simple root* α_i *or else* $\langle \mu, \beta^\vee \rangle = 1$:

(This case, in particular, covers all the roots of simply laced Lie algebras $-$ A_ℓ, D_ℓ, E_6, E_7, and E_8.)

Consider the element $v := X_{-\beta}v_\mu \in [V(\mu)]_{\mu-\beta}$, where $0 \neq X_{-\beta} \in \mathfrak{g}_{-\beta}$ and $0 \neq v_\mu \in [V(\mu)]\mu$. We claim that

(1) $X_\beta v \neq 0$, for $0 \neq X_\beta \in \mathfrak{g}_\beta$ (in particular $v \neq 0$), and (2) $v \in V$:

$X_\beta X_{-\beta}v_\mu = [X_\beta, X_{-\beta}]v_\mu$, since $X_\beta v_\mu = 0$

$= \langle X_\beta, X_{-\beta}\rangle h_\beta v_\mu$, where $\langle\ ,\ \rangle$ is the Killing form on $\mathfrak{g}$ and h_β is the unique element in $\mathfrak{h}$ satisfying $\chi(h_\beta) = \langle \chi, \beta\rangle$ for all $\chi \in \mathfrak{h}^*$

$= \langle X_\beta, X_{-\beta}\rangle \mu(h_\beta)v_\mu.$

Writing $\beta = \Sigma_{1\leq k\leq \ell}\, n_k\alpha_k$, we get that $n_{k_0} > 0$ for some $k_0 \notin F_\beta$. In particular, by the condition P_2, $\mu(\alpha_{k_0}^\vee) > 0$ and hence $\mu(h_\beta) \neq 0$. This proves (1). We now prove (2):

For $i \in F_\beta$, $e_iv = [e_i, X_{-\beta}]v_\mu = 0$ (by the definition of F_β).

For $i \notin F_\beta$ (setting $n = \lambda(\alpha_i^\vee) + 1$), $e_i^n v = ((ad\ e_i)^n X_{-\beta})v_\mu = 0$, for the following reason:

In the case when $\beta - 2\alpha_i$ is not a positive root, $((ad\ e_i)^2X_{-\beta})v_\mu$ is clearly 0. Moreover, since $i \notin F_\beta$, by the assumption P_2, $\lambda(\alpha_i^\vee) \geq 1$.

In the other case, i.e., $\langle \mu, \beta^\vee\rangle = 1$, we easily see that $[V(\mu)]_{\mu-\beta+n\alpha_i} = 0$, since $\|\mu - \beta + n\alpha_i\| > \|\mu\|$:

$$\begin{aligned}\|\mu - \beta + n\alpha_i\|^2 &= \|\mu\|^2 + n^2\|\alpha_i\|^2 + 2n\langle \mu - \beta, \alpha_i\rangle \\ &= \|\mu\|^2 + n^2\|\alpha_i\|^2 + n\|\alpha_i\|^2\langle \mu - \beta, \alpha_i^\vee\rangle \\ &= \|\mu\|^2 + n\|\alpha_i\|^2(\langle \lambda + \mu - \beta, \alpha_i^\vee\rangle + 1) \\ &> \|\mu\|^2;\ \text{since, by } P_1,\ \lambda + \mu - \beta \in D.\end{aligned}$$

So the theorem is established in this case; in particular, for any simply laced $\mathfrak{g}$.

Case (b). $\beta - 2\alpha_j$ *is a positive root for some (and hence, by Lemma* 1.6, *unique); simple root* $\alpha_j = \alpha_j(\beta)$, *and* $\mathfrak{g} \neq G_2$:

Pick a non-zero $X_{-\beta} \in \mathfrak{g}_{-\beta}$ and set $X_{-\beta+\alpha_j} := [e_j, X_{-\beta}]$, and $X_{-\beta+2\alpha_j} := (ad\ e_j)^2 X_{-\beta}$. By our assumption on β, $X_{-\beta+\alpha_j}$ and $X_{-\beta+2\alpha_j}$ are both non-zero. (Observe that none of $\beta + \alpha_j$ and $\beta - 3\alpha_j$ can be a root since, for any $\mathfrak{g} \neq G_2$ and any $\alpha \in \Delta$, $-2 \leq \langle \alpha, \alpha_j^\vee \rangle \leq 2$.) Now take $v = (X_{-\beta+\alpha_j} f_j - 2\langle \mu, \alpha_j^\vee \rangle X_{-\beta}) v_\mu$. Again we claim that

$$(*)\cdots \quad \begin{aligned} &(1)\ X_\beta v \neq 0, \text{ for } 0 \neq X_\beta \in \mathfrak{g}_\beta \\ &(2)\ e_i^2 v = 0, \text{ for all } 1 \leq i \leq \ell, \text{ and} \\ &(3)\ e_i v = 0, \text{ for all } i \in F_\beta \cap F_{\beta-\alpha_j}: \end{aligned}$$

We have

$$\begin{aligned} X_\beta v &= [[X_\beta, X_{-\beta+\alpha_j}], f_j] v_\mu - 2\langle \mu, \alpha_j^\vee \rangle [X_\beta, X_{-\beta}] v_\mu \\ &= [[e_j, [X_\beta, X_{-\beta}]], f_j] v_\mu - 2\langle \mu, \alpha_j^\vee \rangle [X_\beta, X_{-\beta}] v_\mu, \text{ since } [e_j, X_\beta] = 0 \\ &= -\alpha_j([X_\beta, X_{-\beta}]) \mu(\alpha_j^\vee) v_\mu - 2\langle \mu, \alpha_j^\vee \rangle [X_\beta, X_{-\beta}] v_\mu \\ &= -\langle \mu, \alpha_j^\vee \rangle \langle X_\beta, X_{-\beta} \rangle \langle 2\mu + \alpha_j, \beta \rangle v_\mu \\ &\neq 0, \text{ since } \langle \beta, \alpha_j^\vee \rangle = 2 \text{ and } j \notin S_\mu \text{ (by } P_2). \end{aligned}$$

This proves (1). We next show (2):

First take $i \neq j$. Then $e_i^2 X_{-\beta} v_\mu = ((ad\ e_i)^2 X_{-\beta}) v_\mu = 0$, by Lemma (1.6). Also $e_i^2 X_{-\beta+\alpha_j} f_j V_\mu = ((ad\ e_i)^2 X_{-\beta+\alpha_j}) f_j v_\mu = 0$, again by Lemma (1.6). Further

$$\begin{aligned} e_j^2 v &= e_j (X_{-\beta+2\alpha_j} \cdot f_j + \mu(\alpha_j^\vee) X_{-\beta+\alpha_j}) v_\mu - 2\langle \mu, \alpha_j^\vee \rangle X_{-\beta+2\alpha_j} v_\mu \\ &= 2\mu(\alpha_j^\vee) X_{-\beta+2\alpha_j} v_\mu - 2\langle \mu, \alpha_j^\vee \rangle X_{-\beta+2\alpha_j} v_\mu \\ &= 0. \end{aligned}$$

This establishes (2). The assertion (3) is clearly true.

By (*), we get that $v \in V$ in all the cases covered by (b) provided $S_\lambda \subset F_{\beta-\alpha_j}$ (since $S_\lambda \subset F_\beta$ by the assumption P_2), and hence the theorem is established in the case (b) as well provided the following condition is satisfied:

$$(\mathscr{C})\cdots \qquad S_\lambda \subset F_{\beta-\alpha_j}.$$

Two cases are noteworthy when the condition $\mathscr{C}$ is automatically satisfied:

($\mathscr{C}_1$) When $\lambda \in D^0$, so that $S_\lambda = \phi$, and

($\mathscr{C}_2$) Any λ, μ as in Theorem (1.1) but those β (satisfying the condition as in case b) which have $F_\beta = F_{\beta-\alpha_{j(\beta)}}$.

We next observe that for a given $\beta \in \Delta_+$ in any $\mathfrak{g} \neq G_2$, to prove the theorem for arbitrary λ, μ (satisfying the conditions of the theorem), it suffices to assume (in view of Corollary (1.5)) that $\lambda = \mu = \rho_\beta$, where

$$\rho_\beta := \sum_{i \notin F_\beta} \chi_i, \text{ and } \chi_i (1 \leq i \leq \ell)$$

is the i-th fundamental weight: $\chi_i(\alpha_j^\vee) = \delta_{i,j}$.

(Notice that $2\rho_\beta - \beta \in D$, for any $\beta \in \Delta_+$ in any $\mathfrak{g} \neq G_2$.)

Now, by combining the cases (a) and (b) (satisfying the condition $\mathscr{C}_2$), we readily see (from the following chart) that the theorem is established for any $\mathfrak{g}$ and any β (for arbitrary choices of $\lambda, \mu \in D$ satisfying the conditions of the theorem) except the three roots: $\beta = 2\alpha_1 + \alpha_2$ of G_2; $\beta = \alpha_1 + \alpha_2 + 2\alpha_3 + 2\alpha_4$, and $\beta = \alpha_1 + 2\alpha_2 + 4\alpha_3 + 2\alpha_4$ of F_4. So the theorem will be completely proved, if we check its validity for the above three roots and $\lambda = \mu = \rho_\beta$; which we isolate as:

Case c. (1) $\mathfrak{g} = G_2$; $\beta = 2\alpha_1 + \alpha_2$, $\lambda = \mu = \rho_\beta = \chi_1$

(2) $\mathfrak{g} = F_4$; $\beta = \alpha_1 + \alpha_2 + 2\alpha_3 + 2\alpha_4$, $\lambda = \mu = \rho_\beta = \chi_1 + \chi_4$, *and*

(3) $\mathfrak{g} = F_4$; $\beta = \alpha_1 + 2\alpha_2 + 4\alpha_3 + 2\alpha_4$, $\lambda = \mu = \rho_\beta = \chi_3$:

In this case as well, the validity of Theorem (1.1) can be established by an explicit calculation using Proposition (1.4):

In the first subcase (c-(1)) take for $v = (cX_{-\beta} + dX_{-\beta+\alpha_1}f_1)v_{\chi_1}$, in the second $v = (cX_{-\beta} + dX_{-\beta+\alpha_1}f_1)v_{\chi_1+\chi_4}$, and for the third take $v = (cX_{-\beta} + dX_{-\beta+\alpha_3}f_3 + eX_{-\beta+\alpha_3+\alpha_4}X_{-(\alpha_3+\alpha_4)})v_{\chi_3}$. For some appropriate

CHART

The following chart shows the list of all the possible positive non-simple roots β occurring in any simple Lie algebra over $\mathbf{C}$ such that $\beta - 2\alpha_j(\beta)$ is a root, for some (and hence unique) simple root $\alpha_j(\beta)$. Simply laced Lie algebras—$A_\ell(\ell \geq 1)$, $D_\ell(\ell \geq 3)$, E_6, E_7, E_8—have no such roots β. We have not listed the value of ρ_β and $\langle \rho_\beta, \beta^\vee \rangle$ in those cases when $F_\beta = F_{\beta-\alpha_j(\beta)}$ (i.e. when the condition $\mathscr{C}_2$ is satisfied). In the following, the symbol $[n]$ denotes the set $\{1, \ldots, n\}$.

$\mathfrak{g}$	β	$\alpha_j(\beta)$	F_β	$F_{\beta-\alpha_j(\beta)}$	ρ_β	$\langle \rho_\beta, \beta^\vee \rangle$
$B_\ell(\ell \geq 2)$	$(\sum_{i \leq k < \ell} \alpha_k) + 2\alpha_\ell$ (for any $1 \leq i < \ell$)	α_ℓ	$[\ell - 1]\backslash\{i\}$, if $i < \ell - 1$	F_β	—	—
			$[\ell - 1]$, if $i = \ell - 1$	$[\ell - 2]$	χ_ℓ	1
$C_\ell(\ell \geq 2)$	$(2 \sum_{i \leq k < \ell} \alpha_k) + \alpha_\ell$ (for any $1 \leq i < \ell$)	α_i	$[\ell]\backslash\{i\}$	$[\ell]\backslash\{i, i+1\}$	χ_i	1
F_4	$\alpha_2 + 2\alpha_3$	α_3	$\{1, 2, 4\}$	$\{1, 4\}$	χ_3	1
	$\alpha_1 + \alpha_2 + 2\alpha_3$	α_3	$\{2, 4\}$	F_β	—	—
	$\alpha_2 + 2\alpha_3 + 2\alpha_4$	α_4	$\{1, 2, 3\}$	$\{1, 2\}$	χ_4	1
	$\alpha_1 + \alpha_2 + 2\alpha_3 + 2\alpha_4$	α_4	$\{2, 3\}$	$\{2\}$	$\chi_1 + \chi_4$	2
	$\alpha_1 + 2\alpha_2 + 2\alpha_3 + 2\alpha_4$	α_4	$\{1, 3\}$	F_β	—	—
	$\alpha_1 + 2\alpha_2 + 4\alpha_3 + 2\alpha_4$	α_3	$\{1, 2, 4\}$	$\{1, 2\}$	χ_3	2
G_2	$2\alpha_1 + \alpha_2$	α_1	$\{2\}$	ϕ	χ_1	2
	$3\alpha_1 + \alpha_2$	α_1	$\{2\}$	F_β	χ_1	1

choices of c, d, $e \in \mathbf{C}$, v belongs to V in all the cases and $X_\beta v \neq 0$. This completes the proof of the theorem. □

2. Proof of Wahl's conjecture. Following the notation of Section 0, we fix (once and for all in this section) a proper subset $S \subset \{1, \ldots, \ell\}$ (including $S = \phi$), and let $B \subset P = P_S$ be the corresponding parabolic subgroup of G, with Lie algebra $\mathfrak{p} = \mathfrak{p}_S$. We denote the unipotent radical of P by $U = U_S$ and its Lie algebra by $\mathfrak{u} = \mathfrak{u}_S$. Then

$$\mathfrak{u} = \bigoplus_{\alpha \in \Delta_+^S} \mathfrak{g}_\alpha \text{ and } \mathfrak{p} = \mathfrak{b} \oplus (\oplus_{\alpha \in \Delta_+(S)} \mathfrak{g}_{-\alpha}),$$

where $\Delta_+^S := \Delta_+ \backslash \Delta_+(S)$, and $\Delta_+(S) := (\Sigma_{i \in S} \mathbf{Z}\alpha_i) \cap \Delta_+$.

We define $\mathfrak{u}^-$ as the Lie algebra $\oplus_{\alpha \in \Delta_+^S} \mathfrak{g}_{-\alpha}$, and let U^- be the corresponding (unipotent) subgroup of G.

(2.1) *Definition.* For any algebraic P-module M, by $\mathcal{L}(M)$ we mean the vector bundle (locally free sheaf) on G/P associated to the principal P-bundle: $G \to G/P$ by the representation M of P.

For any $\lambda \in \mathfrak{h}_{\mathbf{Z}}^*$, such that $\lambda(\alpha_i^\vee) = 0$ for all $i \in S$, the one dimensional T-module $\mathbf{C}_\lambda$ (given by the character e^λ) admits a unique P-module structure (extending the T-module structure); in particular, we have the line bundle $\mathcal{L}(\mathbf{C}_\lambda)$ on G/P. We abbreviate the line bundle $\mathcal{L}(\mathbf{C}_{-\lambda})$ by $\mathcal{L}(\lambda)$. Given two line bundles $\mathcal{L}(\lambda)$ and $\mathcal{L}(\mu)$ on G/P, we can form their external tensor product to get the line bundle $\mathcal{L}(\lambda \boxtimes \mu)$ on $G/P \times G/P$.

With this definition, we have the following:

(2.2) Proposition. *For any* $\mu \in D_S^0$ (*cf. Section* 0):

(a) $H^p(G/P, \mathcal{I}_e \otimes \mathcal{L}(\mu)) = 0$, *for all* $p > 0$

(b) $H^p(G/P, \mathcal{I}_e^2 \otimes \mathcal{L}(\mu)) = 0$, *for all* $p > 0$, *and*

(c) $H^0(G/P, \mathcal{I}_e^2 \otimes \mathcal{L}(\mu))$ (*resp.* $H^0(G/P, \mathcal{O}/\mathcal{I}_e^2 \otimes \mathcal{L}(\mu))$ *is canonically isomorphic* (*as P-modules*) *with* $(V(\mu)/\mathfrak{g} \cdot v_\mu)^*$ (*resp.* $(\mathfrak{g} \cdot v_\mu)^*$), *where* $\mathcal{I}_e$ *denotes the ideal sheaf of the base point* $e = P/P \subset G/P$, $\mathcal{O}$ *denotes the structure sheaf of* G/P, *and* $v_\mu \neq 0 \in [V(\mu)]_\mu$.

(Observe that, since $\mu(\alpha_i^\vee) = 0$ for $i \in S$, $\mathfrak{g} \cdot v_\mu \subset V(\mu)$ is stable under P, and also the point e being P-fixed $H^0(G/P, \mathcal{I}_e^2 \otimes \mathcal{L}(\mu))$ and $H^0(G/P, \mathcal{O}/\mathcal{I}_e^2 \otimes \mathcal{L}(\mu))$ have canonical P-module structures.)

Proof. (a) Consider the sheaf exact sequence:

$$(\mathcal{S}_1)\cdots \qquad 0 \to \mathcal{I}_e \otimes \mathcal{L}(\mu) \to \mathcal{L}(\mu) \to \mathcal{O}/\mathcal{I}_e \otimes \mathcal{L}(\mu) \to 0.$$

Clearly $H^p(G/P, \mathcal{O}/\mathcal{I}_e \otimes \mathcal{L}(\mu)) = H^p(e, \mathcal{L}(\mu)_{|e}) = 0$, for all $p > 0$. Also $H^p(G/P, \mathcal{L}(\mu)) = 0$, for $p > 0$, by [RR; Theorem 1]; and the restriction map: $H^0(G/P, \mathcal{L}(\mu)) \to H^0(e, \mathcal{L}(\mu)_{|e})$ is surjective (cf. [RR; Theorem 2]). So the long exact cohomology sequence, associated to the sheaf sequence $(\mathcal{S}_1)$, gives (a).

(b) Similarly, consider the sheaf exact sequence:

$$(\mathcal{S}_2)\cdots \qquad 0 \to \mathcal{I}_e^2 \otimes \mathcal{L}(\mu) \to \mathcal{L}(\mu) \to (\mathcal{O}/\mathcal{I}_e^2) \otimes \mathcal{L}(\mu) \to 0.$$

We write a part of the corresponding cohomology exact sequence, and denote it by

$$(\mathcal{S}_3)\cdots \qquad 0 \to H^0(G/P, \mathcal{I}_e^2 \otimes \mathcal{L}(\mu)) \to H^0(G/P, \mathcal{L}(\mu))$$

$$\xrightarrow{\pi} H^0(G/P, (\mathcal{O}/\mathcal{I}_e^2) \otimes \mathcal{L}(\mu)) \to H^1(G/P, \mathcal{I}_e^2 \otimes \mathcal{L}(\mu)) \to 0.$$

The map: $\mathfrak{u}^- \to G/P$, given by $X \mapsto \exp X(\text{mod } P)$, is an open embedding (onto the 'big' open cell through e) and moreover the line bundle $\mathcal{L}(\mu)_{|\text{Image } \mathfrak{u}^-}$ admits a nowhere-vanishing (tautological) section of weight $-\mu$. This allows us to identify $H^0(G/P, \mathcal{O}/\mathcal{I}_e^2 \otimes \mathcal{L}(\mu))$ canonically with $\mathbf{C}_{-\mu} \otimes (\mathbf{C}[t_\beta]_{\beta\in\Delta_+^S}/\langle t_\alpha t_\beta\rangle_{\alpha,\beta\in\Delta_+^S})$, where $\{t_\beta\}_{\beta\in\Delta_+^S}$ are the coordinate functions on $\mathfrak{u}^-$ under a fixed basis $\{X_{-\beta}\}_{\beta\in\Delta_+^S}$ (consisting of root vectors of weight $-\beta$) of $\mathfrak{u}^-$ and $\langle t_\alpha t_\beta\rangle$ denotes the ideal generated by $t_\alpha t_\beta$. Further the Borel-Weil isomorphism canonically identifies $H^0(G/P, \mathcal{L}(\mu))$ with $V(\mu)^*$. So, transporting the map π under these identifications, we get the map

$$\hat{\pi} : V(\mu)^* \to \text{Hom}_{\mathbf{C}}(\mathbf{C}_\mu, \mathbf{C}[t_\beta]_{\beta\in\Delta_+^S}/\langle t_\alpha t_\beta\rangle_{\alpha,\beta\in\Delta_+^S}).$$

It is easy to see that the map $\hat{\pi}$ is given by

$$(\hat{\pi}(\theta)v_\mu)\left(\sum_{\beta\in\Delta_+^S} t_\beta X_{-\beta}\right) = \theta(\exp(\Sigma\, t_\beta X_{-\beta})v_\mu) \bmod \langle t_\alpha t_\beta\rangle,$$

$$\text{for } \theta \in V(\mu)^* \text{ and } v_\mu \in \mathbf{C}_\mu = [V(\mu)]_\mu$$

$$= \theta(v_\mu) + \sum_{\beta\in\Delta_+^S} t_\beta \theta(X_{-\beta}v_\mu).$$

Now we assert that the map $\hat{\pi}$ (and hence π) is surjective: To prove this, it suffices to show (in view of the above expression of $\hat{\pi}$) that the vectors $\{v_\mu, X_{-\beta}v_\mu\}_{\beta\in\Delta_+^S}$ are linearly independent. But since they all have different weights, it suffices to observe that they are all non-zero (which follows since $\mu \in D_S^0$).

So the exact sequence $(\mathcal{S}_3)$ (and the surjectivity of π) establishes the vanishing of $H^1(G/P, \mathcal{I}_e^2 \otimes \mathcal{L}(\mu))$. Vanishing of $H^p(G/P, \mathcal{I}_e^2 \otimes \mathcal{L}(\mu))$, for $p > 1$, follows by considering the cohomology exact sequence associated to the sheaf exact sequence:

$$(\mathcal{S}_4)\cdots \qquad 0 \to \mathcal{I}_e^2 \otimes \mathcal{L}(\mu) \to \mathcal{I}_e \otimes \mathcal{L}(\mu) \to \mathcal{I}_e/\mathcal{I}_e^2 \otimes \mathcal{L}(\mu) \to 0,$$

and the (a)-part of the proposition. So (b) follows.

(c) It is easy to see, from the description of the map $\hat{\pi}$, that

$$\operatorname{Ker} \hat{\pi} = \{\theta \in V(\mu)^* : \theta_{|\mathfrak{g}\cdot v_\mu} = 0\} \approx (V(\mu)/\mathfrak{g}\cdot v_\mu)^*.$$

So, by the exact sequence $(\mathcal{S}_3)$ again, (c) follows. □

Remark. The analogous proof as above also gives that $H^p(G/P, \mathcal{O}/\mathcal{I}_e^k \otimes \mathcal{L}(\mu)) = 0$, for all $p > 0$ and $k = 1, 2$.

(2.3) LEMMA. *For* $\lambda, \mu \in D_S^0$, $k = 1, 2$ *and any* $p \geq 0$, *there are canonical isomorphisms*:

$$H^p(G/P \times G/P, \mathcal{I}_D^k \otimes \mathcal{L}(\lambda \boxtimes \mu))$$
$$\approx H^p(G/P, \mathcal{L}(\lambda) \otimes \mathcal{L}(H^0(G/P, \mathcal{I}_e^k \otimes \mathcal{L}(\mu)))),$$

and

$$H^p(G/P \times G/P, \mathcal{O}/\mathcal{I}_D^k \boxtimes \mathcal{L}(\lambda \otimes \mu))$$
$$\approx H^p(G/P, \mathcal{L}(\lambda) \otimes \mathcal{L}(H^0(G/P, \mathcal{O}/\mathcal{I}_e^k \otimes \mathcal{L}(\mu)))).$$

Further $H^p(G/P \times G/P, \mathcal{I}_D \otimes \mathcal{L}(\lambda \boxtimes \mu)) = 0$, *for all* $p > 0$; *where* $\mathcal{I}_D$ *is the ideal sheaf of the diagonal* $D \subset G/P \times G/P$.

(We will see in Theorem 2.5 that $H^p(G/P \times G/P, \mathcal{I}_D^2 \otimes \mathcal{L}(\lambda \boxtimes \mu)) = 0$, for all $p > 0$.)

Proof. Consider the fibration $\widehat{G/P} := G \times_P G/P \xrightarrow{\sigma} G/P$ (with fiber G/P), where P acts on G/P by left multiplication. Clearly the group

G acts on $\widehat{G/P}$ by multiplication on the first factor. Define a biregular G-equivariant isomorphism $d : \widehat{G/P} \to G/P \times G/P$, by $d(g, g'P) = (gP, gg'P)$, for $g, g' \in G$; where G acts diagonally on $G/P \times G/P$ (cf. [Ku$_1$; Section 1.1]). Clearly $d^{-1}(D) = G \times_P \{e\} \subset G \times_P G/P$. From this it is easy to see that $R^q\sigma_*(d^*(\mathcal{I}_D^k \otimes \mathcal{L}(\lambda \boxtimes \mu)))$ is canonically isomorphic with $\mathcal{L}(\lambda) \otimes \mathcal{L}(H^q(G/P, \mathcal{I}_e^k \otimes \mathcal{L}(\mu)))$, for any $q \geq 0$. (A similar statement is true for $\mathcal{I}_D^k$ replaced by $\mathcal{O}/\mathcal{I}_D^k$.) In particular, by Proposition (2.2) and the (degenerate) Leray spectral sequence, first part of the lemma follows.

To prove the vanishing of $H^p(G/P \times G/P, \mathcal{I}_D \otimes \mathcal{L}(\lambda \boxtimes \mu))$; use the cohomology exact sequence, associated to the sheaf exact sequence:

$$(\mathcal{S}_5)\cdots \qquad 0 \to \mathcal{I}_D \otimes \mathcal{L}(\lambda \boxtimes \mu) \to \mathcal{L}(\lambda \boxtimes \mu)$$

$$\to (\mathcal{O}/\mathcal{I}_D) \otimes \mathcal{L}(\lambda \boxtimes \mu) \to 0,$$

and [Ku$_1$; Theorem 1.5]. □

The following lemma is used in proving our crucial proposition (2.6).

(2.4) LEMMA. *Take* $\lambda, \mu \in D_S^0$ *and* $\theta \in D$. *Then* $\mathrm{Hom}_P(V(\theta)^*, \mathbf{C}_{-\lambda} \otimes (\mathfrak{g} \cdot v_\mu)^*)$ *is one dimensional if and only if* $\theta = \lambda + \mu - \beta$, *for some* $\beta \in \Delta_+ \cup \{0\}$ *such that* (*if* $\beta \neq 0$) $S \subset F_\beta$ (*cf. Section* 1.1). *Otherwise it is zero.*

Proof. It is easy to see that, as a P-module,

$$\mathfrak{g} \cdot v_\mu \approx (\mathfrak{g}/\mathrm{Ann}\, v_\mu) \otimes \mathbf{C}_\mu,$$

where $\mathrm{Ann}\, v_\mu$ is the annihilator of v_μ in $\mathfrak{g}$ and the right side is given the tensor product P-module structure. Hence

$$(\mathfrak{g} \cdot v_\mu)^* \approx (\mathfrak{g}/(\mathfrak{n} \oplus \mathrm{Ker}\, \mu \oplus (\oplus_{\alpha \in \Delta_+(S)} \mathfrak{g}_{-\alpha})))^* \otimes \mathbf{C}_{-\mu},$$

where $\mathrm{Ker}\, \mu := \{h \in \mathfrak{h} : \mu(h) = 0\}$. Using the Killing form on $\mathfrak{g}$, we get:

$$(I_1)\cdots \qquad (\mathfrak{g} \cdot v_\mu)^* \approx (\mathfrak{u} \oplus (\mathrm{Ker}\, \mu)^\perp) \otimes \mathbf{C}_{-\mu},$$

where $(\text{Ker }\mu)^\perp := \{h \in \mathfrak{h} : \langle h, \text{Ker }\mu\rangle = 0\}$ and $\mathfrak{u} \oplus (\text{Ker }\mu)^\perp$ is a P-module under the adjoint representation. By the result of Harish-Chandra (cf. proof of Proposition 1.4), we get:

$$\text{Hom}_B(V(\theta)^*, \mathbf{C}_{-\lambda} \otimes (\mathfrak{g}\cdot v_\mu)^*)$$

$$\approx \text{Hom}_B(V(\theta)^*, \mathbf{C}_{-(\lambda+\mu)} \otimes (\mathfrak{u} \oplus (\text{Ker }\mu)^\perp)), \text{ by } (I_1)$$

$$\approx \{v \in [\mathfrak{u} \oplus (\text{Ker }\mu)^\perp]_{\lambda+\mu-\theta} : (\text{ad } e_i)^{\theta(\alpha_i^\vee)+1} v = 0, \text{ for all } i\}.$$

It is easy to see, from the above description, that

$$(I_2)\cdots \quad \text{Hom}_P(V(\theta)^*, \mathbf{C}_{-\lambda} \otimes (\mathfrak{g}\cdot v_\mu)^*)$$

$$\approx \{v \in [\mathfrak{u} \oplus (\text{Ker }\mu)^\perp]_{\lambda+\mu-\theta} : (\text{ad } e_i)^{\theta(\alpha_i^\vee)+1} v = 0,$$

$$\text{for } 1 \leq i \leq \ell \text{ and } (\text{ad } f_k)v = 0, \text{ for all } k \in S\}.$$

Clearly the right side of I_2 is of dim ≤ 1 and if it is non-zero, the following conditions (1) and (2) are both satisfied:

(1) $\lambda + \mu - \theta = \beta$, for some $\beta \in \Delta_+^S \cup \{0\}$, and
(2) If $\beta \neq 0$, $F_\beta \supset S$.

Conversely, for any dominant θ which satisfies the conditions (1) and (2) as above, the right side of I_2 (and hence $\text{Hom}_P(V(\theta)^*, \mathbf{C}_{-\lambda} \otimes (\mathfrak{g}\cdot v_\mu)^*)$) is one dimensional:

If $\beta = 0$, $(\text{ad } e_i)^{\theta(\alpha_i^\vee)+1}((\text{Ker }\mu)^\perp) = 0$ for all $i \notin S$ (since $\lambda, \mu \in D_S^0$), and $\text{ad } e_k((\text{Ker }\mu)^\perp) = \text{ad } f_k((\text{Ker }\mu)^\perp) = 0$ for all $k \in S$ (since $\mu(\alpha_k^\vee) = 0$).

So assume now that $\beta \in \Delta_+^S$ and pick $0 \neq X_\beta \in \mathfrak{g}_\beta$: By (2), $(\text{ad } f_k)X_\beta = 0$, for $k \in S$. So, by the $s\ell(2)$-theory, $\beta(\alpha_k^\vee) \leq 0$ and $(\text{ad } e_k)^{1-\beta(\alpha_k^\vee)}X_\beta = 0$. For $i \notin S$, we again claim that $(\text{ad } e_i)^{\theta(\alpha_i^\vee)+1}X_\beta = 0$, i.e., $\beta + (\theta(\alpha_i^\vee) + 1)\alpha_i \notin \Delta_+$: We have $[\beta + (\theta(\alpha_i^\vee) + 1)\alpha_i](\alpha_i^\vee) = (\lambda + \mu)(\alpha_i^\vee) + (\lambda + \mu - \beta)(\alpha_i^\vee) + 2 \geq 4$, since $\lambda + \mu - \beta = \theta \in D$ and $\lambda, \mu \in D_S^0$. But this contradicts [B; page 278, Fact 6].

Next observe that any $\beta \in \Delta_+$, with $S \subset F_\beta$, automatically lies in Δ_+^S. This completes the proof of the lemma. □

Following is the main theorem of this section:

(2.5) THEOREM. *For any* $\lambda, \mu \in D_S^0$ (*where* $S \subset \{1, \ldots, \ell\}$ *is arbitrary*)

$$H^p(G/P \times G/P, \mathcal{I}_D^2 \otimes \mathcal{L}(\lambda \boxtimes \mu)) = 0, \text{ for all } p > 0;$$

where (*as in Lemma* 2.3) $\mathcal{I}_D$ *is the ideal sheaf of the diagonal* $D \subset G/P \times G/P$.

Before we come to the proof of the above theorem, we prove the following crucial

(2.6) PROPOSITION. *With the notation and assumptions as in Theorem* (2.5), *the following two are equivalent*:

(a) $H^1(G/P \times G/P, \mathcal{I}_D^2 \otimes \mathcal{L}(\lambda \boxtimes \mu)) = 0$.

(b) *For all* $\beta \in \Delta_+$ *satisfying*

(1) $F_\beta \supset S$, *and*

(2) $\lambda + \mu - \beta \in D$,

there exists a $f_\beta \in \mathrm{Hom}_B(\mathbf{C}_{\lambda+\mu-\beta} \otimes V(\lambda)^*, V(\mu))$ *such that* $X_\beta(f_\beta(\mathbf{C}_{\lambda+\mu-\beta} \otimes v_\lambda^*)) \neq 0$, *for* $X_\beta \neq 0 \in \mathfrak{g}_\beta$; *where* $v_\lambda^* \neq 0 \in [V(\lambda)^*]_{-\lambda}$.

Proof. From the cohomology exact sequence, associated to the sheaf exact sequence:

$$(\mathcal{S}_6)\cdots \quad 0 \to \mathcal{I}_D^2 \otimes \mathcal{L}(\lambda \boxtimes \mu) \to \mathcal{L}(\lambda \boxtimes \mu) \to (\mathcal{O}/\mathcal{I}_D^2) \otimes \mathcal{L}(\lambda \boxtimes \mu) \to 0,$$

and the vanishing of $H^1(G/P \times G/P, \mathcal{L}(\lambda \boxtimes \mu))$ (cf. [Ku$_1$; Theorem 1.5]), we get that (a) is equivalent to the surjectivity of the canonical map $\tau : H^0(G/P \times G/P, \mathcal{L}(\lambda \boxtimes \mu)) \to H^0(G/P \times G/P, (\mathcal{O}/\mathcal{I}_D^2) \otimes \mathcal{L}(\lambda \boxtimes \mu))$.

By Lemma (2.3) and Proposition (2.2) we get:

$$H^0(G/P \times G/P, (\mathcal{O}/\mathcal{I}_D^2) \otimes \mathcal{L}(\lambda \boxtimes \mu)) \approx H^0(G/P, \mathcal{L}(\lambda) \otimes \mathcal{L}((\mathfrak{g} \cdot v_\mu)^*)).$$

Further, by [Ku$_1$; Proof of Theorem 2.2] and Borel-Weil theorem,

$$H^0(G/P \times G/P, \mathscr{L}(\lambda \boxtimes \mu)) \approx H^0(G/P, \mathscr{L}(\lambda) \otimes \mathscr{L}(V(\mu)^*)).$$

Transporting the map τ under these identifications, we get the map

$$\tilde{\tau} : H^0(G/P, \mathscr{L}(\lambda) \otimes \mathscr{L}(V(\mu)^*)) \to H^0(G/P, \mathscr{L}(\lambda) \otimes \mathscr{L}((\mathfrak{g} \cdot v_\mu)^*));$$

which in fact is induced from the canonical restriction: $V(\mu)^* \to (\mathfrak{g} \cdot v_\mu)^*$.

By Peter-Weyl theorem, for any algebraic P-module M, we have a canonical G-module isomorphism:

$$H^0(G/P, \mathscr{L}(M)) \approx \oplus_{\theta \in D} V(\theta)^* \otimes [V(\theta) \otimes M]^P,$$

where G acts trivially on $[V(\theta) \otimes M]^P$. (Even though we do not need, a more general result is obtained by Bott [Bo; Theorem I].) So the surjectivity of the map $\tilde{\tau}$ (and hence of τ) is equivalent to the surjectivity of the canonical restriction maps

$$\gamma_\theta : [V(\theta) \otimes \mathbf{C}_{-\lambda} \otimes V(\mu)^*]^P \approx \mathrm{Hom}_P(\mathbf{C}_\lambda \otimes V(\mu), V(\theta))$$

$$\to \mathrm{Hom}_P(\mathbf{C}_\lambda \otimes (\mathfrak{g} \cdot v_\mu), V(\theta)),$$

for all $\theta \in D$.

For $\theta = \lambda + \mu$, the map γ_θ is clearly non-zero and hence, by Lemma (2.4), is surjective. Making use of Lemma (2.4) again, we only need to check (for the surjectivity of τ) that the map γ_θ is non-zero for $\theta = \lambda + \mu - \beta$, for all those $\beta \in \Delta_+$ satisfying $S \subset F_\beta$ (and $\lambda + \mu - \beta$ dominant):

Choose a positive definite Hermitian form $\{\ ,\ \}$ on $V(\lambda)$ (and $V(\mu)$) satisfying $\{Xv, w\} = -\{v, \sigma(X)w\}$, for $v, w \in V(\lambda)$ and $X \in \mathfrak{g}$; where σ is a conjugate-linear involution of $\mathfrak{g}$ which takes $\mathfrak{g}_\alpha$ to $\mathfrak{g}_{-\alpha}$ for all $\alpha \in \Delta$. Now set the tensor product form (again denoted by) $\{\ ,\ \}$ on $V(\lambda) \otimes V(\mu)$.

Consider the diagram (for $\theta = \lambda + \mu - \beta$):

$$\begin{array}{ccccc}
\mathrm{Hom}_G(V(\theta), V(\lambda)\otimes V(\mu)) & \xrightarrow[\sim]{\gamma_3} & \mathrm{Hom}_G(V(\lambda)\otimes V(\mu), V(\theta)) & \xrightarrow[\sim]{\gamma_4} & \mathrm{Hom}_B(\mathbf{C}_\lambda \otimes V(\mu), V(\theta)) \\
\wr\uparrow \gamma_2 & & \searrow^{\gamma_6}_{\sim} & & \wr\uparrow \gamma_5 \\
\mathrm{Hom}_G(V(\theta)\otimes V(\lambda)^*, V(\mu)) & & & & \mathrm{Hom}_P(\mathbf{C}_\lambda \otimes V(\mu), V(\theta)) \\
\wr\downarrow \gamma_1 & & & & \downarrow \gamma_\theta \\
\mathrm{Hom}_B(\mathbf{C}_\theta \otimes V(\lambda)^*, V(\mu)) & & & & \mathrm{Hom}_P(\mathbf{C}_\lambda \otimes (\mathfrak{g}\cdot v_\mu), V(\theta));
\end{array}$$

where γ_1, γ_4, γ_5, γ_6 are the canonical restriction maps, and γ_2 is the canonical isomorphism. Clearly γ_1 and γ_4 are isomorphisms and moreover γ_5 is injective. But γ_4 being surjective, γ_5 is surjective as well. Now we describe the map γ_3: For any non-zero $\bar{f} \in \mathrm{Hom}_G(V(\theta), V(\lambda) \otimes V(\mu))$, write $V(\lambda) \otimes V(\mu) = \text{Image } \bar{f} \oplus (\text{Image } \bar{f})^\perp$ and set $\gamma_3(\bar{f})$ as the projection on the first factor (identifying it with $V(\theta)$ under $\bar{f}$). It is easy to see that γ_3 is a bijective map. (Observe that it is not a linear map.)

Following through the various isomorphisms in the above diagram, it can be seen that

the map γ_θ is non-zero $\Leftrightarrow$ there exists a B-morphism $f = f_\beta : \mathbf{C}_\theta \otimes V(\lambda)^* \to V(\mu)$ such that $\{(\gamma_2\gamma_1^{-1}f)V(\theta), \mathbf{C}_\lambda \otimes (\mathfrak{g}\cdot v_\mu)\} \neq 0$

$\Leftrightarrow \{U(\mathfrak{n}^-)((\gamma_2\gamma_1^{-1}f)\mathbf{C}_\theta), \mathbf{C}_\lambda \otimes (\mathfrak{g}\cdot v_\mu)\} \neq 0$

$\Leftrightarrow \{(\gamma_2\gamma_1^{-1}f)\mathbf{C}_\theta, \mathbf{C}_\lambda \otimes (\mathfrak{g}\cdot v_\mu)\} \neq 0$, by the invariance of $\{\ ,\ \}$; since $\mathbf{C}_\lambda \otimes (\mathfrak{g}\cdot v_\mu)$ is $U(\mathfrak{n})$-stable

$\Leftrightarrow \{f(\mathbf{C}_\theta \otimes v_\lambda^*), \mathfrak{g}\cdot v_\mu\} \neq 0$

$\Leftrightarrow \{f(\mathbf{C}_\theta \otimes v_\lambda^*), X_{-\beta}v_\mu\} \neq 0$, for $0 \neq X_{-\beta} \in \mathfrak{g}_{-\beta}$

$\Leftrightarrow \{X_\beta(f(\mathbf{C}_\theta \otimes v_\lambda^*)), v_\mu\} \neq 0$, by the invariance of $\{\ ,\ \}$

$\Leftrightarrow X_\beta(f(\mathbf{C}_\theta \otimes v_\lambda^*)) \neq 0.$

This proves the proposition. □

Now we are ready to prove Theorem (2.5):

(2.7) *Proof of Theorem* (2.5). We first prove the vanishing of $H^1(G/P \times G/P, \mathcal{I}_D^2 \otimes \mathcal{L}(\lambda \boxtimes \mu))$: In view of Proposition (2.6), it suffices to check the equivalent condition (b):

By Theorem (1.1), for any $\beta \in \Delta_+$ as in condition (b) of Proposition 2.6, $\mathrm{Hom}_G(V(\lambda + \mu - \beta), V(\lambda) \otimes V(\mu)) \xrightarrow{\sim} \mathrm{Hom}_B(\mathbf{C}_{\lambda+\mu-\beta}, V(\lambda) \otimes V(\mu)) \approx \mathrm{Hom}_B(\mathbf{C}_{\lambda+\mu-\beta} \otimes V(\lambda)^*, V(\mu)) \neq 0$. In fact, if we carefully see the proof of Theorem (1.1) (as given in Section 1.7), we have constructed $f_\beta \in \mathrm{Hom}_B(\mathbf{C}_{\lambda+\mu-\beta} \otimes V(\lambda)^*, V(\mu))$ which satisfies the requirement of Proposition (2.6) viz $X_\beta(f_\beta(\mathbf{C}_{\lambda+\mu-\beta} \otimes v_\lambda^*)) \neq 0$. This completes the proof of the vanishing of $H^1(G/P \times G/P, \mathcal{I}_D^2 \otimes \mathcal{L}(\lambda \boxtimes \mu))$.

To prove the higher cohomology vanishing; consider the cohomology exact sequence, associated to the sheaf exact sequence:

$$(\mathcal{S}_7)\cdots \qquad 0 \to \mathcal{I}_D^2 \otimes \mathcal{L}(\lambda \boxtimes \mu) \to \mathcal{I}_D \otimes \mathcal{L}(\lambda \boxtimes \mu)$$

$$\to (\mathcal{I}_D/\mathcal{I}_D^2) \otimes \mathcal{L}(\lambda \boxtimes \mu) \to 0,$$

together with Lemma (2.3) and [W₄; Proposition 3.9]. (Use the fact that the sheaf $(\mathcal{I}_D/\mathcal{I}_D^2) \otimes \mathcal{L}(\lambda \boxtimes \mu)$ of $\mathcal{O}_{G/P\times G/P}/\mathcal{I}_D \approx \mathcal{O}_D$ modules on $G/P \times G/P$ is supported on the diagonal D and, moreover, restricted to D it is isomorphic with $\Omega^1_{G/P} \otimes \mathcal{L}(\lambda + \mu)$, where $\Omega^1_{G/P}$ is the sheaf of 1-forms on G/P.) This completes the proof of Theorem (2.5). □

Let us recall the definition of the Gaussian map from [W_4]:

(2.8) *Definition.* Let X be a smooth projective variety with two line bundles $\mathcal{L}_1$ and $\mathcal{L}_2$ on X. Let $D \subset X \times X$ be the diagonal. Define the *Gaussian map*

$$\Phi : H^0(X \times X, \mathcal{I}_D \otimes (\mathcal{L}_1 \boxtimes \mathcal{L}_2)) \to H^0(X \times X, (\mathcal{I}_D/\mathcal{I}_D^2) \otimes (\mathcal{L}_1 \boxtimes \mathcal{L}_2)) \approx H^0(X, \Omega^1_X \otimes \mathcal{L}_1 \otimes \mathcal{L}_2),$$

got by the projection: $\mathcal{I}_D \to \mathcal{I}_D/\mathcal{I}_D^2$, where $\mathcal{I}_D$ is the ideal sheaf of $D \subset X \times X$ and Ω^1_X is the sheaf of 1-forms on X.

The following result was conjectured by Wahl [W_4], who proved it for $X = \mathrm{SL}(n, \mathbf{C})/B$ and also for $X = G/P$, where P is a maximal parabolic subgroup (in a semi-simple G) corresponding to a miniscule weight (cf. [W_4; Theorems 6.1 and 6.2]).

(2.9) Theorem. *Let G be a complex semi-simple simply-connected group and $P = P_S$ any parabolic subgroup. Then, for any $\lambda, \mu \in D_S^0$, the Gaussian map $\Phi : H^0(G/P \times G/P, \mathcal{I}_D \otimes \mathcal{L}(\lambda \boxtimes \mu)) \to H^0(G/P, \Omega^1_{G/P} \otimes \mathcal{L}(\lambda) \otimes \mathcal{L}(\mu))$ (defined in Section 2.8) is surjective.*

Proof. Follows trivially from the cohomology sequence associated to the sheaf sequences $\mathcal{S}_7$ (cf. Section 2.7), together with Theorem (2.5). □

(2.10) *Remark.* (a) Surjectivity of the Gaussian map Φ (as in the above theorem) together with Lemma (2.3), in view of the sequence $\mathcal{S}_7$, gives the vanishing of $H^1(G/P \times G/P, \mathcal{I}_D^2 \otimes \mathcal{L}(\lambda \boxtimes \mu))$. Now the vanishing of H^1 implies the vanishing of $H^p(G/P \times G/P, \mathcal{I}_D^2 \otimes \mathcal{L}(\lambda \boxtimes \mu))$, for any $p > 0$ (as is seen in the proof of Theorem 2.5, given in Section 2.7). Hence we see that Wahl's conjecture (proved in Theorem 2.9) is equivalent to Theorem (2.5).

(b) A more geometric proof of Wahl's conjecture (not using Theorem 1.1) is desirable.

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