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A Geometric Realization of Minimal $\mathfrak{k}$-type of Harish-Chandra Modules for Complex S.S. Groups [1]

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0.Introduction

Let G be a semi-simple connected simply-connected complex algebraic group (viewed as a real Lie group), with a fixed Borel subgroup B, a complex maximal torus $T \subset B$, and a maximal compact subgroup K. Let $\mathfrak{g}, \mathfrak{b}, \mathfrak{h}, \mathfrak{k}$ be their (real) Lie algebras (respectively). In this paper we will be concerned with irreducible $(\mathfrak{g}^{\mathbb{C}}, \mathfrak{k}^{\mathbb{C}})$-modules (also called Harish-Chandra modules), where $\mathfrak{g}^{\mathbb{C}} := \mathfrak{g} \otimes_{\mathbb{R}} \mathbb{C}$ (similarly $\mathfrak{k}^{\mathbb{C}}$) is the complexified Lie algebra. Since the Lie algebra pair $(\mathfrak{g}^{\mathbb{C}}, \mathfrak{k}^{\mathbb{C}})$ is canonically isomorphic (as complex Lie algebras) with the pair $(\tilde{\mathfrak{g}}, \Delta(\mathfrak{g}))$ (cf. § 1.1) (where $\tilde{\mathfrak{g}} := \mathfrak{g} \oplus \mathfrak{g}$ is the direct sum Lie algebra, $\Delta(\mathfrak{g})$ is the diagonal subalgebra and, G being a complex group, $\mathfrak{g}$ has the canonical complex structure), we can equivalently consider $(\tilde{\mathfrak{g}}, \Delta(\mathfrak{g}))$-modules. The infinitesimal character of an irreducible $(\tilde{\mathfrak{g}}, \Delta(\mathfrak{g}))$-module is represented by a pair (λ, μ) of dominant (with respect to $\mathfrak{b}$) elements in $\mathfrak{h}^* := \mathrm{Hom}_{\mathbb{C}}(\mathfrak{h}, \mathbb{C})$. In this paper we will only consider irreducible $(\tilde{\mathfrak{g}}, \Delta(\mathfrak{g}))$-modules with integral infinitesimal character (i.e. λ and μ are integral weights).

Let us assume that λ and μ as above are both, in addition, regular. We replace λ (resp. μ) by $\lambda + \rho$ (resp. $\mu + \rho$), where λ and μ are dominant (integral) weights. (The main body of the paper does not have this restriction but we put it here just as a simplifying assumption.) Now it is known (cf. [D] or [BB]; see Theorem 2.2 in this paper) that the Weyl group W (associated to G) parametrizes bijectively the irreducible $(\tilde{\mathfrak{g}}, \Delta(\mathfrak{g}))$-modules with infinitesimal character $(\lambda + \rho, \mu + \rho)$. Let us denote the irreducible $(\tilde{\mathfrak{g}}, \Delta(\mathfrak{g}))$-module thus associated to $w \in W$ by $N_w = N_w(\lambda + \rho, \mu + \rho)$. It is further known that the minimal $\Delta(\mathfrak{g})$-type of N_w is $V(\overline{\mu_w - \lambda})$, where $\mu_w := -w(\mu + \rho) - \rho, V(\overline{\mu_w - \lambda})$ is the (finite dimensional) irreducible G-module with highest weight $\overline{\mu_w - \lambda}$ and, for any $\beta \in \mathfrak{h}^*, \overline{\beta}$ denotes the unique dominant element in the W-orbit of β.

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On the other hand, for any $w \in W$, there is a certain distinguished irreducible $\Delta(\mathfrak{g})$-subquotient E_w (which is isomorphic with $V(\overline{\mu_w - \lambda})$ as a $\mathfrak{g}$-module) of the tensor product $\tilde{\mathfrak{g}}$ -module $V(\lambda+\rho)^* \otimes V(\mu+\rho)^*$ (cf. [Ku₁; § 2]), where $V(\lambda+\rho)^*$ is the dual $\mathfrak{g}$-module. In particular, observe that the minimal $\Delta(\mathfrak{g})$-type of N_w coincides with E_w. The aim of this paper is to explain this coincidence in terms of a 'natural' geometrical construction, which we now describe :

By Beilinson-Bernstein (cf. Theorem 2.2), the module $N_w(\lambda+\rho, \mu+\rho)$ is realized as the space of global sections $H^0(\widetilde{G/B}, \tilde{\mathcal{F}}_w \otimes \mathcal{L}(\lambda \otimes \mu))$, where $\tilde{\mathcal{F}}_w$ is a certain $\mathcal{D}_{\widetilde{G/B}}$-module on the product flag variety $\widetilde{G/B} := G/B \times G/B$, and $\mathcal{L}(\widetilde{\lambda \otimes \mu})$ is a homogeneous line bundle (cf. §§1.3 and 2.1). The module $H^0(\widetilde{G/B}, \tilde{\mathcal{F}}_w \otimes \mathcal{L}(\lambda \otimes \mu))$ embeds as a submodule of the local cohomology module $H^{\ell}_{\tilde{X}_w/\partial\tilde{X}_w}(\widetilde{G/B}, \mathcal{L}(\lambda \otimes \mu))$ (cf. Lemmas 2.3 and 2.4); where $\ell := \dim_{\mathbb{C}} G/B - \ell(w), \tilde{X}_w := \overline{G(e,w)} \subset \widetilde{G/B}$, and $\partial\tilde{X}_w := \tilde{X}_w \setminus G(e,w)$. Now define a Kunneth map (got by taking the tensor product) ψ_w :

$$H^0(\widetilde{G/B}, \mathcal{L}) \otimes H^{\ell}_{\tilde{X}_w/\partial\tilde{X}_w}(\widetilde{G/B}, \mathcal{L}(-\rho \otimes -\rho)) \to H^{\ell}_{\tilde{X}_w/\partial\tilde{X}_w}(\widetilde{G/B}, \mathcal{L}(\lambda \otimes \mu)),$$

where $\mathcal{L} := \mathcal{L}(\lambda+\rho \widetilde{\otimes} \mu+\rho)$. We further show (cf. Corollary 2.12) that the module $H^{\ell}_{\tilde{X}_w/\partial\tilde{X}_w}(\widetilde{G/B}, \mathcal{L}(-\rho \otimes -\rho))$ contains a unique $\Delta(\mathfrak{g})$-invariant ϑ. (Even though we do not need, it is the unique irreducible $(\tilde{\mathfrak{g}}, \Delta(\mathfrak{g}))$-module with infinitesimal character $(0,0)$.) We next prove (cf. Lemma 2.14) that the restricted map

$$\psi_w^{\vartheta} : H^0(\widetilde{G/B}, \mathcal{L}) \to H^{\ell}_{\tilde{X}_w/\partial\tilde{X}_w}(\widetilde{G/B}, \mathcal{L}(\lambda \otimes \mu)),$$

defined by $\psi_w^{\vartheta}(x) = \psi_w(x \otimes \vartheta)$, factors through $H^0(\tilde{X}_w, \mathcal{L})$ giving rise to a map $\overline{\psi}_w^{\vartheta} : H^0(\tilde{X}_w, \mathcal{L}) \to H^{\ell}_{\tilde{X}_w/\partial\tilde{X}_w}(\widetilde{G/B}, \mathcal{L}(\lambda \otimes \mu))$, and moreover the map $\overline{\psi}_w^{\vartheta}$ is injective (cf. Lemma 2.15). But, as proved in [Ku₁], the canonical restriction map : $H^0(\widetilde{G/B}, \mathcal{L}) \approx V(\lambda+\rho)^* \otimes V(\mu+\rho)^* \to H^0(\tilde{X}_w, \mathcal{L})$ is surjective and moreover $H^0(\tilde{X}_w, \mathcal{L})$ contains a unique copy E_w of the $\Delta(\mathfrak{g})$-module $V(\overline{\mu_w - \lambda})$. We next prove that the image of E_w under the map $\overline{\psi}_w^{\vartheta}$ lands inside the irreducible submodule N_w of $H^{\ell}_{\tilde{X}_w/\partial\tilde{X}_w}(\widetilde{G/B}, \mathcal{L}(\lambda \otimes \mu))$ and in fact is the minimal $\Delta(\mathfrak{g})$-type of N_w. (It may be mentioned that we do not use the known information about the minimal $\Delta(\mathfrak{g})$-type of N_w, instead we deduce it as a consequence of the Beilinson-Bernstein realization of irreducible Harish-Chandra modules and our work.)

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1. Notation and preliminaries

(1.1) **Notation.** The notation G is reserved to denote a semi-simple connected simply-connected complex algebraic group with a fixed Borel subgroup B and a (complex) maximal torus $T \subset B$. Let $\mathfrak{g} \supset \mathfrak{b} \supset \mathfrak{h}$ be the (real) Lie algebras of $G \supset B \supset T$ resp. Of course these Lie algebras have canonical complex structures coming from the corresponding groups.

Let $\{\alpha_1, ..., \alpha_\ell\} \subset \mathfrak{h}^*$ (where $\mathfrak{h}^* := \mathrm{Hom}_{\mathbb{C}}(\mathfrak{h}, \mathbb{C})$) be the simple roots for the positive root system determined by $\mathfrak{b}$, and let $\{\alpha_1^\vee, ..., \alpha_\ell^\vee\}$ be the corresponding simple co-roots. Define the set of integral weights $\mathfrak{h}^*_{\mathbb{Z}} := \{\lambda \in \mathfrak{h}^* : \lambda(\alpha_i^\vee) \in \mathbb{Z}, \text{ for all } 1 \leq i \leq \ell\}$. The set of dominant integral weights D is by definition $\{\lambda \in \mathfrak{h}^*_{\mathbb{Z}} : \lambda(\alpha_i^\vee) \geq 0, \text{ for all } i\}$. As usual ρ is the element of D, defined by $\rho(\alpha_i^\vee) = 1$, for all $1 \leq i \leq \ell$. Denote by $D - \rho$ the set $\{\lambda \in \mathfrak{h}^*_{\mathbb{Z}} : \lambda + \rho \in D\}$.

Let $W \approx N(T)/T$ denote the Weyl group, where $N(T)$ is the normalizer of T in G. The group W, which has a canonical representation in $\mathfrak{h}^*$, is generated (as a Coxeter group) by the 'simple' reflections $\{r_i\}_{1 \leq i \leq \ell}$; where $r_i \in \mathrm{Aut}\ \mathfrak{h}^*$ is defined by $r_i(\lambda) = \lambda - \lambda(\alpha_i^\vee)\alpha_i$. In particular, we can talk of the length $\ell(w)$ of any $w \in W$. For any $\lambda \in D$, let $W_\lambda := \{w \in W : w\lambda = \lambda\}$ be the stabilizer of λ. Then W_λ is again a Coxeter group, generated by a certain subset of simple reflections $\{r_i\}$.

We also fix a maximal compact subgroup $K \subset G$, with Lie algebra $\mathfrak{k}$. The complexified Lie algebra $\mathfrak{g}^{\mathbb{C}} := \mathfrak{g} \otimes_{\mathbb{R}} \mathbb{C}$ can be identified with the direct sum (complex) Lie algebra $\tilde{\mathfrak{g}} := \mathfrak{g} \oplus \mathfrak{g}$, under the complex Lie algebra isomorphism $\varphi : \mathfrak{g}^{\mathbb{C}} \to \tilde{\mathfrak{g}}$ (uniquely) defined by $\varphi(X) = (\overline{X}, X)$ for $X \in \mathfrak{g}$; where the bar denotes the conjugate-linear isomorphism of $\mathfrak{g}$ determined by the compact form $\mathfrak{k}$. Clearly $\varphi(\mathfrak{k} \otimes_{\mathbb{R}} \mathbb{C})$ is the diagonal subalgebra $\Delta(\mathfrak{g})$ of $\tilde{\mathfrak{g}}$. *From now onwards, instead of the pair* $(\mathfrak{g}^{\mathbb{C}}, \mathfrak{k}^{\mathbb{C}})$, *we will only consider the isomorphic pair* $(\tilde{\mathfrak{g}}, \Delta(\mathfrak{g}))$ (*under* φ).

(1.2) **Definition.** Let $\mathfrak{g}_1$ be a complex Lie algebra with a complex reductive subalgebra $\mathfrak{k}_1$. A $\mathfrak{g}_1$-module (in a complex vector space) M is called a $(\mathfrak{g}_1, \mathfrak{k}_1)$-*module* (also called *Harish-Chandra module*) if it is locally $\mathfrak{k}_1$-finite and is semi-simple as a $\mathfrak{k}_1$-module. It is called an *admissible* $(\mathfrak{g}_1, \mathfrak{k}_1)$-*module* if all the isotypical components of M (under $\mathfrak{k}_1$) are finite dimensional. If the $(\mathfrak{g}_1, \mathfrak{k}_1)$- module M is irreducible as a $\mathfrak{g}_1$-module, it is called an *irreducible Harish-Chandra module* (for the pair $(\mathfrak{g}_1, \mathfrak{k}_1)$).

Since the centre of the universal enveloping algebra $U(\tilde{\mathfrak{g}})$ can canonically be identified with $Z(\mathfrak{g}) \times Z(\mathfrak{g})$ (where $Z(\mathfrak{g})$ is the centre of $U(\mathfrak{g})$), the *infinitesimal character* of (say) an irreducible $\tilde{\mathfrak{g}}$-module is given by an element $(\lambda, \mu) \in \mathfrak{h}^* \times \mathfrak{h}^*$, and moreover λ and μ can be assumed to be dominant. We follow the standard convention that the trivial (one dimensional) $\tilde{\mathfrak{g}}$-module has infinitesimal character (ρ, ρ).

(1.3) **Definitions.** We denote by $\widetilde{G/B}$ the product flag variety $G/B \times G/B$. The group G acts on $\widetilde{G/B}$ diagonally. For any $w \in W$, we define the *Schubert variety* $X_w \subset G/B$ (resp. the *G-Schubert variety* $\tilde{X}_w \subset \widetilde{G/B}$) as the closure of the B-orbit $\mathcal{B}_w := BwB/B \subset G/B$ (resp. the closure of the G-orbit $\tilde{\mathcal{B}}_w := G(e,w) \subset \widetilde{G/B}$). As is easy to see $\{X_w\}_{w \in W}$ (resp. $\{\tilde{X}_w\}_{w \in W}$) are precisely the B-orbit closures in G/B (resp. G-orbit closures in $\widetilde{G/B}$). We also set $\partial X_w := X_w \setminus \mathcal{B}_w$ (resp. $\partial \tilde{X}_w := \tilde{X}_w \setminus \tilde{\mathcal{B}}_w$) and $Y_w := G/B \setminus \partial X_w$ (resp. $\tilde{Y}_w := \widetilde{G/B} \setminus \partial \tilde{X}_w$). It is easy to see that ∂X_w (resp. $\partial \tilde{X}_w$) is closed in G/B (resp. $\widetilde{G/B}$).

For any $\lambda \in \mathfrak{h}^*_{\mathbb{Z}}$ there is defined a *line bundle* $\mathcal{L}(\lambda)$ on G/B; which is associated to the principal B-bundle: $G \to G/B$ by the 1-dimensional representation $\mathbb{C}_{-\lambda}$ (determined by the character $e^{-\lambda}$ of B). More generally, given an algebraic B-module M (cf. Definition 2.9), we can consider the corresponding *vector bundle* $\mathcal{L}(M) := G \times_B M$ over G/B. For any $\lambda, \mu \in \mathfrak{h}^*_{\mathbb{Z}}$, we define the line bundle $\mathcal{L}(\lambda \otimes \mu)$ on $\widetilde{G/B}$ as the external tensor product of the line bundles $\mathcal{L}(\lambda)$ and $\mathcal{L}(\mu)$ respectively. The restriction of $\mathcal{L}(\lambda)$ to X_w (resp. $\mathcal{L}(\lambda \otimes \mu)$ to $\tilde{X}_w$) is denoted by $\mathcal{L}_w(\lambda)$ (resp. $\mathcal{L}_w(\lambda \otimes \mu)$).

For any topological space X, closed subspaces $Z \subseteq Y \subseteq X$, and an abelian sheaf $\mathcal{S}$ on $X, H^*_{Y/Z}(X, \mathcal{S})$ (resp. $\mathcal{H}^*_{Y/Z}(X, \mathcal{S})$) denotes the *local cohomology* (resp. *local cohomology sheaf*) introduced by Grothendieck ([H_1; page 219, variation 2]). If Z is the empty set $\phi, H^*_{Y/Z}(X, \mathcal{S})$ (resp. $\mathcal{H}^*_{Y/Z}(X, \mathcal{S})$) is abbreviated to $H^*_Y(X, \mathcal{S})$ (resp. $\mathcal{H}^*_Y(X, \mathcal{S})$). The cases of our interest will be when X is an algebraic variety over $\mathbb{C}$ with the Zariski topology and $\mathcal{S}$ is an $\mathcal{O}_X$-module (where $\mathcal{O}_X$ denotes the structure sheaf of X).

For a smooth algebraic variety X over $\mathbb{C}$, $\mathcal{D}_X$ denotes the sheaf of algebraic differential operators on X. A *$\mathcal{D}_X$-module* is, by definition, a sheaf $\mathcal{S}$ of left $\mathcal{D}_X$-modules, which is quasi-coherent as an $\mathcal{O}_X$-module.

We recall the following algebraic analogue of a result of Brylinski-Kashiwara :

(1.4) **Proposition** [BK; Proposition 8.5]. *Let Y be a closed subvariety,* *of pure codimension ℓ, of a smooth algebraic variety X , and let $Z \subset Y$ be a nowhere dense closed subvariety of Y which contains the singular locus of Y. Then there exists a unique holonomic $\mathcal{D}_X$-module with regular singularities* (cf. [BK;§ 1]) $\mathcal{F} = \mathcal{F}(Y, X)$ *($\mathcal{F}$ does not depend upon the choice of Z) satisfying :*

$$(P_1) \qquad \mathcal{F}\,|_{X\setminus Z} \approx \mathcal{H}^{\ell}_{Y\setminus Z}(X \setminus Z, \mathcal{O}_{X\setminus Z})$$

and

$$(P_2) \qquad \mathcal{H}^0_Z(X, \mathcal{F}) = \mathcal{H}^0_Z(X, \mathcal{F}^*) = 0,$$

where

$$\mathcal{F}^* := \mathcal{H}om_{\mathcal{O}_X}(\Omega_X, \mathcal{E}xt^{\dim_{\mathbb{C}} X}_{\mathcal{D}_X}(\mathcal{F}, \mathcal{D}_X)),$$

and Ω_X is the canonical bundle of X.

We also recall the following two results from local cohomology, for their use in Section (2).

(1.5) **Lemma** [K; § 11]. (a) *Let K be an affine algebraic group over $\mathbb{C}$ with Lie algebra $\mathfrak{k}$, let X be a K-variety over $\mathbb{C}$, and let $\mathcal{S}$ be a quasi-coherent K-module on X (also called K-linearized $\mathcal{O}_X$-module). Then, for any closed subspaces $Y \supseteq Z$ of X, the local cohomology $H^p_{Y/Z}(X, \mathcal{S})$ admits a natural $\mathfrak{k}$-module structure, which is functorial in the following sense:*

Let X' be another K-variety over $\mathbb{C}$ with a quasi-coherent K-module $\mathcal{S}'$ on X', a K-morphism $f : X' \to X$, and a K-equivariant sheaf morphism $\hat{f} : f^*(\mathcal{S}) \to \mathcal{S}'$. Then, for any closed subspaces $Y' \supseteq Z'$ of X' such that $Y' \supseteq f^{-1}(Y)$ and $Z' \supseteq f^{-1}(Z)$, the induced map $H^p_{Y/Z}(X, \mathcal{S}) \to H^p_{Y'/Z'}(X', \mathcal{S}')$ (cf. [K; Lemma 11.3]) *is a $\mathfrak{k}$-module map.*

(b) *If we assume, in addition, (in the first paragraph of a) that Y and Z are both K-stable, then the $\mathfrak{k}$-module structure on $H^p_{Y/Z}(X, \mathcal{S})$ "integrates" to give a canonical K-module structure.*

(1.6) **Lemma.** *Let $\mathbf{A}^d$ be the affine space of dimension d over a field k. Then :*

(a) $H^p_{\{0\}}(\mathbf{A}^d, \mathcal{O}_{\mathbf{A}^d}) = 0$, *for all $p \neq d$, and*

(b) $H^d_{\{0\}}(\mathbf{A}^d, \mathcal{O}_{\mathbf{A}^d})$ *is canonically isomorphic with* $\sum_{n_1,\ldots,n_d<0} k x_1^{n_1} \cdots x_d^{n_d}$, *as k-vector spaces; where 0 is the origin of $\mathbf{A}^d$, and $(x_1, \ldots, x_d)$ are the coordinate functions on $\mathbf{A}^d$.*

2. Formulation of the main result and its proof

(2.1). *In this whole section we fix once and for all* $\lambda, \mu \in D - \rho$ (cf. § 1.1), *and* $w \in W$. *Put* $\ell = \ell(w_0) - \ell(w)$, *where* w_0 *is the longest element of* W. We set

$$\begin{aligned} \mathcal{F}_w &= \mathcal{F}(X_w, G/B) \\ \widetilde{\mathcal{F}}_w &= \mathcal{F}(\widetilde{X}_w, \widetilde{G/B}) \\ \mathcal{F}_w(\lambda) &= \mathcal{F}_w \otimes_{\mathcal{O}_{G/B}} \mathcal{L}(\lambda) \\ \widetilde{\mathcal{F}}_w(\lambda \otimes \mu) &= \widetilde{\mathcal{F}}_w \otimes_{\mathcal{O}_{\widetilde{G/B}}} \mathcal{L}(\lambda \otimes \mu) \end{aligned}$$

where $\mathcal{F}(,)$ is as defined in Proposition (1.4). Since X_w is B-stable (resp. $\widetilde{X}_w$ is G-stable, under the diagonal G-action) and the line bundle $\mathcal{L}(\lambda)$ is B-equivariant (resp. the line bundle $\mathcal{L}(\lambda \otimes \mu)$ is G-equivariant), by the uniqueness of $\mathcal{F}$, we obtain that $\mathcal{F}_w$ is a quasi-coherent B-module (resp. $\widetilde{\mathcal{F}}_w$ is a quasi-coherent G-module).

Now we recall the following fundamental result due to Beilinson and Bernstein. (Even though we do not need, a more general result is proved by them.)

(2.2) **Theorem** [BB]. *The map* $w \mapsto H^0(\widetilde{G/B}, \widetilde{\mathcal{F}}_w(\lambda \otimes \mu))$ *sets up a bijective correspondence from* $W'_{\lambda+\rho,\mu+\rho}$ *to the set of isomorphism classes of irreducible* $(\tilde{\mathfrak{g}}, \Delta(\mathfrak{g}))$*-modules with infinitesimal character* $(\lambda + \rho, \mu + \rho)$; *where* $W'_{\lambda+\rho,\mu+\rho} := \{w \in W : w$ *is the (unique) element of minimal length in its double coset* $W_{\lambda+\rho} w W_{\mu+\rho}\}$, *and* $W_{\lambda+\rho}$ *is as defined in* § 1.1.

If $w \notin W'_{\lambda+\rho,\mu+\rho}$, *then* $H^0(\widetilde{G/B}, \widetilde{\mathcal{F}}_w(\lambda \otimes \mu)) = 0$.

As a preparation to prove (or even to formulate) our main result, we prove the following lemmas.

(2.3) **Lemma.** *The canonical restriction map*

$$H^0(\widetilde{G/B}, \widetilde{\mathcal{F}}_w(\lambda \otimes \mu)) \to H^0(\widetilde{Y}_w, \widetilde{\mathcal{F}}_w(\lambda \otimes \mu))$$

is injective, where $\widetilde{Y}_w$ *is as defined in* § 1.3.

Proof. From the long exact sequence for the local cohomology (cf. [H_2; Chap. III, Exercise 2.3]), it suffices to prove that $H^0_{\partial \widetilde{X}_w}(\widetilde{G/B}, \widetilde{\mathcal{F}}_w(\lambda \otimes \mu)) = 0$: By the defining property (P_2) of $\widetilde{\mathcal{F}}_w$ (cf. Proposition 1.4), the sheaf $\mathcal{H}^0_{\partial \widetilde{X}_w}(\widetilde{G/B}, \widetilde{\mathcal{F}}_w(\lambda \otimes \mu)) = 0$. In particular, by [G; page 5, Proposition 1.4], $H^0_{\partial \widetilde{X}_w}(\widetilde{G/B}, \widetilde{\mathcal{F}}_w(\lambda \otimes \mu)) = 0$. □

(2.4) **Lemma.** *There is a canonical isomorphism*

$$\theta_w : H^{\ell}_{\widetilde{X}_w/\partial\widetilde{X}_w}(\widetilde{G/B}, \mathcal{L}(\lambda\otimes\mu)) \to H^0(\widetilde{Y}_w, \widetilde{\mathcal{F}}_w(\lambda\otimes\mu)).$$

Proof. By the defining property (P_1) (cf. Proposition 1.4), the sheaf $\widetilde{\mathcal{F}}_w(\lambda\otimes\mu)\,|_{\widetilde{Y}_w}$ is the local cohomology sheaf $\mathcal{H}^{\ell}_{\widetilde{\mathcal{B}}_w}(\widetilde{Y}_w, \mathcal{L}(\lambda\otimes\mu))$. Further $\mathcal{H}^{i}_{\widetilde{\mathcal{B}}_w}(\widetilde{Y}_w, \mathcal{L}(\lambda\otimes\mu)) = 0$ for all $i \neq \ell$, since $\widetilde{\mathcal{B}}_w$ is a smooth closed subvariety of $\widetilde{Y}_w$ of codimension ℓ. Now the lemma follows from [G; page 5, Proposition 1.4] together with [K; Lemma 7.7]. □

(2.5) **Remark.** Exactly the same proof as above gives an isomorphism: $H^{\ell}_{X_w/\partial X_w}(G/B, \mathcal{L}(\mu)) \xrightarrow{\sim} H^0(Y_w, \mathcal{F}_w(\mu))$, where Y_w is as defined in § 1.3. Similarly the restriction map : $H^0(G/B, \mathcal{F}_w(\mu)) \to H^0(Y_w, \mathcal{F}_w(\mu))$ is injective (cf. Lemma 2.3).

(2.6) **Lemma.** $H^{\ell}_{\widetilde{X}_w/\partial\widetilde{X}_w}(\widetilde{G/B}, \mathcal{L}(\lambda\otimes\mu))$ *is canonically isomorphic with*

$$H^0(G/B, \mathcal{L}(\lambda)\otimes\mathcal{L}(H^{\ell}_{X_w/\partial X_w}(G/B, \mathcal{L}(\mu)))),$$

where $\mathcal{L}()$ *is as in* § 1.3 *and* $X_w, \partial X_w$ *being* B*-stable,* $H^{\ell}_{X_w/\partial X_w}(G/B, \mathcal{L}(\mu))$ *has a canonical* $\mathfrak{g}$*-module structure which restricted to* $\mathfrak{b}$ *integrates to give a* B*-module structure* (cf. Lemma 1.5).

Proof. By the spectral sequence [K; Lemma 8.5(d)], connecting the local cohomology sheaves to local cohomology groups, we get :

$$H^{\ell}_{\widetilde{X}_w/\partial\widetilde{X}_w}(\widetilde{G/B}, \mathcal{L}(\lambda\otimes\mu)) \approx H^0(\widetilde{G/B}, \mathcal{S}),$$

where $\mathcal{S}$ is the local cohomology sheaf $\mathcal{H}^{\ell}_{\widetilde{X}_w/\partial\widetilde{X}_w}(\widetilde{G/B}, \mathcal{L}(\lambda\otimes\mu))$. (The spectral sequence degenerates because $\mathcal{H}^{i}_{\widetilde{X}_w/\partial\widetilde{X}_w}(\widetilde{G/B}, \mathcal{L}(\lambda\otimes\mu)) = 0$, for all $i \neq \ell$; see the proof of Lemma 2.4.)

Further by the definition of the direct image sheaf, applied to the projection on the first factor $\pi_1 : \widetilde{G/B} \to G/B$, we get

$$H^0(\widetilde{G/B}, \mathcal{S}) \approx H^0(G/B, \pi_{1*}(\mathcal{S})).$$

We next assert that the direct image sheaf $\pi_{1*}(\mathcal{S})$ on G/B is isomorphic with $\mathcal{L}(\lambda)\otimes\mathcal{L}(H^{\ell}_{X_w/\partial X_w}(G/B, \mathcal{L}(\mu)))$:

First of all, the sheaf $\pi_{1*}(\mathcal{S})$ is a G-linearized sheaf of $\mathcal{O}_{G/B}$-modules. This is clear because the map π_1 is G-equivariant (under the diagonal action of G on $\widetilde{G/B}$), $\widetilde{X}_w, \partial\widetilde{X}_w$ are G-stable, and $\mathcal{L}(\lambda\otimes\mu)$ is a G-equivariant line bundle. Let us now compute the stalk of $\pi_{1*}(\mathcal{S})$ at the base point $\underline{e} \in G/B$: Consider the affine open subset $U^-\underline{e} \subset G/B$, where U^- is the unipotent subgroup of G with Lie algebra $\oplus_{\alpha\in\Delta_-}\mathfrak{g}_\alpha$, where Δ_+ is the set of roots for $\mathfrak{b}, \Delta_- := -\Delta_+$, and $\mathfrak{g}_\alpha$ is the root space corresponding to the root α. Define a map $m : \pi_1^{-1}(U^-\underline{e}) = U^-\underline{e} \times G/B \to G/B$ by $m(g\underline{e}, x) = g^{-1}x$, for $g \in U^-$ and $x \in G/B$. Then m is an affine morphism. Also, as is easy to see, $m^{-1}(X_w) = \widetilde{X}_w \cap \pi_1^{-1}(U^-\underline{e})$ and $m^{-1}(\partial X_w) = \partial\widetilde{X}_w \cap \pi_1^{-1}(U^-\underline{e})$. In particular, by the spectral sequence [G; Proposition 5.5 and Corollary 5.6] together with [H_2; Chapter III, Exercise 8.2], we get $H^\ell_{\pi_1^{-1}(U^-\underline{e})\cap\widetilde{X}_w/\pi_1^{-1}(U^-\underline{e})\cap\partial\widetilde{X}_w}(\pi_1^{-1}(U^-\underline{e}), \mathcal{L}(\lambda\otimes\mu)) \approx H^\ell_{X_w/\partial X_w}(G/B, m_*\mathcal{L}(\lambda\otimes\mu))$. From this it is not difficult to deduce the assertion that $\pi_{1*}(\mathcal{S}) \approx \mathcal{L}(\lambda)\otimes\mathcal{L}(H^\ell_{X_w/\partial X_w}(G/B, \mathcal{L}(\mu)))$, and hence the lemma is proved. □

(2.7) **Definition.** The Lie algebra $\mathfrak{g}$ admits a unique complex linear involution τ such that $\tau\mid_{\mathfrak{h}} = -1$ and it sends the α-th root space $\mathfrak{g}_\alpha$ to $\mathfrak{g}_{-\alpha}$ for any root α. Given a $\mathfrak{g}$-module M, we get another $\mathfrak{g}$-module structure on M by twisting the original $\mathfrak{g}$-module structure by τ. We denote the twisted $\mathfrak{g}$-module by M^τ.

Let $\widetilde{\mathcal{O}}$ be the category of finitely generated $U(\mathfrak{g})$-modules, which are locally finite as $U(\mathfrak{b})$-modules. Any $N \in \widetilde{\mathcal{O}}$ satisfies $N = \oplus_{\lambda\in\mathfrak{h}^*}N_\lambda$, where N_λ is the λ-th generalized weight space. Set $N^\vee = \{f \in \mathrm{Hom}_{\mathbb{C}}(N, \mathbb{C}) : f(N_\lambda) = 0$, for all but finitely many $\lambda\}$. Then $N^\vee$ has a canonical $\mathfrak{g}$-module structure. Finally we set $N^\sigma := (N^\vee)^\tau$. It is easy to see that $N^\sigma \in \widetilde{\mathcal{O}}$ and moreover $ch(N) = ch(N^\sigma)$, where $ch(N) := \sum(\dim N_\lambda)e^\lambda$ is the formal character of N.

The following lemma is well known (see,e.g., [BK; § 5]), but we recall the proof as it will be used in the proof of Lemma (2.14).

(2.8) **Lemma.** *$H^\ell_{X_w/\partial X_w}(G/B, \mathcal{L}(\mu)) \approx M(\mu_w)^\sigma$, as $\mathfrak{g}$-modules, where $\mu_w := -w(\mu+\rho)-\rho$.*

Proof. Consider the T-equivariant biregular isomorphism (cf. [KL; § 1.4]) $\xi = \xi_w : U_w \times U'_w \xrightarrow{\sim} wU^-B/B$, given by $(g,h) \mapsto ghwB$ (for $g \in U_w$ and $h \in U'_w$); where U_w (resp. U'_w) is the unipotent subgroup of G with Lie algebra $\oplus_{\alpha\in\Delta_+\cap w\Delta_-}\mathfrak{g}_\alpha$(resp. $\oplus_{\alpha\in\Delta_-\cap w\Delta_-}\mathfrak{g}_\alpha$), and T acts by conjugation on U_w and U'_w.

As can be easily seen, there is a nowhere vanishing section s of the line bundle $\mathcal{L}(\mu)\mid_{wU^-B/B}$, which transforms under the canonical T-action via the weight $-w\mu$. Further $\xi(U_w \times e) = \mathcal{B}_w$ and $\mathcal{B}_w$ is closed in the open subset $\xi(U_w \times U'_w)$ of G/B. Hence by [K; Lemmas 7.7 and 7.9],

$$\begin{aligned} H^{\ell}_{X_w/\partial X_w}(G/B, \mathcal{L}(\mu)) &\approx H^{\ell}_{\mathcal{B}_w}(Y_w, \mathcal{L}(\mu)) \\ &\approx H^{\ell}_{U_w \times e}(U_w \times U'_w, \mathcal{O}_{U_w \times U'_w}) \otimes s \end{aligned}$$

$$(I_1)\ldots \qquad H^{\ell}_{X_w/\partial X_w}(G/B, \mathcal{L}(\mu)) \approx H^{\ell}_{\{e\}}(U'_w, \mathcal{O}_{U'_w}) \otimes \mathbb{C}\,[U_w] \otimes s,$$

by [G; Proposition 5.5],

where $\mathbb{C}\,[U_w]$ is the ring of regular functions on U_w. So, by Lemma (1.6),

$$\begin{aligned} ch\ H^{\ell}_{X_w/\partial X_w}(G/B, \mathcal{L}(\mu)) &= ch\ H^{\ell}_{\{e\}}(U'_w, \mathcal{O}_{U'_w}) \cdot ch\, \mathbb{C}\,[U_w] \cdot e^{-w\mu} \\ &= e^{-\sum_{\alpha \in \Delta_+ \cap w\Delta_+} \alpha} \cdot \Big(\prod_{\alpha \in \Delta_+ \cap w\Delta_+} (1 - e^{-\alpha})^{-1} \Big) \\ &\quad \cdot \Big(\prod_{\beta \in \Delta_+ \cap w\Delta_-} (1 - e^{-\beta})^{-1} \Big) \cdot e^{-w\mu} \\ &= e^{\mu_w} \cdot \prod_{\alpha \in \Delta_+} (1 - e^{-\alpha})^{-1} \\ &= ch\ M(\mu_w) \\ &= ch\ (M(\mu_w)^{\sigma}), \quad (\text{cf. } \S 2.7). \end{aligned}$$

So both the modules of the lemma have the same character. From this it is not difficult to establish that they are isomorphic as $\mathfrak{g}$-modules (cf. [BK; § 5] or [Ku$_2$; § 3]). □

(2.9) **Definition.** A B-module M is called *algebraic* if the action of B on M is locally finite and any finite dimensional B-submodule of M is an algebraic B-module.

The following result can easily be deduced from Peter-Weyl theorem. (In fact a more general result is proved by Bott [B; Theorem I].)

(2.10) **Proposition.** *Let M be an algebraic B-module. Then $H^0(G/B, \mathcal{L}(M))$ is G-module isomorphic with $\oplus_{\theta \in D}(V(\theta)^* \otimes_{\mathbb{C}} [M \otimes V(\theta)]^B)$, where we put the trivial G-module structure on the space of B-invariants $[M \otimes V(\theta)]^B$, $V(\theta)$ is the irreducible G-module with highest weight θ and $V(\theta)^*$ is its dual.*

(2.11) **Corollary.** *As G-modules*

$$H^{\ell}_{\widetilde{X}_w/\partial \widetilde{X}_w}(\widetilde{G/B}, \mathcal{L}(\lambda \otimes \mu)) \approx V(\overline{\mu_w - \lambda}) \oplus \Big(\bigoplus_{\substack{\theta \in D \\ \|\theta\| > \|\lambda - \mu_w\|}} V(\theta)^* \otimes [V(\theta)]_{\lambda - \mu_w} \Big),$$

where μ_w is as in Lemma (2.8), $[V(\theta)]_{\lambda-\mu_w}$ denotes the $(\lambda - \mu_w)$-th weight space in $V(\theta)$ and, for any $\chi \in \mathfrak{h}^*$, $\overline{\chi}$ denotes the unique dominant element *in the W-orbit of* χ.

Proof. Combining Lemmas (2.6) and (2.8) we get :

$$\begin{aligned} H^{\ell}_{\widetilde{X}_w/\partial\widetilde{X}_w}(\widetilde{G/B}, \mathcal{L}(\lambda \otimes \mu)) &\approx H^0(G/B, \mathcal{L}(\lambda) \otimes \mathcal{L}(M(\mu_w)^{\sigma})) \\ &\approx \oplus_{\theta \in D}(V(\theta)^* \otimes [V(\theta) \otimes \mathbb{C}_{-\lambda} \otimes M(\mu_w)^{\sigma}]^B), \\ &\qquad \text{by Proposition (2.10)} \\ &\approx \oplus_{\theta \in D}(V(\theta)^* \otimes [V(\theta) \otimes M(\mu_w - \lambda)^{\sigma}]^B) \\ &\approx \oplus_{\theta \in D}(V(\theta)^* \otimes \mathrm{Hom}_{\mathfrak{b}}(M(\mu_w - \lambda)^{\tau}, V(\theta))) \\ &\qquad \text{(cf. Definition 2.7)} \\ &\approx \oplus_{\theta \in D}(V(\theta)^* \otimes [V(\theta)]_{\lambda - \mu_w}), \\ &\quad \text{since } M(\mu_w - \lambda)^{\tau} \text{ is } U(\mathfrak{n})\text{-free.} \end{aligned}$$

We next observe that if any $\chi \in \mathfrak{h}^*$ occurs as a weight in $V(\theta)$, then $\| \chi \| \leq \| \theta \|$ and equality occurs if and only if $\overline{\chi} = \theta$:

We can assume, without loss of generality, that χ is dominant. Write $\theta = \chi + \beta$ for some $\beta \in \sum_{i=1}^{\ell} \mathbb{Z}_+ \alpha_i$, where $\mathbb{Z}_+$ is the set of non-negative integers. Then $\| \theta \|^2 = \| \chi \|^2 + \| \beta \|^2 + 2 < \chi, \beta >$. In particular, $\| \chi \| \leq \| \theta \|$ and equality occurs if and only if $\beta = 0$. This proves the assertion and hence the corollary. □

The following is an immediate consequence of the above corollary.

(2.12) **Corollary.** *For any* $w \in W$, $H^{\ell}_{\widetilde{X}_w/\partial\widetilde{X}_w}(\widetilde{G/B}, \mathcal{L}(-\rho \otimes -\rho))$ *has a unique (up to scalar multiples) G-invariant, where* ℓ *is as in* § 2.1.

(2.13) **The basic map.** For any $w \in W$ and $\lambda, \mu \in D - \rho$, there is defined a canonical Kunneth map (got by taking the tensor product)

$$\psi_w = \psi_w^{\lambda,\mu} : H^0(\widetilde{G/B}, \mathcal{L}(\lambda + \rho \otimes \mu + \rho)) \otimes H^{\ell}_{\widetilde{X}_w/\partial\widetilde{X}_w}(\widetilde{G/B}, \mathcal{L}(-\rho \otimes -\rho))$$

$$\to H^{\ell}_{\widetilde{X}_w/\partial\widetilde{X}_w}(\widetilde{G/B}, \mathcal{L}(\lambda \otimes \mu)),$$

where ℓ is as in § 2.1. (Observe that $H^{p}_{\widetilde{X}_w/\partial\widetilde{X}_w}(\widetilde{G/B}, \mathcal{L}(\lambda \otimes \mu)) = 0$, for all $p \neq \ell$.)

By naturality, the map ψ_w is a $\tilde{\mathfrak{g}}$-module map, where we put the tensor product $\tilde{\mathfrak{g}}$-module structure on the domain (cf. Lemma 1.5). By the above corollary, $H^{\ell}_{\widetilde{X}_w/\partial\widetilde{X}_w}(\widetilde{G/B},\mathcal{L}(-\rho\otimes-\rho))$ contains a unique G-invariant ϑ. Hence by restricting ψ_w, (since ϑ is G-invariant) we get a G-module map

$$\psi_w^{\vartheta}: H^0(\widetilde{G/B},\mathcal{L}(\lambda+\rho\otimes\mu+\rho))\to H^{\ell}_{\widetilde{X}_w/\partial\widetilde{X}_w}(\widetilde{G/B},\mathcal{L}(\lambda\otimes\mu)),$$

given by $\psi_w^{\vartheta}(x)=\psi_w(x\otimes\vartheta)$.

Now we have the following crucial :

(2.14) **Lemma.** *The map ψ_w^{ϑ} factors through $H^0(\widetilde{X}_w,\mathcal{L}_w)$, i.e., there exists a map $\overline{\psi}_w^{\vartheta}: H^0(\widetilde{X}_w,\mathcal{L}_w)\to H^{\ell}_{\widetilde{X}_w/\partial\widetilde{X}_w}(\widetilde{G/B},\mathcal{L}(\lambda\otimes\mu))$ making the following diagram commutative:*

$$\begin{array}{ccc} H^0(\widetilde{G/B},\mathcal{L}) & \xrightarrow{\psi_w^{\vartheta}} & H^{\ell}_{\widetilde{X}_w/\partial\widetilde{X}_w}(\widetilde{G/B},\mathcal{L}(\lambda\otimes\mu)) \\ r_w\searrow & & \nearrow\overline{\psi}_w^{\vartheta} \\ & H^0(\widetilde{X}_w,\mathcal{L}_w) & \end{array}$$

where r_w is the canonical restriction, and $\mathcal{L}:=\mathcal{L}(\lambda+\rho\otimes\mu+\rho)$ ($\mathcal{L}_w$ has a similar meaning).

Proof. From the naturality of the Kunneth map, we get that the following diagram $(\mathcal{D})$ is commutative :

$$\begin{array}{ccc} H^0(\widetilde{G/B},\mathcal{L})\otimes H^{\ell}_{\widetilde{X}_w/\partial\widetilde{X}_w}(\widetilde{G/B},\mathcal{L}(-\rho\otimes-\rho)) & \to & H^{\ell}_{\widetilde{X}_w/\partial\widetilde{X}_w}(\widetilde{G/B},\mathcal{L}') \\ \downarrow\wr & & \downarrow\wr \\ H^0(\widetilde{G/B},\mathcal{L})\otimes H^0(\widetilde{Y}_w,\widetilde{\mathcal{F}}_w(-\rho\otimes-\rho)) & \to & H^0(\widetilde{Y}_w,\widetilde{\mathcal{F}}_w(\lambda\otimes\mu)) \end{array}$$

where $\mathcal{L}':=\mathcal{L}(\lambda\otimes\mu)$, and $\widetilde{Y}_w$ is as defined in § 1.3 and the vertical isomorphisms are induced by the isomorphism of Lemma (2.4).

Define a subsheaf $\mathcal{K}_w=\{x\in\mathcal{F}_w:\mathcal{I}_{X_w}x=0\}$ (resp. $\widetilde{\mathcal{K}}_w=\{x\in\widetilde{\mathcal{F}}_w:\mathcal{I}_{\widetilde{X}_w}x=0\}$), where $\mathcal{I}_{X_w}$ (resp. $\mathcal{I}_{\widetilde{X}_w}$) denotes the ideal sheaf of X_w in G/B (resp. of $\widetilde{X}_w$ in $\widetilde{G/B}$). Set $\mathcal{K}_w(-\rho)=\mathcal{K}_w\otimes_{\mathcal{O}_{G/B}}\mathcal{L}(-\rho)$ and $\widetilde{\mathcal{K}}_w(-\rho\otimes-\rho)=\widetilde{\mathcal{K}}_w\otimes_{\mathcal{O}_{\widetilde{G/B}}}\mathcal{L}(-\rho\otimes-\rho)$.

By the very definition, $\psi_w(Q_w\otimes H^0(\widetilde{Y}_w,\widetilde{\mathcal{K}}_w(-\rho\otimes-\rho)))=0$, where Q_w is the kernel of the restriction map r_w. But, by Kumar [Ku$_1$; Theorem 1.5], the map r_w is surjective and hence, to prove the lemma, it suffices to show that $\vartheta\in H^0(\widetilde{Y}_w,\widetilde{\mathcal{K}}_w(-\rho\otimes-\rho))$:

We first observe that

$$(I_2)\cdots H^0(\widetilde{Y}_w,\widetilde{\mathcal{K}}_w(-\rho\otimes-\rho))\approx H^0(G/B,\mathcal{L}(-\rho)\otimes\mathcal{L}(H^0(Y_w,\mathcal{K}_w(-\rho)))),$$

where Y_w is as defined in § 1.3. By Remark (2.5),

$$H^{\ell}_{X_w/\partial X_w}(G/B, \mathcal{L}(-\rho)) \approx H^0(Y_w, \mathcal{F}_w(-\rho)).$$

Further by (I_1) (cf. proof of Lemma 2.8)

$$\begin{aligned} H^0(Y_w, \mathcal{K}_w(-\rho)) \;\approx\; & \{x \in H^{\ell}_{\{e\}}(U'_w, \mathcal{O}_{U'_w}) : fx = 0, \text{ for all} \\ & f \in \mathbb{C}\,[U'_w] \text{ with } f(e) = 0\} \otimes \mathbb{C}\,[U_w] \otimes s. \end{aligned}$$

Hence

$$(I_3)\cdots \qquad H^0(Y_w, \mathcal{K}_w(-\rho)) \approx (x_1^{-1} \cdots x_\ell^{-1}) \otimes \mathbb{C}\,[U_w] \otimes s,$$

by Lemma (1.6), where $\{x_1, ..., x_\ell\}$ are the coordinate functions on $U'_w \approx$ Lie U'_w (Lie U'_w denotes the Lie algebra of U'_w). In particular, $H^0(Y_w, \mathcal{K}_w(-\rho)) \neq 0$. Now $H^0(Y_w, \mathcal{K}_w(-\rho))$ is a B-stable subspace of $H^0(Y_w, \mathcal{F}_w(-\rho)) \approx M(-\rho)^\sigma$ (cf. Remark 2.5 and Lemma 2.8). As is easy to see, any B-stable non-zero subspace of $M(\lambda)^\sigma$ (for any $\lambda \in \mathfrak{h}^*$) contains the λ-th weight space. So $H^0(Y_w, \mathcal{K}_w(-\rho))$ contains the $(-\rho)$-th weight space. (This can also be obtained from I_3 .) This proves, by (I_2) and Proposition (2.10), that $\vartheta \in H^0(\widetilde{Y}_w, \widetilde{\mathcal{K}}_w(-\rho \otimes -\rho))$; thus proving the lemma. □

(2.15) **Lemma.** *The map*

$$\overline{\psi}^{\vartheta}_w : H^0(\widetilde{X}_w, \mathcal{L}_w(\lambda + \rho \otimes \mu + \rho)) \to H^{\ell}_{\widetilde{X}_w/\partial\widetilde{X}_w}(\widetilde{G/B}, \mathcal{L}(\lambda \otimes \mu))$$

(defined in the above lemma) is injective.

Proof. The sheaf $\widetilde{\mathcal{K}}_w \mid_{\widetilde{Y}_w}$ is supported in the G-orbit $\widetilde{\mathcal{B}}_w$ (cf. Definition 1.3) and moreover (by definition) it is a sheaf of $\mathcal{O}_{\widetilde{\mathcal{B}}_w}$-modules. Since the section $\vartheta \in H^0(\widetilde{Y}_w, \widetilde{\mathcal{K}}_w(-\rho \otimes -\rho))$ is G-invariant, $\vartheta(x) \neq 0$ (as an element of the stalk $\widetilde{\mathcal{K}}_w(-\rho \otimes -\rho)_x$) for any $x \in \widetilde{\mathcal{B}}_w$. Now take any $t \neq 0 \in H^0(\widetilde{X}_w, \mathcal{L}_w(\lambda + \rho \otimes \mu + \rho))$. Then there exists a $x_0 \in \widetilde{\mathcal{B}}_w$ such that $t(x_0) \neq 0$. But then, by the commutative diagram $(\mathcal{D})$ (of Lemma 2.14), $\overline{\psi}^{\vartheta}_w(t)(x_0) \neq 0$. In particular, $\overline{\psi}^{\vartheta}_w(t) \neq 0$. □

We recall the following result due to Kumar.

(2.16) **Theorem** [Ku$_1$; Theorem 2.10 and Proposition 2.9]. *The G-module $H^0(\widetilde{X}_w, \mathcal{L}_w(\lambda + \rho \otimes \mu + \rho))$ contains a unique copy of the irreducible G-module $V(\overline{\mu_w - \lambda})$; where μ_w is as in Lemma (2.8), and the bar is as in Corollary* (2.11).

Now we come to the main result of this paper :

(2.17) **Theorem.** *Let G be a semi-simple connected simply-connected complex algebraic group and fix $\lambda, \mu \in D - \rho$ (cf. § 1.1). Then, for any $w \in W'_{\lambda+\rho,\mu+\rho}$,*

$$\overrightarrow{\psi}^{\vartheta}_{w}(V(\overline{\mu_w - \lambda})) \subset H^0(\widetilde{G/B}, \widetilde{\mathcal{F}}_w(\lambda \otimes \mu)),$$

where $W'_{\lambda+\rho,\mu+\rho}$ is as in Theorem (2.2), μ_w is as in Lemma (2.8), and $\overrightarrow{\psi}^{\vartheta}_{w}$ is the G-module map defined in Lemma (2.14).

In particular, $\overrightarrow{\psi}^{\vartheta}_{w}(V(\overline{\mu_w - \lambda}))$ occurs with multiplicity exactly one in the irreducible Harish-Chandra module $N_w := H^0(\widetilde{G/B}, \widetilde{\mathcal{F}}_w(\lambda \otimes \mu))$ (cf. Theorem 2.2) and is its minimal $\Delta(\mathfrak{g})$-type.

(Recall that, by Lemmas (2.3) and (2.4), N_w canonically embeds inside $H^{\ell}_{\widetilde{X}_w/\partial\widetilde{X}_w}(\widetilde{G/B}, \mathcal{L}(\lambda \otimes \mu))$, and moreover the map $\overrightarrow{\psi}^{\vartheta}_{w}$ is injective by Lemma (2.15).)

Proof. By Corollary (2.11), any irreducible G-submodule $V(\theta)$ of N_w (in fact of $H^{\ell}_{\widetilde{X}_w/\partial\widetilde{X}_w}(\widetilde{G/B}, \mathcal{L}(\lambda \otimes \mu))$) satisfies either $\| \theta \| > \| \lambda - \mu_w \|$ or $\theta = \overline{\mu_w - \lambda}$, and in the later case it occurs with multiplicity one in $H^{\ell}_{\widetilde{X}_w/\partial\widetilde{X}_w}(\widetilde{G/B}, \mathcal{L}(\lambda \otimes \mu))$. So the proof of the theorem will be completed, if we show that $V(\overline{\mu_w - \lambda})$ does occur as a component in N_w :

In view of Lemma (2.4) and the long exact local cohomology sequence [H_2; Chap. III, Exercise 2.3] (cf. proof of Lemma 2.3), it suffices to show that $H^1_{\partial\widetilde{X}_w}(\widetilde{G/B}, \widetilde{\mathcal{F}}_w(\lambda \otimes \mu))$ does not contain $V(\overline{\mu_w - \lambda})$ as a component; which is content of the next lemma. This completes the proof of the theorem (modulo the next lemma). □

(2.18) **Lemma.** *The irreducible G-module $V(\overline{\mu_w - \lambda})$ is not a component of $H^1_{\partial\widetilde{X}_w}(\widetilde{G/B}, \widetilde{\mathcal{F}}_w(\lambda \otimes \mu))$, for any $w \in W'_{\lambda+\rho,\mu+\rho}$.*

Proof. By the defining property (P_2) of the sheaf $\mathcal{F}_w$ (cf. Proposition 1.4), $\mathcal{H}^0_{\partial X_w}(G/B, \mathcal{F}_w(\mu)) = 0$. So, by an analogue of Lemma (2.6),

$$(I_4) \cdots H^1_{\partial\widetilde{X}_w}(\widetilde{G/B}, \widetilde{\mathcal{F}}_w(\lambda \otimes \mu)) \approx H^0(G/B, \mathcal{L}(\lambda) \otimes \mathcal{L}(H^1_{\partial X_w}(G/B, \mathcal{F}_w(\mu)))).$$

Consider the following exact sequence $(\mathcal{T})$:

$$H^0_{\partial X_w}(G/B, \mathcal{F}_w(\mu)) = 0 \to H^0(G/B, \mathcal{F}_w(\mu)) \to H^0(Y_w, \mathcal{F}_w(\mu))$$
$$\to H^1_{\partial X_w}(G/B, \mathcal{F}_w(\mu)) \to H^1(G/B, \mathcal{F}_w(\mu)) = 0,$$

where the vanishing of $H^1(G/B, \mathcal{F}_w(\mu))$ is due to [BB;§ 2]. Further, by [BB] (see also [Ka]), $H^0(G/B, \mathcal{F}_w(\mu))$ is the irreducible highest weight $\mathfrak{g}$-module $L(\mu_w)$ with highest weight μ_w (use the fact that w is of smallest length in its coset $wW_{\mu+\rho}$, since $w \in W'_{\lambda+\rho,\mu+\rho}$ by assumption). Hence, by combining Lemma (2.8) with Remark (2.5), we get (by the exact sequence $\mathcal{T}$)

$$H^1_{\partial X_w}(G/B, \mathcal{F}_w(\mu)) \approx M(\mu_w)^\sigma / L(\mu_w).$$

But then, by (I_4) and Proposition (2.10), we get

$$(I_5) \cdots H^1_{\partial \widetilde{X}_w}(\widetilde{G/B}, \widetilde{\mathcal{F}}_w(\lambda \otimes \mu)) \approx \oplus_{\theta \in D}(V(\theta)^* \otimes [\mathbb{C}_{-\lambda} \otimes K(\mu_w) \otimes V(\theta)]^B),$$

as G-modules, where $K(\mu_w) := M(\mu_w)^\sigma / L(\mu_w)$. So, to complete the proof of the lemma, we need to show that

$$\mathcal{C} := [\mathbb{C}_{-\lambda} \otimes K(\mu_w) \otimes V(\overline{\lambda - \mu_w})]^B = 0 :$$

As is easy to see

$$\mathcal{C} \approx \mathrm{Hom}_{\mathfrak{b}}(A^\tau \otimes \mathbb{C}_\lambda, V(\overline{\lambda - \mu_w})),$$

where A is the kernel of the map : $M(\mu_w) \to L(\mu_w)$. So

$$(I_6) \cdots \qquad \mathcal{C} \approx \mathrm{Hom}_{\mathfrak{b}^-}(A \otimes \mathbb{C}_{-\lambda}, V(\overline{\mu_w - \lambda})),$$

where $\mathfrak{b}^-$ is the opposite Borel subalgebra of $\mathfrak{g}$.

Next we claim that $\mu_{w'} - \lambda$ does not occur as a weight in $V(\overline{\mu_w - \lambda})$, for any $w' \in W$ such that

$$(I_7) \cdots \qquad \mu_{w'} = \mu_w - \beta, \text{ for some } \beta \neq 0 \in \textstyle\sum_{i=1}^{\ell} \mathbb{Z}_+ \alpha_i :$$

We first obtain

$$\| \mu_{w'} - \lambda \|^2 = \| \mu_w - \lambda \|^2 + 2 < \beta, \lambda + \rho > .$$

So if $\mu_{w'} - \lambda$ does occur as a weight in $V(\overline{\mu_w - \lambda})$, then

$$(I_8) \cdots \qquad < \beta, \lambda + \rho > = 0 \text{ (cf. proof of Corollary 2.11).}$$

Rewriting (I_7) we get

$$(I_9) \cdots \qquad w^{-1} w'(\mu + \rho) - (\mu + \rho) = w^{-1} \beta.$$

But, by assumption, $w \in W'_{\lambda+\rho,\mu+\rho}$; in particular, $vw > w$ for any $v \in W_{\lambda+\rho}$. This, together with (I_8), gives that

$$(I_{10}) \cdots \qquad w^{-1}\beta \in \sum \mathbb{Z}_+ \alpha_i \text{ ,and of course } w^{-1}\beta \neq 0.$$

Further by (I_9)

$$(I_{11})\cdots \qquad -w^{-1}\beta \in \sum \mathbb{Z}_+\alpha_i.$$

Now (I_{10}) and (I_{11}) contradict each other, proving the assertion that $\mu_{w'} - \lambda$ does not occur as a weight in $V(\overline{\mu_w - \lambda})$. This proves the vanishing of $\mathcal{C}$, by (I_6). □

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