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Bernstein–Gelfand–Gelfand resolution for arbitrary Kac–Moody algebras

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Dedicated to Professor Bertram Kostant on his sixtieth birthday

1. Introduction

The aim of this paper is to prove the existence of the strong Bernstein–Gelfand–Gelfand (abbreviated as BGG) resolution for arbitrary Kac–Moody algebras, hitherto known only for the class of symmetrizable Kac–Moody algebras (referred to as the symmetrizable case). More specifically we have:

Let $\mathfrak{g}$ be an arbitrary Kac–Moody Lie algebra over $\mathbb{C}$, with Cartan subalgebra $\mathfrak{h}$, and associated Weyl group W. Then, for any dominant integral weight $\lambda \in \mathfrak{h}^*$, there is an exact sequence of $\mathfrak{g}$-modules and $\mathfrak{g}$-module maps:

$$(*) \qquad 0 \leftarrow L^{\max}(\lambda) \leftarrow C_0(\lambda) \leftarrow C_1(\lambda) \leftarrow \cdots \leftarrow C_p(\lambda) \leftarrow \cdots,$$

where $C_p(\lambda) := \bigoplus_{\substack{w \in W \\ l(w)=p}} M(w(\lambda+\rho)-\rho)$ is the direct sum of Verma modules $M(w(\lambda+\rho)-\rho)$ with highest weight $w(\lambda+\rho)-\rho$, and $L^{\max}(\lambda)$ is the maximal integrable highest weight module with highest weight λ (cf. Sect. 2.8).

In particular we obtain Kostant's famous theorem on '$\mathfrak{n}$-homology' for arbitrary Kac–Moody algebras (known, so far, only in the symmetrizable case).

Now let us describe the contents of the paper in more detail:

Section 2 is devoted to preliminaries and setting up the notation.

Section 3. A resolution as in (*) was first obtained by BGG in the finite case (i.e. when $\mathfrak{g}$ is a finite dimensional complex semi-simple Lie algebra); by 'algebraic' methods. (For historical remarks, concerning its extension to the symmetrizable case, see Sect. 3.21.) Later Kempf very beautifully realized, again in the finite case, such a resolution as a particular global Cousin complex (rather its dual) whose terms are certain local cohomology modules introduced by Grothendieck; where recall that given a decreasing filtration $\{X_p\}_{p \geqq 0}$ of a topological space X by closed subspaces and an abelian sheaf $\mathscr{S}$ on X, there is defined a complex known as the global Cousin complex of $\mathscr{S}$ with respect to the filtration $\{X_p\}$ (cf. Sect. 2.9).

Kempf considered a particular filtration of G/B by B-stable (closed) subvarieties and proved the exactness of the corresponding global Cousin complex, when $\mathscr{S}$

is any effective line bundle $\mathscr{L}(\lambda)$ on G/B. The exactness turned out to be a consequence of (P_1) G/B is Cohen–Macaulay (since it is smooth) and (P_2) $H^i(G/B, \mathscr{L}(\lambda))=0$, for all $i>0$. Thus the 'geometric' properties (P_1) and (P_2) together replaced the role of the Casimir operator used crucially by BGG to prove the existence of their resolution.

Now in the general Kac–Moody case; we work at a time with one Schubert variety $X_w := \overline{BwB/B} \subset G/B$ (instead of G/B) and define a certain filtration $\mathscr{F}(w)=\{F_p(w)\}_{p\geq 0}$ of X_w by closed subvarieties (cf. Sect. 3.2) (which is somewhat different from Kempf's filtration). Fix a dominant integral weight λ and let $\mathscr{K}(w)$ be the global Cousin complex of the line bundle $\mathscr{L}_w(\lambda)$ on X_w with respect to the filtration $\mathscr{F}(w)$ (cf. Sect. 3.3). Then the exactness of $\mathscr{K}(w)$ (cf. Theorem 3.4) follows rather easily; since the properties (P_1) and (P_2) (with G/B replaced by X_w) have already been established by the author [Ku5] (and also by Mathieu [M]). Our filtrations $\mathscr{F}(w)$ are 'compatible' with respect to taking 'limits' over $w\in W$, where W is equipped with the usual Bruhat partial order. So taking direct limit of the 'duals' of the exact complexes $\mathscr{K}(w)$, we obtain an exact complex $\mathscr{K}^\vee = \mathscr{K}^\vee(\lambda)$ which we refer as the *Kempf complex*. We further prove that the components of the Kempf complex $\mathscr{K}^\vee(\lambda)$ are nothing but $C_p(\lambda)$ (i.e. the direct sum of appropriate Verma modules); the proof of which is somewhat tricky and follows as a consequence of Lemmas (3.14)–(3.17). Putting these together, the strong BGG resolution (as stated in the beginning) (cf. Theorem 3.20) follows trivially. Of course the BGG resolution immediately gives the Weyl–Kac character formula for arbitrary Kac–Moody algebras (cf. Theorem 3.22), proved in the general case by the author (and also by Mathieu). Generalization of the strong BGG resolution (corresponding) to arbitrary parabolic subalgebras of finite type (cf. Theorem 3.27) can be obtained by a very similar method; considering G/P instead of G/B.

Quite differently, using the combinatorics of the Weyl group, Bernstein–Gelfand–Gelfand have defined (cf. Sect. 3.24) a certain complex $\mathscr{C}=\mathscr{C}(\lambda)$ (with the same $C_p(\lambda)$'s), which we refer as the *BGG complex*:

$$0 \longleftarrow L^{\max}(\lambda) \xleftarrow{\varepsilon} C_0(\lambda) \xleftarrow{s_0} C_1(\lambda) \xleftarrow{s_1} \cdots.$$

Using our (main) theorem (3.20) (on the existence of a strong BGG resolution) and following some ideas due to Rocha–Caridi and Wallach, we show that the BGG complex is exact and moreover any exact complex as in (*) is equivalent to the BGG complex (cf. Theorem 3.25 for a more general statement). In particular, the geometrically defined Kempf complex $\mathscr{K}^\vee$ is equivalent to the combinatorially defined BGG complex.

We have not considered, in the paper, the exactness of the 'local Kempf complex' or the exactness of the Kempf complex $K^\vee$ in char. p (even though both of these are true) because we have no immediate applications for these.

Section 4. Let $\mathfrak{p}=\mathfrak{p}_S$ be a standard parabolic subalgebra of finite type, with nil-radical $\mathfrak{u}^+$ and maximal reductive subalgebra $\mathfrak{r}$ (cf. Sect. 2.4). As a straightforward (and standard) consequence of the BGG resolution, we extend Kostant's famous theorem to arbitrary Kac–Moody algebras; which completely determines the Lie algebra homology $H_*(\mathfrak{u}^-, L^{\max}(\lambda))$ as an $\mathfrak{r}$-module (cf. Theorem 4.1). We recall that it was extended to the symmetrizable case by Garland–Lepowsky.

As another consequence of the BGG resolution (rather Theorem 4.1), we prove (cf. Theorem 4.3) that $H^p(\mathfrak{g},\mathfrak{r})=0$, for p odd and dim. $H^{2p}(\mathfrak{g},\mathfrak{r})=\#\{w\in W_S^1 : l(w)=p\}$ (cf. Sect. 2.3); thus extending Lepowsky's result to arbitrary Kac–Moody algebras. Some other consequences to Lie algebra homology are contained in Proposition (4.2). Finally we conjecture that (appropriately defined) the integration map induces an algebra isomorphism: $H^*(\mathfrak{g})\to H^*(G,\mathbb{C})$, where G is the corresponding Kac–Moody group (cf. Conjecture 4.4).

Our dependence on various ideas in Kempf's paper [K] will be clear to any informed reader.

2. Preliminaries and notation

We will follow the notation from [Ku5, Sect. 1]:

(2.1) **Definition (Kac–Moody algebras).** *A generalized Cartan matrix* (GCM) $A=(a_{ij})_{1\leq i,j\leq l}$ is a matrix of integers, satisfying $a_{ii}=2$ for all i; $a_{ij}\leq 0$ if $i\neq j$; and $a_{ij}=0$ if and only if $a_{ji}=0$.

Choose a triple $(\mathfrak{h},\pi,\pi^\vee)$, unique up to isomorphism, where $\mathfrak{h}$ is a vector space over $\mathbb{C}$ of dimension $2l$-rank A; $\pi=\{\alpha_i\}_{1\leq i\leq l}\subset\mathfrak{h}^*$; and $\pi^\vee=\{h_i\}_{1\leq i\leq l}\subset\mathfrak{h}$ are linearly independent indexed sets satisfying $\alpha_j(h_i)=a_{ij}$.

The *Kac–Moody algebra* $\mathfrak{g}=\mathfrak{g}(A)$ is the Lie algebra (over $\mathbb{C}$) generated by $\mathfrak{h}$ and the symbols e_i and $f_i (1\leq i\leq l)$ with the defining relations R_1-R_4:

(R_1) $[\mathfrak{h},\mathfrak{h}]=0$; $[h,e_i]=\alpha_i(h)e_i$, $[h,f_i]=-\alpha_i(h)f_i$, for $h\in\mathfrak{h}$ and all $1\leq i\leq l$,

(R_2) $[e_i,f_j]=\delta_{i,j}h_j$, for all $1\leq i,j\leq l$,

(R_3) $(ad\, e_i)^{1-a_{ij}}(e_j)=0$, for all $1\leq i\neq j\leq l$, and

(R_4) $(ad\, f_i)^{1-a_{ij}}(f_j)=0$, for all $1\leq i\neq j\leq l$.

The GCM A is called *symmetrizable* if there exists a diagonal matrix $D=\text{dia}(d_1,\dots,d_l)$ with all $d_i>0$ and rational, such that DA is symmetric. The Kac–Moody Lie algebra $\mathfrak{g}(A)$ is called symmetrizable if A is symmetrizable.

(2.2) **Root space decomposition.** There is available the *root space decomposition*: $\mathfrak{g}=\mathfrak{h}\oplus\sum_{\alpha\in\Delta\subset\mathfrak{h}^*\setminus\{0\}}\mathfrak{g}_\alpha$, where $\mathfrak{g}_\alpha$ is the root space corresponding to α and Δ, *the set of roots*, consists of all those $\alpha\in\mathfrak{h}^*\setminus\{0\}$ such that $\mathfrak{g}_\alpha\neq 0$. Moreover $\Delta=\Delta_+\cup\Delta_-$, where $\Delta_+\subset\sum_{i=1}^{l}\mathbb{Z}_+\alpha_i$ and $\Delta_-=-\Delta_+$ ($\mathbb{Z}_+$ is the set of non-negative integers). Elements of Δ_+ (resp. Δ_-) are called *positive* (resp. *negative*) *roots*.

(2.3) **Weyl group.** There is the *Weyl group* $W\subset\text{Aut}\,\mathfrak{h}^*$ (associated to $\mathfrak{g}$); which is generated by the '*simple*' *reflections* $\{r_i\}_{1\leqslant i\leqslant l}$, where $r_i\in\text{Aut}\,\mathfrak{h}^*$ is defined by $r_i(\chi)=\chi-\chi(h_i)\alpha_i$. The set of roots Δ is W-stable. Define the set of *real roots* $\Delta^{re}:=W\cdot\pi$. Further $(W,\{r_i\}_{1\leqslant i\leqslant l})$ is a *Coxeter system*, hence we can talk of the *length* $l(w)$ of any element $w\in W$. Set, for any $p\geq 0$, $W^{(p)}:=\{w\in W: l(w)=p\}$. We also have the standard *Bruhat partial ordering* $\leq$ in W.

We define a shifted action $*$ of W on $\mathfrak{h}^*$ by $w*\chi = w(\chi+\rho)-\rho$, where ρ is any fixed element of $\mathfrak{h}^*$ satisfying $\rho(h_i)=1$, for all $1 \leqq i \leqq l$. (The element $w*\chi$ does not depend upon the choice of ρ.)

For any $S \subset \{1,\ldots,l\}$, let W_S be the subgroup of W generated by $\{r_i\}_{i\in S}$ and define a subset

$$W_S^1 := \{w\in W\colon \Delta_+ \cap w\Delta_- \subset \Delta_+ \setminus \Delta_+^S\} \text{ of } W, \text{ where } \Delta_+^S = \Delta_+ \cap \{\sum_{i\in S} \mathbb{Z}\alpha_i\}.$$

The subset W_S^1 can be characterized as the set of elements of minimal length in the cosets $W_S w (w\in W)$ (each such coset contains a unique element of minimal length).

The subset S is said to be of *finite type* if W_S is a finite group.

(2.4) **Parabolics.** We fix a subset $S \subset \{1,\ldots,l\}$ (including $S=\emptyset$) and define the following Lie subalgebras of $\mathfrak{g}$:

$$\mathfrak{n}^{\pm} = \sum_{\alpha\in\Delta_+} \mathfrak{g}_{\pm\alpha}; \quad \mathfrak{u}^{\pm} = \mathfrak{u}_S^{\pm} = \sum_{\alpha\in\Delta_+\setminus\Delta_+^S} \mathfrak{g}_{\pm\alpha}; \quad \mathfrak{g}_S = \mathfrak{h}_S \oplus \sum_{\alpha\in\Delta_+^S} (\mathfrak{g}_\alpha \oplus \mathfrak{g}_{-\alpha});$$

$$\mathfrak{r} = \mathfrak{r}_S = \mathfrak{h} \oplus \sum_{\alpha\in\Delta_+^S} (\mathfrak{g}_\alpha \oplus \mathfrak{g}_{-\alpha}); \quad \mathfrak{b}^{\pm} = \mathfrak{h} + \mathfrak{n}^{\pm}; \quad \text{and} \quad \mathfrak{p} = \mathfrak{p}_S = \mathfrak{r} \oplus \mathfrak{u}^{+},$$

where $\mathfrak{h}_S$ is the span of $\{h_i\}_{i\in S}$.

Then $\mathfrak{r}$ normalizes $\mathfrak{u}^{\pm}$ and $\mathfrak{r}$ is finite dimensional if and only if S is of finite type.

The algebra $\mathfrak{b} = \mathfrak{b}^+$ (resp. $\mathfrak{p}$) is called the *standard Borel* (resp. *a standard parabolic*) subalgebra. The parabolic subalgebra $\mathfrak{p}_S$ is said to be of *finite type*, if S is of finite type. If S is the singleton $\{i\}$, then $\mathfrak{p}_i := \mathfrak{p}_{\{i\}}$ is called the *minimal parabolic subalgebra corresponding to the simple reflection* r_i. The subalgebra $\mathfrak{h}$ is called the *Cartan subalgebra*.

(2.5) **Kac–Moody groups.** Even though there are several constructions of Kac–Moody groups (giving possibly different groups), and we could have used either of these, we will stick to the construction due to Tits. Instead of recalling his construction here we refer the reader to [Ku5, Sects. 1.2–1.3], where it is given in the form convenient for our purposes and the notation of which we adopt here (without explanation). In particular, recall that G denotes the *Kac–Moody group* associated to the (Kac–Moody) Lie algebra $\mathfrak{g}$ (and a fixed choice of a 'suitable' integral lattice $\mathfrak{h}_{\mathbb{Z}} \subset \mathfrak{h}$); T (resp. B) denotes the *standard maximal torus* (resp. *standard Borel subgroup*) of G; and, for any subset $S \subset \{1,\ldots,l\}$, $P = P_S \supset B$ is the *standard parabolic subgroup corresponding to* S. If we take S to be the singleton $\{i\}$, for any $1 \leqq i \leqq l$, then $P_i := P_{\{i\}}$ is called the *minimal parabolic subgroup corresponding to the simple reflection* r_i. If S is of finite type, the corresponding P_S is said to be *standard parabolic subgroup of finite type.*

(2.6) **Opposite Borel subgroup.** Recall (see, e.g., [S, Sect. 1.7]) that for any real root β, there is a unique additive one parameter subgroup U_β and a homomorphism

u_β: $\mathbb{C} \to G$, such that $U_\beta = u_\beta(\mathbb{C})$ and $tu_\beta(z)t^{-1} = u_\beta(e^\beta(t).z)$, for all $t \in T$ and $z \in \mathbb{C}$. Furthermore, for any $n \in N$ (where $N = N(T)$ is the normalizer of T in G), we have $nU_\beta n^{-1} = U_{w\beta}$, where $w := n \bmod T \in N/T \approx W$. Now let U^- denote the subgroup of G generated by the subgroups $\{U_\beta\}$, where β ranges over all the negative real roots (i.e. $\Delta^{re} \cap \Delta_-$). The torus T normalizes U^- and let B^- be their semi-direct product. We call B^- as the *standard opposite Borel subgroup.*

(2.7) **Dominant weights.** Any element λ of $\mathfrak{h}^*_\mathbb{Z} := \mathrm{Hom}_\mathbb{Z}(\mathfrak{h}_\mathbb{Z}, \mathbb{Z})$ is called an *integral weight*, where $\mathfrak{h}_\mathbb{Z} \subset \mathfrak{h}$ is the integral lattice fixed in Sect. 2.5. It is called *dominant* if it satisfies $\lambda(h_i) \geqq 0$, for all $1 \leqq i \leqq l$.

(2.8) **Generalized Verma modules.** For any $\lambda \in \mathfrak{h}^*$, one defines the *Verma module* $M(\lambda) := U(\mathfrak{g}) \bigotimes_{U(\mathfrak{b})} \mathbb{C}_\lambda$, where $\mathbb{C}_\lambda$ is the one dimensional $\mathfrak{b}$-module such that the Cartan subalgebra $\mathfrak{h}$ acts by the weight λ and $\mathfrak{n}^+$ acts (of course) trivially.

More generally, let $S \subset \{1, \ldots, l\}$ be any subset of finite type. Let $D_S := \{\lambda \in \mathfrak{h}^* : \lambda(h_i) \in \mathbb{Z}_+$, for all $i \in S\}$. There is a natural bijection between D_S and the set of (isomorphism classes of) finite-dimensional irreducible $\mathfrak{r} = \mathfrak{r}_S$-modules which remain irreducible as $\mathfrak{g}_S$-modules; given by: any such $\mathfrak{r}_S$-module $L \mapsto$ its highest weight (cf. [GL, Proposition 3.1]). For any $\lambda \in D_S$, we denote the corresponding irreducible $\mathfrak{r}$-module by $V_S(\lambda)$. Now the *generalized Verma module* $M_S(\lambda)$ (for any $\lambda \in D_S$) is by definition $U(\mathfrak{g}) \bigotimes_{U(\mathfrak{p})} V_S(\lambda)$, where the $\mathfrak{r}$-module structure on $V_S(\lambda)$ is extended to a $\mathfrak{p}$-module structure by demanding $\mathfrak{u}^+$ to act trivially on $V_S(\lambda)$.

Finally for any dominant integral weight λ, define $L^{\max}(\lambda) = M(\lambda)/M'(\lambda)$, where $M'(\lambda)$ is the $U(\mathfrak{g})$-submodule of $M(\lambda)$ generated by the elements $\{f_i^{\lambda(h_i)+1} v_\lambda\}_{1 \leqslant i \leqslant l}$ (v_λ is the highest weight vector in $M(\lambda)$). It can be easily seen that $L^{\max}(\lambda)$ is an integrable (highest weight) $\mathfrak{g}$-module and any integrable highest weight $\mathfrak{g}$-module with highest weight λ is a quotient of $L^{\max}(\lambda)$. In the symmetrizable case, $L^{\max}(\lambda)$ is known to be irreducible but its irreducibility is an open question in the non-symmetrizable case.

(2.9) **Grothendieck's local cohomology ([G, H1, H2, K, ...]).** Let $\mathscr{S}$ be any abelian sheaf on a topological space X and let $Z \subseteq Y \subseteq X$ be closed subspaces. Then $H^p_{Y/Z}(X, \mathscr{S})$ (for any $p \geqq 0$) denotes the *local cohomology* (also called the *cohomology with supports*), as defined in [H1, p. 219, Variation 2]. If Z is the empty set, $H^*_{Y/Z}(X, \mathscr{S})$ will generally be abbreviated as $H^*_Y(X, \mathscr{S})$. Recall that the local cohomology is functorial in the sense made precise in [K, Lemma 11.3].

We also recall (cf. [H1, Chapter IV, Proposition 2.3] or [K, Lemma 7.8]) that for any topological space X with a filtration by closed subspaces: $X = X_0 \supseteq X_1 \supseteq X_2 \supseteq \cdots$, and any abelian sheaf $\mathscr{S}$ on X, there is associated a complex (i.e. composite of any two successive maps is zero) known as the *global Cousin complex of $\mathscr{S}$ with respect to the filtration (X_p) of X*:

$$0 \to H^0(X, \mathscr{S}) \xrightarrow{\varepsilon} H^0_{X_0/X_1}(X, \mathscr{S}) \xrightarrow{d_0} H^1_{X_1/X_2}(X, \mathscr{S}) \xrightarrow{d_1} H^2_{X_2/X_3}(X, \mathscr{S}) \to \cdots.$$

□

We record the following two lemmas for their use in Sect. 3.

(2.10) **Lemma.** *Let* $\mathbb{A}^d$ *be the affine space of* dim. d *over a field* k. *Then*:
(a) $H^p_{\{0\}}(\mathbb{A}^d, \mathcal{O}_{\mathbb{A}^d}) = 0$, *for* $p \neq d$ *and*
(b) $H^d_{\{0\}}(\mathbb{A}^d, \mathcal{O}_{\mathbb{A}^d})$ *is 'canonically' isomorphic with* $\sum_{n_1,\dots,n_d<0} k x_1^{n_1} \cdots x_d^{n_d}$ *as k-vector spaces*; where 0 is the origin of $\mathbb{A}^d$, $\mathcal{O}_{\mathbb{A}^d}$ denotes the structure sheaf and $(x_1, \dots, x_d)$ are the coordinate functions on $\mathbb{A}^d$.

For a proof see, e.g., [G, Corollary 3.10] together with [H2, Chap. III, Exercise 6.11] and [G, Proposition 1.12]. □

(2.11) **Lemma.** (a) *Let* K *be a (finite dimensional) affine algebraic group over* $\mathbb{C}$ *with Lie algebra* $\mathfrak{k}$, *let* X *be a* K*-variety over* $\mathbb{C}$, *and let* $\mathscr{S}$ *be a* K*-equivariant vector bundle (locally free sheaf) on* X. *Then for any closed subspaces* $Y \supseteq Z$ *of* X, *the local cohomology* $H^p_{Y/Z}(X, \mathscr{S})$ *(for any* $p \geq 0$*) admits a natural structure of a* $\mathfrak{k}$*-module, which is functorial in the following sense*:

Let X' *be another* K*-variety over* $\mathbb{C}$ *with closed subspaces* $Y' \supseteq Z'$, *and a* K*-morphism* $f: X' \to X$ *such that* $Y' \supseteq f^{-1}(Y)$ *and* $Z' \supseteq f^{-1}(Z)$. *Then the induced map*: $H^p_{Y/Z}(X, \mathscr{S}) \to H^p_{Y'/Z'}(X', f^*(\mathscr{S}))$ *is a* $\mathfrak{k}$*-module map.*

(b) *If we assume in addition (in the first paragraph of* (a)*) that* Y *and* Z *are both* K*-stable, then the* $\mathfrak{k}$*-module structure on* $H^p_{Y/Z}(X, \mathscr{S})$ *integrates to give a* K*-module structure.*

Even though not stated exactly in this form, a proof of the above lemma can be found in [K, Sect. 11]. (Actually [K, Sect. 11] contains more general results, but we do not need them.) □

3. Kempf and the BGG resolutions

Throughout this section G *is a Kac–Moody group (associated to any Kac–Moody Lie algebra* $\mathfrak{g}$*), with standard Borel subgroup* B, *opposite Borel subgroup* B^-, *maximal torus* T, *and the associated Weyl group* $W \approx N(T)/T$ (cf. Sects. 2.5–2.6).

For any $v \in W$, define the following subsets of G/B:

$$\mathscr{B}_v = BvB/B$$
$$\mathscr{B}^v = B^- vB/B$$
$$X_v = \overline{\mathscr{B}}_v$$
$$X^v = \overline{\mathscr{B}^v}$$

and

$$\partial X^v = X^v \backslash \mathscr{B}^v,$$

where the closure (denoted by —) is taken in G/B with respect to the Zariski topology on G/B (cf. [S]).

The following is well known

(3.1) **Lemma** [KP, Lemma 3.4 and Remarks 3.3–3.4].

$$X_v = \bigcup_{v' \leqq v} \mathscr{B}_{v'},$$

$$X^v = \bigcup_{v \leqq v'} \mathscr{B}^{v'},$$

and

$$\mathscr{B}^v \cap X_w \text{ is non-empty if and only if } v \leqq w. \quad \square$$

(3.2) **Definition.** Fix a $w \in W$ and define a decreasing filtration $\mathscr{F}(w) = \{F_p(w)\}_{p \geqq 0}$ of X_w by (Zariski) closed subspaces, where

$$F_p(w) := \bigcup_{l(v) \geqq p} \mathscr{B}^v \cap X_w.$$

Of course $F_0(w) = X_w$ and moreover, by Lemma (3.1), $F_p(w)$ is the empty set $\emptyset$, for all $p > l(w)$.

As in [Ku5, Sect. 1.8], *we will always endow* X_w *with the 'stable variety structure'; which makes* X_w *into an irreducible projective variety over* $\mathbb{C}$ *of* dim. $l(w)$. *There is a line bundle (invertible sheaf)* $\mathscr{L}_w(\lambda)$ *on* X_w *associated to any integral weight* λ (cf. [Ku5, Sect. 2.2]). *In what follows, we fix once and for all a dominant integral* λ (cf. Sect. 2.7).

(3.3) **Definition.** Recall from Sect. (2.9) that associated to the filtration $\mathscr{F}(w)$ and the line bundle $\mathscr{L}_w(\lambda)$, there is a complex called the *global Cousin complex*

$$\mathscr{K}(w): \qquad 0 \to H^0(X_w, \mathscr{L}_w(\lambda)) \xrightarrow{\varepsilon(w)} H^0_{F_0(w)/F_1(w)}(X_w, \mathscr{L}_w(\lambda)) \xrightarrow{d_0(w)} \cdots$$

$$\cdots \xrightarrow{d_{p-1}(w)} H^p_{F_p(w)/F_{p+1}(w)}(X_w, \mathscr{L}_w(\lambda)) \to \cdots.$$

Now we can state our basic

(3.4) **Theorem.** *For any* $w \in W$ *and any dominant integral weight* λ, *the sequence* $\mathscr{K}(w)$ *defined above is exact.* $\square$

As a preparation for the proof of the above theorem, we recall the following lemma, essentially due to Kazhdan–Lusztig, from [Ku6]:

(3.5) **Lemma** [Ku6, Lemma 3.3]. *Let* $v \leqq w$ *be arbitrary elements of the Weyl group* W. *Then the map*

$$\theta_{v,w}: U_v \times (\mathscr{B}^v \cap X_w) \to (vB^- B/B) \cap X_w,$$

defined by $\theta_{v,w}(g, x) = gx$, *for* $g \in U_v$ *and* $x \in \mathscr{B}^v \cap X_w$, *is a biregular T-equivariant isomorphism*; where U_v is the finite dimensional unipotent algebraic group which is subgroup of G with Lie algebra $\sum_{\alpha \in \Delta_+ \cap v\Delta_-} \mathfrak{g}_\alpha$, $\mathscr{B}^v \cap X_w$ is equipped with the affine variety structure as in [Ku6, Sect. 3.2], and, $vB^- B/B$ being open in G/B, $(vB^- B/B) \cap X_w$ is an open subvariety of X_w. $\square$

(3.6) *Proof of Theorem* (3.4). For any $p \geqq 0$, the locally closed subspace $F_p(w) \backslash F_{p+1}(w)$ (of X_w) with the reduced subscheme structure is a disjoint union

$\coprod_{\substack{v \leq w \\ l(v)=p}} (\mathscr{B}^v \cap X_w)$ of non-empty open affine subschemes (cf. Lemmas 3.1 and 3.5). By the above lemma (U_v being affine) $(vB^- B/B) \cap X_w$ is an (open) affine subvariety of X_w. Hence to prove that the inclusion $\mathscr{B}^v \cap X_w \hookrightarrow X_w$ is an affine morphism, it suffices to observe (in view of the above lemma) that:

(1) For any two affine open subsets U and V in a variety X, $U \cap V$ is also affine (cf. [H3, Chap. II, Exercise 4.3]) and

(2) For any two affine varieties X and Y, the inclusion $Y \hookrightarrow X \times Y$ given by $y \mapsto (x_0, y)$, for some fixed $x_0 \in X$, is an affine morphism.

Further, by the above lemma,

$$\operatorname{codim}_{X_w}(\mathscr{B}^v \cap X_w) = \dim. U_v = l(v).$$

Now the theorem follows from [K, Theorem 10.9] together with [Ku5, Theorems (2.16) and (2.23)]. □

(3.7) **Lemma.** *Fix any $w \in W$ and a simple reflection r_i such that $r_i w < w$. Let P_i (resp. $\mathfrak{p}_i$) denote the corresponding (to r_i) minimal parabolic subgroup of G (resp. minimal parabolic subalgebra of* $\mathfrak{g}$) (cf. Sects. 2.4–2.5). *Then all the modules occurring in the global cousin complex $\mathscr{K}(w)$* (cf. Sect. 3.3) *have natural $\mathfrak{p}_i$-module (in particular $\mathfrak{b}$-module) structures and all the maps are $\mathfrak{p}_i$-module maps. Moreover the action of the Cartan subalgebra $\mathfrak{h} \subset \mathfrak{p}_i$ integrates to give an action of the torus T on all the modules (in the sequence $\mathscr{K}(w)$).*

Such modules are generally called $(\mathfrak{p}_i, T)$-modules.

Proof. Since $r_i w < w$ (by assumption), the group P_i acts on X_w (under the left multiplication). Moreover, by [S, Sect. 1.11, Lemma 2], the action of P_i on X_w factors through the action of a finite dimensional algebraic group, say $P_i(w)$, (which is a quotient group of P_i) to give a regular action of $P_i(w)$ on X_w, i.e., we have a morphism: $P_i(w) \times X_w \to X_w$ making the following triangle commutative:

$$\begin{array}{ccc} P_i \times X_w & \longrightarrow & X_w \\ & \searrow \quad \nearrow & \\ & P_i(w) \times X_w & \end{array}$$

Moreover $P_i(w)$ can be chosen so that it contains T.

Now the lemma is an immediate consequence of Lemma (2.11), if we observe that the action of the torus T on X_w leaves $F_p(w)$ stable for all $p \geq 0$. □

(3.8) **Definitions.** Let V be a T-module. It is called a *weight module* if it is the (direct) sum of all its weight spaces, i.e., $V = \bigoplus_{e^\mu \in X(T)} V_\mu$; where $X(T)$ is the character group of the torus T, and $V_\mu := \{v \in V : t.v = e^\mu(t)v, \text{ for all } t \in T\}$ is the μ-th weight space.

For any weight module V, we define its *restricted dual* $V^\vee$ by

$$V^\vee := \bigoplus_{e^\mu \in X(T)} V_\mu^*,$$

where $V_\mu{}^*$ is the full vector space dual of V_μ. Of course $V^\vee$ is canonically a T-module.

A weight module V is called *admissible* if $\dim V_\mu < \infty$, for all $e^\mu \in X(T)$. An admissible T-module V admits *formal character* $\operatorname{ch} V := \sum_{e^\mu \in X(T)} \dim V_\mu . e^\mu$.

For any (noetherian) T-variety X, a closed T-stable subspace Y, and any T-equivariant locally free sheaf $\mathscr{S}$ on X; it is known that $H^p_Y(X, \mathscr{S})$ are weight modules under the natural T-action. In particular the restricted dual $H^p_Y(X, \mathscr{S})^\vee$ makes sense.

(3.9) **Definitions.** For any $v, w \in W$ and $p \geqq 0$, set

$$H^p_v(X_w, \mathscr{L}_w(\lambda)) := H^p_{X^v \cap X_w/(\partial X^v) \cap X_w}(X_w, \mathscr{L}_w(\lambda)).$$

By Lemma (3.7) (actually by its proof) $H^p_v(X_w, \mathscr{L}_w(\lambda))$ is naturally a $(\mathfrak{p}_i, T)$-module, for any minimal parabolic subalgebra $\mathfrak{p}_i$ such that $r_i w < w$.

Let $w \leqq w'$ and v be arbitrary elements of W. Since $X_w \subset X_{w'}$ (Lemma 3.1) and moreover, for any $p \geqq 0$, $F_p(w') \cap X_w \subset F_p(w)$ (in fact the equality holds), one has the canonical maps:

$$\phi^p_{w,w'} : H^p_{F_p(w')/F_{p+1}(w')}(X_{w'}, \mathscr{L}_{w'}(\lambda)) \to H^p_{F_p(w)/F_{p+1}(w)}(X_w, \mathscr{L}_w(\lambda)),$$

and also

$$\phi^v_{w,w'} : H^{l(v)}_v(X_{w'}, \mathscr{L}_{w'}(\lambda)) \to H^{l(v)}_v(X_w, \mathscr{L}_w(\lambda)).$$

(As a consequence of Lemmas (3.11) and (3.15), $H^n_{F_p(w)/F_{p+1}(w)}(X_w, \mathscr{L}_w(\lambda)) = 0$ for $n \neq p$ and also $H^n_v(X_w, \mathscr{L}_w(\lambda)) = 0$, for $n \neq l(v)$.)

Further, from the naturality, both of the above maps are $\mathfrak{b}$- (in particular T-) module maps. Taking the restricted duals (cf. Sect. 3.8), we get the following maps:

$$\psi^p_{w,w'} : H^p_{F_p(w)/F_{p+1}(w)}(X_w, \mathscr{L}_w(\lambda))^\vee \to H^p_{F_p(w')/F_{p+1}(w')}(X_{w'}, \mathscr{L}_{w'}(\lambda))^\vee,$$

and

$$\psi^v_{w,w'} : H^{l(v)}_v(X_w, \mathscr{L}_w(\lambda))^\vee \to H^{l(v)}_v(X_{w'}, \mathscr{L}_{w'}(\lambda))^\vee.$$

These ($\mathfrak{b}$-module) maps enable us to define the following $\mathfrak{b}$-modules, for any $p \geqq 0$ and $v \in W$:

$$H^p_{F_p/F_{p+1}}(G/B, \mathscr{L}(\lambda))^\vee := \varinjlim_{w \in W} H^p_{F_p(w)/F_{p+1}(w)}(X_w, \mathscr{L}_w(\lambda))^\vee,$$

and

$$H^{l(v)}_v(G/B, \mathscr{L}(\lambda))^\vee := \varinjlim_{w \in W} H^{l(v)}_v(X_w, \mathscr{L}_w(\lambda))^\vee.$$

We also define

$$H^0(G/B, \mathscr{L}(\lambda))^\vee := \varinjlim_{w \in W} H^0(X_w, \mathscr{L}_w(\lambda))^\vee,$$

where W is equipped with the Bruhat partial ordering.

The maps $\psi^p_{w,w'}$ give rise to a chain map (cf. [K, Lemma 11.3]) denoted $\psi_{w,w'}$ from the restricted dual $\mathscr{K}^\vee(w)$ of the sequence $\mathscr{K}(w)$ (cf. Sect. 3.3) to $\mathscr{K}^\vee(w')$, whenever $w \leqq w'$, i.e., the following diagram is commutative:

$$\begin{array}{ccccc}
H^0(X_w, \mathscr{L}_w(\lambda))^\vee \xleftarrow{\varepsilon^\vee(w)} H^0_{F_0(w)/F_1(w)}(X_w, \mathscr{L}_w(\lambda))^\vee \xleftarrow{\delta_0(w)} \cdots \xleftarrow{\delta_{p-1}(w)} H^p_{F_p(w)/F_{p+1}(w)}(X_w, \mathscr{L}_w(\lambda))^\vee \leftarrow \cdots \\
\downarrow \qquad\qquad \downarrow \psi^0_{w,w'} \qquad\qquad \downarrow \psi^p_{w,w'} \\
H^0(X_{w'}, \mathscr{L}_{w'}(\lambda))^\vee \xleftarrow{\varepsilon^\vee(w')} H^0_{F_0(w')/F_1(w')}(X_{w'}, \mathscr{L}_{w'}(\lambda))^\vee \xleftarrow{\delta_0(w')} \cdots \xleftarrow{\delta_{p-1}(w')} H^p_{F_p(w')/F_{p+1}(w')}(X_{w'}, \mathscr{L}_{w'}(\lambda))^\vee \leftarrow \cdots,
\end{array}$$

where the first (unlabelled) vertical map is the canonical map induced from the inclusion $X_w \subset X_{w'}$, and $\delta_p(w)$ (resp. $\varepsilon^\vee(w)$) is the dual of the map $d_p(w)$ (resp. $\varepsilon(w)$) of the complex $\mathscr{K}(w)$.

Taking the direct limit of the chain complexes $\mathscr{K}^\vee(w)$ via the chain maps $\psi_{w,w'}$, we obtain the following fundamental chain complex $\mathscr{K}^\vee$ which we refer as the *Kempf complex*:

$$0 \leftarrow H^0(G/B, \mathscr{L}(\lambda))^\vee \xleftarrow{\varepsilon^\vee} H^0_{F_0/F_1}(G/B, \mathscr{L}(\lambda))^\vee \xleftarrow{\delta_0} \cdots \xleftarrow{\delta_{p-1}} H^p_{F_p/F_{p+1}}(G/B, \mathscr{L}(\lambda))^\vee \leftarrow \cdots.$$

□

As an immediate consequence of Theorem (3.4), we have the following:

(3.10) **Corollary.** *The Kempf complex* $\mathscr{K}^\vee$ *defined above is exact.* □

(3.11) **Lemma.** *For any* $w \in W$ *and* $n, p \geqq 0$, $H^n_{F_p(w)/F_{p+1}(w)}(X_w, \mathscr{L}_w(\lambda))$ *is canonically isomorphic with* $\bigoplus_{v \in W^{(p)}} H^n_v(X_w, \mathscr{L}_w(\lambda))$. (See also Lemma (3.15).)

In particular, taking limits of the restricted duals, we get a canonical isomorphism:

$$H^n_{F_p/F_{p+1}}(G/B, \mathscr{L}(\lambda))^\vee \approx \bigoplus_{v \in W^{(p)}} H^n_v(G/B, \mathscr{L}(\lambda))^\vee.$$

Proof. By the definition, $F_p(w) \backslash F_{p+1}(w)$ is a disjoint union $\coprod_{v \in W^{(p)}} (\mathscr{B}^v \cap X_w)$ of open (possibly empty) subsets. Now

$$H^n_{F_p(w)/F_{p+1}(w)}(X_w, \mathscr{L}_w(\lambda)) \approx H^n_{F_p(w) \backslash F_{p+1}(w)}(X_w \backslash F_{p+1}(w), \mathscr{L}_w(\lambda)),$$

by [K, Lemma 7.7]

$$\approx \bigoplus_{v \in W^{(p)}} H^n_{\mathscr{B}^v \cap X_w}(X_w \backslash F_{p+1}(w), \mathscr{L}_w(\lambda)),$$

by the Mayer–Vietoris sequence (cf. [H3, Chap. III, Exercise 2.4])

$$\approx \bigoplus_{v \in W^{(p)}} H^n_{X^v \cap X_w/(\partial X^v) \cap X_w}(X_w, \mathscr{L}_w(\lambda)),$$

by [K, Lemmas 7.7 and 7.9]. □

From now on we will freely identify $H^n_{F_p/F_{p+1}}(G/B, \mathscr{L}(\lambda))^\vee$ *with* $\bigoplus_{v \in W^{(p)}} H^n_v(G/B, \mathscr{L}(\lambda))^\vee$ *under the canonical isomorphism given in the proof of the above lemma.*

(3.12) **Definition.** For any $v \in W^{(p+1)}$ and $v' \in W^{(p)}$, we define the map $\delta^{v,v'}: H_v^{p+1}(G/B, \mathscr{L}(\lambda))^\vee \to H_{v'}^p(G/B, \mathscr{L}(\lambda))^\vee$ as the restriction of the map δ_p (of the Kempf complex $\mathscr{K}^\vee$) to $H_v^{p+1}(G/B, \mathscr{L}(\lambda))^\vee$ followed by the projection onto the $H_{v'}^p(G/B, \mathscr{L}(\lambda))^\vee$ factor under the above decomposition.

One can similarly define a map (for any $w \in W$) $\delta^{v,v'}(w): H_v^{p+1}(X_w, \mathscr{L}_w(\lambda))^\vee \to H_{v'}^p(X_w, \mathscr{L}_w(\lambda))^\vee$ (cf. Lemma 3.11). □

We have the following crucial proposition on the structure of $H_v^{l(v)}(G/B, \mathscr{L}(\lambda))^\vee$:

(3.13) **Proposition.** *With the notation as in* Sect. 3.9, *the* $\mathfrak{b}$-*module structure on* $H_v^{l(v)}(G/B, \mathscr{L}(\lambda))^\vee$ *extends canonically to give a* $\mathfrak{g}$-*module structure and moreover it is isomorphic with* $M(v*\lambda)$ *as* $\mathfrak{g}$-*modules*; where $M(\mu)$ (for any $\mu \in \mathfrak{h}^*$) is the Verma module with highest weight μ and the notation $*$ stands for the shifted action of the Weyl group on $\mathfrak{h}^*$ (cf. Sect. 2.3). □

As a preparation for the proof of the above proposition, we prove the following four Lemmas (3.14)–(3.17):

(3.14) **Lemma.** *Let V be a* $\mathfrak{g}$-*submodule of a Verma module* $M(\mu)$ (*for any* $\mu \in \mathfrak{h}^*$). *Then V itself is isomorphic to a Verma module* $M(\mu')$ *if and only if* $\operatorname{ch} V = \operatorname{ch} M(\mu')$ (cf. Definition 3.8).

Proof. It suffices to prove the implication $\Leftarrow$:

Choose a non-zero vector $v_0 \in V$ of weight μ' (which exists and is unique upto a non-zero scalar multiple; by the assumption on ch V). Clearly $U(\mathfrak{b})v_0 \subseteq \mathbb{C}v_0$. Let V' be the $\mathfrak{g}$-submodule of V generated by v_0. Since any non-zero homomorphism of one Verma module into another is injective, V' is isomorphic with the Verma module $M(\mu')$. Hence $\operatorname{ch} V' = \operatorname{ch} M(\mu')$

$$= \operatorname{ch} V \text{ (by assumption).}$$

But V' being a submodule of V, this is possible only if $V' = V$, proving the lemma. □

(3.15) **Lemma.** *With the notation as in* Sect. 3.9; *for any* $v, w \in W$ *we have*:

(a) $H_v^p(X_w, \mathscr{L}_w(\lambda)) = 0$, *unless* $v \leqq w$ *and* $p = l(v)$

and

(b) *If* $v \leqq w$, $H_v^{l(v)}(X_w, \mathscr{L}_w(\lambda)) \approx H_{\{e\}}^{l(v)}(U_v, \mathcal{O}_{U_v}) \otimes H^0(\mathscr{B}^v \cap X_w, \mathscr{L}_w(\lambda)|_{\mathscr{B}^v \cap X_w})$

as T-modules; where e is the identity of the unipotent group U_v defined in Lemma (3.5), $\mathcal{O}_{U_v}$ denotes its structure sheaf, $H_{\{e\}}^*(\cdot,\cdot)$ denotes the local cohomology with support in the singleton $\{e\}$, $\mathscr{B}^v \cap X_w$ is equipped with the affine variety structure as in Lemma (3.5), and T acts diagonally on the right side of (b).

Proof. $H_v^p(X_w, \mathscr{L}_w(\lambda)) = H^p_{X^v \cap X_w/(\partial X^v) \cap X_w}(X_w, \mathscr{L}_w(\lambda))$ (cf. Definition 3.9)

$$\approx H^p_{\mathscr{B}^v \cap X_w}((vB^-B/B) \cap X_w, \mathscr{L}_w(\lambda))$$

(by [K, Lemmas 7.7 and 7.9]; since $\mathscr{B}^v \cap X_w$ is closed in the open subset

$(vB^-B/B)\cap X_w$ of X_w by Lemma 3.5)

$$\approx H^p_{\{e\}\times \mathscr{B}^v\cap X_w}(U_v\times(\mathscr{B}^v\cap X_w), \theta^*_{v,w}\mathscr{L}_w(\lambda)).$$

(by Lemma 3.5)

$$\approx H^p_{\{e\}}(U_v, \mathcal{O}_{U_v})\otimes H^0(\mathscr{B}^v\cap X_w, \mathscr{L}_w(\lambda)|_{\mathscr{B}^v\cap X_w})$$

(by [G, Proposition 5.5 and Corollary 5.6] together with [H3, Chap. III, Exercise 8.2]; considering the projection $U_v\times(\mathscr{B}^v\cap X_w)\to U_v$ and observing that $\mathscr{L}_w(\lambda)|_{(vB^-B/B)\cap X_w}$ is trivial).

But by Lemma (2.10), $H^p_{\{e\}}(U_v, \mathcal{O}_{U_v})=0$, unless $p=\dim. U_v=l(v)$.

Now the lemma follows by observing that $\mathscr{B}^v\cap X_w=\varnothing$, unless $v\leqq w$ (cf. Lemma 3.1). □

(3.16) **Lemma.** *For any $v\in W$, $H^{l(v)}_v(G/B, \mathscr{L}(\lambda))^\vee$ is an admissible T-module. Further*

$$\operatorname{ch}(H^{l(v)}_v(G/B, \mathscr{L}(\lambda))^\vee)=e^{v*\lambda}\prod_{\beta\in\tilde{\Delta}_+}(1-e^{-\beta})^{-1}=\operatorname{ch} M(v*\lambda),$$

where $\tilde{\Delta}_+$ is the indexed set of positive roots (consisting of positive roots such that each root occurs exactly as many times as its multiplicity).

Proof. Define a map $\xi_v: U^-\cap vU^-v^{-1}\to\mathscr{B}^v$ by $\xi_v(g)=gv \bmod B$, for $g\in U^-\cap vU^-v^{-1}$. Then ξ_v is a T-equivariant bijective map. Let us denote by $\mathscr{L}(\lambda)$ the (topological) line bundle on G/B associated to the principal B-bundle: $G\to G/B$, by the character $\mathbb{C}_{-\lambda}$ of B (cf. Sect. 2.8). Define a continuous section $s_v:\mathscr{B}^v\to\mathscr{L}(\lambda):=G\times_B\mathbb{C}_{-\lambda}$ by $s_v(\xi_v(g))=(g\bar{v}, 1_{-\lambda}) \bmod B$, for $g\in U^-\cap vU^-v^{-1}$; where $1_{-\lambda}$ is some fixed non-zero element in $\mathbb{C}_{-\lambda}$, and $\bar{v}$ is some fixed element of $N(T)$ such that $\bar{v} \bmod T=v$. (A different choice of $\bar{v}$ and $1_{-\lambda}$ only changes s_v by a non-zero scalar multiple.) The section s_v, on restriction, gives a regular section denoted $s_v(w)$, of the line bundle $\mathscr{L}_w(\lambda)|_{\mathscr{B}^v\cap X_w}$ (cf. Sect. 3.2). Since $s_v(w)$ is a nowhere zero section, we obtain that the line bundle $\mathscr{L}_w(\lambda)|_{\mathscr{B}^v\cap X_w}$ is trivial and hence the map:

$$\mathbb{C}[\mathscr{B}^v\cap X_w]\to H^0(\mathscr{B}^v\cap X_w, \mathscr{L}_w(\lambda)|_{\mathscr{B}^v\cap X_w}),\quad \text{given by } f\mapsto f.s_v(w)$$

for any $f\in\mathbb{C}[\mathscr{B}^v\cap X_w]$ (where $\mathbb{C}[\mathscr{B}^v\cap X_w]$ is the ring of regular functions on the affine variety $\mathscr{B}^v\cap X_w$), is an isomorphism.

Since $\bigcup_w X_w=G/B$, from the T-equivariant bijection ξ_v together with [Ku6, Sect. 3.2], it is easy to see that

$$\text{(I1)}\quad \operatorname{ch}\left(\varinjlim_{w\in W}(\mathbb{C}[\mathscr{B}^v\cap X_w]^\vee)\right)=\prod_{\beta\in\tilde{\Delta}_+\cap v\tilde{\Delta}_+}(1-e^{-\beta})^{-1}.$$

Since the weight of the section $s_v(w)$ is $e^{-v\lambda}$, we get:

$$\text{(I2)}\quad \operatorname{ch}\left(\varinjlim_{w\in W}(H^0(\mathscr{B}^v\cap X_w, \mathscr{L}_w(\lambda)|_{\mathscr{B}^v\cap X_w})^\vee)\right)=e^{v\lambda}\cdot\prod_{\beta\in\tilde{\Delta}_+\cap v\tilde{\Delta}_+}(1-e^{-\beta})^{-1}.$$

Since U_v (defined in Lemma 3.5) is biregular isomorphic to its Lie algebra under the exponential map, we obtain by Lemma (2.10):

$$\mathrm{ch}(H^{l(v)}_{\{e\}}(U_v, \mathcal{O}_{U_v})) = \prod_{\gamma\in\tilde{\Delta}_+\cap v\tilde{\Delta}_-} [e^{\gamma}(1-e^{\gamma})^{-1}]$$

(observe that all the roots in $\Delta_+\cap v\Delta_-$ are real and hence have multiplicity 1), and hence

$$\text{(I3)}\qquad \mathrm{ch}(H^{l(v)}_{\{e\}}(U_v, \mathcal{O}_{U_v})^{\vee}) = e^{v\rho-\rho}\prod_{\gamma\in\tilde{\Delta}_+\cap v\tilde{\Delta}_-}(1-e^{-\gamma})^{-1}$$

$$\left(\text{since } \sum_{\gamma\in\tilde{\Delta}_+\cap v\tilde{\Delta}_-}\gamma = \rho - v\rho, \text{ where } \rho \text{ is defined in Sect. 2.3}\right).$$

By Lemma 3.15(b), combining I2 – I3, we get:

$$\mathrm{ch}(H^{l(v)}_v(G/B, \mathcal{L}(\lambda))^{\vee}) = e^{v*\lambda}\prod_{\beta\in\tilde{\Delta}_+}(1-e^{-\beta})^{-1}.$$

This proves the lemma. □

(3.17) **Lemma.** *For any $v, w\in W$ and any simple reflection r_i such that $vr_i > v$ we have:*

$$H^{l(v)+1}_{X^v\cap X_w/((\partial X^v)\setminus\mathcal{B}^{vr_i})\cap X_w}(X_w, \mathcal{L}_w(\lambda)) = 0.$$

(Observe that by Lemma (3.1), $\mathcal{B}^{vr_i}$ is an open subset of ∂X^v.)

Proof. Let $\pi_i: G/B \to G/P_i$ be the projection, where P_i is the minimal parabolic subgroup corresponding to the reflection r_i (cf. Sect. 2.5). Set $X^{P_i}_w := \pi_i(X_w)$. It is realized as a projective variety as in [Ku5, Sect. 1.8]. Then the restriction of π_i to X_w is a morphism onto $X^{P_i}_w$. The map π_i on restriction gives rise to a surjective map $\tilde{\pi}_i: (vB^-P_i/B)\cap X_w \to (vB^-P_i/P_i)\cap X^{P_i}_w$. Moreover the fibres of $\tilde{\pi}_i$ are either $\mathbb{P}^1$ or single points. By a parabolic analogue of Lemma (3.5) (cf. [Ku6, Lemma 3.3]) $(B^-vP_i/B)\cap X_w$ is closed in the open subspace $(vB^-P_i/B)\cap X_w$ of X_w. Hence by [K, Lemmas 7.7 and 7.9]:

$$\text{(I4)}\qquad \begin{aligned} &H^*_{X^v\cap X_w/((\partial X^v)\setminus\mathcal{B}^{vr_i})\cap X_w}(X_w, \mathcal{L}_w(\lambda))\\ &\approx H^*_{(B^-vP_i/B)\cap X_w}((vB^-P_i/B)\cap X_w, \mathcal{L}_w(\lambda)).\end{aligned}$$

(Observe that $\dot{\mathcal{B}}^v \cup \mathcal{B}^{vr_i} = B^-vP_i/B$; as follows by [PK, Proof of Corollary 2].)

The direct images $R^q\tilde{\pi}_{i*}(\mathcal{L}_w(\lambda)) = 0$, for $q > 0$; since λ is (by assumption) dominant and the fibres of $\tilde{\pi}_i$ are either $\mathbb{P}^1$ or single points. Hence by [G, Corollary 5.6]:

$$\text{(I5)}\qquad \begin{aligned} &H^*_{(B^-vP_i/B)\cap X_w}((vB^-P_i/B)\cap X_w, \mathcal{L}_w(\lambda))\\ &\approx H^*_{(B^-vP_i/P_i)\cap X^{P_i}_w}((vB^-P_i/P_i)\cap X^{P_i}_w, \tilde{\pi}_{i*}\mathcal{L}_w(\lambda)).\end{aligned}$$

Finally by [Ku6, Lemma 3.3(c)] (cf. Proof of Lemma 3.15):

$$\text{(I6)}\qquad H^*_{(B^-vP_i/P_i)\cap X^{P_i}_w}((vB^-P_i/P_i)\cap X^{P_i}_w, \tilde{\pi}_{i*}\mathcal{L}_w(\lambda)) \approx H^*_{\{e\}}(U_v, \mathcal{S}),$$

for some sheaf $\mathcal{S}$ on U_v.

But then by [G, Proposition 1.12], $H^p_{\{e\}}(U_v, \mathscr{S}) = 0$, for all $p > \dim U_v = l(v)$. Combining I4–I6, the Lemma follows. □

(3.18) *Proof of Proposition (3.13).* We first extend the $\mathfrak{b}$-module structure to a $\mathfrak{p}_i$-module structure, for any minimal parabolic subalgebra $\mathfrak{p}_i \supset \mathfrak{b}$ $(1 \leq i \leq l)$:

Let $W_i := \{w \in W : r_i w < w\}$. For any $w \in W_i$, by Lemma (3.7) (See also Definition 3.9), $H^p_v(X_w, \mathscr{L}_w(\lambda))$ (and hence its restricted dual) is naturally a $(\mathfrak{p}_i, T)$-module and moreover, by the naturality, the map

$$\psi^v_{w,w'} : H^{l(v)}_v(X_w, \mathscr{L}_w(\lambda))^\vee \to H^{l(v)}_v(X_{w'}, \mathscr{L}_{w'}(\lambda))^\vee \quad \text{(defined in Sect. 3.9)}$$

is a $\mathfrak{p}_i$-module map for any $w \leq w' \in W_i$. But then, the subset $W_i \subset W$ being cofinal with respect to the partial ordering $\leq$, the $\mathfrak{b}$-module structure on $H^{l(v)}_v(G/B, \mathscr{L}(\lambda))^\vee$ extends to a $\mathfrak{p}_i$-module structure.

Further define a Lie algebra $\tilde{\mathfrak{g}} = \tilde{\mathfrak{g}}(A)$ as in Sect. 2.1 by exactly the same generators and the same relations as $\mathfrak{g}(A)$, except the relation (R_4). Clearly the Lie algebra $\mathfrak{g}$ is quotient of $\tilde{\mathfrak{g}}$. From the defining relations for $\tilde{\mathfrak{g}}$ it easily follows that any $\mathfrak{b}$-module V, such that its $\mathfrak{b}$-module structure extends to a $\mathfrak{p}_i$-module structure for all $1 \leq i \leq l$, acquires a unique $\tilde{\mathfrak{g}}$-module structure extending all the $\mathfrak{p}_i$-structures. In particular $H^{l(v)}_v(G/B, \mathscr{L}(\lambda))^\vee$ is canonically a $\tilde{\mathfrak{g}}$-module. The same argument also shows that $H^p_{F_p/F_{p+1}}(G/B, \mathscr{L}(\lambda))^\vee$ is canonically a $\tilde{\mathfrak{g}}$-module and moreover all the maps ε^v and $\{\delta_p\}_{p \geq 0}$ of the Kempf complex $K^\vee$ (cf. Sect. 3.9) are $\tilde{\mathfrak{g}}$-module maps.

Now we prove, by induction on $l(v)$, that the $\tilde{\mathfrak{g}}$-module structure on $H^{l(v)}_v(G/B, \mathscr{L}(\lambda))^\vee$ descends to a $\mathfrak{g}$-module structure and moreover as a $\mathfrak{g}$-module it is isomorphic with the Verma module $M(v * \lambda)$:

The case $l(v) = 0$ (i.e. $v = e$) is the content of the next Lemma (3.19). Write $v = v' r_i$, for some simple reflection r_i such that $v > v'$. We first assert that the map

$$\delta^{v,v'} : H^{p+1}_v(G/B, \mathscr{L}(\lambda))^\vee \to H^p_{v'}(G/B, \mathscr{L}(\lambda))^\vee \quad \text{(defined in Sect. 3.12)}$$

is injective, where $p = l(v')$. In fact, more strongly, we prove that the map $\delta^{v,v'}(w) : H^{p+1}_v(X_w, \mathscr{L}_w(\lambda))^\vee \to H^p_{v'}(X_w, \mathscr{L}_w(\lambda))^\vee$ (cf. Sect. 3.12) is injective, for any $w \in W$:

Consider the triple $X^{v'} \cap X_w \supset (\partial X^{v'}) \cap X_w \supset ((\partial X^{v'}) \backslash \mathscr{B}^{v' r_i}) \cap X_w$ of closed subspaces of X_w (call these subspaces $Y_1 \supset Y_2 \supset Y_3$ resp.). This gives rise to a long exact sequence as in [K, Lemma 7.6]:

$$\cdots \to H^p_{v'}(X_w, \mathscr{L}_w(\lambda)) \xrightarrow{d} H^{p+1}_{Y_2/Y_3}(X_w, \mathscr{L}_w(\lambda)) \to H^{p+1}_{Y_1/Y_3}(X_w, \mathscr{L}_w(\lambda)) \to \cdots.$$

But, by Lemma (3.17), $H^{p+1}_{Y_1/Y_3}(X_w, \mathscr{L}_w(\lambda)) = 0$. Further, by [K, Lemmas 7.7 and 7.9], $H^{p+1}_{Y_2/Y_3}(X_w, \mathscr{L}_w(\lambda))$ can be canonically identified with $H^{p+1}_v(X_w, \mathscr{L}_w(\lambda))$ and moreover the dual map $d^\vee : H^{p+1}_{Y_2/Y_3}(X_w, \mathscr{L}_w(\lambda))^\vee \to H^p_{v'}(X_w, \mathscr{L}_w(\lambda))^\vee$ can be easily seen to be the same map as $\delta^{v,v'}(w)$. This establishes the assertion that $\delta^{v,v'}(w)$ is injective.

This, in particular, implies (by the induction hypothesis) that the $\tilde{\mathfrak{g}}$-module structure on $H^{l(v)}_v(G/B, \mathscr{L}(\lambda))^\vee$ descends to a $\mathfrak{g}$-module structure and moreover, by Lemmas (3.14) and (3.16), it is isomorphic (as a $\mathfrak{g}$-module) with the Verma module

$M(v*\lambda)$. This completes the induction (modulo the next lemma) and hence proof of the proposition will be completed after we prove the next lemma.

(3.19) **Lemma.** *The* $\tilde{\mathfrak{g}}$*-module structure on* $H^0_e(G/B, \mathscr{L}(\lambda))^\vee$, *defined in* Sect. 3.18, *descends to give a* $\mathfrak{g}$*-module structure and moreover, as a* $\mathfrak{g}$*-module, it is isomorphic with the Verma module* $M(\lambda)$.

Proof. This lemma is well known in the finite case. We recall one of the proofs below (in this case) and the point is that the same proof works in the general case:

Put $M = H^0_e(G/B, \mathscr{L}(\lambda))^\vee$. By [K, Lemma 7.7], M can be canonically identified with $H^0(\mathscr{B}^e, \mathscr{L}(\lambda)|_{\mathscr{B}^e})^\vee$. Further, as in the proof of Lemma (3.16), $H^0(\mathscr{B}^e, \mathscr{L}(\lambda)|_{\mathscr{B}^e})$ is identified with the ring of regular functions $\mathbb{C}[U^-]$; which is a U^--module under the left regular representation.

Let $\tilde{\mathfrak{n}}^- \subset \tilde{\mathfrak{g}}$ be the subalgebra generated by $\{f_i\}_{1 \le i \le l}$. Then $\mathfrak{n}^-$ is naturally a quotient of $\tilde{\mathfrak{n}}^-$; in particular $\mathbb{C}[U^-]^\vee$ is a $\tilde{\mathfrak{n}}^-$-module. Now the isomorphism of M with $\mathbb{C}[U^-]^\vee$, given in the above paragraph, can be easily seen to be a $\tilde{\mathfrak{n}}^-$-module isomorphism. This, in particular, shows that the $\tilde{\mathfrak{g}}$-module structure on M descends to give a $\mathfrak{g}$-module structure. It remains to show that it is isomorphic with the Verma module $M(\lambda)$:

By Lemma (3.16), there exists a unique (upto non-zero scalar multiples) weight vector v_0 of weight λ in M and moreover $U(\mathfrak{b})v_0 \subseteq \mathbb{C}v_0$. In particular, there is a $\mathfrak{g}$-module map $f: M(\lambda) \to M$, taking a fixed highest weight vector in $M(\lambda)$ to v_0. We claim that f is surjective:

It suffices to show that $M/U^+(\mathfrak{n}^-).M$ (where $U^+(\mathfrak{n}^-)$ is the standard augmentation ideal) is one dimensional; which is equivalent to showing that the space of $\mathfrak{n}^-$-invariants in $H^0_e(G/B, \mathscr{L}(\lambda)) \approx \mathbb{C}[U^-]$ is one dimensional. But since U^- acts on $\mathbb{C}[U^-]$ under the left regular representation, the only U^-(or $\mathfrak{n}^-$)-invariants are the constant functions. This proves that f is surjective. But by Lemma (3.16) $\operatorname{ch} M = \operatorname{ch} M(\lambda)$, which forces f to be an isomorphism. This proves the lemma and thereby completes the proof of Proposition (3.13). □

Combining Corollary (3.10), Lemma (3.11), Proposition (3.13), and [Ku5, Corollary 3.11]; we immediately obtain the following main theorem of this paper: (The fact, that the maps $\varepsilon^\vee$ and δ_p of the Kempf complex $\mathscr{K}^\vee$ are $\mathfrak{g}$-module maps, follows from the proof of Proposition (3.13) given in Sect. 3.18.)

(3.20) **Theorem (Strong BGG resolution).** *Let* $\mathfrak{g}$ *be an arbitrary Kac–Moody Lie algebra and let* λ *be any dominant integral weight. Then there is an exact sequence of* $\mathfrak{g}$*-module maps:*

$$0 \leftarrow L^{\max}(\lambda) \leftarrow C_0(\lambda) \leftarrow C_1(\lambda) \leftarrow \cdots \leftarrow C_p(\lambda) \leftarrow \cdots,$$

where $C_p(\lambda) := \bigoplus_{v \in W^{(p)}} M(v*\lambda)$ *is the sum of Verma modules, and* $L^{\max}(\lambda)$ *is the maximal integrable module with highest weight* λ (cf. Sect. 2.8). (The notation $*$ and $W^{(p)}$ are as explained in Sect. 2.3.) □

The above result is sharpened (resp. generalized) in Theorem (3.25) (resp. Theorem 3.27).

(3.21) *Historical remarks.* As is well known (and mentioned in the introduction) a resolution of $L^{\max}(\lambda)$, as in the above theorem, was first obtained by Bernstein–Gelfand–Gelfand [BGG, Theorem 10.1] in the finite case in 1971, by 'algebraic' methods; making crucial use of the centre of $U(\mathfrak{g})$.

Later around 1978 (again in the finite case) Kempf [K] realized such a resolution as a particular global Cousin complex. In the meantime in 1976 Garland–Lepowsky [GL] proved the existence of the BGG resolution for symmetrizable Kac–Moody algebras. But they proved only a 'weaker' resolution; in which the modules $C_p(\lambda)$ (a priori) only admitted a filtration such that the set of successive quotients coincides with $\{M(v*\lambda)\}_{v\in W(p)}$, which was later shown by Rocha–Caridi and Wallach [RW] (by proving 'algebraically' an appropriate Ext vanishing result) to be actually the direct sum of these Verma modules. The proof of Garland–Lepowsky again was 'algebraic' and followed in spirit the work of Bernstein et al [BGG]. They (Garland–Lepowsky) also crucially used the centre of $U(\mathfrak{g})$ (rather its one 'special' element—the Casimir operator; which of course exists only in the symmetrizable case). In contrast, we have followed the geometric line of Kempf. □

It will be very interesting to find an algebraic proof for the existence of BGG resolution in the general Kac–Moody case treated in this paper.

As an immediate (and standard) consequence of Theorem (3.20), we can rederive the extension of Weyl–Kac character formula to arbitrary Kac–Moody algebras, proved by Kumar [Ku5, Theorem 3.5] and also by Mathieu [M, Theorem 1]: (Of course it is a celebrated result due to H. Weyl in the finite case which was generalized to the symmetrizable Kac–Moody algebras by V. Kac making use of the casimir operator.)

(3.22) **Theorem.** *With the assumptions and notation as in Theorem* (3.20), *we have*:

$$\left(\prod_{\beta\in\tilde{\Delta}_+}(1-e^{-\beta})\right).\operatorname{ch} L^{\max}(\lambda)=\sum_{w\in W}\varepsilon(w)e^{w*\lambda};$$

where the notation $\tilde{\Delta}_+$ is as in Lemma (3.16), and $\varepsilon(w)$ denotes the signature of w.

In particular (*taking* $\lambda=0$) *we have the denominator formula*:

$$\prod_{\beta\in\tilde{\Delta}_+}(1-e^{-\beta})=\sum_{w\in W}\varepsilon(w)e^{w\rho-\rho}. \quad \square$$

Let us recall a purely combinatorial construction of a chain complex, as in Theorem (3.20); given by Bernstein–Gelfand–Gelfand. But before that we need to recall the following elementary lemma essentially from [BGG]:

(3.23) **Lemma.** *Let W be the Weyl group associated to any Kac–Moody algebra. Then*:

(a) *Let $w_1, w_2\in W$ are such that $l(w_2)=l(w_1)+2$. Then the number of elements $w\in W$ satisfying $w_1\to w\to w_2$ is either zero or two*; where the notation $v\to w$ means that there exists a $\beta \in \Delta^{re}$ (cf. Sect. 2.3) such that $w = \gamma_\beta v$ and $l(w) = l(v) + 1$ (γ_β denotes the reflection in W defined by $\gamma_\beta(\chi) = \chi - <\chi, \beta^\vee> \beta$).

(b) *To each arrow* $w_1 \to w_2$, *one can assign a number* $s(w_1, w_2) = \pm 1$ *in such a way that for every square* (w_1, w_2, w_3, w_4), *the product of the numbers assigned to the four arrows occurring in it is* -1; where one calls a quadruple (w_1, w_2, w_3, w_4) of elements of W a square if $w_1 \to w_2 \to w_4$, $w_1 \to w_3 \to w_4$, and $w_2 \neq w_3$.

(c) *For any dominant integral weight* λ *and any* $v, w \in W$, $\dim \operatorname{Hom}_{\mathfrak{g}}(M(v*\lambda), M(w*\lambda)) \leqq 1$ *and equality occurs if and only if* $v \geqq w$.

Of course any non-zero homomorphism of one Verma module into another is injective.

Proof. (a) and (b) are nothing but Lemmas (10.3) and (10.4) (respectively) in [BGG]. (Even though they prove in the finite case, their proof works in the general situation without any change.) A proof of (c) is given in [RW, Sect. 8] in the symmetrizable case, but the same proof is valid in our general situation. □

Fix a dominant integral weight λ and also, for any $w \in W$, we fix an embedding $i_w : M(w*\lambda) \hookrightarrow M(\lambda)$ (cf. Lemma 3.23(c)). The embeddings $\{i_w\}_{w \in W}$ give rise to uniquely defined embeddings $i_{v,v'} : M(v'*\lambda) \hookrightarrow M(v*\lambda)$, for any $v \leqq v'$. So, by Lemma 3.23(c), any $\mathfrak{g}$-module map $f : C_{p+1}(\lambda) \to C_p(\lambda)$, for any $p \geqq 0$, (where $C_*(\lambda)$ is as in Theorem 3.20) can be written in the form

$$(*) \qquad f = \sum_{\substack{v \to w \\ l(v) = p}} f(v, w) i_{v,w},$$

for some unique complex numbers $\{f(v, w)\}_{\substack{v \leqq w \\ l(v) = p}}$ and conversely, i.e., given any complex numbers $\{f(v, w)\}_{\substack{v \leqq w \\ l(v) = p}}$, $(*)$ defines a $\mathfrak{g}$-module map: $C_{p+1}(\lambda) \to C_p(\lambda)$.

(3.24) **Definition.** Taking $f(v, w) = s(v, w)$ in $(*)$ (where $s(v, w)$ is as in Lemma 3.23(b)), we get a map $s_p : C_{p+1}(\lambda) \to C_p(\lambda)$. As a consequence of the definition of $s(.,.)$ and Lemma 3.23(a), one can easily see that the following sequence $\mathscr{C}$ is a complex:

$$0 \leftarrow L^{\max}(\lambda) \xleftarrow{\hat{\varepsilon}} C_0(\lambda) \xleftarrow{s_0} C_1(\lambda) \leftarrow \cdots \xleftarrow{s_p} C_{p+1}(\lambda) \leftarrow \cdots,$$

where $\hat{\varepsilon}$ is the standard quotient map.

We will refer to $\mathscr{C}$ as the *BGG complex*. □

By making *essential use* of Theorem (3.20), we prove the following sharpening of Theorem (3.20). This (sharpened) result in the symmetrizable case is due to Rocha–Caridi and Wallach [RW, Sect. 9] and our proof is adopted from theirs.

(3.25) **Theorem.** *With the assumptions and notation as in Theorem* (3.20); *any chain complex*

$$\tilde{\mathscr{C}}: \qquad 0 \leftarrow L^{\max}(\lambda) \xleftarrow{\tilde{\varepsilon}} C_0(\lambda) \xleftarrow{\tilde{s}_0} C_1(\lambda) \leftarrow \cdots \xleftarrow{\tilde{s}_p} C_{p+1}(\lambda) \leftarrow \cdots$$

is exact if and only if the following holds:

$\mathscr{P} \cdots$ *For any* $p \geqq 0$ *and any* $w \in W^{(p+1)}$, $\tilde{s}_{p|_{M(w*\lambda)}} \neq 0$ *and also* $\tilde{\varepsilon} \neq 0$.

Further any exact complex $\tilde{\mathscr{C}}$ *as above is equivalent to the* BGG *complex* $\mathscr{C}$ (*defined in* Sect. 3.24), *i.e., there are* $\mathfrak{g}$-module isomorphisms $\{\mu_p\}_{p\geq 0}$, *making the following diagram commutative*:

$$\mathscr{D}\cdots\qquad \begin{array}{ccccccc} L^{\max}(\lambda) & \xleftarrow{\tilde{\varepsilon}} & \tilde{C}_0(\lambda) & \xleftarrow{\tilde{s}_0} & \tilde{C}_1(\lambda) & \xleftarrow{\tilde{s}_1} & \cdots \\ \|{\scriptstyle Id} & & \wr\downarrow{\scriptstyle \mu_0} & & \wr\downarrow{\scriptstyle \mu_1} & & \\ L^{\max}(\lambda) & \xleftarrow{\varepsilon} & C_0(\lambda) & \xleftarrow{s_0} & C_1(\lambda) & \xleftarrow{s_1} & \cdots. \end{array}$$

In particular the BGG *complex* $\mathscr{C}$ *is exact.*

Proof. The assertion that, for any exact complex $\tilde{\mathscr{C}}, \mathscr{P}$ holds, follows from an argument identical to the proof of [R, Lemma 10.1]. Further, by an argument identical to the proof of [R, Lemma 10.5 and Corollary 10.7], it follows that for any chain complex $\tilde{\mathscr{C}}$ satisfying $\mathscr{P}$, there are $\mathfrak{g}$-module isomorphisms $\{\mu_p\}_{p\geq 0}$ making the diagram $\mathscr{D}$ commutative. In particular taking for $\tilde{\mathscr{C}}$ an exact complex, guaranteed by Theorem (3.20), we obtain that the BGG complex $\mathscr{C}$ itself is exact. Now using the exactness of $\mathscr{C}$ and the existence of the isomorphisms $\{\mu_p\}$, we obtain that any complex $\tilde{\mathscr{C}}$ satisfying $\mathscr{P}$ is exact. This completes the proof of the theorem. □

(3.26) *Remark.* As a consequence of the above theorem, the geometrically defined Kempf's complex $\mathscr{K}^\vee$ (cf. Sect. 3.9) is equivalent to the combinatorially defined BGG complex $\mathscr{C}$. In particular (in the notation of Definition 3.12) the map $\delta^{v,v'}$: $H_v^{p+1}(G/B, \mathscr{L}(\lambda))^\vee \to H_{v'}^p(G/B, \mathscr{L}(\lambda))^\vee$ (for any $v\in W^{(p+1)}$ and $v'\in W^{(p)}$) is non-zero (and hence injective) if and only if $v'\to v$. A particular case of this when $v=v'r_i$, for some simple reflection r_i, was established geometrically in Sect. 3.18. □

The following theorem provides an extension of Theorems (3.20) and (3.25) to arbitrary parabolic subalgebras of finite type. In the symmetrizable case it was proved by Rocha–Caridi and Wallach [RW, Theorem 12] in 1981, making an essential use of a parabolic analogue of the weak BGG resolution (in this case) proved by Garland–Lepowsky [GL, Theorem 8.7]. Earlier in 1976, in the finite case, Lepowsky [L1] had proved the first part of the following theorem again by making crucial use of the presently stated result of Garland–Lepowsky.

Proof of the following theorem is similar to the proof of Theorems (3.20) and (3.25); if we work with G/P (instead of G/B). The details are omitted.

(3.27) **Theorem** (*Parabolic extension of the strong* BGG *resolution*). *Let* $\mathfrak{g}$ *be an arbitrary Kac–Moody algebra* (*associated to a* $l\times l$ *GCM*) *and let* $S\subseteq\{1,\dots,l\}$ *be any subset of finite type* (cf. Sect. 2.3). *Then, for any dominant integral* λ, *there is an exact sequence of* $\mathfrak{g}$-*module maps*:

$$0\leftarrow L^{\max}(\lambda)\leftarrow C_0^S(\lambda)\leftarrow C_1^S(\lambda)\leftarrow\cdots\leftarrow C_p^S(\lambda)\leftarrow\cdots,$$

where $C_p^S(\lambda)=\bigoplus_{\substack{w\in W_S^1\\ l(w)=p}} M_S(w*\lambda)$; $W_S^1, M_S(\cdot)$ *are as defined in* Sect. 2. (*Observe that for any* $w\in W_S^1$, $w*\lambda\in D_S$.)

Further any exact complex, as above, is equivalent to the parabolic analogue $\mathscr{C}^S$ (*defined by Lepowsky* [L1, Sect. 4]) *of the* BGG *complex* $\mathscr{C}$ (*defined in* Sect. 3.24).

In particular the complex $\mathscr{C}^S$ *itself is exact.* □

(3.28) *Remark.* It is likely that the restriction in the above theorem, that S is of finite type, is unnecessary; provided we define $M_S(w*\lambda) := U(\mathfrak{g}) \bigotimes_{U(\mathfrak{p}_S)} V_S^{\max}(w*\lambda)$.

4. Applications-determination of certain Lie algebra homologies

As consequences of Theorem (3.27), we derive some results on Lie algebra homology:

The following theorem generalizes a result of Garland–Lepowsky [GL, Theorem 8.6] from symmetrizable to arbitrary Kac–Moody algebras. As is well known, in the finite case, this is a famous result due to Kostant [Ko]. It may be mentioned that the author gave a proof of the result of Garland–Lepowsky (i.e. the following theorem in the symmetrizable case) in the spirit of Kostant's proof; by proving an expression for the 'Laplacian' [Ku1, Theorem (2.1) and Corollary 2.3(a)].

(4.1) **Theorem.** *With the assumptions and notation as in Theorem* (3.27), *the Lie algebra homology* $H_p(\mathfrak{u}^-, L^{\max}(\lambda))$ (*for any* $p \geqq 0$) *is* $\mathfrak{r}$*-module isomorphic with the direct sum* $\bigoplus_{\substack{w \in W_S^1 \\ l(w)=p}} V_S(w*\lambda)$ *of* (*inequivalent*) *irreducible* $\mathfrak{r}$*-modules* $V_S(w*\lambda)$ (*defined in* Sect. 2.8) *with highest weight* $w*\lambda$; *where* $\mathfrak{u}^- = \mathfrak{u}_S^-$ *and* $\mathfrak{r} = \mathfrak{r}_S$ *are defined in* Sect. 2.4.

(Observe that since $\mathfrak{r}$ normalizes $\mathfrak{u}^-$ and $L^{\max}(\lambda)$ is a $\mathfrak{g}$-module, the Lie algebra homology $H_*(\mathfrak{u}^-, L^{\max}(\lambda))$ has a canonical $\mathfrak{r}$-module structure.)

Proof. Follows by a standard argument from Theorem (3.27). □

As a consequence of the above theorem, we obtain the following: (We omit the details of the proof as the proof of (a) and (b) parts below is identical to the proof of the corresponding results in the symmetrizable case [Ku3, Proposition 1.5]; and the (c) part follows by first proving the vanishing of $\mathrm{Ext}_{\mathfrak{g}}^*(M(w*\lambda), L^{\max}(\mu)^\sigma)$ for all $w \in W$, by using [RW, Sect. 7, Theorem 2] and Theorem (4.1), and then using Theorem 3.20.)

(4.2) **Proposition.** (a) *For any* $\lambda \in \mathfrak{h}^*$ *such that* $\lambda \neq w*0$ (*for any* $w \in W$), $H_p(\mathfrak{g}, M(\lambda)) = 0$, *for all* $p \geqq 0$.

(b) *For any* $w \in W$, $H_p(\mathfrak{g}, M(w*0)) \approx \Lambda^{p-l(w)}(\mathfrak{h})$, *as* $\mathbb{C}$*-vector spaces*; where $\Lambda^n(\mathfrak{h})$ denotes the n-th exterior power of $\mathfrak{h}$.

In particular $H_p(\mathfrak{g}, M(w*0)) = 0$, *for* $p < l(w)$.

(c) *For any dominant integral* $\lambda \neq \mu$, *we have*:

$$\mathrm{Ext}_{\mathfrak{g}}^p(L^{\max}(\lambda), L^{\max}(\mu)^\sigma) = 0, \quad \textit{for all} \quad p \geqq 0;$$

where, for a module N, N^σ is as defined in [DGK, Sect. 4].

In particular, if $\mu \neq 0$,

$$H^p(\mathfrak{g}, L_{\max}(\mu)^\sigma) = H_p(\mathfrak{g}, L^{\max}(\mu)) = 0, \quad \textit{for all} \quad p \geqq 0. \quad \square$$

As another consequence of Theorem (4.1), we obtain the following result which was proved by Lepowsky [L2, Corollary 6.7] in the symmetrizable case.

(4.3) **Theorem.** *Let* $\mathfrak{g}$ *be an arbitrary Kac–Moody Lie algebra, and let* $\mathfrak{r} = \mathfrak{r}_S$ *be the subalgebra* (*defined in* Sect. 2.4) *corresponding to any finite type S. Then, the Lie algebra cohomology with trivial coefficients,* $H^p(\mathfrak{g}, \mathfrak{r}) = 0$ *for p odd, and*

$$\dim._{\mathbb{C}} H^{2p}(\mathfrak{g}, \mathfrak{r}) = \#\{w \in W_S^1 : l(w) = p\}.$$

Proof. Follows by an argument identical to the one given in [Ku1, Remark 3.3]; in view of Theorem 4.1. $\square$

Finally it seems reasonable to make the following conjecture; which was proved in the symmetrizable case by the author [Ku2, Theorem 1.6]:

(4.4) **Conjecture.** *Let* $\mathfrak{g}$ *and* $\mathfrak{r}_S$ *be as in the above theorem. Then the integration map* $\int : C(\mathfrak{g}, \mathfrak{r}_S) \to C_{c^\infty}(G/P_S, \mathbb{C})$, *defined in* [Ku2, Sect. 1.3], *is a co-chain map which induces isomorphism in cohomology.*

The validity of the conjecture, in particular, will imply that the integration map induces an algebra isomorphism

$$[\textstyle\int] : H^*(\mathfrak{g}) \to H^*(G, \mathbb{C}). \quad \square$$

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