Existence of Certain Components in the Tensor Product of Two Integrable Highest Weight Modules for Kac–Moody Algebras.

SHRAWAN KUMAR

ABSTRACT. The aim of this note is to show the existence of certain components in the tensor product of two integrable highest weight $\mathfrak{g}$–modules, where $\mathfrak{g}$ is any symmetrizable Kac–Moody algebra.

1. Introduction

In [Ku_2] we proved the Parthasarathy–Ranga Rao–Varadarajan (henceforth called the PRV) conjecture, in fact its strengthened form due to Kostant, for any (finite dimensional) semi–simple Lie algebra $\mathfrak{g}$. The aim of this paper is to show that the analogous result is true for any symmetrizable Kac–Moody Lie algebra. More precisely, we have the following theorem:

Let $\mathfrak{g}$ *be a symmetrizable Kac–Moody Lie algebra with associated Weyl group* W *and let* $V(\lambda)$ *and* $V(\mu)$ *be two integrable highest weight (hence irreducible)* $\mathfrak{g}$*–modules (with highest weights* λ *and* μ *respectively). We assume that* λ *is regular (see remark 3.8(a)). Then for any* $w \in W$, *the integrable highest weight* $\mathfrak{g}$*–module* $V(\overline{\lambda + w\mu})$ *occurs with multiplicity exactly one inside the* $\mathfrak{g}$*–submodule* $U(\mathfrak{g}) \cdot (e_\lambda \otimes e_{w\mu})$ *(cf. §3.1) of* $V(\lambda) \otimes V(\mu)$, *where* $\overline{\lambda + w\mu}$ *denotes the unique dominant weight in the* W*–orbit of* $\lambda + w\mu$ *(cf. §3.2).*

Throughout the paper, we follow the notation of [Ku_1; §§1–2] and [Ku_2; §§0–1] often without explanation. But we make one deviation in that the maximal integrable highest weight $\mathfrak{g}$–module with highest weight λ will be denoted by $V^{max}(\lambda)$ (in contrast to the notation $L^{max}(\lambda)$ introduced in [Ku_1; §1.5]). Of course, as is well known, in the symmetrizable case (i.e., $\mathfrak{g}$ is symmetrizable) $V^{max}(\lambda)$ is irreducible and in this case we will just write $V(\lambda)$ for

$V^{max}(\lambda)$. The weights in this paper will be implicitly assumed to be integral. We will assume familiarity with the contents of [Ku_2].

The main results of this paper (with only a brief sketch of the proofs) were communicated to P. Polo in a letter dated September 20, 1987, in response to his letter. I take this opportunity to thank him for his letter. Earlier I had not intended to publish the detailed proofs, since they are similar to the proofs in the finite case as in [Ku_2]. However, the interest shown by some mathematicians in seeing the proofs has prompted me to write this note.

§2. Cohomology of Certain Line Bundles on $G/B \times G/B$

In this section we work in the general (not necessarily symmetrizable) Kac–Moody setting.

(2.1) *The varieties* $Z_{\mathfrak{v},\mathfrak{w}}$ *and the line bundles* $\mathcal{L}_{\mathfrak{v},\mathfrak{w}}(\lambda \boxtimes \mu)$. For any two sequences (not necessarily reduced) $\mathfrak{v} = (r_{i_1},\dots,r_{i_m})$ and $\mathfrak{w} = (r_{j_1},\dots,r_{j_n})$ of simple reflections, define $Z_{\mathfrak{v},\mathfrak{w}}$ as the Bott–Samelson–Demazure–Hansen variety (as in [Ku_1; §2.1]) got from the sequence $(\mathfrak{v},\mathfrak{w}) := (r_{i_1},\dots,r_{i_m}, r_{j_1},\dots,r_{j_n})$. There is a map $\theta_{\mathfrak{v},\mathfrak{w}} : Z_{\mathfrak{v},\mathfrak{w}} \longrightarrow G/B \times G/B$ defined by

$$(p_{i_1},\dots,p_{i_m}, p_{j_1},\dots,p_{j_n}) \bmod B^{m+n} \longmapsto ((p_{i_1}\cdots p_{i_m}) \bmod B, (p_{i_1}\cdots p_{i_m} p_{j_1}\cdots p_{j_n}) \bmod B),$$

for $p_{i_s} (1 \le s \le m) \in P_{i_s}$ and $p_{j_{s'}} (1 \le s' \le n) \in P_{j_{s'}}$.

For any integral weights λ and μ, we have a line bundle $\mathcal{L}_{\mathfrak{v},\mathfrak{w}}(\lambda \boxtimes \mu)$ got by taking the tensor product of the line bundles $\pi_{\mathfrak{v}}^* \mathcal{L}_{\mathfrak{v}}(\lambda)$ and $\mathcal{L}_{(\mathfrak{v},\mathfrak{w})}(\mu)$ on $Z_{\mathfrak{v},\mathfrak{w}}$ (cf. [Ku_1; §2.2]), where $\pi_{\mathfrak{v}}$ is the canonical projection: $Z_{\mathfrak{v},\mathfrak{w}} \longrightarrow Z_{\mathfrak{v}}$ (cf. [Ku_1; §2.1]).

The line bundle $\mathcal{L}_{\mathfrak{v},\mathfrak{w}}(\lambda \boxtimes \mu)$ (as a topological line bundle) can also be thought of as the pull back via $\theta_{\mathfrak{v},\mathfrak{w}}$ of the line bundle $\mathcal{L}(\lambda \boxtimes \mu)$ on $G/B \times G/B$ (cf. [Ku_2; §1.1]).

Now we can state the following crucial:

(2.2) Proposition. *Let* $\mathfrak{v} = (r_{i_1},\dots,r_{i_m})$ *and* $\mathfrak{w} = (r_{j_1},\dots,r_{j_n})$ *be arbitrary sequences of simple reflections. Let* $\{s, s+1,\dots,t\} \subset \{1,\dots,m\}$ *and* $\{s', s'+1,\dots,t'\} \subset \{1,\dots,n\}$ *be (possibly empty) subsets such that* $(r_{i_s},\dots,r_{i_t})$ *and* $(r_{j_{s'}},\dots,r_{j_{t'}})$ *are reduced sequences. Then for any dominant regular* λ, *dominant* μ, *and* $p > 0$; *we have:*

$$H^p(Z_{\mathfrak{v},\mathfrak{w}}, \mathcal{L}_{\mathfrak{v},\mathfrak{w}}(\lambda \boxtimes \mu) \otimes \mathcal{O}_{Z_{\mathfrak{v},\mathfrak{w}}}[-(\bigcup_{q=s}^{t} Z_{\mathfrak{v}(q),\mathfrak{w}} \cup \bigcup_{q'=s'}^{t'} Z_{\mathfrak{v},\mathfrak{w}(q')})]) = 0.$$

Recall from [Ku_1; §2.1] that $Z_{\mathfrak{v}(q),\mathfrak{w}}$ (and $Z_{\mathfrak{v},\mathfrak{w}(q')}$) is a divisor in $Z_{\mathfrak{v},\mathfrak{w}}$. ■

(2.3) Remark. The restriction in the above proposition, that λ is regular, is essential. Consider, e.g., $\lambda = 0$, $\mathfrak{v}$ is reduced, and $s = 1$, $t = m$. ■

Proof of the above proposition is similar to the proof (given in [Ku_1; §4]) of the analogous proposition. We indicate some of the necessary changes:

<u>Step I</u>. The canonical bundle

$$K_{Z_{\mathfrak{v},\mathfrak{w}}} \approx \mathcal{L}_{\mathfrak{v},\mathfrak{w}}(0 \boxtimes -\rho) \otimes \mathcal{O}_{Z_{\mathfrak{v},\mathfrak{w}}}[-(\bigcup_{q=1}^{m} Z_{\mathfrak{v}(q),\mathfrak{w}} \cup \bigcup_{q'=1}^{n} Z_{\mathfrak{v},\mathfrak{w}(q')})].$$

This is essentially [Ku_1; Lemma 4.4]. We just need to observe that $\mathcal{L}_{\mathfrak{v},\mathfrak{w}}(0 \boxtimes -\rho) \approx \mathcal{L}_{(\mathfrak{v},\mathfrak{w})}(-\rho)$.

<u>Step II</u>. First prove the proposition in the case when $\mathfrak{v}$ is reduced and $s = 1$, $t = m$; the proof in this case being almost identical to the one in [Ku_1; §4].

Step III. We first observe that the line bundle $\mathcal{O}_{Z_{\mathfrak{v},\mathfrak{w}}}[-Z_{\mathfrak{v}(q),\mathfrak{w}}]$ is the pull back $\pi_{\mathfrak{v}}^{*}(\mathcal{O}_{Z_{\mathfrak{v}}}[-Z_{\mathfrak{v}(q)}])$, where $\pi_{\mathfrak{v}} : Z_{\mathfrak{v},\mathfrak{w}} \longrightarrow Z_{\mathfrak{v}}$ is the canonical projection (cf. §2.1). Hence, by the projection formula, for any $p \geq 0$:

$$R^{p}\pi_{\mathfrak{v}*}(\mathcal{L}_{\mathfrak{v},\mathfrak{w}}(\lambda \boxtimes \mu) \otimes \mathcal{O}_{Z_{\mathfrak{v},\mathfrak{w}}}[-(\bigcup_{q=s}^{t} Z_{\mathfrak{v}(q),\mathfrak{w}} \cup \bigcup_{q'=s'}^{t'} Z_{\mathfrak{v},\mathfrak{w}(q')})])$$

$$\approx \mathcal{O}_{Z_{\mathfrak{v}}}[-\bigcup_{q=s}^{t} Z_{\mathfrak{v}(q)}] \otimes R^{p}\pi_{\mathfrak{v}*}(\mathcal{L}_{\mathfrak{v},\mathfrak{w}}(\lambda \boxtimes \mu) \otimes \mathcal{O}_{Z_{\mathfrak{v},\mathfrak{w}}}[-\bigcup_{q'=s'}^{t'} Z_{\mathfrak{v},\mathfrak{w}(q')}]).$$

But by [Ku$_1$; Proposition 2.3], applied to $Z_{\mathfrak{w}}$, we obtain that the sheaf $R^{p}\pi_{\mathfrak{v}*}(\mathcal{L}_{\mathfrak{v},\mathfrak{w}}(\lambda \boxtimes \mu) \otimes \mathcal{O}_{Z_{\mathfrak{v},\mathfrak{w}}}[-\bigcup_{q'=s'}^{t'} Z_{\mathfrak{v},\mathfrak{w}(q')}]) = 0$, for all $p > 0$. Further, by the "invariance", it is easy to see that $\pi_{\mathfrak{v}*}(\mathcal{L}_{\mathfrak{v},\mathfrak{w}}(\lambda \boxtimes \mu) \otimes \mathcal{O}_{Z_{\mathfrak{v},\mathfrak{w}}}[-\bigcup_{q'=s'}^{t'} Z_{\mathfrak{v},\mathfrak{w}(q')}])$ is the locally free sheaf $\mathcal{S}$ on $Z_{\mathfrak{v}}$ associated to the standard principal B–bundle with base $Z_{\mathfrak{v}}$ by the representation $S := \mathbb{C}_{-\lambda} \otimes H^{0}(Z_{\mathfrak{w}}, \mathcal{L}_{\mathfrak{w}}(\mu) \otimes \mathcal{O}_{Z_{\mathfrak{w}}}[-\bigcup_{q'=s'}^{t'} Z_{\mathfrak{w}(q')}])$ of B. In particular, the Leray spectral sequence associated to the morphism $\pi_{\mathfrak{v}}$ degenerates at E_2 and hence we have

$$H^{p}(Z_{\mathfrak{v},\mathfrak{w}}, \mathcal{L}_{\mathfrak{v},\mathfrak{w}}(\lambda \boxtimes \mu) \otimes \mathcal{O}_{Z_{\mathfrak{v},\mathfrak{w}}}[-(\bigcup_{q=s}^{t} Z_{\mathfrak{v}(q),\mathfrak{w}} \cup \bigcup_{q'=s'}^{t'} Z_{\mathfrak{v},\mathfrak{w}(q')})]) \approx$$

$$H^{p}(Z_{\mathfrak{v}}, \mathcal{S} \otimes \mathcal{O}_{Z_{\mathfrak{v}}}[-\bigcup_{q=s}^{t} Z_{\mathfrak{v}(q)}]), \text{ for all } p \geq 0.$$

Now follow the inductive argument exactly as in [Ku$_1$; §4], but replace the line bundle $\mathcal{L}(\lambda)$ throughout by the locally free sheaf $\mathcal{S}$ which is (by definition) associated to the B–module S.

This completes the proof of the proposition. ■

(2.4) Corollary. *The map* $\theta_{\mathfrak{v},\mathfrak{w}} : Z_{\mathfrak{v},\mathfrak{w}} \longrightarrow G/B \times G/B$, *defined in* §2.1, *is a rational resolution onto its image, provided we assume that* $\mathfrak{v}$ *and* $\mathfrak{w}$ *are reduced sequences.*

(We will see in the proof below that the image of $\theta_{\mathfrak{v},\mathfrak{w}}$ denoted $X_{v,w}$, where $v = m(\mathfrak{v})$ and $w = m(\mathfrak{w})$ [Ku$_1$; 2.6], does not depend upon the particular choice of the reduced decompositions $\mathfrak{v}$ and $\mathfrak{w}$ of v and w respectively and moreover $X_{v,w}$ acquires a natural projective variety structure.).

In particular, for any locally free sheaf $\mathcal{T}$ *on* $X_{v,w}$, *we have:*

$$H^{p}(X_{v,w}, \mathcal{T}) \approx H^{p}(Z_{\mathfrak{v},\mathfrak{w}}, \theta^{*}_{\mathfrak{v},\mathfrak{w}}(\mathcal{T})),$$

for all $p \geq 0$.

Proof. For any reduced sequence $\mathfrak{v}$, the canonical map $\theta_{\mathfrak{v}} : Z_{\mathfrak{v}} \longrightarrow X_{v}$ is a birational morphism [Ku$_1$; §2.1], where $X_{v} := \overline{BvB/B} \subset G/B$ is the Schubert variety. Further we have the following commutative diagram:

$$\begin{array}{ccc} Z_{\mathfrak{v},\mathfrak{w}} & \xrightarrow{\theta_{\mathfrak{v},\mathfrak{w}}} & G/B \times G/B \\ \pi_{\mathfrak{v}} \downarrow & & \downarrow \pi_1 \\ Z_{\mathfrak{v}} & \xrightarrow[\theta_{\mathfrak{v}}]{} & G/B \end{array}$$

, where π_1 is the projection on the first factor. By the definition of the map $\theta_{\mathfrak{v},\mathfrak{w}}$, $X_{v,w} = \bigcup_{\bar{g} \in X_{v}} (\bar{g}, gX_{w})$, where $\bar{g} = g \bmod B$. In particular $X_{v,w}$ does not depend upon the choice of the reduced $\mathfrak{v}$ and $\mathfrak{w}$ and moreover by the Tits property $X_{v,w} \subset X_{v} \times X_{w'}$, for some large enough w' (depending upon v and w). Hence $X_{v,w}$ acquires a (natural) projective variety structure as the subvariety of $X_{v} \times X_{w'}$.

From the above diagram $X_{v,w}$ fibers over X_{v} with fiber X_{w}. In particular $\theta_{\mathfrak{v},\mathfrak{w}}$ is a birational morphism and $X_{v,w}$ is a normal variety (since X_v and X_w are normal varieties; cf. [Ku_1; Theorem 2.16] or [M]). Now the assertion that the map $\theta_{\mathfrak{v},\mathfrak{w}}$ is a rational resolution follows easily from proposition (2.2) in view of the following lemma due to Kempf:

(2.5) **Lemma.** *Let* X *and* Y *be two proper schemes over a Noetherian ring and let* $f: X \longrightarrow Y$ *be a morphism. Suppose further that* $f_*\mathcal{O}_X = \mathcal{O}_Y$ *and there exists an ample line bundle* $\mathcal{L}$ *on* Y *such that* $H^p(X, f^*(\mathcal{L}^n)) = 0$, *for all* $p > 0$ *and all sufficiently large* n *then* $R^p f_*(\mathcal{O}_X) = 0$, *for all* $p > 0$. ∎

(2.6) **Definition.** For any $w \in W$, fix a reduced sequence $\mathfrak{w}$ with $m(\mathfrak{w}) = w$ and define for any integral weights λ, μ, and any $p \geq 0$:

$$H^p(\tilde{X}_w, \mathcal{L}_w(\lambda \boxtimes \mu))^{\check{}} := \underset{\mathfrak{v} \in \mathfrak{W}}{\underrightarrow{\lim}}\, H^p(Z_{\mathfrak{v},\mathfrak{w}}, \mathcal{L}_{\mathfrak{v},\mathfrak{w}}(\lambda \boxtimes \mu))^*,$$

where the directed set $\mathfrak{W}$ is as in [Ku_1; §2.6] and $*$ denotes the full dual.

Using [Ku_1; Lemma 4.6], it can be seen that $H^p(\tilde{X}_w, \mathcal{L}_w(\lambda \boxtimes \mu))^{\check{}}$ does not depend upon the particular reduced decomposition $\mathfrak{w}$ of w. As in [Ku_1; §§2.6 and 2.11] it is easy to see that $H^p(\tilde{X}_w, \mathcal{L}_w(\lambda \boxtimes \mu))^{\check{}}$ acquires a natural integrable $\mathfrak{g}$–module structure. Further, for any $w' \leq w$, there is a canonical $\mathfrak{g}$–module map (got from the restriction) : $H^p(\tilde{X}_{w'}, \mathcal{L}_{w'}(\lambda \boxtimes \mu))^{\check{}} \longrightarrow H^p(\tilde{X}_w, \mathcal{L}_w(\lambda \boxtimes \mu))^{\check{}}$. With this notation, as a consequence of proposition (2.2), we obtain the following:

(2.7) **Theorem.** *For any dominant regular* λ, *dominant* μ, *and any* $w' \leq w \in W$; *we have:*

(a) $H^p(\tilde{X}_w, \mathcal{L}_w(\lambda \boxtimes \mu))^{\check{}} = 0$, *for all* $p > 0$.

(b) *The canonical map:* $H^0(\tilde{X}_{w'}, \mathcal{L}_{w'}(\lambda \boxtimes \mu))^{\check{}} \longrightarrow H^0(\tilde{X}_w, \mathcal{L}_w(\lambda \boxtimes \mu))^{\check{}}$ *is injective.*

Proof. (a) of course follows immediately from proposition (2.2). To prove (b); we can assume that $\ell(w') = \ell(w) - 1$, where ℓ denotes the length. Hence it suffices to show that the restriction map:

$$H^0(Z_{\mathfrak{v},\mathfrak{w}}, \mathcal{L}_{\mathfrak{v},\mathfrak{w}}(\lambda \boxtimes \mu)) \longrightarrow H^0(Z_{\mathfrak{v},\mathfrak{w}(i)}, \mathcal{L}_{\mathfrak{v},\mathfrak{w}(i)}(\lambda \boxtimes \mu))$$

is surjective, for any sequence $\mathfrak{v}$ and any $1 \leq i \leq n$; where $\mathfrak{w}$ is any reduced sequence of length n with $m(\mathfrak{w}) = w$: Now considering the long exact sequence associated to the sheaf exact sequence (tensored over $\mathcal{O}_{Z_{\mathfrak{v},\mathfrak{w}}}$ with the locally free sheaf $\mathcal{L}_{\mathfrak{v},\mathfrak{w}}(\lambda \boxtimes \mu)$):

$$0 \longrightarrow \mathcal{O}_{Z_{\mathfrak{v},\mathfrak{w}}}[-Z_{\mathfrak{v},\mathfrak{w}(i)}] \longrightarrow \mathcal{O}_{Z_{\mathfrak{v},\mathfrak{w}}} \longrightarrow \mathcal{O}_{Z_{\mathfrak{v},\mathfrak{w}(i)}} \longrightarrow 0$$

and using proposition (2.2), we get (b). ∎

3. Proof of the Main Theorem

The following basic proposition provides a bridge between representation theory and algebraic geometry.

(3.1) **Proposition.** *Let* $\mathfrak{g}$ *be any (not necessarily symmetrizable) Kac–Moody algebra. Fix a dominant regular* λ, *dominant* μ, *and* $w \in W$. *Then* $H^0(\tilde{X}_w, \mathcal{L}_w(\lambda \boxtimes \mu))^{\check{}}$ *(defined in §2.6) is* $\mathfrak{g}$*–module isomorphic with the* $\mathfrak{g}$*–submodule* $U(\mathfrak{g})\,(e_\lambda \otimes e_{w\mu}) \subset V^{max}(\lambda) \otimes V^{max}(\mu)$, *generated by the element* $e_\lambda \otimes e_{w\mu}$; *where* e_λ *(resp.* $e_{w\mu}$*) denotes any non–zero highest weight vector in* $V^{max}(\lambda)$ *(resp. an extremal weight vector in* $V^{max}(\mu)$ *of weight* $w\mu$*).*

Proof. For any $v \in W$, there exist large enough (depending upon v and w) $w \le w' \le w^* \in W$ such that:

$$(*) \qquad X_{v,w} \subset X_v \times X_{w'} \subset X_{v,w^*}.$$

The first inclusion is already observed in the proof of corollary (2.4). After making the choice of w', choose a w^* such that $X_{w'} \subset gX_{w^*}$, for any $g \in G$ such that $g \bmod B \in X_v$, which again is possible by the Tits property. With such a choice of w^*, we have $X_v \times X_{w'} \subset X_{v,w^*}$ (see the proof of corollary 2.4), thus establishing $(*)$.

Observe that $V_{\lambda,\mu} := [V^{max}(\lambda) \otimes V^{max}(\mu)]^*$ is canonically a $G \times G$–module. For any $f \in V_{\lambda,\mu}$ define a map $\tilde{\psi}(f) : G \times G \longrightarrow \mathrm{Hom}_{\mathbb{C}}(\mathbb{C}_\lambda, \mathbb{C}) \otimes \mathrm{Hom}_{\mathbb{C}}(\mathbb{C}_\mu, \mathbb{C})$ by

$$(\tilde{\psi}(f)(g_1, g_2))(e_\lambda \otimes e_\mu) = ((g_1^{-1}, g_2^{-1})f)(e_\lambda \otimes e_\mu),$$

for $e_\lambda \in \mathbb{C}_\lambda \subset V^{max}(\lambda)$ and $e_\mu \in \mathbb{C}_\mu \subset V^{max}(\mu)$; where $\mathbb{C}_\lambda$(resp. $\mathbb{C}_\mu$) is the highest weight space in $V^{max}(\lambda)$ (resp. $V^{max}(\mu)$). As is easy to see, the map $\tilde{\psi}(f)$ gives rise to a continuous section $\psi(f)$ of the line bundle $\mathcal{L}(\lambda \boxtimes \mu)$ on $G/B \times G/B$. The pull back of $\psi(f)$ via $\theta_{\mathfrak{v},\mathfrak{w}}$ (for any $\mathfrak{v}$, $\mathfrak{w} \in \mathfrak{W}$), induces a map

$$\psi_{\mathfrak{v},\mathfrak{w}} : V_{\lambda,\mu} \longrightarrow H^0(Z_{\mathfrak{v},\mathfrak{w}}, \mathcal{L}_{\mathfrak{v},\mathfrak{w}}(\lambda \boxtimes \mu)) \ .$$

(The fact that $\psi_{\mathfrak{v},\mathfrak{w}}(f)$, for any $f \in V_{\lambda,\mu}$, is indeed a regular section is easy to see.)

Also consider the map $\theta_{\mathfrak{v}} \times \theta_{\mathfrak{w}} : Z_{\mathfrak{v}} \times Z_{\mathfrak{w}} \longrightarrow G/B \times G/B$ defined as the Cartesian product of the maps $\theta_{\mathfrak{v}} : Z_{\mathfrak{v}} \longrightarrow G/B$ and $\theta_{\mathfrak{w}} : Z_{\mathfrak{w}} \longrightarrow G/B$ (cf. [Ku_1; §2.1]). Analogous to the definition of the map $\psi_{\mathfrak{v},\mathfrak{w}}$, we get a map $\psi_{\mathfrak{v}} \times \psi_{\mathfrak{w}} : V_{\lambda,\mu} \longrightarrow H^0(Z_{\mathfrak{v}} \times Z_{\mathfrak{w}}, \mathcal{L}_{\mathfrak{v}}(\lambda) \boxtimes \mathcal{L}_{\mathfrak{w}}(\mu))$, where $\mathcal{L}_{\mathfrak{v}}(\lambda)$ is the line bundle on $Z_{\mathfrak{v}}$ defined in [Ku_1; §2.2].

Choose a reduced sequence $\mathfrak{v}$ (resp. $\mathfrak{w}^*$) with $m(\mathfrak{v}) = v$ (resp. $m(\mathfrak{w}^*) = w^*$). Then there are reduced subsequences $\mathfrak{w} \le \mathfrak{w}' \le \mathfrak{w}^*$ such that $m(\mathfrak{w}') = w'$ and $m(\mathfrak{w}) = w$.

By virtue of corollary (2.4) and [Ku_1; Proposition 2.14], there are canonical restriction maps γ_1 and γ_2 (got from the inclusions $(*)$) making the following diagram commutative:

$$\begin{array}{ccccc} & & V_{\lambda,\mu} & & \\ & \swarrow^{\psi_{\mathfrak{v},\mathfrak{w}^*}} & \downarrow{\scriptstyle \psi_{\mathfrak{v}} \times \psi_{\mathfrak{w}'}} & \searrow^{\psi_{\mathfrak{v},\mathfrak{w}}} & \\ H^0(Z_{\mathfrak{v},\mathfrak{w}^*}, \mathcal{L}_{\mathfrak{v},\mathfrak{w}^*}(\lambda \boxtimes \mu)) & \xrightarrow[\gamma_1]{} & H^0(Z_{\mathfrak{v}} \times Z_{\mathfrak{w}'}, \mathcal{L}_{\mathfrak{v}}(\lambda) \boxtimes \mathcal{L}_{\mathfrak{w}'}(\mu)) & \xrightarrow[\gamma_2]{} & H^0(Z_{\mathfrak{v},\mathfrak{w}}, \mathcal{L}_{\mathfrak{v},\mathfrak{w}}(\lambda \boxtimes \mu)) \end{array}$$

The composite map $\gamma_2 \circ \gamma_1$ is surjective by proposition (2.2) (see the proof of Theorem 2.7) and hence γ_2 is surjective. Further the map $\psi_{\mathfrak{v}} \times \psi_{\mathfrak{w}'}$ is surjective by [Ku_1; Proposition 2.14] and hence the map $\psi_{\mathfrak{v},\mathfrak{w}}$ is surjective.

Now we determine the kernel of the map $\psi_{\mathfrak{v},\mathfrak{w}}$:

The subset $\{(bv \bmod B, bvb'w \bmod B) : b, b' \in B\}$ of $X_{v,w}$ is open and dense. From this it is easy to see that the kernel $K_{\mathfrak{v},\mathfrak{w}}$ of the map $\psi_{\mathfrak{v},\mathfrak{w}}$ (since $\mathfrak{v}$ and $\mathfrak{w}$ are reduced) is given by: $K_{\mathfrak{v},\mathfrak{w}} = \{f \in [V^{max}(\lambda) \otimes V^{max}(\mu)]^* : f$ restricted to the linear span $\Sigma(BvB)(e_\lambda \otimes e_{w\mu})$ of $BvB(e_\lambda \otimes e_{w\mu})$ is identically zero$\}$.

Hence by dualizing the surjective map $\psi_{\mathfrak{v},\mathfrak{w}}$, we get that $H^0(Z_{\mathfrak{v},\mathfrak{w}}, \mathcal{L}_{\mathfrak{v},\mathfrak{w}}(\lambda \boxtimes \mu))^* \approx \Sigma(BvB)(e_\lambda \otimes e_{w\mu})$, for any reduced $\mathfrak{v}$ and $\mathfrak{w}$.

Further by an analogue of [Ku_1; Corollary 4.7], for any sequence $\mathfrak{v}$, $H^0(Z_{\mathfrak{v},\mathfrak{w}}, \mathcal{L}_{\mathfrak{v},\mathfrak{w}}(\lambda \boxtimes \mu))$ is isomorphic with $H^0(Z_{\mathfrak{v}_1,\mathfrak{w}}, \mathcal{L}_{\mathfrak{v}_1,\mathfrak{w}}(\lambda \boxtimes \mu))$, for some reduced subsequence $\mathfrak{v}_1$ of $\mathfrak{v}$. Hence $H^0(\tilde{X}_w, \mathcal{L}_w(\lambda \boxtimes \mu))^{\check{}} := \varinjlim_{\mathfrak{v} \in \mathfrak{W}} H^0(Z_{\mathfrak{v},\mathfrak{w}}, \mathcal{L}_{\mathfrak{v},\mathfrak{w}}(\lambda \boxtimes \mu))^* \approx U(\mathfrak{g}) \cdot (e_\lambda \otimes e_{w\mu})$. ∎

(3.2) **Definition.** Let $C := \{\chi \in \mathfrak{h}_{\mathbb{R}}^* : \chi(\alpha_i^\vee) \ge 0,$ for all the simple co–roots $\alpha_i^\vee\}$ denote the dominant chamber and let $Y := \bigcup_{w \in W} wC$. Then Y is a convex cone and moreover for any $\chi \in Y$, $(W\chi) \cap C$ is a single point denoted $\bar{\chi}$ (cf. [K; Proposition 3.12]). In particular, for any dominant integral weights λ, μ, and any $w \in W$, there is a unique dominant integral weight $\overline{\lambda + w\mu}$ in the W–orbit of $\lambda + w\mu$. ■

We recall the following complete reducibility result:

(3.3) Theorem [K; Theorem 10.7(b)]. *Let* $\mathfrak{g}$ *be a symmetrizable Kac–Moody algebra. Then any integrable* $\mathfrak{g}$*–module* V *in the category* $\mathcal{O}$ *is completely reducible, i.e.,* V *can be (uniquely) written as (a* $\mathfrak{g}$*–module):*

$$V \approx \oplus\, n_\theta V(\theta),$$

where the sum runs over all the dominant integral weights θ *and* $n_\theta V(\theta)$ *denotes the direct sum of* $V(\theta)$, n_θ*–times.*

We call n_θ (which is a non–negative integer) *the multiplicity of* $V(\theta)$ *in* V.

Observe that for any two integrable highest weight $\mathfrak{g}$–modules $V(\lambda)$ and $V(\mu)$, the tensor product $V(\lambda) \otimes V(\mu)$ is in the category $\mathcal{O}$ and of course it is integrable. In particular, the multiplicity of any $V(\theta)$ in $V(\lambda) \otimes V(\mu)$ makes sense. ■

The following generalization of Joseph's result to arbitrary (not necessarily symmetrizable) Kac–Moody algebras is essentially due to P. Polo (unpublished; letter to the author):

(3.4) Theorem. *For any* $w \in W$ *and dominant integral* μ*; the* $U(\mathfrak{n})$*–module map:* $U(\mathfrak{n}) \longrightarrow V_w^{max}(\mu)$, *defined by* $x \longmapsto x\, e_{w\mu}$ *has kernel precisely equal to the left* $U(\mathfrak{n})$*–ideal*

$$\sum_{\alpha \in \Delta_+^{re}} U(\mathfrak{n})\, x_\alpha^{k_\alpha + 1},$$

where x_α *is any non–zero root vector in* $\mathfrak{g}$ *corresponding to the positive real root* α (observe that the real root spaces are one dimensional), $e_{w\mu}$ *is an extremal weight vector of weight* $w\mu$ *in* $V^{max}(\mu)$, $V_w^{max}(\mu)$ *is* $U(\mathfrak{b})$*–submodule of* $V^{max}(\mu)$ *generated by* $e_{w\mu}$, *and* k_α *is defined as follows:*

$$k_\alpha = k_\alpha^\mu(w) = 0 \ , \quad \textit{if } \langle \alpha^\vee, w\mu \rangle \geq 0$$
$$= -\langle \alpha^\vee, w\mu \rangle, \ \textit{otherwise.}$$ ■

By a proof identical to the proof of the corresponding result in the finite case [Ku$_2$; Proposition 2.4], we obtain the following (as a consequence of the above theorem):

(3.5) Proposition. *For any dominant weights* λ, μ *and any* $w \in W$

$$\mathrm{Hom}_{\mathfrak{b}}(\mathbb{C}_\lambda \otimes V_w^{max}(\mu),\ V^{max}(\overline{\lambda + w\mu}))$$

is one dimensional. ■

(3.6) Proposition. *Assume that* $\mathfrak{g}$ *is symmetrizable. Then for any integral weight* λ, *dominant integral* μ, *and* $w \in W$ *we have:*

$$H^0(\tilde{X}_w, \mathcal{L}_w(\lambda \boxtimes \mu))^* \approx \oplus_\theta\, V(\theta) \otimes \mathrm{Hom}_{\mathfrak{b}}(\mathbb{C}_\lambda \otimes V_w(\mu), V(\theta))$$

(as $\mathfrak{g}$*–modules), where the sum runs over all the dominant integral* θ, *and we put the trivial*

$\mathfrak{g}$-module structure on $\mathrm{Hom}_{\mathfrak{b}}(\mathbb{C}_\lambda \otimes V_w(\mu), V(0))$.

(Actually there is an analogous result valid for any H^p, but we will have no occasion to use it.)

Proof. By the definition of the direct image sheaf, there is a natural isomorphism (for any $\mathfrak{v}, \mathfrak{w}$)

$$F_{\mathfrak{v}} : H^0(Z_{\mathfrak{v},\mathfrak{w}}, \mathcal{L}_{\mathfrak{v},\mathfrak{w}}(\lambda \boxtimes \mu)) \approx H^0(Z_{\mathfrak{v}}, \pi_{\mathfrak{v}}{*}(\mathcal{L}_{\mathfrak{v},\mathfrak{w}}(\lambda \boxtimes \mu))),$$

where $\pi_{\mathfrak{v}}$ is as in §2.1.

Now take $\mathfrak{w}$ reduced such that $m(\mathfrak{w}) = w$. Then by [Ku_1; Theorem 2.16 and Lemma 4.5] the sheaf $\pi_{\mathfrak{v}}{*}(\mathcal{L}_{\mathfrak{v},\mathfrak{w}}(\lambda \boxtimes \mu))$ is the locally free sheaf $\mathcal{V}_w(\mathfrak{v})$ on $Z_{\mathfrak{v}}$ associated to the standard principal B–bundle with base $Z_{\mathfrak{v}}$, by the B–module $M_w := \mathbb{C}_{-\lambda} \otimes H^0(X_w, \mathcal{L}_w(\mu))$. Taking the direct limit of the dual of the isomorphisms $F_{\mathfrak{v}}$ we get an isomorphism $\check{F} : H^0(\tilde{X}_w, \mathcal{L}_w(\lambda \boxtimes \mu))^{\check{}} \approx H^0(G/B, \mathcal{V}_w)^{\check{}}$, where $H^0(G/B, \mathcal{V}_w)^{\check{}}$ is by the definition $\varinjlim_{\mathfrak{v} \in \mathfrak{W}} H^0(Z_{\mathfrak{v}}, \mathcal{V}_w(\mathfrak{v}))^*$. Further by [$\mathrm{Ku}_1$; Proposition 2.14], $H^0(X_w, \mathcal{L}_w(\mu))$ is isomorphic with $V_w(\mu)^*$. Now the proposition follows from [M; Proposition 15] (see also [KP; Theorem 1]). ■

Combining propositions (3.1), (3.5), and (3.6) we readily obtain the following main theorem of this paper:

(3.7) Theorem. *Let $\mathfrak{g}$ be any symmetrizable Kac–Moody Lie algebra. Fix any dominant regular λ and dominant μ. Then, for any $w \in W$, the $\mathfrak{g}$–module $V(\overline{\lambda + w\mu})$ occurs in the $\mathfrak{g}$–submodule $U(\mathfrak{g}) \cdot (e_\lambda \otimes e_{w\mu})$ of $V(\lambda) \otimes V(\mu)$ (cf. §3.1) with multiplicity exactly one.* ■

(3.8) **Remarks.**

(a) It is very likely that the restriction, in Theorems (2.7) and (3.7), that λ is regular can be removed by suitably modifying our proposition (2.2). Observe that in the finite case we did not have such a restriction.

(b) A note "Construction du groupe de Kac–Moody et applications," written by O. Mathieu has appeared in C.R. Acad. Sci Paris, t. 306 série I (1988), where some of the main results of this paper are announced. ■

Finally by an argument identical to the proof of [Ku_2; Proposition 2.13], we obtain the following:

(3.9) Proposition. *If in Theorem (3.7), we further assume that μ also is regular. Then $V(\overline{\lambda + w\mu})$ does not occur in $U(\mathfrak{g}) \cdot (e_\lambda \otimes e_{w'\mu})$, for any $w' < w$.* ■

References

[K] Kac, V.G.: "Infinite dimensional Lie algebras." Progress in Math. vol 44. Boston: Birkhäuser (1983).

[Ku_1] Kumar, S.: Demazure character formula in arbitrary Kac–Moody setting. Invent. Ma 89, 395–423 (1987).

[Ku_2] ————: Proof of the Parthasarathy–Ranga Rao–Varadarajan conjecture. Invent. Math. 93, 117–130 (1988).

[KP] Kac, V.G. and Peterson, D.H.: Regular functions on certain infinite–dimensional grou In: Arithmetic and Geometry II (Artin, M. and Tate, J. eds.), pp. 141–166. Boston: Birkhäuser (1983).

[M] Mathieu, O.: Formules de Weyl et de Demazure, et théorèmes de Borel–Weil–Bott po les algèbres de Kac–Moody générales II (preprint).

School of Mathematics
Tata Institute of Fundamental Research
Homi Bhabha Road
Bombay–400005
INDIA

and

The Institute for Advanced Study
School of Mathematics
Princeton, New Jersey
08540
U.S.A.