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Demazure character formula in arbitrary Kac-Moody setting

Shrawan Kumar

School of Mathematics, Tata Institute of Fundamental Research, Homi Bhabha Road, Bombay 400 005, India

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0. Introduction

Let $\mathfrak{g}$ be an arbitrary (not necessarily symmetrizable) Kac-Moody Lie-algebra with Cartan subalgebra $\mathfrak{h}$, associated group G, Borel subgroup B, maximal torus T (Lie $T \approx \mathfrak{h}$), and Weyl group W (see Sect. 1). For any $w \in W$, let X_w denote the Schubert variety $\overline{BwB/B} \subset G/B$. Associated to any sequence $\mathfrak{w} = (r_{i_1}, \ldots, r_{i_n})$ of simple reflections, there is Bott-Samelson-Demazure-Hansen variety $Z_{\mathfrak{w}}$ and a map $\theta_{\mathfrak{w}}: Z_{\mathfrak{w}} \to G/B$ (cf. § 2.1). Also, for any integral $\lambda \in \mathfrak{h}^*$, there is a line bundle $\mathscr{L}(\lambda)$ associated to the principal B-bundle: $G \to G/B$ by the character $e^{-\lambda}$, in particular, by restriction to X_w (resp. pulling back to $Z_{\mathfrak{w}}$ via $\theta_{\mathfrak{w}}$), we get a line bundle $\mathscr{L}_w(\lambda)$ on X_w (resp. $\mathscr{L}_{\mathfrak{w}}(\lambda)$ on $Z_{\mathfrak{w}}$) (cf. § 2.2).

Our main interest in the paper is to understand the algebraic geometry of $X_w'^s$, over char. 0, and to prove the Demazure character formula. More specifically, we prove the following results (cf. Theorems 2.16, 2.23 and 3.4). For any $w \in W$:

(1) X_w is a normal variety.

(2) For any dominant integral λ, $H^p(X_w, \mathscr{L}_w(\lambda)) = 0$, for all $p > 0$ and the canonical restriction map: $H^0(X_w, \mathscr{L}_w(\lambda)) \to H^0(X_v, \mathscr{L}_v(\lambda))$ is surjective, for any $v \leqq w$.

(3) For any reduced expression $w = r_{i_1} \ldots r_{i_n}$, the map $\theta_{\mathfrak{w}}: Z_{\mathfrak{w}} \to X_w$ is a rational resolution (cf. § 2.20), where $\mathfrak{w}$ is the sequence $(r_{i_1}, \ldots, r_{i_n})$. In particular, X_w is Cohen-Macaulay and $H^p(Z_{\mathfrak{w}}, \theta_{\mathfrak{w}}^* \mathscr{S}) \approx H^p(X_w, \mathscr{S})$, for all $p \geqq 0$ and any locally free sheaf $\mathscr{S}$ on X_w.

(4) *Demazure character formula*: For any integral λ, $\chi(X_w, \mathscr{L}_w(\lambda)) = \bar{D}_w(e^\lambda)$, as elements in $\mathbb{Z}[R(T)]$, where $\chi(X_w, \mathscr{L}_w(\lambda))$ denotes the alternating sum $\sum(-1)^p \operatorname{ch} H^p(X_w, \mathscr{L}_w(\lambda))$, $\mathbb{Z}[R(T)]$ denotes the group algebra on the group of characters $R(T)$ of T, $\operatorname{ch} H^p(X_w, \mathscr{L}_w(\lambda)) \in \mathbb{Z}[R(T)]$ is the formal T-character of the (canonical) T-module $H^p(X_w, \mathscr{L}_w(\lambda))$, and $\bar{D}_w$ is the Demazure operator: $\mathbb{Z}[R(T)] \to \mathbb{Z}[R(T)]$ defined in § 3.3.

We also prove projective normality and arithmetic Cohen-Macaulay property of $X_w'^s$, in the case when $\mathfrak{g}$ is symmetrizable. One easily extends all the above results-(1) through (4)- to any parabolic subgroup of finite type (cf. § 1.3).

It should be mentioned that, in the case when $\mathfrak{g}$ is finite-dimensional (henceforth called the finite case), all these results are known (in fact over arbitrary char.). See [J, S, RR, R and A]. (Demazure had earlier "proved" these results in this case but, in trying to generalize these results to infinite dimensional situation, V. Kac noticed a serious gap in his paper.) All these proofs proceed, except in [RR], via downward induction on $l(w)$ starting from the element of maximal length. In [RR], smoothness (actually only normality) of G/B has crucially been used. *Our work provides (I believe) a new proof in the finite case as well.* In particular, we do not make use of any char. p methods (e.g. Frobenius splitting) at all.

Amusingly, as a particular case of (1) (cf. § 2.25), we can derive (using a result due to Lusztig) the following famous result due to Kostant:

'The nilpotent cone $\mathcal{N} \subset \operatorname{End} V$, where V is a finite dimensional vector space over $\mathbb{C}$, is a normal variety.'

As a consequence of these results, we extend the Weyl-Kac character formula and the denominator formula to arbitrary Kac-Moody algebras (cf. Theorem 3.5). More specifically, for any dominant integral λ and the 'maximal' integrable highest weight $\mathfrak{g}$-module $L^{\max}(\lambda)$ (cf. § 1.5), we have:

(a)
$$\operatorname{ch} L^{\max}(\lambda) = \frac{\sum_{w \in W} \varepsilon(w)\, e^{w(\lambda+\rho)}}{\sum_{w \in W} \varepsilon(w)\, e^{w\rho}}$$

and

(b)
$$\sum_{w \in W} \varepsilon(w)\, e^{w\rho - \rho} = \prod_{\beta \in \Delta_+} (1 - e^{-\beta})^{\operatorname{mult} \beta}$$

(mult β is the dim of the β-th root space).

(Of course, in the symmetrizable case, (a) and (b) above was proved by Kac using the Casimir operator.)

So far we don't know if $L^{\max}(\lambda)$ is irreducible in the non-symmetrizable case. In case this is true, there will (obviously) be a unique integrable highest weight $\mathfrak{g}$-module (with highest weight λ) and 'radical' would be zero in the non-symmetrizable case as well, thus extending the result of Gabber-Kac.

We also get an extension of Borel-Weil-Bott theorem in arbitrary Kac-Moody setting. More precisely, we prove (Theorem 3.10) that, for any integral λ such that $\lambda + \rho$ is dominant (appropriately defined, cf. § 3.8) the G-module $H^p(G/B, \mathscr{L}(\lambda))^*$ is isomorphic with the G-module $H^{p+l(w)}(G/B, \mathscr{L}(w(\lambda+\rho)-\rho))$, for all $p \in \mathbb{Z}$ and $w \in W$. In particular, the G-module $H^p(G/B, \mathscr{L}(\lambda))^*$ gets determined for all $p \geq 0$ and all λ belonging to the Tits cone (cf. § 3.8 and § 3.11).

There are essentially three main ideas in our paper: One is our cohomology vanishing proposition 2.3, which is very crucial to our work. I believe that this proposition may be of interest elsewhere as well. Our proof of the proposition uses, among others, a result of Grauert-Riemenschneider (Theorem 4.1) and a precise knowledge of the canonical bundle of $Z_{\mathfrak{w}}$.

Now a construction of Borel-Weil (cf. § 2.5) gives, for any dominant integral λ and any integrable highest weight $\mathfrak{g}$-module $V(\lambda)$ (with highest weight λ), an injective map $\bar{\psi}_w$: $V_w(\lambda)^* \to H^0(X_w, \mathscr{L}_w(\lambda))$, where $V_w(\lambda) \subset V(\lambda)$ is the Schubert module defined in § 2.5. Further let $w = r_{i_1} \dots r_{i_n}$ be any reduced expression and define $\mathfrak{w}$ to be the sequence $(r_{i_1}, \dots, r_{i_n})$. Dualizing the map $\bar{\psi}_w$ gives rise to the following commutative triangle:

$$\begin{array}{ccc} H^0(Z_{\mathfrak{w}}, \mathscr{L}_{\mathfrak{w}}(\lambda))^* & \xrightarrow{\phi_{\mathfrak{w}}} & V_w(\lambda) \\ & {\scriptstyle \tilde{\theta}_{\mathfrak{w}}} \searrow \quad \nearrow {\scriptstyle \bar{\psi}_w^*} & \\ & H^0(X_w, \mathscr{L}_w(\lambda))^* & \end{array}$$

where $\bar{\psi}_w^*$ is the dual of the map $\bar{\psi}_w$, $\tilde{\theta}_{\mathfrak{w}}$ is induced from the map $\theta_{\mathfrak{w}}: Z_{\mathfrak{w}} \to X_w$, and $\phi_{\mathfrak{w}}$ is defined to be the composite $\bar{\psi}_w^* \circ \tilde{\theta}_{\mathfrak{w}}$. It is easy to see that all the maps in the above triangle are surjective.

Now our second observation is that if $\phi_{\mathfrak{w}}$ is not an isomorphism for some w, then $\phi_{\mathfrak{v}}$ also fails to be an isomorphism for all $v \geqq w$ (cf. § 2.14).

Further define $H^0(Z_\infty, \mathscr{L}(\lambda))^*$ as the direct limit of $H^0(Z_{\mathfrak{w}}, \mathscr{L}_{\mathfrak{w}}(\lambda))^*$ over an appropriate directed set (cf. § 2.6). Finally we prove (cf. proposition 2.11) that $H^0(Z_\infty, \mathscr{L}(\lambda))^*$ is an integrable highest weight $\mathfrak{g}$-module with highest weight λ. In particular, we can take for $V(\lambda)$ the $\mathfrak{g}$-module $H^0(Z_\infty, \mathscr{L}(\lambda))^*$. But then if $\phi_{\mathfrak{w}}$ is not an isomorphism for some w, then this would lead to a surjective $\mathfrak{g}$-module map: $H^0(Z_\infty, \mathscr{L}(\lambda))^*$ onto itself, which is not an isomorphism. A contradiction! So $\phi_{\mathfrak{w}}$ is an isomorphism, for all $w \in W$. In particular, the canonical map: $H^0(X_w, \mathscr{L}_w(\lambda)) \to H^0(Z_{\mathfrak{w}}, \mathscr{L}_{\mathfrak{w}}(\lambda))$ is an isomorphism and so is the map: $L_w^{\max}(\lambda)^* \to H^0(X_w, \mathscr{L}_w(\lambda))$ (cf. § 2.14).

These observations, together with some standard facts in algebraic geometry, prove all the assertions (1) through (4).

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1. Preliminaries and notations

(1.1) *Kac-Moody Lie-algebra* $\mathfrak{g}$, *its root space decomposition, and Weyl group* [K_1, M]. *A generalized Cartan matrix* $A = (a_{ij})_{1 \leqq i, j \leqq l}$ is a matrix of integers, satisfying $a_{ii} = 2$ for all i; $a_{ij} \leqq 0$ if $i \neq j$; and $a_{ij} = 0$ if and only if $a_{ji} = 0$.

Choose a triple $(\mathfrak{h}, \pi, \pi^\vee)$, unique upto isomorphism, where $\mathfrak{h}$ is a vector space over $\mathbb{C}$ of dimension $l+$co-rank A, $\pi=\{\alpha_i\}_{1\leqq i\leqq l}\subset\mathfrak{h}^*$, and $\pi^\vee=\{h_i\}_{1\leqq i\leqq l}\subset\mathfrak{h}$ are linearly independent indexed sets satisfying $\alpha_j(h_i)=a_{ij}$.

The *Kac-Moody algebra* $\mathfrak{g}=\mathfrak{g}(A)$ is the Lie-algebra (over $\mathbb{C}$) generated by $\mathfrak{h}$ and the symbols e_i and f_i $(1\leqq i\leqq l)$ with the defining relations: $[\mathfrak{h},\mathfrak{h}]=0$; $[h,e_i]=\alpha_i(h)\,e_i, [h,f_i]=-\alpha_i(h)f_i$ for $h\in\mathfrak{h}$ and all $1\leqq i\leqq l$; $[e_i,f_j]=\delta_{ij}h_j$ for all $1\leqq i,j\leqq l$; and $(\operatorname{ad} e_i)^{1-a_{ij}}(e_j)=0=(\operatorname{ad} f_i)^{1-a_{ij}}(f_j)$, for $1\leqq i\neq j\leqq l$.

We will, as usual, denote the *universal enveloping algebra* of $\mathfrak{g}$ by $U(\mathfrak{g})$.

There is available the *root space decomposition* $\mathfrak{g}=\mathfrak{h}\oplus\sum_{\alpha\in\Delta\subset\mathfrak{h}^*\setminus\{0\}}\mathfrak{g}_\alpha$, where $\mathfrak{g}_\alpha$ is the root space corresponding to $\alpha\in\mathfrak{h}^*\setminus\{0\}$ and Δ, the set of roots, consists of all those $\alpha\in\mathfrak{h}^*\setminus\{0\}$ such that $\mathfrak{g}_\alpha\neq 0$. Moreover $\Delta=\Delta_+ U\Delta_-$, where $\Delta_+\subset\sum_{i=1}^{l}\mathbb{Z}_+\alpha_i$ and $\Delta_-=-\Delta_+$ ($\mathbb{Z}_+$ is the set of non-negative integers). Elements of Δ_+ (resp. Δ_-) are called positive (resp. negative) roots.

There is a *Weyl group* $W\subset\operatorname{Aut}\mathfrak{h}^*$ (associated to $\mathfrak{g}$), generated by the 'simple' reflections $\{r_i\}_{1\leqq i\leqq l}$, where $r_i\in\operatorname{Aut}\mathfrak{h}^*$ is defined by $r_i(\chi)=\chi-\chi(h_i)\,\alpha_i$. Also Δ is W-stable. Define $\Delta^{re}=W\cdot\pi$. Further $(W,\{r_i\}_{1\leqq i\leqq l})$ is a *Coxeter system*, hence we can talk of the length of elements of W. We denote the length of $w\in W$ by $l(w)$. We also have the standard Bruhat partial ordering $\leqq$ in W.

(1.2) *Group associated to* $\mathfrak{g}$ [G, K$_2$, KP$_1$, KP$_2$, Ma, MT, Sl$_1$, Sl$_2$, T$_1$, T$_2$]. Garland, Kac-Peterson, Marcuson, Moody-Teo, Slodowy, Tits,... have constructed groups associated to the Kac-Moody Lie-algebra $\mathfrak{g}$. Although these groups could be different, the associated flag varieties ('G/B or G/P') are essentially the 'same'. Since we would mainly be interested in the flag varieties G/P (rather than G itself), we can use either of these constructions of G. However, we will stick to the associated group G given by Tits.

We fix (once and for all) an integral lattice $\mathfrak{h}_\mathbb{Z}\subset\mathfrak{h}$ (i.e. $\mathfrak{h}_\mathbb{Z}\otimes_\mathbb{Z}\mathbb{C}=\mathfrak{h}$), satisfying: (1) $h_i\in\mathfrak{h}_\mathbb{Z}$, for all $1\leqq i\leqq l$ and (2) $\mathfrak{h}_\mathbb{Z}^*=\operatorname{Hom}(\mathfrak{h}_\mathbb{Z},\mathbb{Z})(\subset\mathfrak{h}^*)$ contains $\{\alpha_i\}_{1\leqq i\leqq l}$. Clearly $\mathfrak{h}_\mathbb{Z}^*$ is W-stable.

Of course, $\mathfrak{h}_\mathbb{Z}$ canonically gives rise to a *root base* as in [Sl$_2$]. One can define a group G with subgroups B and N satisfying the following properties (cf. [S1$_2$]):

(1) The pair (B, N) is a Tits system in G, i.e.,

(a) G is generated by B and N

(b) The intersection $T=B\cap N$ is normal in N

(c) The quotient $\tilde{W}=N/T$ is generated by a set $S=\{s_i\}_{1\leqq i\leqq l}$ of involutions, such that $s_i B\tilde{w}\subset B\tilde{w}B\,U\,Bs_i\tilde{w}B$ and $s_iBs_i\neq B$ for any $1\leqq i\leqq l$ and $\tilde{w}\in\tilde{W}$.

(2) The group T is isomorphic with $\mathfrak{h}_\mathbb{Z}\otimes_\mathbb{Z}\mathbb{C}^*$.

(3) The system $(\tilde{W},S)$ is isomorphic to the Coxeter system $(W,\{r_i\}_{1\leqq i\leqq l})$ (defined in § 1.1). Under this isomorphism, the action of $\tilde{W}$ on T is induced by the action of W on $\mathfrak{h}_\mathbb{Z}$ (via(2)) and s_i corresponds with r_i. So, from now on, we can (and often will) identify $(\tilde{W},S)$ with $(W,\{r_i\})$.

(4) The group T acts naturally on the subalgebra $\mathfrak{n}=\sum_{\alpha\in\Delta_+}\mathfrak{g}_\alpha$ of $\mathfrak{g}$ as well as on the completion $\bar{\mathfrak{n}}$ of $\mathfrak{n}$ with respect to the filtration $\{\mathfrak{n}^p\}_{p\in\mathbb{N}}$, where

$\mathfrak{n}^p = \sum_{\text{height}\,\alpha > p} \mathfrak{g}_\alpha$. (For $\alpha = \sum n_i \alpha_i \in \Delta_+$, we define height $\alpha = \sum n_i$.) Thus there is a natural action of T on the prounipotent proalgebraic group U corresponding to $\bar{\mathfrak{n}}$:

$$U = \varprojlim_p U_p,$$

where U_p is the unipotent algebraic group with Lie-algebra $\mathfrak{n}/\mathfrak{n}^p$.

The group B is now the semi-direct product $B = T \ltimes U$.

Moreover any such group G is unique up to isomorphism.

(1.3) *Parabolic subgroups.* For any $S \subset \{1, \ldots, l\}$, put $P = P_S = BW_SB$, where W_S is the subgroup of W generated by $\{r_i\}_{i \in S}$. In particular when $S = \emptyset$, $P_\emptyset = B$. The subgroup T (resp. B) is called the *standard torus* (resp. *standard Borel*), and P_S the *standard parabolic subgroup corresponding to the subset* $S \subset \{1, \ldots, l\}$. If W_S is a finite group, we call P_S *a standard parabolic subgroup of finite type.*

(1.4) *Bruhat decomposition.* For any $S \subset \{1, \ldots, l\}$, define a subset $W'_S = \{w \in W$: $\Delta_+ \cap w^{-1} \Delta_- \subset \Delta_+ \setminus \Delta_+^S\}$, where $\Delta_+^S = \Delta_+ \cap \sum_{i \in S} \mathbb{Z} \alpha_i$. Now W'_S can be characterized as the set of elements of minimal length in the cosets wW_S ($w \in W$) (each such coset contains a unique element of minimal length). The group G can be written as a disjoint union:

$$G = \bigcup_{w \in W'_S} U w P_S,$$

so that

$$G/P_S = \bigcup_{w \in W'_S} U w P_S/P_S.$$

For any $w \in W$ and $S \subset \{1, \ldots, l\}$, denote $X_w^P = \bigcup_{v \leqq w} U v P/P$, where $P = P_S$.

(1.5) *Analytic and Zariski topologies* on X_w^P [Sl$_2$]. (See also [K$_2$, KP$_1$, KP$_2$ and T$_1$].) For any $\lambda \in \mathfrak{h}_{\mathbb{Z}}^*$ (cf. § 1.2) which is dominant, i.e., $\lambda(h_i) \geqq 0$ for all $1 \leqq i \leqq l$, define $L^{\max}(\lambda) = M(\lambda)/M'(\lambda)$, where $M(\lambda)$ is the Verma module (associated to $\mathfrak{g}$) with highest weight λ and $M'(\lambda)$ is the $U(\mathfrak{g})$-submodule of $M(\lambda)$ generated by the elements $\{f_i^{\lambda(h_i)+1} v_\lambda\}_{1 \leq i \leq l} \subset M(\lambda)$ (v_λ is the highest weight vector in $M(\lambda)$). It can be easily seen that $L^{\max}(\lambda)$ is an integrable (highest weight) $\mathfrak{g}$-module and any integrable highest weight $\mathfrak{g}$-module $V(\lambda)$, with highest weight λ, is a quotient of $L^{\max}(\lambda)$ [GL; Proposition 6.2]. Of course, as is well known, in the case when $\mathfrak{g}$ is symmetrizable, $L^{\max}(\lambda)$ is the (unique) irreducible quotient $L(\lambda)$ of $M(\lambda)$.

Fix any dominant $\lambda \in \mathfrak{h}_{\mathbb{Z}}^*$, which further satisfies: (1) $\lambda(h_i) = 0$, for all $i \in S$ and (2) $\lambda(h_i) > 0$, for all $i \in \{1, \ldots, l\} \setminus S$. We will call any such λ dominant regular with respect to S (or $P = P_S$). *The set of all the dominant regular (with respect to S) λ will be denoted by D_S^0.*

Fix $\lambda \in D_S^0$ as above. Let $\mathbb{P}(\lambda)$ denote the projective space associated to the vector space $L^{\max}(\lambda)$. Define an embedding $i(\lambda)$: $G/P \hookrightarrow \mathbb{P}(\lambda)$, by $g \bmod P \mapsto g\, v_\lambda$, for $g \in G$. It is shown [Sl$_2$] that, for any $w \in W$, $i(\lambda)(X_w^P)$ is closed in $\mathbb{P}(\lambda)$ with respect to the analytic (as well as the Zariski) limit topology on $\mathbb{P}(\lambda)$ and $i(\lambda)(X_w^P)$ is contained in a finite dim. projective space $\mathbb{P}' \subset \mathbb{P}(\lambda)$. It is further shown [Sl$_2$; Proposition 2.5] that the analytic (as well as the Zariski) topology on X_w^P (i.e. the topology got by restricting the topology of $\mathbb{P}(\lambda)$ via the map $i(\lambda)$) does not depend upon the particular choice of $\lambda \in D_S^0$.

(1.6) *Realizing X_w^P as projective varieties* [Sl$_2$]. (See also [K$_2$, KP$_1$, KP$_2$ and T$_1$].) For any $\lambda \in D_S^0$ and $w \in W$, we put a complex (projective) algebraic variety structure on X_w^P as the subvariety of $\mathbb{P}(\lambda)$ via the embedding $i(\lambda)$. Equipped with this variety structure, we denote X_w^P by $X_w^P(\lambda)$.

For any dominant $\lambda, \mu \in \mathfrak{h}_{\mathbb{Z}}^*$, there is a (uniquely defined) $\mathfrak{g}$-module map

$$d_{\lambda,\mu}\colon L^{\max}(\lambda+\mu) \to L^{\max}(\lambda) \underset{\mathbb{C}}{\otimes} L^{\max}(\mu),$$

which takes $v_{\lambda+\mu} \mapsto v_\lambda \otimes v_\mu$, where v_λ (similarly v_μ and $v_{\lambda+\mu}$) is some fixed non-zero highest weight vector in $L^{\max}(\lambda)$. (In the non-symmetrizable case, we do not know, if $d_{\lambda,\mu}$ is an embedding.) This gives rise, for any $\mu, \mu' \in D_S^0$ and $w \in W$, to a variety morphism $\tilde{d}^{(w)}_{\mu,\mu'}\colon X_w^P(\mu+\mu') \to X_w^P(\mu) \times X_w^P(\mu')$, and hence, by projection, a (variety) morphism: $X_w^P(\mu+\mu') \to X_w^P(\mu)$, which is set theoretically the identity map. In particular, for any λ, $\mu \in D_S^0$, $\lambda > \mu$ (i.e. $\lambda(h_i) > \mu(h_i)$ for all $i \notin S$) we get that the (set theoretically) identity map: $X_w^P(\lambda) \to X_w^P(\mu)$, which is of course a homeomorphism in Zariski as well as analytic topology, is a (variety) morphism. In particular, the morphism: $X_w^P(\lambda) \to X_w^P(\mu)$ is birational and they have the same normalization, i.e., there is a (unique) normal variety $\tilde{X}_w^P$ together with finite birational morphisms: $\tilde{X}_w^P \to X_w^P(\lambda)$ and $\tilde{X}_w^P \to X_w^P(\mu)$, making the following diagram commutative:

$$\begin{array}{ccc} & \tilde{X}_w^P & \\ \swarrow & & \searrow \\ X_w^P(\lambda) & \longrightarrow & X_w^P(\mu) \end{array}$$

(1.7) **Lemma.** *Fix $w \in W$. Then there is a positive integer $n(w)$ such that, for $\lambda, \mu \in D_S^0$ satisfying $\lambda(h_i)$, $\mu(h_i) > n(w)$ for all $i \in \{1, \dots, l\} \setminus S$, the variety structures $X_w^P(\lambda)$ and $X_w^P(\mu)$ on X_w^P are the same.*

Proof. Define $\rho_S \in D_S^0$ to be any element satisfying $\rho_S(h_i) = 1$, for all $i \notin S$ (of course $\rho_S(h_i) = 0$, for $i \in S$). Consider the sequence of varieties and morphisms (cf. § 1.6):

$$\dots \to X_w^P(3\rho_S) \to X_w^P(2\rho_S) \to X_w^P(\rho_S).$$

By the Noetherian property and the fact that $X_w^P(k\rho_S)$ have the same normalization for all $k \geqq 1$, it is clear that there exists a $n(w) > 0$ such that the variety structure $X_w^P(n\rho_S)$ on X_w^P is the same for all $n \geqq n(w)$.

Now let $\lambda \in D_S^0$ be arbitrary, satisfying $\lambda(h_i) > n(w)$ for all $i \notin S$. Choose $n (> n(w))$ such that $n\rho_S > \lambda$. We have the commutative triangle:

$$\begin{array}{ccc} X_w^P(n\rho_S) & \longrightarrow & X_w^P(\lambda) \\ {\scriptstyle \wr}\searrow & & \swarrow \\ & X_w^P(n(w)\rho_S) & \end{array}$$

This proves the lemma. $\square$

(1.8) *Stable variety structure on* X_w^P. The variety structure on X_w^P given by $X_w^P(\lambda)$, for sufficiently large $\lambda \in D_S^0$ (i.e. sufficiently large $\lambda(h_i)$, for all $i \notin S$), is called the *stable variety structure on* X_w^P. *From now on, we will always equip the space* X_w^P *with the stable variety structure, unless otherwise explicitly stated.*

The complex algebraic variety X_w^P is an irreducible, finite dimensional, projective variety. Further, $\dim X_w^P$ is equal to the length of the (unique) element $\bar{w} \in W_S'$ such that $\bar{w} \in w W_S$.

Recall that, in the case when $\mathfrak{g}$ is symmetrizable, Tits [T$_1$] has proved that the variety structure $X_w^P(\lambda)$ on X_w^P does not depend upon $\lambda \in D_S^0$. It is likely that the same is true in the non-symmetrizable case as well, but I don't know a proof. In any case, as observed in § 1.6, the normalization of $X_w^P(\lambda)$ does not depend upon $\lambda \in D_S^0$.

(1.9) **Convention.** *Unless otherwise stated,* $\mathfrak{g}$ *will denote an arbitrary (not necessarily symmetrizable) Kac-Moody algebra with its associated group G over* $\mathbb{C}$ *(as defined in* § 1.2), *(standard) Borel subgroup B, and (standard) maximal torus T. The group of characters of the torus T will be denoted by R(T) and the group algebra* $\mathbb{Z}[R(T)]$ *will be denoted by A(T). When S is the empty set, we generally abbreviate* X_w^B *by* X_w.

2. Algebraic geometry of Schubert varieties – The main results

(2.1) *Construction of Bott-Samelson-Demazure-Hansen desingularization* [T$_1$] and [S1$_2$]. For any sequence $\mathfrak{w} = (r_{i_1}, \ldots, r_{i_n})$ of simple reflections, we denote by $Z_\mathfrak{w}$ the *Bott-Samelson-Demazure-Hansen variety* (over $\mathbb{C}$) defined below.

For any $1 \leqq j \leqq n$, let P_{i_j} be the minimal parabolic subgroup of G containing B and the simple reflection r_{i_j}. (In the notation of § 1.3, it is the parabolic subgroup corresponding to the singleton set $\{i_j\}$.) The group $B^n = BX \ldots XB$ acts on $P_{i_1} X \ldots X P_{i_n}$ from the right as follows:

$$(p_1, \ldots, p_n)(b_1, \ldots, b_n) = (p_1 b_1, b_1^{-1} p_2 b_2, \ldots, b_{n-1}^{-1} p_n b_n), \quad \text{for } p_j \in P_{i_j}$$

and $b_j \in B$. Now put

$$Z_\mathfrak{w} = P_{i_1} X \ldots X P_{i_n} / B^n.$$

The group P_{i_1} (in particular the Borel subgroup B) acts (from the left) on $Z_\mathfrak{w}$ as the left multiplication (only) on the first factor.

For any $1 \leqq j \leqq n$, denote by $\mathfrak{w}[j]$ (resp. $\mathfrak{w}(j)$) the subsequence $(r_{i_1}, \ldots, r_{i_j})$ (resp. the subsequence $(r_{i_1}, \ldots, \hat{r}_{i_j}, \ldots, r_{i_n})$). There is a canonical projection $\pi = \pi_{\mathfrak{w}[j]}: Z_\mathfrak{w} \to Z_{\mathfrak{w}[j]}$ given by $(p_1, \ldots, p_n) \bmod B^n \mapsto (p_1, \ldots, p_j) \bmod B^j$. Also there is a canonical inclusion $i = i_{\mathfrak{w}(j)}: Z_{\mathfrak{w}(j)} \hookrightarrow Z_\mathfrak{w}$ defined by $(p_1, \ldots, \hat{p}_j, \ldots, p_n) \bmod B^{n-1} \mapsto (p_1, \ldots, 1, \ldots, p_n) \bmod B^n$. It is easy to see that the above maps π and i are well defined, i is an embedding, and π is a surjective smooth morphism of smooth projective varieties. In fact $\pi_{\mathfrak{w}[n-1]}$ is a $\mathbb{P}^1$-fibration. As a consequence, $Z_\mathfrak{w}$ is the total space of successive (totally n) $\mathbb{P}^1$-fibrations, starting from a point. (In particular, $Z_\mathfrak{w}$ is a smooth projective variety of $\dim n$.)

Define a map $\theta_\mathfrak{w}: Z_\mathfrak{w} \to G/B$ by $\theta_\mathfrak{w}((p_1, \ldots, p_n) \bmod B^n) = (p_1 \ldots p_n) \bmod B$. The map $\theta_\mathfrak{w}$ is P_{i_1}-equivariant, where P_{i_1} acts on G/B as the left multiplication. If $\mathfrak{w}$

is a *reduced sequence*, i.e., $w = r_{i_1} \dots r_{i_n}$ is a reduced expression then $\theta_{\mathfrak{w}}(Z_{\mathfrak{w}}) = X_w$ and $\theta_{\mathfrak{w}}$ is a birational morphism of $Z_{\mathfrak{w}}$ onto X_w.

(2.2) *Line bundles on G/B and $Z_{\mathfrak{w}}$*. Fix $\lambda \in \mathfrak{h}_{\mathbb{Z}}^*$ (cf. § 1.2) and let $\mathscr{L}(\lambda)$ be the line bundle on G/B associated to the principal B-bundle: $G \to G/B$ by the character $e^{-\lambda}$: $T \to \mathbb{C}^*$. (Although $e^{-\lambda}$ is a character defined on T, we extend it to the whole of B by defining it to be identically 1 on the 'unipotent radical', the commutator subgroup $[B, B]$, of B.) (The twist $\lambda \to e^{-\lambda}$ is introduced only to keep the correspondence of dominant regular with 'ample bundles'.) More generally, for any finite dimensional B-module M, we denote by $\mathscr{L}(M)$ the vector bundle on G/B associated to the B-module M. For any $w \in W$, by $\mathscr{L}_w(\lambda)$ (resp. $\mathscr{L}_w(M)$) we mean the line bundle $\mathscr{L}(\lambda)$ (resp. the vector bundle $\mathscr{L}(M)$) restricted to the Schubert variety $X_w \subset G/B$. (The constructions in this paragraph are only in the topological category.)

By [Sl_2; § 2.7], for any $\lambda \in D_\phi^0$ and any $w \in W$, the line bundle $\mathscr{L}_w(\lambda)$ is an algebraic line bundle with respect to the variety structure $X_w(\lambda)$ on X_w, in particular, $\mathscr{L}_w(\lambda)$ is an algebraic line bundle on the (stable) variety X_w. Now write any $\lambda \in \mathfrak{h}_{\mathbb{Z}}^*$ as a difference $\lambda = \mu - \mu'$, for some $\mu, \mu' \in D_\phi^0$. By [Sl_2; § 2.7], as topological line bundles, $\mathscr{L}_w(\lambda) \approx \mathscr{L}_w(\mu) \otimes (\mathscr{L}_w(\mu')^*)$, where $\mathscr{L}_w(\mu')^*$ denotes the dual of the line bundle $\mathscr{L}_w(\mu')$, and hence (for any $\lambda \in \mathfrak{h}_{\mathbb{Z}}^*$) $\mathscr{L}_w(\lambda)$ is an algebraic line bundle on the (stable) Schubert variety X_w. It is easy to see (see the proof of [Sl_2; Lemma 3 of § 2.7]) that if we write $\lambda = \nu - \nu'$ (for ν, $\nu' \in D_\phi^0$), then the algebraic line bundles $\mathscr{L}_w(\mu) \otimes \mathscr{L}_w(\nu')$ and $\mathscr{L}_w(\nu) \otimes \mathscr{L}_w(\mu')$ are isomorphic. So the algebraic line bundle $\mathscr{L}_w(\lambda)$ does not depend upon the particular decomposition of λ.

Now, for any sequence $\mathfrak{w} = (r_{i_1}, \dots, r_{i_n})$, we define $\mathscr{L}_{\mathfrak{w}}(\lambda)$ to be the line bundle on $Z_{\mathfrak{w}}$ got by pulling the line bundle $\mathscr{L}(\lambda)$ via the map $\theta_{\mathfrak{w}}$. (The notation $\mathscr{L}_{\mathfrak{w}}(M)$ will have a similar meaning.) Since $\mathscr{L}_{\mathfrak{w}}(M)$ can also be defined as an associated bundle (associated to a finite dimensional representation of B^n) on $Z_{\mathfrak{w}}$, it is easy to see that $\mathscr{L}_{\mathfrak{w}}(M)$ is an algebraic vector bundle on $Z_{\mathfrak{w}}$, which is P_{i_1}-equivariant (in the algebraic category), see [Sl_2; § 1.9]. Hence $H^p(Z_{\mathfrak{w}}, \mathscr{L}_{\mathfrak{w}}(M))$ is canonically a P_{i_1} (in particular a B)-module, for all $p \geqq 0$.

Now we can state one of *the most crucial propositions of this paper*, the proof of which is slightly long and will be taken up in the fourth section.

(2.3) **Proposition.** *Let $\mathfrak{w} = (r_{i_1}, \dots, r_{i_n})$ be any sequence and let $1 \leqq j \leqq k \leqq n$ be such that the sequence $(r_{i_j}, \dots, r_{i_k})$ is reduced*[1] *(cf. § 2.1). Choose any dominant $\lambda \in \mathfrak{h}_{\mathbb{Z}}^*$* (i.e. $\lambda(h_i) \geqq 0$, *for all $1 \leqq i \leqq l$). Then, with the notations as above, we have:*

$$H^p\left(Z_{\mathfrak{w}}, \mathscr{L}_{\mathfrak{w}}(\lambda) \otimes \mathcal{O}_{Z_{\mathfrak{w}}}\left[-\bigcup_{q=j}^{k} Z_{\mathfrak{w}(q)}\right]\right) = 0, \quad \textit{for all } p > 0.$$

We also have:

$$H^p(Z_{\mathfrak{w}}, \mathscr{L}_{\mathfrak{w}}(\lambda)) = 0, \quad \textit{for all } p > 0.$$

(As is standard, $\mathcal{O}_{Z_{\mathfrak{w}}}$ denotes the structure sheaf of $Z_{\mathfrak{w}}$ and $\mathcal{O}_{Z_{\mathfrak{w}}}[-Y]$, for any hypersurface $Y \subset Z_{\mathfrak{w}}$, can canonically be identified with the ideal sheaf of Y inside $Z_{\mathfrak{w}}$.) □

[1] This assumption is essential. Take, e.g., $\mathfrak{w} = (r_i, r_i)$ and $j = 1$, $k = 2$

(2.4) **Corollary.** *Let $\mathfrak{w}$ be any sequence. Then, for any $1 \leqq j \leqq n$ and dominant $\lambda \in \mathfrak{h}_{\mathbb{Z}}^*$, the canonical map*: $H^0(Z_{\mathfrak{w}}, \mathscr{L}_{\mathfrak{w}}(\lambda)) \to H^0(Z_{\mathfrak{w}(j)}, \mathscr{L}_{\mathfrak{w}(j)}(\lambda))$ *is surjective.*

Proof. Consider the exact sheaf sequence (corresponding to the hypersurface $Z_{\mathfrak{w}(j)} \subset Z_{\mathfrak{w}}$):

$$0 \to \mathcal{O}_{Z_{\mathfrak{w}}}[-Z_{\mathfrak{w}(j)}] \to \mathcal{O}_{Z_{\mathfrak{w}}} \to \mathcal{O}_{Z_{\mathfrak{w}(j)}} \to 0.$$

On tensoring with the (locally free) sheaf $\mathscr{L}_{\mathfrak{w}}(\lambda)$, we get the exact sequence:

$$0 \to \mathscr{L}_{\mathfrak{w}}(\lambda) \otimes \mathcal{O}_{Z_{\mathfrak{w}}}[-Z_{\mathfrak{w}(j)}] \to \mathscr{L}_{\mathfrak{w}}(\lambda) \to \mathscr{L}_{\mathfrak{w}(j)}(\lambda) \to 0.$$

Considering the corresponding long exact cohomology sequence, together with the above proposition, we get the corollary. □

(2.5) *The Borel-Weil homomorphism.* Fix a dominant $\lambda \in \mathfrak{h}_{\mathbb{Z}}^*$ and fix any integrable highest weight $\mathfrak{g}$-module $V(\lambda)$ with highest weight λ. (For symmetrizable $\mathfrak{g}$, there is only one such $V(\lambda)$, but for non-symmetrizable $\mathfrak{g}$ it is not known to be unique, cf. § 3.7.) The representation $V(\lambda)$ can be integrated to give a representation of the group G. For $w \in W$, denote by $V_w(\lambda)$ the 'Schubert module' $U(\mathfrak{b})\, v_{w\lambda}$ (which is a $U(\mathfrak{b})$-submodule of $V(\lambda)$), where $v_{w\lambda}$ is the unique (up to a non-zero scalar multiple) vector $\varepsilon\, V(\lambda)$ of weight $w\lambda$ and $\mathfrak{b} = \mathfrak{h} \oplus \mathfrak{n}$ is the Borel subalgebra.

Recall that the line bundle $\mathscr{L}(\lambda)$ on G/B can be identified with $G X_B \mathbb{C}_\lambda^*$, where $\mathbb{C}_\lambda$ is the one dimensional B-module with character e^λ, $\mathbb{C}_\lambda^*$ is its dual, and B acts (from the right) on $G \times \mathbb{C}_\lambda^*$ by $(g,f) \cdot b = (gb, b^{-1} f)$.

For any $w \in W$, define a map ψ_w: $V(\lambda)^* \to H^0(X_w, \mathscr{L}_w(\lambda))$ by (where $V(\lambda)^*$ denotes the full dual of $V(\lambda)$):

$$\psi_w(f)(g \bmod B) = (g, \tilde{\psi}(f)\, g) \bmod B, \text{ for } f \in V(\lambda)^* \text{ and } g \bmod B \in X_w,$$

where $\tilde{\psi}(f)$: $G \to \mathbb{C}_\lambda^*$ is the map defined by:

$$(\tilde{\psi}(f)\, g)(v) = (g^{-1} f)(v), \text{ for } g \in G \text{ and } v \in \mathbb{C}_\lambda \subset V(\lambda).$$

(As a B-module, $\mathbb{C}_\lambda$ can be thought of as the highest weight space in $V(\lambda)$.)

It is easy to see that ψ_w is well defined, i.e., if $g \bmod B = g' \bmod B$ then $\psi_w(f)$ $(g \bmod B) = \psi_w(f)$ $(g' \bmod B)$. Also, for any $f \in V(\lambda)^*$, $\psi_w(f)$ gives a regular section of the bundle $\mathscr{L}_w(\lambda)$ with respect to the variety structure $X_w(\lambda)$ on X_w, in particular, $\psi_w(f)$ gives a regular section of the bundle $\mathscr{L}_w(\lambda)$ with respect to the (stable) variety structure on X_w. Further, let $g_0 \in G$ be any element such that the left multiplication by g_0 keeps X_w stable and g_0 lies in some parabolic P of finite type (e.g. $g_0 \in B$) then g_0 acts on $H^0(X_w, \mathscr{L}_w(\lambda))$ (see [Sl_2; § 1.9]). In this case, it is easy to see that $\psi_w(g_0 f) = g_0(\psi_w f)$, for any $f \in V(\lambda)^*$.

Let K_w be the kernel of ψ_w. Then BwB/B being dense in X_w, $K_w = \{f \in V(\lambda)^* \colon (w^{-1} b f)\, v_\lambda = 0$, for all $b \in B\}$, where v_λ is a non-zero highest weight vector of $V(\lambda)$, i.e.,

$$K_w = \{f \in V(\lambda)^* \colon f|_{V_w(\lambda)} \equiv 0\}.$$

In other words, the map ψ_w factors through $V_w(\lambda)^*$ giving rise to an injective map

$$\bar{\psi}_w\colon\ V_w(\lambda)^* \hookrightarrow H^0(X_w, \mathscr{L}_w(\lambda)).$$

(2.6) **Definition of $H^0(Z_\infty, \mathscr{L}(\lambda))^*$.** Let $\mathfrak{W}$ denote the set of all possible sequences $\mathfrak{w}=(r_{i_1},\dots,r_{i_n})$ (as in §2.1). We define a map m: $\mathfrak{W}\to W$ by $m(r_{i_1},\dots,r_{i_n})=r_{i_1}\dots r_{i_n}$. Further we define a partial order $\leqq$ in $\mathfrak{W}$ by demanding $\mathfrak{v}\leqq\mathfrak{w}$ if $\mathfrak{v}$ is obtained from $\mathfrak{w}$ by deleting some entries from $\mathfrak{w}$. Clearly $(\mathfrak{W}, \leqq)$ is a directed set, i.e., given $\mathfrak{w}_1, \mathfrak{w}_2\in\mathfrak{W}$, there exists a $\mathfrak{w}\in\mathfrak{W}$ such that $\mathfrak{w}_1\leqq\mathfrak{w}$ and $\mathfrak{w}_2\leqq\mathfrak{w}$.

For any $\mathfrak{v}\leqq\mathfrak{w}$, there is a canonical embedding $Z_\mathfrak{v}\hookrightarrow Z_\mathfrak{w}$, which is composite of maps of the type $i_{\mathfrak{w}(j)}$ (cf. §2.1), and this embedding is B-equivariant. Hence this embedding gives rise to a (canonical) B-module map: $H^0(Z_\mathfrak{v}, \mathscr{L}_\mathfrak{v}(\lambda))^*\to H^0(Z_\mathfrak{w}, \mathscr{L}_\mathfrak{w}(\lambda))^*$ (which is injective, for dominant λ, by successively using corollary (2.4)). Now define:

$$H^0(Z_\infty, \mathscr{L}(\lambda))^* = \varinjlim_{\mathfrak{w}\in\mathfrak{W}} H^0(Z_\mathfrak{w}, \mathscr{L}_\mathfrak{w}(\lambda))^*.$$

Of course $H^0(Z_\infty, \mathscr{L}(\lambda))^*$ is a B-module. Moreover, for any fixed $1\leqq i\leqq l$, the sequences $\mathfrak{w}=(r_{i_1},\dots,r_{i_n})$ such that $r_{i_1}=r_i$ are cofinal in $\mathfrak{W}$ and for any such $\mathfrak{w}$, the line bundle $\mathscr{L}_\mathfrak{w}(\lambda)$ on $Z_\mathfrak{w}$ is P_i-equivariant (see §2.2) and hence $H^0(Z_\infty, \mathscr{L}(\lambda))^*$ is a P_i-module compatible with the B-module structure on it. In fact, we have a stronger proposition (2.11).

(2.7) **Definition.** Let M be a finite dimensional $\mathfrak{b}$-module. For any simple reflection r_i, $1\leqq i\leqq l$, define a finite dimensional $\mathfrak{p}_i$ (in particular a $\mathfrak{b}$)-module $\mathscr{D}_{r_i}(M)$ [J; §2] to be the largest finite dimensional $U(\mathfrak{p}_i)$-module quotient of the induced module $U(\mathfrak{p}_i)\underset{U(\mathfrak{b})}{\otimes} M$, where $\mathfrak{p}_i$ is the parabolic subalgebra $\mathfrak{b}+\mathbb{C}f_i$.

Now take an arbitrary sequence $\mathfrak{w}=(r_{i_1},\dots,r_{i_n})$ and define:

$$\mathscr{D}_\mathfrak{w}(M)=\mathscr{D}_{r_{i_1}}\dots\mathscr{D}_{r_{i_n}}(M).$$

The following is an interesting fact to know.

(2.8) **Lemma.** *For any finite dimensional* $\mathfrak{b}$*-module M and any sequence $\mathfrak{w}$ (not necessarily reduced) $H^0(Z_\mathfrak{w}, \mathscr{L}_\mathfrak{w}(M))$ is $U(\mathfrak{b})$-isomorphic with the dual module* $[\mathscr{D}_\mathfrak{w}(M^*)]^*$.

Proof. We first observe that for any simple reflection r_i, the B-module $H^0(Z_{r_i}, \mathscr{L}_{r_i}(M))$ is isomorphic with the B-module $[\mathscr{D}_{r_i}(M^*)]^*$. To prove this; it suffices to assume that $\mathfrak{b}$ is the standard Borel subalgebra $\mathfrak{b}_{\alpha_i}$ (spanned by $\{e_i, h_i\}$) of the Lie algebra $sl_2(i)$ (which is a subalgebra of $\mathfrak{g}$ spanned by $\{e_i, f_i, h_i\}$) and M is the cyclic $\mathfrak{b}_{\alpha_i}$-module (also called a string module) $F(\mu,\nu)$, uniquely determined by its highest weight μ and lowest weight ν [J; §2]. It can be easily seen that $F(\mu,\nu)\approx E(\mu-\nu/2)\otimes\mathbb{C}_{\mu+\nu/2}$, where $E(\mu-\nu/2)$ is the (finite dimensional) irreducible $sl_2(i)$-module with highest weight $\mu-\nu/2$ and $\mathbb{C}_{\mu+\nu/2}$ is the one-dimensional $\mathfrak{b}_{\alpha_i}$-module associated to the character $\mu+\nu/2$. Now, by [J; Lemma 2.5],

$$[\mathscr{D}_{r_i}(F(\mu,\nu)^*)]^*\approx E(\mu-\nu/2)\otimes[\mathscr{D}_{r_i}(\mathbb{C}^*_{\mu+\nu/2})]^*.$$

Similarly

$$H^0(Z_{r_i}, \mathscr{L}_{r_i}(F(\mu,\nu)))\approx E(\mu-\nu/2)\otimes H^0(Z_{r_i}, \mathscr{L}_{r_i}(-(\mu+\nu/2))).$$

But then, combining [J; §2.4] and [A; §4], it is fairly easy to see that $[\mathscr{D}_{r_i}(\mathbb{C}^*_{\mu+\nu/2})]^*$ is $s\,l_2(i)$-isomorphic with $H^0(Z_{r_i}, \mathscr{L}_{r_i}(-(\mu+\nu/2)))$. This proves the lemma in the case when $\mathfrak{w}$ consists of a single reflection.

Now the general case, i.e. $\mathfrak{w}=(r_{i_1}, \ldots, r_{i_n})$ is arbitrary, follows easily by induction on the length n of $\mathfrak{w}$ and the Leray spectral sequence, corresponding to the map $\pi_{\mathfrak{w}[1]}\colon Z_{\mathfrak{w}} \to Z_{r_{i_1}}$ (cf. lemma 4.5).

(2.9) *Remark.* As a consequence of the above lemma, proposition (2.3), and lemma (4.6), we deduce that, for any finite dimensional $\mathfrak{b}$-module M, $\mathscr{D}_{\mathfrak{w}}(M)$ is $\mathfrak{b}$-isomorphic with $\mathscr{D}_{\mathfrak{v}}(M)$, provided $\mathfrak{v}$, $\mathfrak{w}$ are reduced sequences with $m(\mathfrak{v}) = m(\mathfrak{w})$. An 'algebraic' proof of this, in the finite case, was given by Joseph [J; Proposition 2.15].

(2.10) **Lemma.** *Let $\mathfrak{w}$ be a reduced sequence, then $H^0(Z_{\mathfrak{w}}, \mathscr{L}_{\mathfrak{w}}(\lambda))^*$ is a cyclic $U(\mathfrak{b})$-module generated by an element of weight $m(\mathfrak{w})\,\lambda$ (where $m(\mathfrak{w})$ is as defined in* §2.6).

First Proof. An argument exactly similar to [J; §2.10] (see also [J; §2.17]), together with lemma (2.8), provide a proof of the above lemma. However, we will give a different proof in more detail:

Second Proof. The affine space BwB/B sits canonically as a Zariski open subset of $Z_{\mathfrak{w}}$ and hence the canonical map: $H^0(BwB/B,\ \mathscr{L}_w(\lambda))^* \to H^0(Z_{\mathfrak{w}},\ \mathscr{L}_{\mathfrak{w}}(\lambda))^*$ $(w=m(\mathfrak{w}))$ is surjective.

Let U_w be the subgroup of U (cf. §1.2) generated by the one parameter subgroups $\{U_\beta\}_{\beta\in\Delta_+\cap w\Delta_-}$, where $U_\beta \subset U$ is the one parameter group corresponding to the positive (real) root β. It is known [Sl_2; §1.8] that U_w is a closed subgroup of U isomorphic, as an algebraic variety, to the affine space of $\dim l(w)$. Moreover the map: $U_w \to BwB/B$, defined by $g \mapsto g\,w \bmod B$ is an (algebraic) isomorphism.

Define a section $s_0 \in H^0(B\,w\,B/B,\ \mathscr{L}_w(\lambda))$ by $s_0(u)=(u\,w,\ v_{-\lambda}) \in (B\,w\,B \times \mathbb{C}_{-\lambda}) \bmod B$, for $u\in U_w$ (where $v_{-\lambda}$ is some fixed nonzero vector of $\mathbb{C}_{-\lambda}$). It is easy to see that $g\,s_0 = s_0$, for all $g\in U_w$ and moreover s_0 is of weight $-w(\lambda)$. Now clearly $H^0(B\,w\,B/B,\ \mathscr{L}_w(\lambda))$ can be identified (as a U_w-module) with $\mathbb{C}[U_w] \underset{\mathbb{C}}{\otimes} \mathbb{C}\,s_0$, where $\mathbb{C}[U_w]$ is the affine ring of the variety U_w which is thought of as a U_w-module via the left regular representation and U_w acts trivially on $\mathbb{C}\,s_0$. Now the restricted dual $\mathbb{C}[U_w]^\wedge$, which is finite linear span of all the T-weight spaces of the full dual $\mathbb{C}[U_w]^*$ (the torus T normalizes U_w and hence there is a natural action of T on $\mathbb{C}[U_w]$ and therefore on its dual $\mathbb{C}[U_w]^*$), can be canonically identified with the universal enveloping algebra $U(\mathfrak{u}_w)$ $(\mathfrak{u}_w = \mathrm{Lie}\,U_w)$. Moreover, under this identification, the $\mathfrak{u}_w$-module structure is nothing but the left multiplication. See, e.g., J.C. Jantzen's Bonn lecture notes 'Representations of Algebraic groups I' – Chapter 7. (Of course $\mathbb{C}[U_w]^*$ is a U_w-module. Although the restricted dual $\mathbb{C}[U_w]^\wedge$ is not a U_w-submodule, it can be seen that it is a $\mathfrak{u}_w$-module.) In particular $\mathbb{C}[U_w]^\wedge$ is $U(\mathfrak{u}_w)$-cyclic, generated by 1. Hence $H^0(Z_{\mathfrak{w}}, \mathscr{L}_{\mathfrak{w}}(\lambda))^*$ is a cyclic $U(\mathfrak{u}_w)$-module, generated by an element of weight $w\,\lambda$. In particular $H^0(Z_{\mathfrak{w}}, \mathscr{L}_{\mathfrak{w}}(\lambda))^*$ is $U(\mathfrak{b})$-cyclic. □

(2.11) **Proposition.** *Fix a dominant λ. Then $H^0(Z_\infty, \mathscr{L}(\lambda))^*$ is an integrable highest weight $\mathfrak{g}$-module, with highest weight λ, compatible with the P_i-module* *structure on it, as described in* §2.6. *In particular, in the case when* $\mathfrak{g}$ *is symmetrizable,* $H^0(Z_\infty, \mathscr{L}(\lambda))^*$ *is the irreducible quotient* $L(\lambda)$ *of the Verma module* $M(\lambda)$, *of highest weight* λ.

Proof. By Kac-Peterson [KP_2; Corollary 1.1] (see also [CPS; Proposition 3.5]), $N = H^0(Z_\infty, \mathscr{L}(\lambda))^*$ is an integrable $\mathfrak{g}$-module, compatible with the P_i-module structure, for all $1 \leqq i \leqq l$, on it given in §2.6 (Although in [KP_2], symmetrizability is tacitly assumed, this particular corrollary goes through without the symmetrizability assumption.)

It remains to show that N is a highest weight module with highest weight λ. From lemma (2.8), it is easy to see that N has at the most one (upto a nonzero scalar multiple) vector $v_\lambda \neq 0$ of weight λ and moreover the weights of N are contained in the cone $\lambda - \sum_{i=1}^{l} \mathbb{Z}_+ \alpha_i$. Further, by corollary (2.4), N has at least one vector $v_\lambda \neq 0$ of weight λ. Let $M = U(\mathfrak{g}) \cdot v_\lambda \subset N$. Fix $w \in W$ and a reduced decomposition $w = r_{i_1} \dots r_{i_n}$. Denote by $\mathfrak{w}$ the sequence $(r_{i_1}, \dots, r_{i_n})$. Since N (resp. M) is integrable, the weight space $N_{w\lambda}$ (resp. $M_{w\lambda}$) corresponding to the weight $w\lambda$ is one dimensional and hence $N_{w\lambda} \subset M$. But, by lemma (2.10), $H^0(Z_\mathfrak{w}, \mathscr{L}_\mathfrak{w}(\lambda))^*$ is a cyclic $U(\mathfrak{b})$-module generated by an element of weight $w\lambda$. Further, by corollary (2.4), $H^0(Z_\mathfrak{w}, \mathscr{L}_\mathfrak{w}(\lambda))^*$ injects inside N, and hence we get $H^0(Z_\mathfrak{w}, \mathscr{L}_\mathfrak{w}(\lambda))^* \subset M$.

But now let $\mathfrak{w} = (r_{i_1}, \dots, r_{i_n})$ be any sequence (not necessarily reduced) and let $\mathfrak{v} = (r_{i_{j_1}}, \dots, r_{i_{j_k}})$ be a maximal reduced subsequence of $\mathfrak{w}$, such that the canonical map: $H^0(Z_\mathfrak{w}, \mathscr{L}_\mathfrak{w}(\lambda)) \to H^0(Z_\mathfrak{v}, \mathscr{L}_\mathfrak{v}(\lambda))$ is an isomorphism. The existence of such a $\mathfrak{v}$ is guaranteed by a subsequent corollary (4.7). Now since N is a direct limit of $H^0(Z_\mathfrak{w}, \mathscr{L}_\mathfrak{w}(\lambda))^*$, the proposition follows. □

As an immediate consequence of the above proposition, we get the following:

(2.11)′ **Corollary.** *The weight spaces of* $H^0(Z_\infty, \mathscr{L}(\lambda))^*$ *are all finite dimensional.*

(2.12) For any reduced sequence $\mathfrak{w}$, the map $\theta_\mathfrak{w}: Z_\mathfrak{w} \to X_w (w = m(\mathfrak{w}))$, defined in §2.1, is surjective and hence the canonical map $H^0(X_w, \mathscr{L}_w(\lambda)) \to H^0(Z_\mathfrak{w}, \mathscr{L}_\mathfrak{w}(\lambda))$ is injective. Dualizing and composing with the dual of the map $\bar{\psi}_w$ (defined in §2.5), we get a surjective map

$$\phi_\mathfrak{w}: H^0(Z_\mathfrak{w}, \mathscr{L}_\mathfrak{w}(\lambda))^* \twoheadrightarrow V_w(\lambda) \subset V(\lambda).$$

Taking the direct limit, we get a $\mathfrak{g}$-module map

$$(*) \qquad \phi: H^0(Z_\infty, \mathscr{L}(\lambda))^* \to V(\lambda).$$

But since $V(\lambda)$ is (by assumption) a highest weight module, ϕ is in fact a surjective map. Now we can determine the structure of the $\mathfrak{g}$-module $H^0(Z_\infty, \mathscr{L}(\lambda))^*$. We have the following:

(2.13) **Lemma.** *The* $\mathfrak{g}$*-module* $H^0(Z_\infty, \mathscr{L}(\lambda))^*$ *is isomorphic with the* $\mathfrak{g}$*-module* $L^{\max}(\lambda)$ *defined in* §1.5.

Proof. First of all the g-module $L^{\max}(\lambda)$ is integrable. Further any integrable highest weight g-module $V(\lambda)$ with highest weight λ is a quotient of $L^{\max}(\lambda)$ (cf. § 1.5). Now, by (∗), we get a surjective g-module map: $H^0(Z_\infty, \mathscr{L}(\lambda))^* \twoheadrightarrow L^{\max}(\lambda)$. Hence, from character consideration together with proposition (2.11), we get the lemma. □

(2.14) **Proposition.** *Fix any dominant $\lambda \in \mathfrak{h}_{\mathbb{Z}}^*$ and take for $V(\lambda)$ the integrable highest weight g-module $L^{\max}(\lambda)$. Then, for any reduced sequence $\mathfrak{w}$ with $m(\mathfrak{w}) = w$, the surjective map (defined in § 2.12) $\phi_{\mathfrak{w}}$: $H^0(Z_{\mathfrak{w}}, \mathscr{L}_{\mathfrak{w}}(\lambda))^* \to L_w^{\max}(\lambda)$ is an isomorphism, where $L_w^{\max}(\lambda)$ denotes $V_w(\lambda)$ for $V(\lambda) = L^{\max}(\lambda)$.*

In particular, the canonical map: $H^0(X_w, \mathscr{L}_w(\lambda)) \to H^0(Z_{\mathfrak{w}}, \mathscr{L}_{\mathfrak{w}}(\lambda))$ and the map $\bar{\psi}_w$: $L_w^{\max}(\lambda)^* \to H^0(X_w, \mathscr{L}_w(\lambda))$, defined in § 2.5, are both isomorphisms.

Proof. Assume, if possible, that $\phi_{\mathfrak{v}}$ is not an isomorphism, for some reduced $\mathfrak{v}$. Then there exists a $\mu \in \mathfrak{h}^*$ such that the dimension of the μ-weight space

$$\dim [H^0(Z_{\mathfrak{v}}, \mathscr{L}_{\mathfrak{v}}(\lambda))^*]_\mu > \dim [L_v^{\max}(\lambda)]_\mu,$$

where $v = m(\mathfrak{v})$. Now, for any reduced $\mathfrak{w} \geqq \mathfrak{v}$, the canonical map (induced by $\phi_{\mathfrak{w}}$ and $\phi_{\mathfrak{v}}$):

$$\frac{H^0(Z_{\mathfrak{w}}, \mathscr{L}_{\mathfrak{w}}(\lambda))^*}{H^0(Z_{\mathfrak{v}}, \mathscr{L}_{\mathfrak{v}}(\lambda))^*} \to \frac{L_w^{\max}(\lambda)}{L_v^{\max}(\lambda)} \quad \text{is surjective.}$$

And hence we have $\dim [H^0(Z_{\mathfrak{w}}, \mathscr{L}_{\mathfrak{w}}(\lambda))^*]_\mu > \dim [L_w^{\max}(\lambda)]_\mu$, for all reduced $\mathfrak{w} \geqq \mathfrak{v}$. From this, together with lemma (4.6), it is easy to see that $\dim [H^0(Z_\infty, \mathscr{L}(\lambda))^*]_\mu > \dim [L^{\max}(\lambda)]_\mu$, which is a contradiction to lemma (2.13).

The second assertion follows trivially from the following commutative triangle:

$$\begin{array}{ccc} H^0(Z_{\mathfrak{w}}, \mathscr{L}_{\mathfrak{w}}(\lambda))^* & \xrightarrow[\sim]{\phi_{\mathfrak{w}}} & L_w^{\max}(\lambda) \\ & \hat{\theta}_{\mathfrak{w}} \searrow \quad \nearrow \bar{\psi}_w^* & \\ & H^0(X_w, \mathscr{L}_w(\lambda))^* & \end{array}$$

where $\bar{\psi}_w^*$ is the dual of $\bar{\psi}_w$ and the map $\hat{\theta}_{\mathfrak{w}}$ is induced from the map $\theta_{\mathfrak{w}}$: $Z_{\mathfrak{w}} \to X_w$ defined in § 2.1. (The map $\hat{\theta}_{\mathfrak{w}}$ is clearly surjective.) □

(2.15) *The line bundles $\mathscr{L}_w^P(\lambda)$.* Given any parabolic $P = P_S$ and any $\lambda \in \mathfrak{h}_{\mathbb{Z}}^*$ satisfying $\lambda(h_i) = 0$, for all $i \in S$, we define a line bundle $\mathscr{L}^P(\lambda)$ on G/P as the line bundle associated to the principal P-bundle: $G \to G/P$ via the character $e^{-\lambda}$: $P \to \mathbb{C}^*$. For any $w \in W$, we denote by $\mathscr{L}_w^P(\lambda)$, the restriction of $\mathscr{L}^P(\lambda)$ to the variety $X_w^P \subset G/P$ (cf. § 1.4). It is easy to see that $\mathscr{L}_w^P(\lambda)$ is an algebraic line bundle on X_w^P (cf. § 2.2).

With these notations, as a consequence of the above proposition, we deduce the following one of the main theorems of the paper.

(2.16) **Theorem.** *Let g be an arbitrary Kac-Moody Lie-algebra with associated group G (cf. § 1.2) and Weyl group W and let $P = P_S$ be any standard parabolic subgroup of finite type of G (cf. § 1.3) (e.g. P is the Borel subgroup B). Then for any $v \leqq w \in W$ and any dominant $\lambda \in \mathfrak{h}_{\mathbb{Z}}^*$ satisfying $\lambda(h_i) = 0$, for all $i \in S$, we have:*

(1) *The canonical map:* $H^0(X^P_w, \mathcal{L}^P_w(\lambda)) \to H^0(X^P_v, \mathcal{L}^P_v(\lambda))$ *is surjective.*

(2) (a) X^P_w *is a normal variety.*

(b) *Further, in the case when* $\mathfrak{g}$ *is symmetrizable, the linear system on* X^P_w *given by any* $\mathcal{L}^P(\lambda)$, *for dominant regular* λ *with respect to* P *(i.e.* $\lambda(h_i)>0$ *for* $i\notin S$*) (cf. § 1.5) imbeds* X^P_w *as a projectively normal variety.*

(3) *For any* $p\geqq 0$ *and any locally free sheaf* $\mathcal{S}$ *on* X^P_w, *the canonical map:* $H^p(X^P_w, \mathcal{S}) \to H^p(Z_{\mathfrak{w}}, \theta^{P*}_{\mathfrak{w}}(\mathcal{S}))$ *is an isomorphism for any reduced sequence* $\mathfrak{w}$ *with* $m(\mathfrak{w})=w$ *(where* $\theta^P_{\mathfrak{w}}$ *is the morphism* $\theta_{\mathfrak{w}}\colon Z_{\mathfrak{w}} \to X_w$ *followed by the canonical morphism* $X_w \to X^P_w$*).*

In particular, $H^p(X^P_w, \mathcal{L}^P_w(\lambda))=0$, *for all* $p>0$ *and the canonical* map

$$\bar{\psi}^P_w\colon [L^{\max}_w(\lambda)]^* \to H^0(X^P_w, \mathcal{L}^P_w(\lambda))$$

is an isomorphism.

(The map $\bar{\psi}^P_w$ is defined analogous to the definition of the map $\bar{\psi}_w$ in § 2.5.)

Proof. First we consider the case when $P=B$. Since $L^{\max}_v(\lambda)\subset L^{\max}_w(\lambda)$, by proposition (2.14), the assertion (1) follows.

In view of proposition (2.14), normality of X_w follows immediately from the following lemma:

(2.17) **Lemma.** *Let* $f\colon X\to Y$ *be a desingularization of a projective variety* Y *(i.e.* X *is a smooth projective variety and* f *is a birational surjective morphism). Assume that there is an ample line bundle* $\mathcal{L}$ *on* Y *such that the canonical map:* $H^0(Y, \mathcal{L}^n)\to H^0(X, f^*\mathcal{L}^n)$ *is an isomorphism for all* $n\geqq n_0$, *where* n_0 *is some fixed positive integer. Then* Y *is a normal variety.*

Proof. Consider the exact sheaf sequence on Y:

$$0\to \mathcal{O}_Y \to f_*\mathcal{O}_X \to \mathcal{Q} \to 0,$$

where $\mathcal{Q}$, by definition, is the quotient sheaf $f_*\mathcal{O}_X/\mathcal{O}_Y$. Tensoring this sequence with (locally free sheaf) $\mathcal{L}^n$ and taking cohomology (and using the projection formula), we get:

$$0\to H^0(Y,\mathcal{L}^n)\to H^0(X, f^*\mathcal{L}^n)\to H^0(Y, \mathcal{Q}\otimes\mathcal{L}^n)\to H^1(Y,\mathcal{L}^n)\to \dots.$$

But $\mathcal{L}$ being ample, by a theorem of Serre [H; Chap. III, § 5], there exists a $\bar{n}_0>0$ such that $H^1(Y,\mathcal{L}^n)=0$, for all $n\geqq \bar{n}_0$. In particular, by the assumption, $H^0(Y, \mathcal{Q}\otimes\mathcal{L}^n)=0$, for all $n\geqq \max(n_0, \bar{n}_0)$. But then, $\mathcal{L}$ being ample, we conclude that $\mathcal{Q}$ itself is 0, i.e., $\mathcal{O}_Y \approx f_*\mathcal{O}_X$, proving the lemma. □

To prove the projective normality; it suffices to show that for dominant $\lambda, \lambda'\in\mathfrak{h}^*_{\mathbb{Z}}$, the canonical map

$$H^0(X_w, \mathcal{L}_w(\lambda))\otimes H^0(X_w, \mathcal{L}_w(\lambda'))\to H^0(X_w, \mathcal{L}_w(\lambda+\lambda'))$$

is surjective. We have the following commutative diagram:

$$\text{(D)}\ldots\qquad \begin{array}{ccc} H^0(X_w, \mathscr{L}_w(\lambda))\otimes H^0(X_w, \mathscr{L}_w(\lambda')) & \longrightarrow & H^0(X_w, \mathscr{L}_w(\lambda+\lambda')) \\ \uparrow\wr & & \uparrow\wr \\ [L_w^{\max}(\lambda)]^*\otimes[L_w^{\max}(\lambda')]^* & \longrightarrow & [L_w^{\max}(\lambda+\lambda')]^* \\ \uparrow & & \uparrow \\ [L^{\max}(\lambda)\otimes L^{\max}(\lambda')]^* & \longrightarrow & [L^{\max}(\lambda+\lambda')]^* \end{array}$$

where the two top vertical maps are induced by $\bar{\psi}_w$ and are isomorphisms by proposition (2.14); the two bottom vertical maps are the canonical restriction maps and hence are surjective; and the bottom horizontal map is given by dualizing the canonical $\mathfrak{g}$-module map (cf. § 1.6)

$$d=d_{\lambda,\lambda'}\colon L^{\max}(\lambda+\lambda')\to L^{\max}(\lambda)\otimes L^{\max}(\lambda'),$$

induced by

$$v_{\lambda+\lambda'}\mapsto v_\lambda\otimes v_{\lambda'},$$

where v_λ is some non-zero highest weight vector in $L^{\max}(\lambda)$.

But the map d is injective in the case when the Lie-algebra $\mathfrak{g}$ is symmetrizable since, in this case, $L^{\max}(\lambda+\lambda')$ is known to be an irreducible $\mathfrak{g}$-module. So in this case (i.e. $\mathfrak{g}$ is symmetrizable) the bottom map in the diagram (D) is surjective. This proves 2(b).

To prove (3); we recall the following lemma of Kempf (in the form convenient for our purposes). See, e.g., [D$_1$; § 5, Proposition 2].

(2.18) **Lemma.** *Let X and Y be two proper schemes over a Noetherian ring and let $f\colon X\to Y$ be a morphism. Suppose further that $f_*\mathcal{O}_X=\mathcal{O}_Y$ and there exists an ample line bundle $\mathscr{L}$ on Y such that $H^p(X, f^*(\mathscr{L}^n))=0$, for all $p>0$ and all sufficiently large n then $R^p f_*(\mathcal{O}_X)=0$, for all $p>0$.* □

Proof of Theorem (2.16) *continued:* Applying the above lemma to the map $\theta_{\mathfrak{w}}$: $Z_{\mathfrak{w}}\to X_w$ and taking some dominant regular $\lambda\in\mathfrak{h}_{\mathbb{Z}}^*$ (i.e. $\lambda(h_i)\geqq 1$ for all i) we get, by proposition (2.3) and part 2(a) of this theorem, that $R^p\theta_{\mathfrak{w}*}\mathcal{O}_{Z_{\mathfrak{w}}}=0$, for all $p>0$. Now (3) follows easily from the projection formula and the Leray spectral sequence associated to the map $\theta_{\mathfrak{w}}$: $Z_{\mathfrak{w}}\to X_w$ together with proposition (2.14).

Now we come to the general case; when P is an arbitrary parabolic of finite type. Consider the projection $\tilde{\pi}\colon G/B\to G/P$. Since P is of finite type, P/B (which is the same as X_{w_0}, where w_0 is the maximal element of $W_S\subset W$) is a finite dimensional smooth projective variety. Clearly $\tilde{\pi}^{-1}(X_w^P)$ is (left) B-stable and is irreducible and hence is equal to X_v, for some $v\in W$. Further since $X_v\to X_w^P$ is an algebraic P/B-bundle, we have the normality of X_w^P. Moreover, $\tilde{\pi}_*\mathcal{O}_{X_v}=\mathcal{O}_{X_w^P}$ and $R^p\tilde{\pi}_*\mathcal{O}_{X_v}=0$ for all $p>0$. So we get, from the Leray spectral sequence and the projection formula, that $H^p(X_w^P, \mathscr{S})\approx H^p(X_v, \tilde{\pi}^*\mathscr{S})$, for any locally free sheaf $\mathscr{S}$ on X_w^P and any $p\geqq 0$. From this, the assertion (1) and the projective normality of X_w^P follow. This argument also proves the assertion (3), provided w is such that $\tilde{\pi}^{-1}(X_w^P)=X_w$. We further observe that for any sequences $\mathfrak{w}, \mathfrak{w}'$, such that entries of $\mathfrak{w}'$ are in $\{r_i\}_{i\in S}$, we have

$H^p(Z_{\mathfrak{v}}, \theta_{\mathfrak{v}}^{P*}(\mathcal{S})) \approx H^p(Z_{\mathfrak{w}}, \theta_{\mathfrak{w}}^{P*}(\mathcal{S}))$, for all $p \geqq 0$, where $\mathfrak{v}$ is the sequence $(\mathfrak{w}, \mathfrak{w}')$. This follows from the Leray spectral sequence for the map $Z_{\mathfrak{v}} \to Z_{\mathfrak{w}}$, together with the fact that the image of $\theta_{\mathfrak{w}'}^P$ is a single point. From this the assertion (3) follows for arbitrary w. □

(2.19) *Remark.* It is very likely that the restriction in the above theorem, that P is of finite type, can be removed. In any case when G is an affine (including twisted affine) group, any proper parabolic subgroup is of finite type.

(2.20) **Definition** [KKMS; p. 50]. Let X (resp. Y) be a smooth (resp. arbitrary) projective variety and let $f: X \to Y$ be a birational morphism. Then f is called a *rational resolution* of Y if:

(a) $f_* \mathcal{O}_X = \mathcal{O}_Y$ and $R^p f_* \mathcal{O}_X = 0$ for $p > 0$ and

(b) $R^p f_* K_X = 0$, for $p > 0$, where K_X is the canonical line bundle of X.

(2.21) *Remark.* In Char. 0, (b) is automatic because of a result of Grauert and Riemenschneider [GR]. Since we are always working (tacitly) over the base field $\mathbb{C}$, (b) is no restriction for us.

(2.22) **Definitions** [H]. A local Noetherian ring A is said to be *Cohen-Macaulay* if depth $A = \dim A$. A scheme is Cohen-Macaulay if all of its local rings are Cohen-Macaulay.

Of course a projective variety $X \subset \mathbb{P}^n$ is said to be *arithmetically Cohen-Macaulay* (inside $\mathbb{P}^n$), if the cone over X (in $\mathbb{A}^{n+1}$) is Cohen-Macaulay.

From Theorem (2.16), we deduce the following:

(2.23) **Theorem.** *With the notations and assumptions as in Theorem* (2.16), *we have the following:*

For $w \in W_S'$ *and reduced* $\mathfrak{w}$ *with* $m(\mathfrak{w}) = w$, *the resolution* $\theta_{\mathfrak{w}}^P: Z_{\mathfrak{w}} \to X_w^P$ *is a rational resolution. In particular,* X_w^P *is Cohen-Macaulay.*

Further, in the case when $\mathfrak{g}$ *is symmetrizable,* X_w^P *is arithmetically Cohen-Macaulay in any projective embedding given by* $\mathcal{L}^P(\lambda)$ *for dominant regular* λ *with respect to P.* (Recall that W_S' is defined in § 1.4.)

Proof. Since $w \in W_S'$, the map $\theta_{\mathfrak{w}}^P: Z_{\mathfrak{w}} \to X_w^P$ is a birational morphism. Moreover, by Theorem (2.16), X_w^P is normal and hence by proposition (2.3) and lemma (2.18), we deduce that $\theta_{\mathfrak{w}}^P$ is a rational resolution. Now X_w^P is Cohen-Macaulay follows from the following:

(2.24) **Proposition** [R; Proposition 4]. *Let* $f: X \to Y$ *be a rational resolution of the projective variety Y then Y is Cohen-Macaulay.* □

Proof of Theorem (2.23) *continued:* To prove the arithmetic Cohen-Macaulay property of X_w^P, in view of Theorem 2.16-2(b), it suffices to show that $H^p(X_w^P, \mathcal{L}_w^P(n\lambda)) = 0$, for all $0 < p < \dim X_w^P$ and all $n \in \mathbb{Z}$. By Theorem 2.16(3), $H^p(X_w^P, \mathcal{L}_w^P(n\lambda)) = 0$, for all $p > 0$ and all $n \geqq 0$. So assume that $n < 0$. The map $Z_{\mathfrak{w}} \to X_w^P$ being birational, we again get the vanishing of $H^p(Z_{\mathfrak{w}}, \mathcal{L}_{\mathfrak{w}}(n\lambda))$ for $0 \leqq p < \dim X_w^P$, by Theorem 4.1 (in this paper) due to Grauert and Riemenschneider. □

Let V be a finite dimensional vector space/$\mathbb{C}$ of $\dim n$ and let $\mathcal{N} \subset \operatorname{End} V$ be the nilpotent cone, i.e., $\mathcal{N}$ is the set of all the nilpotent linear transformations of V. Then, by a result of Lusztig [L; §2], $\mathcal{N}$ occurs as an open subset of a Schubert variety $\subset \mathcal{S}\mathcal{L}_n/\mathcal{P}$, where $\mathcal{S}\mathcal{L}_n$ denotes the affine group associated to the group SL_n and $\mathcal{P}$ is the 'standard' maximal parabolic subgroup (which is a central extension by $\mathbb{C}^*$ of $SL_n(\mathbb{C}[t])$). In particular, as an amusing consequence of our Theorem (2.16), we deduce the following famous result due to Kostant:

(2.25) **Theorem** [Ko; §5]. *With the notations as above, $\mathcal{N}$ is a normal variety.* □

(2.26) *Remark.* Although, in this paper, we are only interested in the case when the base field is $\mathbb{C}$ (see also remark 4.8) it is quite possible that most of the results are true in arbitrary char. (We, together with C. Procesi, have given 'explicit' local description of the Schubert varieties around the base point e, in the case when G is the affine group corresponding to SL_n, extending a result of Lusztig [L; §2]. In particular, this implies that many results of the paper are indeed true in arbitrary char. for this particular case.)

3. The character formulae – Demazure character formula and generalization of Weyl-Kac Character formula to arbitrary Kac-Moody algebras

(3.1) *The Demazure operators.* For any simple reflection r_i, $1 \leqq i \leqq l$, Demazure has defined a $\mathbb{Z}$-linear operator D_{r_i}: $A(T) \to A(T)$ by

$$D_{r_i} e^\lambda = \frac{e^\lambda - e^{-\alpha_i} e^{r_i \lambda}}{1 - e^{-\alpha_i}}, \quad \text{for } e^\lambda \in R(T),$$

where (recall from §1.9) $A(T) = \mathbb{Z}[R(T)]$ is the group algebra on the character group $R(T)$ and α_i is the (positive) simple root associated to the reflection r_i. It is easy to see that $D_{r_i}(e^\lambda) \in A(T)$. In fact, one has the following simple:

(3.2) **Lemma.**

$$\begin{aligned} D_{r_i} e^\lambda &= e^\lambda + e^{\lambda - \alpha_i} + \ldots + e^{r_i \lambda}, && \text{if } \lambda(h_i) \geqq 0 \\ &= 0, && \text{if } \lambda(h_i) = -1 \\ &= -(e^{\lambda + \alpha_i} + \ldots + e^{r_i \lambda - \alpha_i}), && \text{if } \lambda(h_i) < -1. \quad \square \end{aligned}$$

Now let $w \in W$ be arbitrary. Choose any reduced expression $w = r_{i_1} \ldots r_{i_n}$ and define $D_w = D_{r_{i_1}} \circ \ldots \circ D_{r_{i_n}}$: $A(T) \to A(T)$. The following lemma justifies the notation D_w.

(3.3) **Lemma.** *The operator D_w: $A(T) \to A(T)$, defined above, does not depend upon the particular reduced expression of w.*

This lemma follows easily by combining (a subsequent) lemma (4.6) with the proof of Theorem (3.4). A purely algebraic proof can also be given by using a result of Matsumoto. □

There is a conjugation in the ring $A(T)$, defined by $\overline{e^\lambda} = e^{-\lambda}$. We denote $\overline{D_w e^\lambda}$ by $\bar{D}_w e^\lambda$.

Recall that any Schubert variety X_w is (left) B-stable and the line bundle $\mathscr{L}_w(\lambda)$ is a B-equivariant bundle on X_w (cf. § 2.2) and hence $H^p(X_w, \mathscr{L}_w(\lambda))$ is canonically a B (in particular a T)-module.

Now we are ready to prove the following generalization of Demazure character formula in the arbitrary Kac-Moody setting:

(3.4) **Theorem.** *Let* $\mathfrak{g}$ *be an arbitrary (not necessarily symmetrizable) Kac-Moody algebra with the associated group* G *and Weyl group* W (cf. § 1.2). *Then, with the notations as in Sect.* 2, *for any* $w \in W$ *and any* $\lambda \in \mathfrak{h}^*_{\mathbb{Z}}$, *we have*

$$\chi(X_w, \mathscr{L}_w(\lambda)) = \bar{D}_w(e^\lambda) \quad \text{as elements in } A(T),$$

where $\chi(X_w, \mathscr{L}_w(\lambda))$ *is defined to be* $\sum_p (-1)^p \operatorname{ch} H^p(X_w, \mathscr{L}_w(\lambda)) \in A(T)$ *and* $\operatorname{ch} H^p(X_w, \mathscr{L}_w(\lambda))$ *denotes the formal T-character of the T-module* $H^p(X_w, \mathscr{L}_w(\lambda))$.

In particular, if λ *is dominant then* $\operatorname{ch} H^0(X_w, \mathscr{L}_w(\lambda)) = \bar{D}_w(e^\lambda)$ *and hence, by proposition* (2.14), $\operatorname{ch} L_w^{\max}(\lambda) = D_w(e^\lambda)$.

Proof. Let $\mathfrak{w} = (r_{i_1}, \ldots, r_{i_n})$ be a reduced sequence with $m(\mathfrak{w}) = w$. In view of Theorem (2.16), it suffices to prove that $\chi(Z_{\mathfrak{w}}, \mathscr{L}_{\mathfrak{w}}(\lambda)) = \bar{D}_w(e^\lambda)$. The validity of the theorem is clear when $l(w) = 1$, since

$$\chi(Z_{r_i}, \mathscr{L}_{r_i}(\lambda)) = \chi(P_i/B, \mathscr{L}_{r_i}(\lambda)) = \bar{D}_{r_i}(e^\lambda),$$

by lemma (3.2).

Now we prove the theorem by induction on $l(w)$. (The proof given here is identical to the one given in [A; § 4].) By Leray spectral sequence (cf. lemma 4.5),

$$\begin{aligned} H^p(Z_{\mathfrak{w}}, \mathscr{L}_{\mathfrak{w}}(\lambda)) &\approx H^p(Z_{\mathfrak{w}(n)}, \mathscr{L}_{\mathfrak{w}(n)}(H^0(P_{i_n}/B, \mathscr{L}_{r_{i_n}}(\lambda)))), && \text{if } \lambda(h_{i_n}) \geqq 0 \\ &\approx H^{p-1}(Z_{\mathfrak{w}(n)}, \mathscr{L}_{\mathfrak{w}(n)}(H^1(P_{i_n}/B, \mathscr{L}_{r_{i_n}}(\lambda)))), && \text{if } \lambda(h_{i_n}) < 0. \end{aligned}$$

Now, for any exact sequence of finite dimensional B-modules:

$$0 \to M_1 \to M \to M_2 \to 0,$$

we have

$$\chi(Z_{\mathfrak{w}}, \mathscr{L}_{\mathfrak{w}}(M)) = \chi(Z_{\mathfrak{w}}, \mathscr{L}_{\mathfrak{w}}(M_1)) + \chi(Z_{\mathfrak{w}}, \mathscr{L}_{\mathfrak{w}}(M_2)).$$

This, together with the induction hypothesis, proves the first part of the theorem.

For dominant λ, $H^p(X_w, \mathscr{L}_w(\lambda)) = 0$, for all $p > 0$, by Theorem (2.16). This completes the proof. □

We generalize the Weyl-Kac character formula, as well as the denominator formula, to arbitrary Kac-Moody algebras. The case when $\mathfrak{g}$ is finite dimensional is, of course, due to Weyl and the symmetrizable case is due to Kac. More precisely, we have the following:

(3.5) **Theorem.** *Let* $\mathfrak{g}$ *be an arbitrary Kac-Moody algebra with Cartan subalgebra* $\mathfrak{h}$ *and Weyl group* W *and let* λ *be any dominant element in* $\mathfrak{h}^*_{\mathbb{Z}}$. *Then the formal character* (*with respect to* $\mathfrak{h}$), *denoted by* ch, *of the integrable highest weight* $\mathfrak{g}$*-module* $L^{\max}(\lambda)$ (see lemma 1.5) *is given by:*

$$\Big(\sum_{w\in W}\varepsilon(w)\,e^{w\rho}\Big)\cdot \operatorname{ch}(L^{\max}(\lambda))=\sum_{w\in W}\varepsilon(w)\,e^{w(\lambda+\rho)},$$

where $\varepsilon(w)$ *denotes the signature of* w *and* ρ *is any element of* $\mathfrak{h}^*$ *satisfying* $\rho(h_i)=1$ *for all*, $1\leqq i\leqq l$.

Moreover, we also have the denominator formula:

$$\sum_{w\in W}\varepsilon(w)\,e^{w\rho-\rho}=\prod_{\beta\in\Delta_+}(1-e^{-\beta})^{\operatorname{mult}\beta},$$

where mult β *denotes the dimension of the* β*-th root space.*

Proof. Recall the definition of $L_w^{\max}(\lambda)$ from proposition (2.14). Write, for any $w\in W$,

$$\operatorname{ch} L_w^{\max}(\lambda)=\sum_{\mu} m_\mu(w)\,e^\mu$$

and

$$\operatorname{ch} L^{\max}(\lambda)=\sum_{\mu} m_\mu\,e^\mu.$$

Then, clearly $m_\mu=\max\limits_{w\in W}\{m_\mu(w)\}$.

Define a 'shifted' action of W on $\mathfrak{h}^*$ by $w*\mu=w(\mu+\rho)-\rho$, for $w\in W$ and $\mu\in\mathfrak{h}^*$.

Fix any $v\in W$ and $\mu\in\mathfrak{h}^*$. Then we have the following:

(3.6) **Lemma.**

(a) $$\sum_{w\in W}\varepsilon(w)\,m_{w*\mu}(v)=\begin{cases}\varepsilon(w_0), & \text{if } \mu=w_0*\lambda, \text{ for some } w_0\in W\\ 0, & \text{if } \mu\notin W*\lambda.\end{cases}$$

In particular,

(b) $$\sum_{w\in W}\varepsilon(w)\,m_{w*\mu}=\begin{cases}\varepsilon(w_0), & \text{if } \mu=w_0*\lambda\\ 0, & \text{if } \mu\notin W*\lambda.\end{cases}$$

Proof. For (any fixed) $\mu\in\mathfrak{h}^*$, there exists only finitely many $w_1,\dots,w_n\in W$ (depending upon μ) such that $m_{w_i*\mu}\neq 0$, for $1\leqq i\leqq n$. In particular, $m_{w*\mu}(v)=0$ for any $v\in W$, and any $w\notin\{w_1,\dots,w_n\}$. To prove this; observe that, $L^{\max}(\lambda)$ being integrable, $m_{w*\mu}=m_{\mu+\rho-w^{-1}\rho}$. Hence if $l(w)$ is sufficiently large, $\mu+\rho-w^{-1}\rho$ is not a weight of $L^{\max}(\lambda)$.

Hence the sums in the lemma make sense. We prove the lemma by induction on $l(v)$. The lemma is obvious for $l(v)=0$. So write $v=r_iv'$ with $l(v')<l(v)$. By proposition (2.14) and Theorem (3.4), we get:

$$\begin{aligned}\sum_{\nu} m_\nu(v)\,e^\nu &= D_{r_i}\Big(\sum_{\nu} m_\nu(v')\,e^\nu\Big)\\ &=\sum_{\nu} m_\nu(v')\left(\frac{e^\nu-e^{-\alpha_i}e^{r_i\nu}}{1-e^{-\alpha_i}}\right)\\ &=\sum_{\nu} m_\nu(v')\left(e^\nu\sum_{k=0}^{\infty}e^{-k\alpha_i}-e^{r_i\nu}\sum_{k=1}^{\infty}e^{-k\alpha_i}\right).\end{aligned}$$

Hence, for any $\nu \in \mathfrak{h}_{\mathbb{Z}}^*$,

$$m_\nu(v) = \sum_{k=0}^{\infty} m_{\nu+k\alpha_i}(v') - \sum_{k=1}^{\infty} m_{r_i\nu-k\alpha_i}(v').$$

Hence

$$\sum_w \varepsilon(w)\, m_{w*\mu}(v) = \sum_w \varepsilon(w) \sum_{k=0}^{\infty} m_{w*\mu+k\alpha_i}(v') - \sum_w \varepsilon(w) \sum_{k=0}^{\infty} m_{(r_i w)*\mu-k\alpha_i}(v')$$

(since $r_i\,\nu = r_i * \nu + \alpha_i$)

$$= \sum_w \varepsilon(w) \sum_{k=0}^{\infty} m_{w*\mu+k\alpha_i}(v') + \sum_w \varepsilon(w) \sum_{k=0}^{\infty} m_{w*\mu-k\alpha_i}(v')$$

(replacing w by $r_i w$ in the second sum)

$$= \sum_w \varepsilon(w)\, m_{w*\mu}(v') + \sum_w \varepsilon(w) \sum_{k\in\mathbb{Z}} m_{w*\mu+k\alpha_i}(v').$$

But $\sum_w \varepsilon(w) \sum_{k\in\mathbb{Z}} m_{w*\mu+k\alpha_i}(v') = 0$, since it is also equal to

$$-\sum_w \varepsilon(w) \sum_{k\in\mathbb{Z}} m_{(r_i w)*\mu+k\alpha_i}(v') = -\sum_w \varepsilon(w) \sum_{k\in\mathbb{Z}} m_{w*\mu+k\alpha_i}(v').$$

This proves (a).

To prove (b); For any w_j $(1 \leqq j \leqq n)$ choose v_j such that $m_{w_j*\mu} = m_{w_j*\mu}(v_j')$, for all $v_j' \geqq v_j$. If we now choose any $v \in W$ such that $v \geqq v_j$ for all $1 \leqq j \leqq n$, then $m_{w_j*\mu} = m_{w_j*\mu}(v)$, for all $1 \leqq j \leqq n$. □

Proof of Theorem (3.5) *continued:*

$$\Big(\sum_{w\in W} \varepsilon(w)\, e^{w\rho-\rho}\Big) \cdot \operatorname{ch} L^{\max}(\lambda) = \sum_w \varepsilon(w)\, e^{w\rho-\rho} \cdot \sum_\mu m_\mu\, e^\mu$$

$$= \sum_{w,\mu} \varepsilon(w)\, m_\mu\, e^{\mu+w\rho-\rho}$$

$$= \sum_{w,\mu} \varepsilon(w)\, m_{\mu-w\rho+\rho}\, e^\mu$$

$$= \sum_{w,\mu} \varepsilon(w)\, m_{w^{-1}\mu-\rho+w^{-1}\rho}\, e^\mu$$

(since $m_\nu = m_{w\nu}$)

$$= \sum_{w,\mu} \varepsilon(w^{-1})\, m_{w*\mu}\, e^\mu$$

$$= \sum_\mu \Big(\sum_w \varepsilon(w)\, m_{w*\mu}\Big) e^\mu$$

$$= \sum_{w_0} \varepsilon(w_0)\, e^{w_0*\lambda} \quad \text{(by lemma 3.6(b))}$$

(observe that for $v \neq w$, $v*\lambda \neq w*\lambda$).

This proves the first part of the theorem.

To prove the second part; take any $\mu \in \mathfrak{h}^*$ satisfying

$$\mu = \lambda - \sum_i n_i\,\alpha_i, \quad \text{where, for all } 1 \leqq i \leqq l,\ 0 \leqq n_i \leqq \lambda(h_i). \tag{$*$}$$

Then the μ-weight spaces in the Verma module $M(\lambda)$ and $L^{\max}(\lambda)$ are the same (by the definition of $L^{\max}(\lambda)$). Hence the coefficient of e^{μ} in

$$\frac{e^{\lambda}}{\prod_{\beta\in\Delta_+}(1-e^{-\beta})^{\operatorname{mult}\beta}} \quad\text{and}\quad \frac{\sum_w \varepsilon(w)\, e^{w(\lambda+\rho)-\rho}}{\sum_w \varepsilon(w)\, e^{w\rho-\rho}}$$

are the same, i.e., the coefficient of $e^{\mu-\lambda}$ in

$$\frac{\sum_w \varepsilon(w)\, e^{w\rho-\rho}}{\prod_{\beta\in\Delta_+}(1-e^{-\beta})^{\operatorname{mult}\beta}} \quad\text{and}\quad \sum_w \varepsilon(w)\, e^{w(\lambda+\rho)-(\lambda+\rho)}$$

are the same (since $\sum_w \varepsilon(w)\, e^{w\rho-\rho} = \sum_{\beta\in-\sum_i \mathbb{Z}_+\alpha_i} n_\beta e^\beta$, for some $n_\beta\in\mathbb{Z}$).

But, because of the condition $(*)$ on μ, the coefficient $n_{\mu,\lambda}$ of $e^{\mu-\lambda}$ in the second term is the same as the coefficient of $e^{\mu-\lambda}$ in 1 (i.e. $n_{\mu,\lambda}=\delta_{\mu,\lambda}$). Now for any $\beta\in-\sum_{i=1}^{l}\mathbb{Z}_+\alpha_i$, we can choose sufficiently large λ such that $\mu=\beta+\lambda$ satisfies $(*)$ with respect to λ. This completes the proof. □

(3.7) *Remark.* In the symmetrizable case, it is (of course) known that given any dominant $\lambda\in\mathfrak{h}^*_{\mathbb{Z}}$, there is a unique highest weight integrable module (with highest weight λ), in particular, it is irreducible. But in the non-symmetrizable case the uniqueness is not known (as far as I know). One consequence of the uniqueness would be that the 'radical' is 0 (a result due to Gabber-Kac in the symmetrizable case).

The character formula and the denominator formula proved above are new in the non-symmetrizable case. □

We come to the Borel-Weil-Bott theorem.

(3.8) **Definitions.** (a) For any $\lambda\in\mathfrak{h}^*_{\mathbb{Z}}$ and any $p\geqq 0$, define:

$$H^p(G/B, \mathscr{L}(\lambda))^* = \varinjlim_{w\in W} H^p(X_w, \mathscr{L}_w(\lambda))^*.$$

By an argument similar to the one in § 2.6 and the first paragraph of the proof of proposition (2.11), $H^p(G/B, \mathscr{L}(\lambda))^*$ is an integrable $\mathfrak{g}$-module.

(b) Let C be the set of all the dominant elements in $\mathfrak{h}^*_{\mathbb{Z}}$. Define the (integral) *Tits cone* $X=[\bigcup_{w\in W} wC]-\rho$, i.e., $\{\nu-\rho\colon \nu\in\bigcup_{w\in W} wC\}$.

The following lemma is essentially due to Demazure [D_2].

(3.9) **Lemma.** *Fix a simple reflection r_i and $\lambda\in\mathfrak{h}^*_{\mathbb{Z}}$ such that $\lambda(h_i)\geqq -1$. Then, for all $p\in\mathbb{Z}$, $H^p(X_w, \mathscr{L}_w(\lambda))$ is 'canonically' isomorphic with $H^{p+1}(X_w, \mathscr{L}_w(r_i*\lambda))$ as B-modules, for any $w\in W$ such that $wr_i<w$.*

In fact, this isomorphism is a P_j-module isomorphism for all those $1\leqq j\leqq l$ such that X_w is left P_j-stable, i.e., $r_jw<w$.

(The notation $r_i*\lambda$ is defined just above lemma 3.6.) □

Now we can prove the following theorem, which is a generalization of Borel-Weil-Bott theorem to arbitrary Kac-Moody situation. See also [KP_2; Corollary 2.2].

(3.10) **Theorem.** *Let* $\mathfrak{g}$ *be an arbitrary Kac-Moody algebra with associated group G, Borel subgroup B, and Weyl group W. Then, for any* $\lambda \in C - \rho$, $v \in W$, *and* $p \in \mathbb{Z}$, *we have:*

$$H^p(G/B, \mathscr{L}(\lambda))^* \approx H^{p+l(v)}(G/B, \mathscr{L}(v * \lambda))^*, \quad \text{as } G\text{-modules.}$$

Proof. We prove the theorem by induction on $l(v)$. Write $v = r_i v'$ with $l(v') < l(v)$. We first observe that $(v' * \lambda) h_i = (v'(\lambda + \rho)) h_i - 1 = (\lambda + \rho)(v'^{-1} h_i) - 1 \geqq -1$, since $\lambda + \rho \in C$ (by assumption) and $v'^{-1} h_i = \sum n_j h_j$ with $n_j \geqq 0$. Hence, by the previous lemma (3.9), $H^p(X_w, \mathscr{L}_w(v' * \lambda)) \approx H^{p+1}(X_w, \mathscr{L}_w(v * \lambda))$ as B-modules, provided $w r_i < w$. But since the set W_i, of all those Weyl group elements w such that $w r_i < w$, is cofinal in W, we get that $H^p(G/B, \mathscr{L}(v' * \lambda))^* \approx H^{p+1}(G/B, \mathscr{L}(v * \lambda))^*$, as B-modules. It remains to show that the isomorphism is indeed a G-module isomorphism.

Take any simple reflection r_j and define $W_{j,i} = \{w \in W_i : r_j w < w\}$. In view of lemma (3.9), it suffices to show that $W_{j,i}$ also is cofinal in W. So, pick any $w \in W_i$. If $r_j w > w$, we claim that $r_j w \in W_{j,i}$, i.e., $r_j w r_i < r_j w$. By [De; Theorem 1.1], either $r_j w r_i \leqq w$ or $r_j w r_i < r_j w$. But since $r_j w > w$, if $r_j w r_i \leqq w$ then $l(r_j w r_i) = l(w)$ and hence $r_j w r_i = w$, i.e., $w r_i = r_j w$. But $l(w r_i) < l(w)$, a contradiction! Hence we are left with the only possibility that $r_j w r_i < r_j w$. Hence $W_{j,i}$ is cofinal in W_i and hence in W. □

(3.11) **Corollary.** *With the notations as in the above Theorem* (3.10), *we have:*

(a) *For* $\lambda \in C$ *and any* $v \in W$, $H^p(G/B, \mathscr{L}(v * \lambda))^* = 0$, *unless* $p = l(v)$ *and*

$$H^{l(v)}(G/B, \mathscr{L}(v * \lambda))^* \approx H^0(G/B, \mathscr{L}(\lambda))^* \approx L^{\max}(\lambda) \quad (\text{as } G\text{-modules}).$$

(b) *For any* $\lambda \in C - \rho$ *but* $\lambda \notin C$ *and any* $v \in W$, $H^p(G/B, \mathscr{L}(v * \lambda))^* = 0$, *for all* $p \geqq 0$.

Proof. (a) follows trivially from Theorems (3.10) and (2.16).

By the above Theorem (3.10), $H^p(G/B, \mathscr{L}(v * \lambda))^* \approx H^{p-l(v)}(G/B, \mathscr{L}(\lambda))^*$. By assumption on λ, there exists $1 \leqq i \leqq l$ such that $\lambda(h_i) = -1$. For any $w \in W_i$, the canonical map $\tilde{\pi}_i : X_w \to X_w^{P_i} \subset G/P_i$ is a $\mathbb{P}^1$-fibration with fibre $\approx P_i/B$. But since $\lambda(h_i) = -1$, we get that the restriction of $\mathscr{L}_w(\lambda)$ to the fibres of $\tilde{\pi}_i$ is of degree -1. Hence all the cohomologies of $\mathscr{L}_w(\lambda)$ restricted to the fibers of $\tilde{\pi}_i$ vanish. So (b) follows from the Leray spectral sequence. □

The following remark may be useful to add.

(3.12) *Remark.* If X is a projective variety over $\mathbb{C}$, such that $X = X_k \supset X_{k-1} \supset \ldots \supset X_0 \supset X_{-1} = \emptyset$, where each X_n is a Zariski closed subset of X. Assume further that $X_n \backslash X_{n-1}$ is a disjoint union of affine spaces, for all n. Then, with respect to any mixed Hodge structure on X, $H^{p,q}(X) = 0$, unless $p = q$.

In particular, for $w \in W$ and any parabolic P, $H^{p,q}(X_w^P) = 0$, unless $p = q$.

4. Proof of the main 'vanishing' Proposition (2.3)

We recall proposition (2.3):

Proposition. *Let* $\mathfrak{w}=(r_{i_1}, \ldots, r_{i_n})$ *be any sequence and let* $1 \leqq j \leqq k \leqq n$ *be such that the sequence* $(r_{i_j}, \ldots, r_{i_k})$ *is reduced. Then, for any dominant* $\lambda \in \mathfrak{h}_{\mathbb{Z}}^*$,

$$H^p\left(Z_{\mathfrak{w}}, \mathscr{L}_{\mathfrak{w}}(\lambda) \otimes \mathcal{O}_{Z_{\mathfrak{w}}}\left[-\bigcup_{q=j}^{k} Z_{\mathfrak{w}(q)}\right]\right)=0, \quad \text{for all } p>0.$$

We also have $H^p(Z_{\mathfrak{w}}, \mathscr{L}_{\mathfrak{w}}(\lambda))=0$, *for all* $p>0$. □

We continue to use the same notations as in Sect. 2. We crucially use the following result due to Grauert and Riemenschneider.

(4.1) **Theorem** [Ra; §4]. *Let X be a projective variety over* $\mathbb{C}$, *L a line bundle on X, such that there is an integer* $N>0$ *and a birational morphism* $\phi: X \to Y \subset \mathbb{CP}^{N_0}$ *such that* $\phi^*(\mathcal{O}_Y(1)) \approx L^N$. *Then* $H^p(X, L^{-1})=0$, *for* $0 \leqq p < \dim X$. □

Towards the preparation for the proof of above proposition, we recall the following simple facts. (We do not state the lemmas (4.2) and (4.3) in full generality. The versions given below are sufficient for our purposes.)

(4.2) **Lemma.** *Let Z be a smooth proj. variety and let Y, D be smooth irreducible hypersurfaces in Z such that Y intersects D transversally (in Z). Then*

$$\mathcal{O}_Y \otimes_{\mathcal{O}_Z} \mathcal{O}_Z[D] \approx \mathcal{O}_Y[Y \cap D] \quad \text{as sheaves on } Y.$$

(4.3) **Lemma.** *Let* $f: X \to Y$ *be a surjective smooth morphism of smooth proj. varieties and let D be an irreducible hypersurface in Y then the pull back bundle* $f^* \mathcal{O}_Y[D]$ *is isomorphic with* $\mathcal{O}_X[f^*(D)]$ *and since f is surjective smooth,* $f^*(D)$ *is the reduced scheme* $f^{-1}(D)$.

(4.4) **Lemma.** *For any* $\mathfrak{w}=(r_{i_1}, \ldots, r_{i_n})$, *the canonical bundle* $K_{Z_{\mathfrak{w}}}$ *of* $Z_{\mathfrak{w}}$ *is isomorphic with* $\mathscr{L}_{\mathfrak{w}}(-\rho) \otimes \mathcal{O}_{Z_{\mathfrak{w}}}\left[-\bigcup_{q=1}^{n} Z_{\mathfrak{w}(q)}\right]$, *where* $\rho \in \mathfrak{h}_{\mathbb{Z}}^*$ *is any element satisfying* $\rho(h_i)=1$ *for all* $1 \leqq i \leqq l$.

For a proof, see [R; §1]. (Although he gives a proof under the assumption that $\mathfrak{w}$ is reduced and the group G is finite dimensional, the proof goes through in our general set up without difficulty.) □

(4.5) **Lemma** [A; Lemma 1.4(ii)]. *Let M be a finite dimensional B-module and let* $\mathfrak{w}=(r_{i_1}, \ldots, r_{i_n})$ *be any sequence. Fix* $1 \leqq j \leqq n$. *Consider the projection* $\pi: Z_{\mathfrak{w}} \to Z_{\mathfrak{w}[j]}$ (cf. §2.1). *Then, for any* $p \geqq 0$, *the sheaf* $R^p \pi_*(\mathscr{L}_{\mathfrak{w}}(M))$ *is canonically isomorphic with the vector bundle* $\mathscr{L}_{\mathfrak{w}[j]}(N)$, *where N is the canonical B-module* $H^p(Z_{\mathfrak{v}}, \mathscr{L}_{\mathfrak{v}}(M))$ *and* $\mathfrak{v}$ *is the sequence* $(r_{i_{j+1}}, \ldots, r_{i_n})$. (See §2.2 for the notation $\mathscr{L}_{\mathfrak{w}}(M)$.)

Proof. Roughly the idea of the proof is as follows: Let $\mathfrak{p}=(p_1, \ldots, p_j)$ be any sequence such that $p_q \in P_{i_q}$, for all $1 \leqq q \leqq j$. Define a map $i_{\mathfrak{p}}: Z_{\mathfrak{v}} \to Z_{\mathfrak{w}}$ by $i_{\mathfrak{p}}((p_{j+1}, \ldots, p_n) \bmod B^{n-j})=(p_1, \ldots, p_j, p_{j+1}, \ldots, p_n) \bmod B^n$. It is easy to see that $i_{\mathfrak{p}}$

is an embedding (of smooth varieties) with its image exactly equal to the fiber of π over the point $(p_1, \ldots, p_j) \bmod B^j \in Z_{\mathfrak{w}[j]}$. Now use the P_{i_1}-equivariance of the vector bundle $\mathscr{L}_{\mathfrak{w}}(M)$ (cf. § 2.2) and an easy induction on j. □

(4.6) **Lemma.** *Let* $\mathfrak{w}$ *and* $\mathfrak{w}'$ *be two reduced sequences such that* $m(\mathfrak{w}) = m(\mathfrak{w}')$. *Assume further that* $H^p(Z_{\mathfrak{w}}, \mathscr{L}_{\mathfrak{w}}(\lambda)) = 0 = H^p(Z_{\mathfrak{w}'}, \mathscr{L}_{\mathfrak{w}'}(\lambda))$, *for all* $p > 0$ *and all dominant* λ. *Then for any finite* dim *B-module* M, $H^p(Z_{\mathfrak{w}}, \mathscr{L}_{\mathfrak{w}}(M)) \approx H^p(Z_{\mathfrak{w}'}, \mathscr{L}_{\mathfrak{w}'}(M))$, *as B-modules, for all* $p \geqq 0$.

Proof. Let $\beta: \tilde{X}_\theta \to X_\theta$ be the normalization, where $w = m(\mathfrak{w})$. Since $\theta_{\mathfrak{w}}: Z_{\mathfrak{w}} \to X_\theta$ is a birational map, there exists a (unique) map $\tilde{\theta}_{\mathfrak{w}}: Z_{\mathfrak{w}} \to \tilde{X}_\theta$, making the following diagram commutative:

$$\begin{array}{ccc} Z_{\mathfrak{w}} & \xrightarrow{\tilde{\theta}_{\mathfrak{w}}} & \tilde{X}_w \\ & {\scriptstyle \theta_{\mathfrak{w}}} \searrow \quad \swarrow {\scriptstyle \beta} & \\ & X_w & \end{array}$$

Since β is a finite map, any ample line bundle on X_w lifts to an ample line bundle on $\tilde{X}_w$. Now by Kempf's lemma (2.18), applied to the map $\tilde{\theta}_{\mathfrak{w}}$, and the Leray spectral sequence, we get that the canonical map: $H^p(\tilde{X}_w, \tilde{\theta}_{\mathfrak{w}*} \mathscr{L}_w(M)) \to H^p(Z_{\mathfrak{w}}, \mathscr{L}_{\mathfrak{w}}(M))$ is an isomorphism, for all $p \geqq 0$. (Observe that, by induction on dim M and the fact that the action of B on M is solvable, $R^p \tilde{\theta}_{\mathfrak{w}*}(\mathscr{L}_{\mathfrak{w}}(M)) = 0$ for all $p > 0$. Moreover, since X_w is a B-variety, $\tilde{X}_w$ is also canonically a B-variety. Further, by the uniqueness, the lift $\tilde{\theta}_{\mathfrak{w}}$ is a B-equivariant map. Now it can be seen that the sheaf $\tilde{\theta}_{\mathfrak{w}*}(\mathscr{L}_{\mathfrak{w}}(M))$ is canonically isomorphic with the sheaf $\tilde{\theta}_{\mathfrak{w}'*}(\mathscr{L}_{\mathfrak{w}'}(M))$.) This proves the lemma. □

With these preliminaries, we come to the proof of the main proposition.

It is easy to see that the proposition is true for $n = 1$. Now we assume, by induction, that the proposition is true for any sequence $\mathfrak{v} = (r_{j_1}, \ldots, r_{j_{n'}})$ with $n' < n$ and any choices of $1 \leqq j' \leqq k' \leqq n'$ as stated in the proposition.

Now we work with the fixed sequence $\mathfrak{w} = (r_{i_1}, \ldots, r_{i_n})$. We claim that if we know the validity of the proposition for some $1 \leqq j < k \leqq n$, then the proposition is valid also for the pair $j \leqq k - 1$. To prove this; consider the sheaf exact sequence corresponding to the hypersurface $Z_{\mathfrak{w}(k)} \subset Z_{\mathfrak{w}}$:

$$0 \to \mathcal{O}_{Z_{\mathfrak{w}}}[-Z_{\mathfrak{w}(k)}] \to \mathcal{O}_{Z_{\mathfrak{w}}} \to \mathcal{O}_{Z_{\mathfrak{w}(k)}} \to 0. \qquad (S_1)$$

Tensoring this with the locally free sheaf $\mathscr{L}_{\mathfrak{w}}(\lambda) \otimes \mathcal{O}_{Z_{\mathfrak{w}}}\left[-\bigcup_{q=j}^{k-1} Z_{\mathfrak{w}(q)}\right]$, we get:

$$\begin{aligned} 0 &\to \mathscr{L}_{\mathfrak{w}}(\lambda) \otimes \mathcal{O}_{Z_{\mathfrak{w}}}\left[-\bigcup_{q=j}^{k} Z_{\mathfrak{w}(q)}\right] \to \mathscr{L}_{\mathfrak{w}}(\lambda) \otimes \mathcal{O}_{Z_{\mathfrak{w}}}\left[-\bigcup_{q=j}^{k-1} Z_{\mathfrak{w}(q)}\right] \\ &\to \mathcal{O}_{Z_{\mathfrak{w}(k)}} \underset{\mathcal{O}_{Z_{\mathfrak{w}}}}{\otimes} \left(\mathscr{L}_{\mathfrak{w}}(\lambda) \otimes \mathcal{O}_{Z_{\mathfrak{w}}}\left[-\bigcup_{q=j}^{k-1} Z_{\mathfrak{w}(q)}\right]\right) \to 0. \qquad (S_2) \end{aligned}$$

But $Z_{\mathfrak{w}(k)}$ intersects transversally (inside $Z_{\mathfrak{w}}$) all the hypersurfaces $Z_{\mathfrak{w}(q)}$ for any $j \leqq q \leqq k-1$ and hence the last sheaf in the above sequence (S_2) can be identified with the sheaf $\mathscr{L}_{\mathfrak{w}(k)}(\lambda) \otimes \mathcal{O}_{Z_{\mathfrak{w}(k)}}\left[-\bigcup_{q=j}^{k-1}\left(Z_{\mathfrak{w}(q)} \cap Z_{\mathfrak{w}(k)}\right)\right]$ (by lemma 4.2). Now by induction hypothesis and by the long exact cohomology sequence corresponding to the sheaf sequence (S_2), the assertion (validity of the proposition for $j \leqq k-1$) follows. Also from the long exact cohomology sequence, corresponding to the sheaf sequence (S_1) tensored with $\mathscr{L}_{\mathfrak{w}}(\lambda)$, it follows that $H^p(Z_{\mathfrak{w}}, \mathscr{L}_{\mathfrak{w}}(\lambda)) = 0$ for all $p > 0$, provided we know the validity of the proposition for the pair $j \leqq j$.

So we can assume now that the pair $j \leqq k$ is such that the sequence $(r_{i_j}, \ldots, r_{i_k})$ is (of course) reduced and either (a) $k = n$ or else

(b) $k < n$ and the sequence $(r_{i_j}, \ldots, r_{i_k}, r_{i_{k+1}})$ is not reduced.

We deal with these two cases separately.

Case (a), $k = n$: By an argument exactly similar to the one used above, we can assume that we have two subcases:

(a_1) $j = 1$ or

(a_2) $j > 1$ and $(r_{i_{j-1}}, r_{i_j}, \ldots, r_{i_k})$ is not reduced.

In the case (a_1), i.e., $j = 1$; the proposition follows from Theorem (4.1) together with lemma (4.4), since

$$H^p\left(Z_{\mathfrak{w}}, \mathscr{L}_{\mathfrak{w}}(\lambda) \otimes \mathcal{O}_{Z_{\mathfrak{w}}}\left[-\bigcup_{q=1}^{n} Z_{\mathfrak{w}(q)}\right]\right) = H^p(Z_{\mathfrak{w}}, \mathscr{L}_{\mathfrak{w}}(\lambda+\rho) \otimes K_{Z_{\mathfrak{w}}}) \quad \text{(by lemma 4.4)}$$
$$\approx H^{n-p}(Z_{\mathfrak{w}}, \mathscr{L}_{\mathfrak{w}}(-(\lambda+\rho)))^* \quad \text{(by Serre duality)}$$
$$= 0, \quad \text{by Theorem 4.1}$$

(since $\mathscr{L}(\lambda+\rho)$ is ample on X_w and $\theta_{\mathfrak{w}}\colon Z_{\mathfrak{w}} \to X_w$ is a birational morphism).

Case (a_2): Since $v = r_{i_j} \ldots r_{i_n}$ is reduced and $r_{i_{j-1}} r_{i_j} \ldots r_{i_n}$ is not reduced, we can write $v = r_{i_{j-1}} \cdot s_{j+1} \ldots s_n$, for some simple reflections $s_{j+1}, \ldots, s_n$. Define the following sequences:

$$\mathfrak{u} = (r_{i_1}, \ldots, r_{i_{j-1}})$$
$$\mathfrak{v} = (r_{i_j}, \ldots, r_{i_n})$$

and

$$\mathfrak{v}' = (r_{i_{j-1}}, s_{j+1}, \ldots, s_n).$$

We have the canonical projection $\pi\colon Z_{\mathfrak{w}} \to Z_{\mathfrak{u}}$. By induction hypothesis,

$$H^p\left(Z_{\mathfrak{v}}, \mathscr{L}_{\mathfrak{v}}(\lambda) \otimes \mathcal{O}_{Z_{\mathfrak{v}}}\left[-\bigcup_{q=1}^{n-j+1} Z_{\mathfrak{v}(q)}\right]\right) = 0, \qquad \text{for all } p > 0.$$

Hence by Leray spectral sequence for the map π, together with lemma (4.5) and the facts that for any $\mathfrak{p} = (p_1, \ldots, p_{j-1}) \in P_{i_1} \times \ldots \times P_{i_{j-1}}$, $i_{\mathfrak{p}}(Z_{\mathfrak{v}})$ is transverse to any $Z_{\mathfrak{w}(q)}$ for $j \leqq q \leqq n$ and

$$Z_{\mathfrak{v}} \cap i_{\mathfrak{p}}^{-1}\left(\bigcup_{q=j}^{n} Z_{\mathfrak{w}(q)}\right) = \bigcup_{q=1}^{n-j+1} Z_{\mathfrak{v}(q)}$$

(i_p is defined in the proof of lemma 4.5), we get:

$$\begin{aligned} &H^p\left(Z_{\mathfrak{w}}, \mathscr{L}_{\mathfrak{w}}(\lambda)\otimes\mathcal{O}_{Z_{\mathfrak{w}}}\left[-\bigcup_{q=j}^{n} Z_{\mathfrak{w}(q)}\right]\right)\\ &\quad\approx H^p\left(Z_{\mathfrak{u}}, \mathscr{L}_{\mathfrak{u}}\left(H^0\left(Z_{\mathfrak{v}}, \mathscr{L}_{\mathfrak{v}}(\lambda)\otimes\mathcal{O}_{Z_{\mathfrak{v}}}\left[-\bigcup_{q=1}^{n-j+1} Z_{\mathfrak{v}(q)}\right]\right)\right)\right)\\ &\quad= H^p(Z_{\mathfrak{u}}, \mathscr{L}_{\mathfrak{u}}(H^0(Z_{\mathfrak{v}}, \mathscr{L}_{\mathfrak{v}}(\lambda+\rho)\otimes K_{Z_{\mathfrak{v}}}))) \quad \text{(by lemma 4.4).} \qquad (\mathrm{I}_1)\end{aligned}$$

Now we claim that the B-module structure on $H^0(Z_{\mathfrak{v}}, \mathscr{L}_{\mathfrak{v}}(\lambda+\rho)\otimes K_{Z_{\mathfrak{v}}})$ admits an extension to $P_{i_{j-1}}$-module structure:

By Serre duality, $H^0(Z_{\mathfrak{v}}, \mathscr{L}_{\mathfrak{v}}(\lambda+\rho)\otimes K_{Z_{\mathfrak{v}}})\approx H^{n-j+1}(Z_{\mathfrak{v}}, \mathscr{L}_{\mathfrak{v}}(-(\lambda+\rho)))^*$. (Observe that Serre duality is a B-equivariant isomorphism.) But by lemma (4.6), $H^{n-j+1}(Z_{\mathfrak{v}}, \mathscr{L}_{\mathfrak{v}}(-(\lambda+\rho)))$ is isomorphic with (as B-modules) $H^{n-j+1}(Z_{\mathfrak{v}'}, \mathscr{L}_{\mathfrak{v}'}(-(\lambda+\rho)))$, because, by induction, the proposition (2.3) is true for the sequences $\mathfrak{v}$ and $\mathfrak{v}'$. But $\mathfrak{v}'$ is the sequence $(r_{i_{j-1}}, s_{j+1}, \dots, s_n)$ and hence the claim is established (cf. § 2.2).

Now considering the Leray spectral sequence corresponding to the projection: $Z_{\mathfrak{u}}\to Z_{\mathfrak{u}[j-2]}$, we conclude, by lemma (4.5), that

$$\begin{aligned} H^p(Z_{\mathfrak{u}}, \mathscr{L}_{\mathfrak{u}}(H^0(Z_{\mathfrak{v}}, \mathscr{L}_{\mathfrak{v}}(\lambda+\rho)\otimes K_{Z_{\mathfrak{v}}}))) &\approx H^p(Z_{\mathfrak{u}[j-2]}, \mathscr{L}_{\mathfrak{u}[j-2]}(H^0(Z_{\mathfrak{v}}, \mathscr{L}_{\mathfrak{v}}(\lambda+\rho)\otimes K_{Z_{\mathfrak{v}}})))\\ &\approx H^p\left(Z_{\mathfrak{w}'}, \mathscr{L}_{\mathfrak{w}'}(\lambda)\otimes\mathcal{O}_{Z_{\mathfrak{w}'}}\left[-\bigcup_{q=j-1}^{n-1} Z_{\mathfrak{w}'(q)}\right]\right),\end{aligned}$$

where $\mathfrak{w}'=\mathfrak{w}(j-1)=(r_{i_1}, \dots, r_{i_{j-2}}, r_{i_j}, \dots, r_{i_n})$. (The last isomorphism is got by considering the isomorphism (I_1) for the sequence $\mathfrak{w}$ replaced by $\mathfrak{w}'$.) Hence, by induction, this case (a_2) is taken care of.

Finally we come to the case (b), i.e., $k<n$ and the sequence $(r_{i_j}, \dots, r_{i_k}, r_{i_{k+1}})$ is not reduced. Define the following sequences:

$$\mathfrak{v}=(r_{i_j}, \dots, r_{i_k})$$
$$\mathfrak{x}=(r_{i_1}, \dots, r_{i_k})$$
$$\mathfrak{y}=(r_{i_{k+1}}, \dots, r_{i_n})$$

and

$$\mathfrak{u}=(r_{i_1}, \dots, r_{i_{j-1}}).$$

Let $\sigma=\pi_{\mathfrak{w}[k]}\colon Z_{\mathfrak{w}}\to Z_{\mathfrak{x}}$ be the canonical projection. Then by lemma (4.3),

$$\mathcal{O}_{Z_{\mathfrak{w}}}\left[-\bigcup_{q=j}^{k} Z_{\mathfrak{w}(q)}\right]\approx\sigma^*\left(\mathcal{O}_{Z_{\mathfrak{x}}}\left[-\bigcup_{q=j}^{k} Z_{\mathfrak{x}(q)}\right]\right).$$

Hence by the projection formula and Leray spectral sequence corresponding to the map σ (since $H^p(Z_{\mathfrak{y}}, \mathscr{L}_{\mathfrak{y}}(\lambda))=0$, for all $p>0$ by induction), we have:

$$H^p\left(Z_{\mathfrak{w}}, \mathscr{L}_{\mathfrak{w}}(\lambda)\otimes\mathcal{O}_{Z_{\mathfrak{w}}}\left[-\bigcup_{q=j}^{k} Z_{\mathfrak{w}(q)}\right]\right)\approx H^p\left(Z_{\mathfrak{x}}, \mathscr{L}_{\mathfrak{x}}(M)\otimes\mathcal{O}_{Z_{\mathfrak{x}}}\left[-\bigcup_{q=j}^{k} Z_{\mathfrak{x}(q)}\right]\right) \quad (\mathrm{I}_2)$$

where the B-module $M=H^0(Z_{\mathfrak{y}}, \mathscr{L}_{\mathfrak{y}}(\lambda))$.

Now by an argument exactly analogous to the one used to derive (I_1), replacing the B-module $e^{-\lambda}$ by M, we get:

$$H^p\left(Z_x, \mathscr{L}_x(M)\otimes\mathcal{O}_{Z_x}\left[-\bigcup_{q=j}^{k} Z_{x(q)}\right]\right)$$
$$\approx H^p(Z_u, \mathscr{L}_u(H^0(Z_v, \mathscr{L}_v(\rho)\otimes\mathscr{L}_v(M)\otimes K_{Z_v}))) \qquad (I_3)$$

But

$$H^0(Z_v, \mathscr{L}_v(\rho)\otimes\mathscr{L}_v(M)\otimes K_{Z_v})$$
$$\approx H^{k-j+1}(Z_v, \mathscr{L}_v(-\rho)\otimes\mathscr{L}_v(M^*))^* \quad \text{(by Serre duality).} \qquad (I_4)$$

Since $(r_{i_j}, \ldots, r_{i_k}, r_{i_{k+1}})$ is not reduced, we can write $r_{i_j}\ldots r_{i_k}=s_j\ldots s_{k-1}r_{i_{k+1}}$, for some simple reflections $s_j, \ldots, s_{k-1}$. But by lemma (4.6), for all $p\geqq 0$,

$$H^p(Z_v, \mathscr{L}_v(-\rho)\otimes\mathscr{L}_v(M^*))\approx H^p(Z_{v'}, \mathscr{L}_{v'}(-\rho)\otimes\mathscr{L}_{v'}(M^*)),$$

where v' is the reduced sequence $(s_j, \ldots, s_{k-1}, r_{i_{k+1}})$.

But M (and hence M^*) is, of course, a $P_{i_{k+1}}$-module and hence by Leray spectral sequence for the map $Z_{v'}\to Z_{v'[k-j]}$ and [D$_2$; § 2], $H^p(Z_{v'}, \mathscr{L}_{v'}(-\rho)\otimes\mathscr{L}_{v'}(M^*))=0$, for all $p\geqq 0$. Hence $H^p(Z_v, \mathscr{L}_v(-\rho)\otimes\mathscr{L}_v(M^*))=0$, for all $p\geqq 0$. Now by (I_2), (I_3), and (I_4), we get the desired vanishing. (In fact, in this case, $H^0\left(Z_w, \mathscr{L}_w(\lambda)\otimes\mathcal{O}_{Z_w}\left[-\bigcup_{q=j}^{k} Z_{w(q)}\right]\right)$ also is 0.)

This completes the proof of proposition (2.3). □

(4.7) **Corollary.** *Let w be any sequence. Then there exists a maximal reduced subsequence v of w (i.e. v is reduced and if u is any reduced sequence $v\leqq u\leqq w$, then $u=v$) such that the canonical map: $H^0(Z_w, \mathscr{L}_w(\lambda))\to H^0(Z_v, \mathscr{L}_v(\lambda))$, induced from the inclusion of Z_v in Z_w via the maps $i_{w(j)}$ (§ 2.1), is an isomorphism.*

Proof. We first prove that if w_1 is any reduced sequence and $w_2=(r_i, w_1)$ is not a reduced sequence, then the canonical map: $H^0(Z_{w_2}, \mathscr{L}_{w_2}(\lambda))\to H^0(Z_{w_1}, \mathscr{L}_{w_1}(\lambda))$ is an isomorphism. By the Leray spectral sequence corresponding to $\pi_{w_2[1]}$: $Z_{w_2}\to Z_{r_i}$, we get that $H^0(Z_{w_2}, \mathscr{L}_{w_2}(\lambda))\approx H^0(Z_{r_i}, \mathscr{L}_{r_i}(H^0(Z_{w_1}, \mathscr{L}_{w_1}(\lambda))))$. But $H^0(Z_{w_1}, \mathscr{L}_{w_1}(\lambda))$ being a P_i-module (by proposition 2.3 and lemma 4.6), we get that $H^0(Z_{w_2}, \mathscr{L}_{w_2}(\lambda))\xrightarrow{\sim} H^0(Z_{w_1}, \mathscr{L}_{w_1}(\lambda))$.

Now, we prove the corollary by induction on the length n of w. Write $w=(r_{i_1}, \ldots, r_{i_n})$. Choose (by induction) a maximal reduced subsequence $v'\leqq w'$, where $w'=(r_{i_2}, \ldots, r_{i_n})$, such that $H^0(Z_{w'}, \mathscr{L}_{w'}(\lambda))\to H^0(Z_{v'}, \mathscr{L}_{v'}(\lambda))$ is an isomorphism. Now there are two cases to consider:

(1) (r_{i_1}, v') is reduced,

(2) (r_{i_1}, v') is not reduced.

In the case (1); it is easy to see that (r_{i_1}, v') is a maximal reduced subsequence of w. By the Leray spectral sequence, this case is taken care of. In the case (2); it is easy to see that v' is a maximal reduced subsequence of w. Now $H^0(Z_{w'}, \mathscr{L}_{w'}(\lambda))\xrightarrow{\sim} H^0(Z_{v'}, \mathscr{L}_{v'}(\lambda))$ (by the choice of v'). But, by the argument in the first paragraph of the proof, $H^0(Z_w, \mathscr{L}_w(\lambda))\xrightarrow{\sim} H^0(Z_{w'}, \mathscr{L}_{w'}(\lambda)$. □

In fact, it can be shown that *any* maximal reduced subsequence $\mathfrak{v}$ of $\mathfrak{w}$ satisfies the assertion in the corollary. But we don't need this stronger fact.

(4.8) *Remark.* Although, through the paper, we tacitly were working over the base field $\mathbb{C}$, most of the results of the paper go through (with the same proofs) over an arbitrary algebraically closed field of char. 0.

(4.9) *Remark.* After this paper was submitted, we learnt from, among others, the Referee that O. Mathieu has recently announced quite similar results in C.R. Acad. Sc. Paris, t.303, Série I, n°9 (1986), pp. 391–394. Although his proofs have not yet appeared, presumably his methods are quite different from ours.

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