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Mathematics

T-equivariant K-theory of generalized flag varieties

(Kac–Moody algebra and the associated group/maximal torus/smash product/Hecke ring)

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ABSTRACT **Let G be a Kac–Moody group with Borel subgroup B and compact maximal torus T. Analogous to Kostant and Kumar [Kostant, B. & Kumar, S. (1986) *Proc. Natl. Acad. Sci. USA* 83, 1543–1545], we define a certain ring Y, purely in terms of the Weyl group W (associated to G) and its action on T. By dualizing Y we get another ring Ψ, which, we prove, is "canonically" isomorphic with the T-equivariant K-theory $K_T(G/B)$ of G/B. Now $K_T(G/B)$, apart from being an algebra over $K_T(\text{pt.}) \approx A(T)$, also has a Weyl group action and, moreover, $K_T(G/B)$ admits certain operators $\{D_w\}_{w\in W}$ similar to the Demazure operators defined on $A(T)$. We prove that these structures on $K_T(G/B)$ come naturally from the ring Y. By "evaluating" the $A(T)$-module Ψ at 1, we recover $K(G/B)$ together with the above-mentioned structures. We believe that many of the results of this paper are new in the finite case (i.e., G is a finite-dimensional semisimple group over $\mathbb{C}$) as well.**

Section 1

To any (not necessarily symmetrizable) generalized $l \times l$ Cartan matrix A, one associates a Kac–Moody algebra $\mathfrak{g} = \mathfrak{g}(A)$ over $\mathbb{C}$ (1) and group $G = G(A)$. (Actually G has as its "Lie algebra," the commutator subalgebra $\mathfrak{g}'$ of $\mathfrak{g}$.) G has a "standard unitary form" K. If A is a classical Cartan matrix, then G is a finite-dimensional semisimple simply connected algebraic group over $\mathbb{C}$ and K is a maximal compact subgroup of G. We refer to this as the finite case. In general, one has subalgebras of $\mathfrak{g}$; $\mathfrak{h} \subset \mathfrak{b} \subseteq \mathfrak{p}$, the Cartan subalgebra, the Borel subalgebra, and a parabolic subalgebra, respectively. One also has the corresponding subgroups: $H \subset B \subseteq P$, the complex maximal torus, the Borel subgroup, and a parabolic subgroup, respectively. H has as its Lie algebra $\mathfrak{h}' = \mathfrak{h} \cap \mathfrak{g}'$, which is linear span of the simple co-roots $\{h_i\}_{1\le i\le l}$. We denote by T the compact maximal torus $H \cap K$ of K. Let W be the Weyl group associated to $(\mathfrak{g}, \mathfrak{h})$ and let $\{r_i\}_{1\le i\le l}$ denote the set of simple reflections. The group W operates on the compact maximal torus T (as well as on H) and hence on the group algebra $A(T) = \mathbb{Z}[X(T)]$ of the character group $X(T)$ of T and also on the quotient field $Q(T)$ of $A(T)$.

For any W-field F, we can form the smash product F_W of the group algebra $\mathbb{Z}[W]$ with F. Now in ref. 2 we took, for F, the field $Q = Q(\mathfrak{h}^*)$ of all the rational functions on $\mathfrak{h}$ and defined an appropriate subring $R \subset Q_W$ and showed that R and its "appropriate" dual Λ, along with a certain R-module structure on Λ, replace the study of the cohomology *algebra* of G/B together with the various operators defined on $H^*(G/B)$. Hence, the problem of understanding $H^*(G/B)$, especially the cup product structure and other operators on $H^*(G/B)$, reduced to a purely combinatorial (and hopefully more tractable) problem of understanding the ring R and its "dual" Λ, defined purely and explicitly in terms of the Coxeter group W and its representation on $\mathfrak{h}^*$.

Our aim in this paper is to announce similar results for T-equivariant K-theory of G/B as well as the K-theory of G/B, where T acts on G/B by left multiplication.

We replace $Q(\mathfrak{h}^*)$ by the W-field $Q(T)$ and analogously define a certain subring Y of $Q(T)_W$, again purely and explicitly, in terms of the Coxeter group W and its action on the torus T. We prove a crucial structure theorem for Y analogous to the corresponding structure theorem for R (theorem 2.4 of ref. 2). Our next main result is that the dual Ψ of Y, which is also a Y-module, is "canonically" isomorphic with $K_T(G/B)$ and, moreover, under this isomorphism, the Weyl group action as well as certain operators $\{D_w\}_{w\in W}$ on $K_T(G/B)$, which are similar to the Demazure operators defined on $A(T)$, correspond to the action of certain well-defined elements in Y. The ring Ψ "evaluated" at 1 does the same for $K(G/B)$. Similar results are true for any G/P and in fact for any Schubert subvariety of G/P.

As a particular case, we obtain the above-mentioned results in the finite case. As an application of our results in this case, we can easily deduce some of the important (though known) results: For any compact simply connected group G_0 with a maximal torus T, (*i*) $K^*(G_0)$ is torsion free; (*ii*) the Atiyah–Hirzebruch homomorphism: $A(T) \to K(G_0/T)$ is surjective; and (*iii*) the Hodgkin's conjecture, that a certain map,

$$A(T) \underset{R(G_0)}{\otimes} A(T) \to K_T(G_0/T)$$

is an isomorphism.

This is merely an announcement of results. The detailed paper will appear elsewhere, but let us mention that the proof of *Theorem 3.9* involves, as main ingredients, the localization theorem of Atiyah and Segal and the equivariant Thom isomorphism.

Section 2

The treatment in this section is parallel to the one in section 2 of ref. 2.

The Weyl group W operates as a group of automorphisms on the field $Q = Q(T)$. Let $Q_W = Q(T)_W$ be the smash product of $Q(T)$ with the group algebra $\mathbb{Z}[W]$; i.e., Q_W is a right Q-module (under right multiplication by Q) with a (free) basis $\{\delta_w\}_{w\in W}$ and the multiplicative structure is given by

$$(\delta_{w_1}q_1)(\delta_{w_2}q_2) = \delta_{w_1w_2}(w_2^{-1}q_1)q_2,$$

for $q_1, q_2 \in Q$ and $w_1, w_2 \in W$.

Observe that $\delta_e Q = Q\delta_e$ is not central in Q_W.

(The notations Q and Q_W in this paper, and also the subsequent notation Ω, should not be confused with the corresponding notations in ref. 2, where they have somewhat different meaning.)

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The ring Q_W has an involutary anti-automorphism, defined by

$$(\delta_w q)^t = \delta_{w^{-1}}(wq), \text{ for } q \in Q \text{ and } w \in W.$$

Let $A_W \subset Q_W$ be defined in the same way as Q_W with $A(T)$ replacing $Q(T)$.

We define, for $i = 1, \ldots, l$, certain elements y_i in Q_W by

$$y_i = y_{r_i} = [1/(1 - e^{-\alpha_i})](\delta_e - e^{-\alpha_i}\delta_{r_i}) = (\delta_e + \delta_{r_i})[1/(1 - e^{-\alpha_i})],$$

where $e^{\alpha_i} \in X(T)$ is the character corresponding to the simple root α_i (associated to the simple reflection r_i).

Let $\mathfrak{l}: W \to \mathbb{Z}_+$ be the length function. We have the following:

PROPOSITION 2.1. (i) $y_i^2 = y_i$, *for any* $1 \le i \le l$.

(ii) $y_i q = (r_i q) y_i + \left(\dfrac{q - r_i q}{1 - e^{-\alpha_i}}\right)\delta_e$, *for any* $q \in Q$.

(iii) *For any* $w \in W$, *let* $w = r_{i_1} \ldots r_{i_n}$ *be a reduced decomposition. Then the element* $y_w = y_{i_1} \cdot \ldots \cdot y_{i_n} \in Q_W$ *does not depend on the particular choice of the reduced decomposition of* w. *Hence, for* v, w $\in$ W,

$$y_v \cdot y_w = y_{vw} \quad \textit{if} \quad \mathfrak{l}(vw) = \mathfrak{l}(v) + \mathfrak{l}(w). \qquad \square$$

Let Δ_+ (resp Δ_-) denote the set of positive (resp negative) roots and let $\le$ denote the Bruhat partial ordering on W. Of course the elements $\{\delta_w\}_{w \in W}$ are a right (as well as a left) Q-basis of Q_W. But also, we have the following:

PROPOSITION 2.2. *The elements* $\{y_w\}_{w \in W}$ *form a right (as well as a left)* Q-*basis of* Q_W. *Write* $\delta_{w^{-1}} = \Sigma_v e_{v,w} y_{v^{-1}}$, *for some (unique)* $e_{v,w} \in Q$. *Then,*

(i) $e_{v,w} \in A(T)$, *for any* v,w $\in$ W.

(ii) $e_{v,w} = 0$, *unless* $v \le w$.

(iii) $e_{w,w} = \Pi_{\nu \in w^{-1}\Delta_- \cap \Delta_+} (1 - e^{\nu})$. $\square$

In particular the invertible (over Q) matrix $E = (e_{v,w})_{v,w \in W}$, which relates the two bases $\{\delta_w\}$ and $\{y_w\}$, is upper triangular (with nonzero diagonal entries).

Now, clearly, Q has the structure of a left Q_W-module, defined by $(\delta_w q)q' = w(qq')$, for $w \in W$ and $q, q' \in Q$.

We define the following basic subring $Y \subset Q_W$ by $Y = \{y \in Q_W: y \cdot A(T) \subset A(T)\}$.

It is easy to see that $y_i (1 \le i \le l)$ and hence y_w, for any $w \in W$, belongs to Y. Also $A_W \subset Y$. Conversely, we have the following crucial structure theorem for Y, which is our first main theorem.

THEOREM 2.3. Y *is free as a right (as well as a left)* A(T)-*module. In fact the elements* $\{y_w\}_{w \in W}$ *form a right (as well as a left)* A(T)-*basis of* Y.

Remarks 2.4. (*i*) Note that Y is a finitely generated ring over $\mathbb{Z}$, since it is generated by $\{y_i\}_{1 \le i \le l}$ and $A(T)$.

(*ii*) Observe that the "homology" analogue of the above structure theorem (theorem 4.6 of ref. 3) was proved only "over $\mathbb{C}$" (or $\mathbb{Q}$), not "over $\mathbb{Z}$."

Definitions 2.5. Dualizing Q_W *and* Y: Regarding Q_W as a right Q-module, let $\Omega = \Omega(T) = \mathrm{Hom}_Q(Q_W, Q)$. Since any $\psi \in \Omega$ is determined by its restriction to the base $\{\delta_w\}$ (and conversely), we can regard Ω as the Q-module of *all* the functions : $W \to Q$ with pointwise addition and scalar multiplication. Furthermore, Ω inherits a commutative algebra (over Q) structure, with the product as pointwise multiplication of functions on W.

More subtly, Ω also admits the structure of a left Q_W-module defined by

$$(x \cdot \psi)w = \psi(x^t \cdot \delta_w),$$

for $x \in Q_W$, $\psi \in \Omega$, and $w \in W$. Observe that the action of x is Q-linear.

Now let us define the $A(T)$-dual of Y:

$$\Psi = \{\psi \in \Omega: \psi(Y^t) \subset A(T)\}.$$

Notice the difference in the definition of Ψ with the definition of the analogous ring Λ in ref. 2, where we put, in addition, some finiteness condition.

Define certain elements $\psi^w \in \Psi$ (for any $w \in W$) by

$$\psi^w(y^t_{v^{-1}}) = \delta_{v,w}, \text{ for } v, w \in W.$$

Observe that $\psi^w(\delta_v) = e_{w,v}$, where $e_{w,v}$ is as defined in *Proposition 2.2*. In particular, by item *ii* of *Proposition 2.2*, $\psi = \Sigma_w q_w \psi^w$ is well defined for arbitrary (infinitely of them could be nonzero) choices of $q_w \in Q$. Of course, if all the q_w's belong to $A(T)$ then $\psi \in \Psi$.

One has the following:

PROPOSITION 2.6. (i) Ψ *is an* A(T)-*subalgebra of* Ω.

(ii) Ψ *is stable under the left action of* $Y \subset Q_W$. *In particular, the elements* δ_w *and* y_w *(for any* w $\in$ W) *act on* Ψ.

We have, $y_{r_i} \cdot \psi^w = \psi^w + \psi^{r_i w}$ *if* $r_i w < w$
$= 0$ *otherwise.*

(iii) $\Psi \cong \Pi_{w \in W} A(T)\psi^w$; *i.e., any element of* Ψ *can be uniquely written as* $\Sigma_w a_w \psi^w$ *with* $a_w \in A(T)$, *where infinitely many of* a_w'*s are allowed to be nonzero.* $\square$

The following proposition determines the product in the ring Ψ in terms of the "basis" $\{\psi^w\}_w$. Recall the definition of the *E*-matrix from *Section 2*.

PROPOSITION 2.7. *For any* u, v $\in$ W, *write (by* Proposition 2.6*)*

$$\psi^u \cdot \psi^v = \sum_w a^w_{u,v}\psi^w, \textit{ for some unique } a^w_{u,v} \in A(T).$$

Now for any fixed w $\in$ W, *define two matrices* A_w *and* E_w *by* $A_w(u, v) = a^v_{w,u}$ *and* $E_w(u, v) = \delta_{u,v} e_{w,v}$. *Then,*

(i) $a^w_{u,v} = 0$ *unless* $u \le w$ *and* $v \le w$.

(ii) $A_w = E \cdot E_w \cdot E^{-1}$. $\square$

A similar expression can be given for the action of the Weyl group element δ_w on ψ^u.

Remark 2.8. We consider $\mathbb{Z}$ as an $A(T)$-module under the augmentation (i.e., the evaluation at the identity of T) map: $A(T) \to \mathbb{Z}$. By item *i* of *Proposition 2.6*, the tensor product $\mathbb{Z} \otimes_{A(T)} \Psi$ is a $\mathbb{Z}$-algebra. Moreover, the action of Y on Ψ being $A(T)$-linear, we obtain an action of Y on $\mathbb{Z} \otimes_{A(T)} \Psi$.

Section 3

Definition 3.1. Recall the Bruhat decomposition $G/B = \cup_{w \in W} B\,w\,B/B$. Now define $X_n = \cup_{\mathfrak{l}(w) \le n} B\,w\,B/B \subset G/B$. Then X_n is a compact subspace of G/B and the topology on G/B is the direct limit topology induced from the sequence:

$$X_{-1} = \phi \subset X_0 \subset X_1 \subset \ldots, \qquad \cup X_n = G/B.$$

The group G acts on G/B by the left multiplication, in particular, the compact maximal torus T acts on G/B and (clearly) X_n is T-stable.

Now define $K_T(G/B) = \mathrm{Inv}\lim_{n \to \infty} K_T(X_n)$, where $K_T(X_n)$ is the T-equivariant K-group of X_n as defined in section 2 of ref. 4.

It may be remarked that $K_T(G/B)$ does not depend on the particular choice of a filtration of G/B by compact subspaces.

Fix any simple reflection r_i, $1 \le i \le l$, and let $P_i \supset B$ be the minimal parabolic subgroup containing r_i. Let $\chi_i \in (\mathfrak{h}')^*$ be the ith fundamental weight [defined by $\chi_i(h_j) = \delta_{i,j}$] and let

V_i be the (two-dimensional) representation of P_i, which is trivial restricted to the "nil radical" of P_i and the "standard maximal reductive subgroup" of P_i acts on V_i with highest weight χ_i. We have the following:

LEMMA 3.2. *The $\mathbb{P}^1$-fibration π_i:G/B → G/P$_i$ is canonically isomorphic to the projective bundle of the (rank two) vector bundle on* G/P$_i$ *associated to the representation* V$_i$ *of* P$_i$. □

Now by proposition 3.9 of ref. 4 (which is a consequence of Thom isomorphism), applied to the map π_i of the above lemma, we obtain the following:

PROPOSITION 3.3. *For any* $n \geq 0$, $K_T(\pi_i^{-1}(\pi_i X_n))$ *is a free module over* $K_T(\pi_i X_n)$ *with free generators* 1 *and the Hopf bundle* $H_i(n)$.

Definition 3.4. Define an operator $D_{r_i}(n)$: $K_T(\pi_i^{-1}(\pi_i X_n))$ into itself by $D_{r_i}(n)(x + H_i(n)y) = x$, for $x, y \in \pi_i^*(K_T(\pi_i X_n))$. The operators $D_{r_i}(n)$ make the following diagram commutative:

$$\begin{array}{ccc} K_T(\pi_i^{-1}(\pi_i X_{n+1})) & \rightarrow & K_T(\pi_i^{-1}(\pi_i X_n)) \\ \downarrow D_{r_i}(n+1) & & \downarrow D_{r_i}(n) \\ K_T(\pi_i^{-1}(\pi_i X_{n+1})) & \rightarrow & K_T(\pi_i^{-1}(\pi_i X_n)). \end{array}$$

In particular, we obtain an operator D_{r_i}:$K_T(G/B) \rightarrow K_T(G/B)$. We have the following:

LEMMA 3.5. (i) $D_{r_i}^2 = D_{r_i}$, *for any simple reflection* r_i.

(ii) *Fix* $w \in W$ *and take a reduced expression* $w = r_{i_1} \ldots r_{i_n}$. *Then the operator* $D_{r_{i_1}} \circ \ldots \circ D_{r_{i_n}}$:$K_T(G/B) \rightarrow K_T(G/B)$ *does not depend upon the particular reduced expression of* w. Set $D_w = D_{r_{i_1}} \circ \ldots \circ D_{r_{i_n}}$.

Remark 3.6. A similar operator on $A(T)$ (see *Definition 4.1*), introduced by Demazure in section 5.5 of ref. 5, provided motivation for our definition of D_w.

Definition 3.7. Weyl group action on $K_T(G/B)$: Since the Weyl group W acts on $G/B \approx K/T$ (where K is the standard unitary form of G) by (n mod T). (k mod T) = kn^{-1} mod T, for n mod $T \in N_K(T)/T \approx W$ and $k \in K$ [where $N_K(T)$ denotes the normalizer of T in K]. Moreover, this action of W commutes with the action of T on G/B and hence we obtain an action of W on $K_T(G/B)$.

Exactly similarly, we get an action of W on $K(G/B)$ and also the operators, again denoted by, $\{D_w\}_{w \in W}$ on $K(G/B)$.

Definition 3.8. The localization map: For any $n \geq 0$, let $\hat{\gamma}_n$:$K_T(X_n) \rightarrow K_T(X_n^T)$ be the canonical restriction map, where X_n^T is the set of all the T-fixed points in X_n. Since the maps $\{\hat{\gamma}_n\}_{n \geq 0}$ are compatible, we get a map $\hat{\gamma}$:$K_T(G/B) \rightarrow K_T(G/B^T)$. Now the map i:$W \approx N_K(T)/T \rightarrow G/B^T$, given by $w \mapsto w^{-1}$ mod B, induces a homeomorphism, provided we put the discrete topology on W. Moreover, by proposition 2.2 of ref. 4, $K_T(W)$ can be canonically identified [as an algebra over $A(T)$] with the $A(T)$-subalgebra of Ω (see *Definitions 2.5*) consisting of precisely those maps: $W \rightarrow Q$, which have image $\subset A(T)$. Hence, on composition, we obtain a map

$$\bar{\gamma}:K_T(G/B) \rightarrow \Omega.$$

Now we come to our second main theorem.

THEOREM 3.9. *Let* G *be an arbitrary (not necessarily symmetrizable) Kac–Moody group with Borel subgroup* B. *Then the map* $\bar{\gamma}$:$K_T(G/B) \rightarrow \Omega$, *defined above, has its image precisely equal to* Ψ *(see* Definitions 2.5*)*.

Let γ *be the map* $\bar{\gamma}$, *considered as a map*: $K_T(G/B) \rightarrow \Psi$. *Then the map* γ *is an* A(T)-*algebra isomorphism. Further the action of the Weyl group element* $w \in W$ (Definition 3.7) *and the operator* D_w (Lemma 3.5) *correspond under* γ *to the action of* δ_w *and* y_w, *respectively* (Proposition 2.6).

Remark 3.10. Arabia has recently identified the "cohomological analogue" of our ring Ψ (i.e., the ring Λ defined in ref. 2) with the equivariant cohomology $H_T(G/B)$.

As an easy consequence of *Theorem 3.9*, we deduce the following theorem (with the same assumptions and notations):

THEOREM 3.11. *The map* γ :$K_T(G/B) \rightarrow \Psi$ *induces a unique map* γ_1:$K(G/B) \rightarrow \mathbb{Z} \otimes_{A(T)} \Psi$ (*cf.* Remark 2.8) *making the following diagram commutative (the vertical maps being the canonical maps)*:

$$\begin{array}{ccc} K_T(G/B) & \xrightarrow{\gamma} & \Psi \\ \downarrow & & \downarrow \\ K(G/B) & \xrightarrow{\gamma_1} & \mathbb{Z} \underset{A(T)}{\otimes} \Psi. \end{array}$$

Now the map γ_1 *is a* $\mathbb{Z}$-*algebra isomorphism. Further, the (Weyl group) action of* $w \in W$ *and the operator* D_w *on* $K(G/B)$ *correspond (under* γ_1*) to the action of* $1 \otimes \delta_w$ *and* $1 \otimes y_w$, *respectively*.

Remarks 3.12. (*i*) We can prove an appropriate analogue of *Theorems 3.9* and *3.11* for G/P, where P is an arbitrary parabolic subgroup of G, in fact even for an arbitrary left B-stable closed subvariety of G/P.

(*ii*) We will identify the "basis" $\{b^w = \gamma^{-1}\psi^w\}_{w \in W}$ of $K_T(G/B)$ in *Section 4*. In particular, by *Proposition 2.7*, the product in $K_T(G/B)$ can be "explicitly" determined, in the $\{b^w\}$ basis, in terms of the matrix E. A similar remark applies for the Weyl group action.

Section 4

In this section, we assume that we are in the finite case; i.e., G is a finite-dimensional semisimple simply connected algebraic group $\mathbb{C}$ and we denote by G_0 (instead of K) any maximal compact subgroup of G with a maximal torus T.

Definition 4.1. The Demazure operators (5): For any simple reflection r_i, define $L_{r_i}(e^\lambda) = (e^\lambda - e^{r_i\lambda - \alpha_i})/1 - e^{-\alpha_i}$, for $e^\lambda \in X(T)$ and extend linear to $A(T)$. Now set, for any $w \in W$, $L_w = L_{r_{i_1}} o \ldots o L_{r_{i_n}}$, where $w = r_{i_1} \ldots r_{i_n}$ is any reduced decomposition. (As is well known, L_w does not depend on the reduced expression.)

Definition 4.2. The Atiyah–Hirzebruch homomorphism: We recall the definition of the Atiyah–Hirzebruch homomorphism χ:$A(T) \rightarrow K(G/B)$, which takes e^λ to the line bundle on G/B associated to the character e^λ of B, for any $e^\lambda \in X(T)$. We have the following:

LEMMA 4.3. *The homomorphism* χ:$A(T) \rightarrow K(G/B)$ *is a* $\mathbb{Z}$-*algebra homomorphism, which commutes with Weyl group actions and* $\chi \circ L_w = D_w \circ \chi$, *for any* $w \in W$. □

Now, as fairly easy consequences of *Theorem 3.9*, we can deduce the following (known) results *(Theorems 4.4–4.6)*:

THEOREM 4.4. *The map* χ *(defined above) is surjective*.

THEOREM 4.5. *The map* ϕ:$R(T) \otimes_{R(G_0)} R(T) \rightarrow K_T(G_0/T)$, *defined on page 11 of ref. 6, is an isomorphism*.

THEOREM 4.6. $K^*(G_0)$ *is a torsion-free* $\mathbb{Z}$-*module, in fact is an exterior algebra over* $\mathbb{Z}$ *on a free* $\mathbb{Z}$-*module of rank = rank* G_0. □

Now we give a characterization of the basis $\{b^w = \gamma^{-1}(\psi^w)\}_{w \in W}$ (cf. *Theorem 3.9)* of $K_T(G/B)$. For any projective variety X, denote by $K^0(X)$ (resp $K_0(X)$) the Grothendieck group of algebraic vector bundles—i.e., locally free sheaves (resp the coherent sheaves) on X. Since G/B is smooth, the canonical map: $K^0(G/B) \rightarrow K_0(G/B)$ is an isomorphism. Moreover, as is known, the canonical map: $K^0(G/B) \rightarrow K(G/B)$ is also an isomorphism. Similar definitions and remarks apply for T-equivariant K-groups of G/B. In particular, we can assume that any (topological) T-equivariant vector bundle on G/B is algebraic (at least in

$K_T(G/B)$). For any T-equivariant algebraic vector bundle V on G/B, and any $w \in W$, denote $\chi(X_w, V) = \Sigma_p(-1)^p \mathrm{ch}_T(H^p(X_w, V)) \in A(T)$, where $\mathrm{ch}_T(H^p(X_w, V))$ denotes its character as a T-module. Now we state the following:

PROPOSITION 4.7. $\{b^w\}_{w\in W}$ *is the unique* A(T)-*basis of* $K_T(G/B)$ *satisfying* $\chi(X_v, b^{w*}) = \delta_{v^{-1},w} \in A(T)$, *for all* v, w $\in$ W, *where* $*$ *denotes the involution of* $K_T(G/B)$ *obtained by taking the dual of the vector bundles*.

Similarly, $\{b_1^w = \gamma_1^{-1}(1 \otimes \psi^w)\}_{w\in W}$ *is the unique* $\mathbb{Z}$-*basis of* K(G/B), *satisfying* $\Sigma_p(-1)^p dim\ H^p(X_v, b_1^{w*}) = \delta_{v^{-1},w}$, *for all* v, w $\in$ W.

Remarks 4.8. (*i*) A similar result, as above, holds good in the general Kac–Moody case, using some results of Kumar (7).

(*ii*) The basis $\{b_1^w\}$ is precisely the basis $\{a_w\}_w$, given by Demazure (proposition 7 of section 5 of ref. 5). Actually, $b_1^w = a_{w^{-1}}$ for all $w \in W$.

(*iii*) In the finite case, there is at least one other interesting $\mathbb{Z}$-basis of $K(G/B)$ given by $\{i_{w*}\mathcal{O}_{X_w}\}_{w\in W}$, where $i_w: X_w \hookrightarrow G/B$ is the canonical embedding and i_{w*} is the standard push-for ward map in K_0. Though the two bases differ, it is possible to write down "explicitly" one in terms of the other.

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1. Kac, V. G. (1983) *Infinite Dimensional Lie Algebras*, Progress in Mathematics (Birkhäuser), No. 44.
2. Kostant, B. & Kumar, S. (1986) *Proc. Natl. Acad. Sci. USA* **83**, 1543–1545.
3. Kostant, B. & Kumar, S. (1986) *Adv. Math.* **62**, 187–237.
4. Segal, G. (1968) *Publ. Math. IHES* **34**, 129–151.
5. Demazure, M. (1974) *Ann. Sci. Ec. Norm. Super. 4^e Ser.* **t.7**, 53–88.
6. Hodgkin, L. H. & Snaith, V. P. (1975) *Topics in K-theory*, Lecture Notes in Mathematics (Springer), No. 496.
7. Kumar, S. (1987) *Invent. Math.*, in press.