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The Nil Hecke Ring and Cohomology of G/P for a Kac–Moody Group G*

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INTRODUCTION

To any (not necessarily symmetrizable) generalized $l \times l$ Cartan matrix A, one associates a Kac–Moody algebra $\mathbf{g} = \mathbf{g}(A)$ over $\mathbb{C}$ and group $G = G(A)$. If A is a classical Cartan matrix, then G is a finite dimensional semi-simple algebraic group over $\mathbb{C}$. We refer to this as the finite case. In general, one has subalgebras of $\mathbf{g}$; $\mathbf{h} \subset \mathbf{b} \subset \mathbf{p}$, the Cartan subalgebra, Borel subalgebra, and a parabolic subalgebra, respectively. One also has the corresponding subgroups $H \subset B \subseteq P$. Let W be the Weyl group associated to $(\mathbf{g}, \mathbf{h})$ and let $\{r_i\}_{1 \leqslant i \leqslant l}$ denote the set of simple reflections. The group W operates on $\mathbf{h}$ (and hence on its dual space $\mathbf{h}^*$).

W parametrizes the Schubert cell decomposition of the generalized flag variety $G/B = \bigcup_{w \in W} V_w (= Bw^{-1}B/B)$. (A suitable subset $W^1 \subset W$ does the same for G/P.) Our principal concern is the cohomology ring $H(G/B)$ (more generally $H(G/P)$) and in fact the cohomology ring of arbitrary (left) B-stable closed subspaces of G/P.

Now besides having a ring structure and having a distinguished basis consisting of Schubert classes (given by the dual of the closures of Schubert cells, $H(G/B)$ is also a module for W. In addition, in the finite case, a ring

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of operators $\mathscr{A}$ (with $\mathbb{C}$-basis $\{A_w\}_{w\in W}$) on $H(G/B)$ was introduced in [3], where A_{r_i} $(1 \leqslant i \leqslant l)$, although defined algebraically, correspond topologically to the integration on fiber for the fibration $G/B \to G/P_i$ (P_i is the minimal parabolic containing r_i). Kac and Peterson have extended the definition of the ring of operators $\mathscr{A}$ on $H(G/B)$ to the general case and they have used these operators to study the topology of G (as well as G/B).

The problems, we wish to deal with, are to describe $H(G/B)$:

(1) as a ring, in particular the cup product of two arbitrary Schubert classes, and

(2) as a module for W and $\mathscr{A}$.

Our main result is that all these structures arise very naturally from a single ring R, which admits a simple and concrete definition, using only the Weyl group W and its representation on $\mathfrak{h}^*$ and which has some rather remarkable properties. We refer to R as the *nil Hecke ring*, corresponding to the pair $(W, \mathfrak{h}^*)$ (see (4.12)).

We would like to remark that there are a number of serious obstacles in trying to directly pass from the finite to the general infinite case, and as a consequence we have sought a new approach. Among the obstacles are (1) the characteristic homomorphism: $S(\mathfrak{h}^*) \to H(G/B)$ fails to be surjective in general, (2) the failure of complete reducibility of the W-modules $S(\mathfrak{h}^*)$ and $H(G/B)$, (3) the absence of "harmonics," and (4) the absence of the fundamental (top) cohomology class and as a consequence, the absence of Poincaré duality. An approach, which remains valid in the general case, was motivated from [3, Theorem 5.9] a result of the first author. This theorem arises from the correspondence of the Lie algebra cohomology $H(\mathfrak{n})$ ($\mathfrak{n}$ is the nil-radical of $\mathfrak{b}$) and $H(G/B)$ established by the first author in [17] in the finite case and was established, in the general case, by the second author in [23].

It may be mentioned that different aspects of the topology of G, G/B (and G/P) in the infinite case have been studied, among others, by Bott (who has done an extensive work on the topology of loop groups via Morse theory), Iwahori and Matsumoto [12] and more recently by Garland and Raghunathan [10], Tits [30], Kac and Peterson [21, 22], Gutkin and Slodowy [11], Kumar [23, 24], Kac [14], and Pressley and Segal [27].

We describe the contents of the paper in more detail. Section 1 is devoted to recalling some standard facts from Kac–Moody theory and setting up notations to be used throughout the paper. In Section 2 we establish a certain relationship between the cohomology of G/B with $\mathrm{End}_{\mathfrak{h}}\, H^*(\mathfrak{n})$, where $\mathfrak{n}$ is the nil radical of the Borel subalgebra $\mathfrak{b}$. The main result of this chapter is Theorem (2.12) (see also Remark (2.13)(a)), which asserts that there is a graded algebra isomorphism from $H(G/B)$ with $\mathrm{Gr}\,\mathbb{C}\{W\}$, where

$\mathbb{C}\{W\}$ is the algebra (under pointwise addition and pointwise multiplication) of *all* the functions: $W \to \mathbb{C}$ and $\operatorname{Gr} \mathbb{C}\{W\}$ denotes the associated graded algebra, with respect to some "natural" filtration of $\mathbb{C}\{W\}$. Let us recall that $\operatorname{End}_{\mathbf{h}} H^*(\mathbf{n})$ can be identified with $\mathbb{C}\{W\}$ by a result of Garland and Lepowsky [9]. The filtration of $\mathbb{C}\{W\}$ arises from a filtration of $\operatorname{End} C(\mathbf{n})$, which in turn is a "super" analog of the usual filtration of differential operators on a manifold. Our proof of this theorem is based upon the correspondence of the (Lie algebra) cohomology of $\mathbf{n}$ with $H^*(\mathbf{g}, \mathbf{h})$, as given by the "$d, \partial$-Hodge theory," proved by the first author in the finite case [17] and established by the second author, in the general case [23].

In Section 3 we construct [23] certain d, ∂ harmonic forms $\{s^w\}_{w \in W} \subset C(\mathbf{g}, \mathbf{h})$, which are dual to (up to a positive real number depending upon w) the Schubert varieties $\{\bar{V}_w\}$. It is further shown [23] that (properly defined) $\int_{\bar{V}_w} s^w = (-1)^{p(p-1)/2} 2^{2p} \int_{wUw^{-1} \cap U^-} \exp(2(w\rho - \rho)\, h(g))\, dg$, where $p = l(w)$ and U (resp. U^-) is the commutator subgroup of B (resp. the opposite Borel subgroup B^-). We explicitly compute the above integral in this chapter and show (Theorem 3.1) it to be $(-1)^{p(p-1)/2}(4\pi)^p \prod_{v \in w^{-1}\Delta_- \cap \Delta_+} \sigma(\rho, v)^{-1}$. Our proof of Theorem (3.1) occupies the whole of Section 3 and proceeds via an induction on $l(w)$. Interestingly, we use information about the cup product in $H(G/B)$, as given in Corollary (3.12), to compute the integral.

Section 4 can be viewed as the main algebraic part of this paper. We denote by Q_W (resp. S_W), the smash product of the group ring $\mathbb{C}[W]$ with the W-field $Q = Q(\mathbf{h}^*)$, the field of rational functions on $\mathbf{h}$ (resp. the W-ring $S = S(\mathbf{h}^*)$). The ring Q_W admits an involutary anti-automorphism t (see (I_{23})). With the help of Proposition (4.2), we define certain elements $\{x_w\}_{w \in W}$ in Q_W, which form a right (as well as left) Q-basis (Corollary (4.5)). The basis x_w behaves like a "degenerate" Hecke basis (see Proposition (4.3)(a)). Further, Q has a canonical (left) Q_W-module structure, given by (I_{33}). We define (a basic ring) R as the subring of Q_W, consisting of all those elements $x \in Q_W$ such that x keeps S stable. We prove a crucial structure theorem (Theorem 4.6) for the ring R. This asserts that R is a free (left as well as right) S-module, with the $\{x_w\}_{w \in W}$ as basis and $R \cap R^t = S_W$. In the finite case, part (a) of this theorem admits a simple proof using the theory of "harmonics." We refer to R as the *nil Hecke ring* (4.12). We also put a co-product structure Δ on Q_W (4.14) and prove Proposition (4.15), which describes Δ in terms of the $\{\bar{x}_w\}_{w \in W}$ basis of Q_W, where $\bar{x}_w = x^t_{w^{-1}}$.

We dualize the above objects and define $\Omega = \operatorname{Hom}_Q(Q_W, Q)$ (Q_W considered as a right Q-module) and the S-subalgebra $\Lambda = \{\psi \in \Omega \colon \psi(R^t) \subset S$ and $\psi(\bar{x}_w) = 0$, for all but a finite number of $w \in W\}$. Since Q_W has $\{\delta_w\}_{w \in W}$ as a right Q-basis, we can (and often will) view Ω as the space of all the maps: $W \to Q$. Of course, Ω is an algebra under pointwise addition and pointwise multiplication (of maps: $W \to Q$). More subtly, since Q_W has an involutary anti-automorphism t, we can also put the structure of a left Q_W-module on Ω defined by (I_{49}). It is easy to see that Λ is stable under (the left action of) R. In particular, the elements δ_w, x_w act on Λ. We refer to the action of the δ_w as the Weyl group action and the action of the x_w is referred to as the Hecke operators, for reasons which will be clear in Section 5. As a consequence of (structure) Theorem (4.6)(a), we deduce (Proposition (4.20)) that Λ is a free S-module with basis $\{\xi^w\}_{w \in W}$, where ξ^w is defined by $\xi^w(\bar{x}_v) = \delta_{v,w}$, for all $v \in W$. We collect various properties of $\{\xi^w\}$ in Proposition (4.24), and define a matrix $D = (d_{v,w})_{v,w \in W}$ by $d_{v,w} = \xi^v(w)$. The matrix D is fundamental to our paper. In the finite case, it can be extracted from [3, Theorem 5.9]. We give an explicit formula (Proposition (4.32)) for the arbitrary product $\xi^u \cdot \xi^v$ (as well as $\delta_u \cdot \xi^v$) in the $\{\xi^w\}$ basis, purely in terms of the matrix D. We give a different formula for the arbitrary product $\xi^u \cdot \xi^v$ in Proposition (4.31), in terms of the action of the ring R on Λ.[1]

Section 5 synthesizes Section 2 and Section 4. Since it is basically an application of Section 2 and since in Section 2 we have assumed A is symmetrizable we assumed the same in Section 5. However, using the algebraic results of Kac–Peterson the main theorem in Section 5 (as pointed out by Peterson) may be proved without the symmetrizability assumption. See Remark 5.7. We show that the (a priori very complicated) filtration of $\mathbb{C}\{W\}$ (given in Sect. 2), obtained by purely geometrical considerations, also arises from a very explicit combinatorial construction, using only the Weyl group W (associated to $\mathbf{g}$) and its representation on the Cartan subalgebra $\mathbf{h}$. By using d, ∂ harmonic forms $s_0^w = s^w / \int_{V_w} s^w$ and the map $\bar{\eta}$ (given in (2.7), we define a map $\tilde{D}$: $W \times W \to \mathbb{C}$ in Section 5.1. The map $\tilde{D}$, in turn, gives rise to the filtration (of Theorem (2.12)) of $\mathbb{C}\{W\}$. Moreover, the matrix D can also be thought of as a map: $W \times W \to S$. By evaluating D at $h(\rho)$ ($h(\rho)$ is defined in Proposition (5.2)), we get a map $D_{h(\rho)}$: $W \times W \to \mathbb{C}$. We further prove a characterization of the matrix $D_{h(\rho)}$ in Proposition (5.5), which roughly asserts that any $W \times W$ matrix E (over $\mathbb{C}$), which is upper triangular, has the same diagonal values as that of $D_{h(\rho)}$, and satisfies a suitable "cup product formula" is the same as $D_{h(\rho)}$. Finally, we show (Proposition (5.2)) that $\tilde{D}$ does satisfy all the characterizing properties (given in (5.5)) of $D_{h(\rho)}$ and hence we have the Corollary (5.6); $\tilde{D} = D_{h(\rho)}$. (We should mention that our calculation of the integral, in Sect. 3, is crucially used to establish that the diagonal entries of

[1] *Note added in proof.* Our later applications to $H(G/B)$ involve a quotient ring of $\wedge$. However, recently A. Arabia has proved that $\wedge$ itself is isomorphic to the T-equivariant cohomology of $G/B = K/T$. (See [31].)

D are precisely equal to those of $D_{h(\rho)}$.) This leads to our main theorem (5.12), which asserts that $H(G/B)$ is canonically isomorphic (as a graded algebra) with $\mathbb{C}_0 \otimes_S \Lambda$. Moreover, under this isomorphism, the (Weyl group) action of $w \in W$ and the Hecke operator A_w on $H(G/B)$ correspond respectively to the action of δ_w and x_w on $\mathbb{C}_0 \otimes_S \Lambda$, where $\mathbb{C}_0$ is the quotient of S by the augmentation ideal S^+. This, in particular, gives (by Proposition (4.32)) an expression, by a fairly explicit formula, for the cup product of two arbitrary Schubert classes (as well as the Weyl group action of arbitrary w) on $H(G/B)$, purely in terms of the matrix D. Of course, as a particular case, it gives the cup product of two arbitrary cohomology classes of the based loop group $\Omega_e(K_0)$ (for a finite dimensional compact s.s. simply connected group K_0), in terms of the Schubert basis. One can easily generalize Theorem (5.12) so that an arbitrary parabolic subgroup P replaces the Borel subgroup B, as done in Corollary (5.13)(c). Very interestingly, we can prove an analog of Theorem (5.12) for an arbitrary (left) B-stable closed subspace of G/P. This is the content of our Theorem (5.16). Recently, Akyildiz, Carrell, and Lieberman [1] (see also [5]) have, quite independently and by a different method, proved an analog of Theorem (5.16) in the particular case of Schubert varieties $\subset G/B$, where G is finite. It is not clear if their proofs can be extended to the general (infinite) case.

In Section 6 we prove that in the finite case, the matrix D can be obtained from the W-Harmonic polynomials on $\mathbf{h}$, as shown in Theorem (6.3) (see also Remark (6.4)). This says that the matrix D is nothing but the "upper triangular part" of the matrix obtained by the W-translates of the harmonic polynomials.

The main results of this paper have been announced in [15].

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1. Preliminaries and Notations

(1.1) *Definitions and Basic Properties* [13, 25].

Let $A = (a_{ij})_{1 \leqslant i,j \leqslant l}$ be any *generalized Cartan matrix* (i.e., $a_{ii} = 2$ and $-a_{ij} \in \mathbb{Z}_+$ for all $i \neq j$). A is called *symmetrizable* if DA is symmetric for some diagonal matrix $D = \operatorname{diag}(q_1, \ldots, q_l)$ with $q_i > 0$ and rational.

Choose a triple $(\mathbf{h}, \pi, \check{\pi})$, unique up to isomorphism, where $\mathbf{h}$ is a vector space over $\mathbb{C}$ of $\dim(l + \text{co-rank } A)$, $\pi = \{\alpha_i\}_{1 \leqslant i \leqslant l} \subset \mathbf{h}^*$ and $\check{\pi} = \{h_i\}_{1 \leqslant i \leqslant l} \subset \mathbf{h}$ are linearly independent indexed sets satisfying $\alpha_j(h_i) = a_{ij}$. The *Kac–Moody algebra* $\mathbf{g} = \mathbf{g}(A)$ is the Lie algebra over $\mathbb{C}$, generated by $\mathbf{h}$ and symbols e_i and f_i $(1 \leqslant i \leqslant l)$ with the defining relations $[\mathbf{h}, \mathbf{h}] = 0$; $[h, e_i] = \alpha_i(h)\, e_i$, $[h, f_i] = -\alpha_i(h)\, f_i$ for $h \in \mathbf{h}$ and all $1 \leqslant i \leqslant l$;

$[e_i, f_j] = \delta_{ij} h_j$ for all $1 \leqslant i, j \leqslant l$; $(\operatorname{ad} e_i)^{1-a_{ij}}(e_j) = 0 = (\operatorname{ad} f_i)^{1-a_{ij}}(f_j)$ for all $1 \leqslant i \neq j \leqslant l$. $\mathbf{h}$ is canonically embedded in $\mathbf{g}$ and is called the *Cartan subalgebra* of $\mathbf{g}$.

One has a root space decomposition $\mathbf{g} = \mathbf{h} \oplus \sum_{\alpha \in \Delta_+} (\mathbf{g}_\alpha \oplus \mathbf{g}_{-\alpha})$, where $\mathbf{g}_\alpha = \{x \in \mathbf{g}\colon [h, x] = \alpha(h)\, x$, for all $h \in \mathbf{h}\}$ and $\Delta_+ = \{\alpha \in \sum_{i=1}^{l} \mathbb{Z}_+ \alpha_i\colon \mathbf{g}_\alpha \neq 0\}$. Define $\Delta = \Delta_+ \cup \Delta_-$ $(\Delta_- = -\Delta_+)$.

We fix a subset X (including $X = \emptyset$) of $\{1, ..., l\}$. Put $\Delta_+^X = \Delta_+ \cap \{\sum_{i \in X} \mathbb{Z}\alpha_i\}$ and define the following Lie subalgebras:

$$\mathbf{n} = \sum_{\alpha \in \Delta_+} \mathbf{g}_\alpha, \qquad \mathbf{n}^- = \sum_{\alpha \in \Delta_+} \mathbf{g}_{-\alpha};$$

$$\mathbf{u} = \mathbf{u}_X = \sum_{\alpha \in \Delta_+ \setminus \Delta_+^X} \mathbf{g}_\alpha, \qquad \mathbf{u}^- = \mathbf{u}_X^- = \sum_{\alpha \in \Delta_+ \setminus \Delta_+^X} \mathbf{g}_{-\alpha};$$

$$\mathbf{r} = \mathbf{r}_X = \mathbf{h} \oplus \sum_{\alpha \in \Delta_+^X} (\mathbf{g}_\alpha \oplus \mathbf{g}_{-\alpha});$$

$$\mathbf{b} = \mathbf{h} \oplus \mathbf{n} \text{ and } \mathbf{p} = \mathbf{p}_X = \mathbf{r} \oplus \mathbf{u}.$$

In the case when X is of finite type (i.e., $\mathbf{r}_X$ is finite dimensional) $\mathbf{r}_X$ is a reductive subalgebra and since $[\mathbf{r}_X, \mathbf{u}_X] \subset \mathbf{u}_X$ (resp. $[\mathbf{r}_X, u_X^-] \subset \mathbf{u}_X^-$), $\mathbf{r}_X$ acts on $\mathbf{u}_X$ (resp. $\mathbf{u}_X^-$).

There is a *Weyl group* $W \subset \operatorname{Aut}(\mathbf{h}^*)$ generated by the "simple" reflections $\{r_i\}_{1 \leqslant i \leqslant l} (r_i(\chi) = \chi - \chi(h_i)\, \alpha_i$, for any $\chi \in \mathbf{h}^*)$, associated to the Lie algebra $\mathbf{g}$. $(W, \{r_i\}_{1 \leqslant i \leqslant l})$ is a Coxeter group, and hence we can speak of the Bruhat ordering $\leqslant$ and lengths of elements of W. We denote the length of w by $l(w)$. W preserves Δ. Δ^{re} is defined to be $W \cdot \pi$ and $\Delta^{\text{im}} = \Delta \setminus \Delta^{\text{re}}$. For $\alpha \in \Delta^{\text{re}}$, $\dim \mathbf{g}_\alpha = 1$. We set $\Delta_+^{\text{re}} = \Delta^{\text{re}} \cap \Delta_+$, similarly $\Delta_-^{\text{re}} = \Delta^{\text{re}} \cap \Delta_-$. By dualizing, we get a representation of W in $\mathbf{h}$. Explicitly, $r_i(h) = h - \alpha_i(h)\, h_i$, for $h \in \mathbf{h}$ and any $1 \leqslant i \leqslant l$.

For any $X \subset \{1, ..., l\}$, let W_X be the subgroup of W generated by $\{r_i\}_{i \in X}$ and define a subset W_X^1, of the Weyl group W, by $W_X^1 = \{w \in W\colon \Delta_+ \cap w\Delta_- \subset \Delta_+ \setminus \Delta_+^X\}$. W_X^1 can be characterized as the set of elements of minimal length in the cosets $W_X w (w \in W)$ (each such coset contains a unique element of minimal length).

There is a ($\mathbb{C}$-linear) involution ω of $\mathbf{g}$ defined (uniquely) by $\omega(f_i) = -e_i$ for all $1 \leqslant i \leqslant l$ and $\omega(h) = -h$, for all $h \in \mathbf{h}$. It is easy to see that ω leaves $\mathbf{g}(\mathbb{R})$ (="real points" of $\mathbf{g}$) stable. Let ω_0 be the *conjugate linear* involution of $\mathbf{g}$, which coincides with ω on $\mathbf{g}(\mathbb{R})$. In the case when g is symmetrizable, there is a nondegenerate, W-invariant, symmetric $\mathbb{C}$-bilinear form σ on $\mathbf{h}^*$. σ gives rise to a nondegenerate, $\mathbf{g}$-invariant, symmetric $\mathbb{C}$-bilinear form (called the Killing form) $\langle\ ,\ \rangle$ on $\mathbf{g}$ (see [13]).

We fix, once and for all, one such σ (and hence $\langle\ ,\ \rangle$). The symmetric form $\langle\ ,\ \rangle$ on $\mathbf{g}$ gives rise to a Hermitian form $\{\ ,\ \}$ on $\mathbf{g}$, defined by

$\{x, y\} = -\langle x, \omega_0(y)\rangle$, for $x, y \in \mathbf{g}$. The Hermitian form $\{\ ,\ \}$ is positive definite on $\mathbf{n}^-$ (and $\mathbf{n}$) (see [8, Sect. 12] and [21, Remark IV, p. 1782].)

(1.2) *Algebraic Group Associated to a Kac–Moody Lie algebra* $\mathbf{g}$ [30, 21, 22].

A $\mathbf{g}^1$ ($=$the commutator subalgebra $[\mathbf{g}, \mathbf{g}]$) module (V, π) (π: $\mathbf{g}^1 \to \operatorname{End} V$) is called integrable, if $\pi(x)$ is locally nilpotent whenever $x \in \mathbf{g}_\alpha$, for $\alpha \in \Delta^{\mathrm{re}}$. Let G^* be the free product of the additive groups $\{g_\alpha\}_{\alpha \in \Delta^{\mathrm{re}}}$, with canonical inclusions i_α: $\mathbf{g}_\alpha \to G^*$. For any integrable $\mathbf{g}^1$-module (V, π), define a homomorphism π^*: $G^* \to \operatorname{Aut}_{\mathbb{C}} V$ by $\pi^*(i_\alpha(x)) = \exp(\pi(x))$, for $x \in \mathbf{g}_\alpha$. Let N^* be the intersection of all $\operatorname{Ker} \pi^*$. Put $G = G^*/N^*$. Let q be the canonical homomorphism: $G^* \to G$. For $x \in \mathbf{g}_\alpha$ ($\alpha \in \Delta^{\mathrm{re}}$), put $\exp(x) = q(i_\alpha x)$, so that $U_\alpha = \exp \mathbf{g}_\alpha$ is an additive one parameter subgroup of G. Denote by U (resp. U^-) the subgroup of G generated by the U_α's with $\alpha \in \Delta^{\mathrm{re}}_+$ (resp. $\alpha \in \Delta^{\mathrm{re}}_-$). We put a topology on G as given in [22, 4(G)]. G may be viewed as, possibly infinite dimensional, affine algebraic group in the sense of Šafarevič with Lie-algebra $\mathbf{g}^1$. For a proof, see [22, Sect. 4]. We call G the *group associated to the Kac–Moody Lie algebra* $\mathbf{g}$.

The conjugate linear involution ω_0 of $\mathbf{g}$, on "integration," gives rise to an involution $\tilde{\omega}_0$ of G. Let K denote the fixed point set of this involution. K is called the *standard real form of G*.

For each $1 \leqslant i \leqslant l$, there exists a unique homomorphism β_i: $SL_2(\mathbb{C}) \to G$, satisfying $\beta_i\left(\begin{smallmatrix} 1 & z \\ 0 & 1 \end{smallmatrix}\right) = \exp(ze_i)$ and $\beta_i\left(\begin{smallmatrix} 1 & 0 \\ z & 1 \end{smallmatrix}\right) = \exp(zf_i)$ (for all $z \in \mathbb{C}$), where e_i, f_i is as in (1.1). Define $H_i = \beta_i\{\left(\begin{smallmatrix} z & 0 \\ 0 & z^{-1} \end{smallmatrix}\right): z \in \mathbb{C}^*\}$; $H_i^+ = \beta_i\{\left(\begin{smallmatrix} z & 0 \\ 0 & z^{-1} \end{smallmatrix}\right): z$ is real and $>0\}$; $G_i = \beta_i(SL_2(\mathbb{C}))$; $N_i =$ Normalizer of H_i in G_i; H (resp. H^+) $=$ the subgroup (of G) generated by all H_i (resp. H_i^+); $N =$ the subgroup (of G) generated by all N_i. There is an isomorphism τ: $W \xrightarrow{\sim} N/H$, such that $\tau(r_i)$ is the coset $N_i H \backslash H$ mod H [21, Sect. 2]. *We would, sometimes, identify* W *with* N/H *under* τ.

Put $B = HU$ and $P = P_X = BW_X B$. B is called the *standard Borel subgroup* and P_X the *standard parabolic subgroup of* G, associated to the subset X. Denote by K_X the subgroup $K \cap P_X$. We denote by $T = K \cap B$, the "*maximal torus*" of K. It is easy to see that the canonical inclusion $K/K_X \to G/P_X$ is a (surjective) homeomorphism. Use [22, Theorem 4(d)]. ($K \subset G$ is given the subspace topology.)

(1.3) *Bruhat decomposition* [12, 30, 21, 22].

G can be written as disjoint union $G = \bigcup_{w \in W_X^1} (\bigcup w^{-1} P_X)$, so that $G/P_X = \bigcup_{w \in W_X^1} (\bigcup w^{-1} P_X/P_X)$. G/P_X is a C–W complex with cells $\{V_w = \bigcup w^{-1} P_X/P_X\}_{w \in W_X^1}$ and $\dim_{\mathbb{R}} V_w = 2l(w)$.

(1.4) *Notations.* Unless otherwise stated, vector spaces will be over $\mathbb{C}$ and linear maps will be $\mathbb{C}$-linear maps. For a vector space V, V^* denotes

$\mathrm{Hom}_{\mathbb{C}}(V, \mathbb{C})$, $\bigwedge(V)$ denotes the exterior algebra, and $S(V)$ denotes the symmetric algebra. Tensor product without a subscript will mean over $\mathbb{C}$. Modules will be left, unless stated otherwise.

For a Lie algebra pair $(\mathbf{g}, \mathbf{r})$, $C(\mathbf{g}, \mathbf{r})$ (resp. $\bigwedge(\mathbf{g}, \mathbf{r})$) denotes the standard co-chain (resp. chain) complex associated to the pair $(\mathbf{g}, \mathbf{r})$. ρ, as usual, means any element $\in \mathbf{h}^*$ satisfying $\rho(h_i) = 1$, for all $1 \leqslant i \leqslant l$. We fix elements χ_i $(1 \leqslant i \leqslant l) \in \mathbf{h}^*$ satisfying $\chi_i(h_j) = \delta_{i,j}$, for all $1 \leqslant j \leqslant l$.

Complex manifolds are oriented by its complex structure. Explicitly, on $\mathbb{C}^n$ (with complex coordinates $z_1 = x_1 + \sqrt{-1}\, y_1, \ldots, z_n = x_n + \sqrt{-1}\, y_n$) $\{\partial/\partial x_1, \partial/\partial y_1, \ldots, \partial/\partial x_n, \partial/\partial y_n\}$ will be declared as positively oriented basis of $T(\mathbb{C}^n)$.

2. Identification of the Cohomology Algebra $H^*(G/P_X, \mathbb{C})$ with $\mathrm{Gr}\, \mathbb{C}\{W_X^1\}$

We recall the following well-known

(2.1) Definition. A differential graded algebra/$\mathbb{C}$ (abbreviated as DGA) is a graded associative algebra (over $\mathbb{C}$), having identity, $\mathscr{A} = \sum_{i \geqslant 0} \mathscr{A}^i$, with a differential d: $\mathscr{A} \to \mathscr{A}$ of degree $+1$, such that

(1) $\mathscr{A}$ is graded commutative, i.e., $\alpha \cdot \beta = (-1)^{ij} \beta \cdot \alpha$, for $\alpha \in \mathscr{A}^i$ and $\beta \in \mathscr{A}^j$ and

(2) d is a derivation, i.e., $d(\alpha \cdot \beta) = d\alpha \cdot \beta + (-1)^i \alpha \cdot d\beta$, for $\alpha \in \mathscr{A}^i$.

Let $\mathscr{A}$ be a DGA with differential d. Denote by

$\mathrm{End}(\mathscr{A}) = \mathrm{Hom}_{\mathbb{C}}(\mathscr{A}, \mathscr{A})$ (all the $\mathbb{C}$-linear maps from $\mathscr{A}$ into itself)

$\mathrm{End}^{-p,q}(\mathscr{A}) = \mathrm{Hom}_{\mathbb{C}}(\mathscr{A}^p, \mathscr{A}^q)$, for all $p, q \geqslant 0$, and

$\mathrm{End}^i(\mathscr{A}) = \prod_{p \geqslant 0} \mathrm{End}^{-p, p+i}(\mathscr{A})$, for $i \in \mathbb{Z}$. We clearly have

$$\prod_{i \in \mathbb{Z}} \mathrm{End}^i(\mathscr{A}) \supset \mathrm{End}(A) \supset \sum_{i \in \mathbb{Z}} \mathrm{End}^i(\mathscr{A}).$$

d induces a derivation (of degree $+1$) δ in $\mathrm{End}(\mathscr{A})$ ($\mathrm{End}(\mathscr{A})$ is viewed as an algebra, with product as composition of maps) defined by

$$\delta\tau = d\tau - (-1)^i \tau d \qquad \text{for} \quad \tau \in \mathrm{End}^i(\mathscr{A}).$$

From the definition, it is easy to check that

$$\delta^2 = 0. \tag{I$_1$}$$

Further, $\tau \in \operatorname{End}(\mathscr{A})$, with $\delta\tau = 0$, takes $Z(\mathscr{A})$ $(=\{a \in \mathscr{A}\colon da = 0\})$ into itself and the same is true with $Z(\mathscr{A})$ replaced by $d(\mathscr{A})$. Hence, there is a canonical homomorphism of $\mathbb{Z}$-graded algebras (obtained by restriction)

$$\gamma\colon H(\operatorname{End}(\mathscr{A}), \delta) \to \operatorname{End}(H(\mathscr{A})).$$

A proof of the following proposition can be seen, e.g., in [4, Theorem 31(a), p. 114].

(2.2) Proposition. *The map γ, defined above, is an isomorphism of graded algebras.*

(2.3) Let $\mathbf{g} = \mathbf{g}(A_l)$ be any symmetrizable Kac–Moody Lie algebra. Fix a subset $X \subset \{1, ..., l\}$ of finite type. Recall the Lie algebras $\mathbf{u}_X, \mathbf{u}_X^-, \mathbf{r}_X$ defined in Section 1.1. We would often abbreviate $\mathbf{u}_X, \mathbf{u}_X^-, \mathbf{r}_X$ (respectively) by $\mathbf{u}, \mathbf{u}^-, \mathbf{r}$.

Let $C(\mathbf{g}, \mathbf{r}) = \sum_{i \geq 0} \operatorname{Hom}_{\mathbf{r}}(\bigwedge^i(\mathbf{u} \oplus \mathbf{u}^-), \mathbb{C})$ denote the standard co-chain complex associated to the Lie algebra pair $(\mathbf{g}, \mathbf{r})$ and $C(\mathbf{u}) = \sum_{i \geq 0} \operatorname{Hom}(\Lambda^i(\mathbf{u}), \mathbb{C})$ is the DGA associated to the Lie algebra $\mathbf{u}$.

As in [23, Sect. 3], we put the topology of pointwise convergence on $C(\mathbf{g}, \mathbf{r})$ and $C(\mathbf{u})$. We also put the topology of pointwise convergence on $\operatorname{End} C(\mathbf{u})$, i.e., $\{\tau_n\} \subset \operatorname{End} C(\mathbf{u})$ converges to $\tau \in \operatorname{End} C(\mathbf{u})$ if and only if $\tau_n(\alpha) \to \tau(\alpha)$ (in the topology of $C(\mathbf{u})$), for all $\alpha \in C(\mathbf{u})$.

$\delta\colon \operatorname{End} C(\mathbf{u}) \to \operatorname{End} C(\mathbf{u})$ is continuous under this topology. Further, it can be easily seen that δ commutes with the canonical $\mathbf{r}$ action on $\operatorname{End} C(\mathbf{u})$. We denote by δ_0, the restriction of δ to $\operatorname{End}_{\mathbf{r}} C(u)$.

The map η (defined below) is basic to this section.

(2.4) Lemma. *There exists a (unique) continuous map $\eta\colon C(\mathbf{g}, \mathbf{r}) \to \operatorname{End}_{\mathbf{r}} C(\mathbf{u})$, such that*

$$\eta\left(\sum_{n=1}^{k} \boldsymbol{\alpha}^n \otimes e(\mathbf{a}^n)\right) = \left(\frac{2\pi}{\sqrt{-1}}\right)^q \sum_n \varepsilon(\boldsymbol{\alpha}^n)\, i(\mathbf{a}^n) \tag{I$_2$}$$

for $\boldsymbol{\alpha}^n \in C(\mathbf{u})$ and $\mathbf{a}^n \in \bigwedge^q(\mathbf{u})$ (with $\sum \boldsymbol{\alpha}^n \otimes e(\mathbf{a}^n) \in C(\mathbf{g}, \mathbf{r})$), where $\varepsilon\colon C(\mathbf{u}) \to \operatorname{End} C(\mathbf{u})$ is exterior multiplication, $i\colon \Lambda(\mathbf{u}) \to \operatorname{End} C(\mathbf{u})$ is interior multiplication, and $e\colon \bigwedge(\mathbf{u}) \to C(\mathbf{u}^-)$ is induced from the Killing form. Moreover η is injective. ▌

It may be remarked that, though η is defined on a dense subspace of $\sum_{i \geq 0} \operatorname{Hom}_{\mathbb{C}}(\bigwedge^i(\mathbf{u} \oplus \mathbf{u}^-), \mathbb{C})$, it does not extend to a continuous map on $\sum_{i \geq 0} \operatorname{Hom}_{\mathbb{C}}(\bigwedge^i(\mathbf{u} \oplus \mathbf{u}^-), \mathbb{C})$, in general. Further, η does not commute with the differentials. In fact, we have

(2.5) LEMMA. *$\eta d_0 = \delta_0 \eta$, where the operator d_0 on $C(\mathbf{g}, \mathbf{r})$ is defined to be $d' + (2\pi/\sqrt{-1})\,\partial''$. ($d'$ and ∂'' are defined in* [23, Sect. 3]. *One has $d_0^2 = 0$.*

Proof. For $\mathbf{a} = a_1 \wedge \cdots \wedge a_q \varepsilon \bigwedge^q(\mathbf{u})$, we have

$$d_{\mathbf{u}} i(\mathbf{a}) + (-1)^{q+1} i(\mathbf{a})\, d_{\mathbf{u}} = \sum_{k=1}^{q} (-1)^{k+1} \operatorname{ad}(a_k) \circ i(a_1) \circ \cdots \circ \widehat{i(a_k)} \circ \cdots \circ i(a_q)$$
$$+ i(\partial_{\mathbf{u}} \mathbf{a}) \qquad \text{as operators on } C(\mathbf{u})) \tag{I$_3$}$$

where $\partial_{\mathbf{u}}$ (resp. $d_{\mathbf{u}}$) denotes the chain (resp. co-chain) map of the standard complex $\bigwedge(\mathbf{u})$ (resp. $C(\mathbf{u})$), associated to the Lie algebra $\mathbf{u}$ and ad: $\mathbf{u} \to \operatorname{End} C(\mathbf{u})$ is induced from the adjoint representation.

The identity (I_3) can be easily proved by induction on q, using the well-known identities (I_4) and (I_5): For any $a_1, a_2 \in \mathbf{u}$,

$$d_{\mathbf{u}} i(a_1) + i(a_1)\, d_{\mathbf{u}} = \operatorname{ad}(a_1) \tag{I$_4$}$$

$$i(a_1) \operatorname{ad}(a_2) - \operatorname{ad}(a_2)\, i(a_1) = i[a_1, a_2] \tag{I$_5$}$$

as operators on $C(\mathbf{u})$. Further, for any $a \in \mathbf{u}$,

$$\operatorname{ad}(a) = \sum_{\phi \in I} \varepsilon(a_\phi^*)\, i[a_\phi, a], \tag{I$_6$}$$

where $\{a_\phi\}_{\phi \in I}$ is an orthonormal basis of $\mathbf{u}$, consisting of weight vectors, and $a_\phi^* \in \operatorname{Hom}(\mathbf{u}, \mathbb{C})$ is the element satisfying $a_\phi^*(a_{\phi'}) = \delta_{\phi, \phi'}$ for all $\phi' \in I$.

To prove identity (I_6), observe that both the sides of (I_6) are derivations and, moreover, both are continuous maps: $C(\mathbf{u}) \to C(\mathbf{u})$. So it suffices to prove that $\operatorname{ad}(a)\, \alpha = \sum_{\phi \in I} \varepsilon(a_\phi^*)\, i[a_\phi, a]\, \alpha$, for $\alpha \in C^1(\mathbf{u})$, which is easy to verify.

We are ready to prove the lemma. Since all the maps η, δ_0, d_0 are continuous, it suffices to prove that $\eta d_0 = \delta_0 \eta$ on $e([\bigwedge(\mathbf{u}^-) \otimes \bigwedge(\mathbf{u})]^{\mathbf{r}})$. Fix an element $\sum_n \mathbf{b}^n \otimes \mathbf{a}^n \in [\bigwedge(\mathbf{u}^-) \otimes \bigwedge(\mathbf{u})]^{\mathbf{r}}$, where $\mathbf{b}^n \in \bigwedge^p(\mathbf{u}^-)$ and $\mathbf{a}^n = a_1^n \wedge \cdots \wedge a_q^n \in \bigwedge^q(\mathbf{u})$. Then

$$\left(\frac{\sqrt{-1}}{2\pi}\right)^q \delta_0 \eta(e \Sigma \mathbf{b}^n \otimes \mathbf{a}^n)$$
$$= \sum_n [d_{\mathbf{u}} \varepsilon(e\mathbf{b}^n)\, i(\mathbf{a}^n) - (-1)^{p-q} \varepsilon(e\mathbf{b}^n)\, i(\mathbf{a}^n)\, d_{\mathbf{u}}]$$
$$= \sum_n [\varepsilon(d_{\mathbf{u}}(e\mathbf{b}^n))\, i(\mathbf{a}^n) + (-1)^p \varepsilon(e\mathbf{b}^n)\, d_{\mathbf{u}} i(\mathbf{a}^n) - (-1)^{p-q} \varepsilon(e\mathbf{b}^n)\, i(\mathbf{a}^n)\, d_{\mathbf{u}}]$$

$$= \sum_n \Big[\varepsilon(d_{\mathbf{u}}(e\mathbf{b}^n))\, i(\mathbf{a}^n) + (-1)^p \varepsilon(e\mathbf{b}^n) \Big\{ \sum_{k=1}^{q} (-1)^{k+1} \operatorname{ad}(a_k^n)$$

$$\circ\, i(a_1^n) \circ \cdots \circ \widehat{i(a_k^n)} \circ \cdots \circ i(a_q^n) + i(\partial_{\mathbf{u}} \mathbf{a}^n) \Big\} \Big] \qquad \text{using } (\mathrm{I}_3)$$

$$= \sum_n \Big[\varepsilon(d_{\mathbf{u}}(e\mathbf{b}^n))\, i(\mathbf{a}^n) + (-1)^p \varepsilon(e\mathbf{b}^n)\, i(\partial_{\mathbf{u}} \mathbf{a}^n) + (-1)^p \varepsilon(e\mathbf{b}^n)$$

$$\times \sum_{k=1}^{q} (-1)^{k+1} \sum_{\phi \in I} \varepsilon(a_\phi^*)\, i[a_\phi, a_k^n] \circ i(a_1^n) \circ \cdots \circ \widehat{i(a_k^n)} \circ \cdots \circ i(a_q^n) \Big]$$

using (I_6)

$$\left(\frac{\sqrt{-1}}{2\pi}\right)^q \delta_0 \eta \left(e \sum_n \mathbf{b}^n \otimes \mathbf{a}^n \right) = \sum_n \Big[\varepsilon(d_{\mathbf{u}}(e\mathbf{b}^n))\, i(\mathbf{a}^n) + (-1)^p \varepsilon(e\mathbf{b}^n)\, i(\partial_{\mathbf{u}} \mathbf{a}^n)$$

$$+ (-1)^p \sum_{\phi \in I} \varepsilon(e\mathbf{b}^n)\, \varepsilon(a_\phi^*)\, i(\operatorname{ad} a_\phi \mathbf{a}^n) \Big]. \qquad (\mathrm{I}_7)$$

Also,

$$\eta d_0 \left(e \sum_n \mathbf{b}^n \otimes \mathbf{a}^n \right)$$

$$= \eta \Bigg(\sum_n \Big[(d_{\mathbf{u}}(e\mathbf{b}^n)) \otimes (e\mathbf{a}^n) + (-1)^p \sum_{\phi \in I} e(\mathbf{b}^n) \cdot a_\phi^* \otimes e(\operatorname{ad} a_\phi \mathbf{a}^n)$$

$$+ (-1)^p \left(\frac{2\pi}{\sqrt{-1}} \right) (e\mathbf{b}^n) \otimes e(\partial_{\mathbf{u}} \mathbf{a}^n) \Big] \Bigg)$$

(See the proof of [23, Lemma (3.1)].) So, we have

$$\left[\frac{\sqrt{-1}}{2\pi}\right]^q \eta\, d_0 \left(e \sum_n \mathbf{b}^n \otimes \mathbf{a}^n \right)$$

$$= \sum_n \Big[\varepsilon(d_{\mathbf{u}}(e\mathbf{b}^n))\, i(\mathbf{a}^n) + (-1)^p \sum_{\phi \in I} \varepsilon(e(\mathbf{b}^n)\, a_\phi^*)\, i(\operatorname{ad} a_\phi \mathbf{a}^n)$$

$$+ (-1)^p \varepsilon(e\mathbf{b}^n)\, i(\partial_{\mathbf{u}} \mathbf{a}^n) \Big] \qquad (\mathrm{I}_8)$$

Comparing (I_7) and (I_8), we get the lemma. ∎

Recall the definition of the operator $S = d\partial + \partial d$, acting on $C(\mathbf{g}, \mathbf{r})$, from [23, Sect. 3]. From [23, Lemma (3.5), Theorem 3.13, and Remark 3.14] we have, $\operatorname{Ker} S = \operatorname{Ker} d' \cap \operatorname{Ker} \partial' \cap \operatorname{Ker} d'' \cap \operatorname{Ker} \partial''$. As an immediate corollary of lemma (2.5), we get

(2.6) LEMMA. $\eta(\operatorname{Ker} S) \subset \operatorname{Ker} \delta_0$.

By virtue of this lemma, η gives rise to a map $\tilde{\eta}$: $\operatorname{Ker} S \to H(\operatorname{End}_{\mathbf{r}} C(\mathbf{u}), \delta_0)$. Recall the definition of the map γ from Proposition (2.2). Obviously, γ gives rise to a map γ_0: $H(\operatorname{End}_{\mathbf{r}} C(\mathbf{u}), \delta_0) \to \operatorname{End}_{\mathbf{r}} H^*(\mathbf{u})$, where $H^*(\mathbf{u}) = \sum_{i \geqslant 0} H^i(\mathbf{u})$ is the Lie algebra cohomology of $\mathbf{u}$ (with trivial coefficients $\mathbb{C}$).

Let us recall the structure of $H^*(\mathbf{u})$, as an $\mathbf{r}$-module, from [9].

(2.7) THEOREM. [9, Theorem 8.6]. *As* $\mathbf{r}$-*modules,*

$$H^i(\mathbf{u}) \sim \sum_{\substack{w \in W^1_X \text{ with} \\ l(w) = i}} L(w\rho - \rho),$$

where $L(w\rho - \rho)$ *is an irreducible* $\mathbf{r}$-*module with highest weight* $w\rho - \rho$. *In particular, any irreducible r-module occurs with multiplicity at the most one in* $H^*(\mathbf{u})$. ▌

So, we get

$$\operatorname{End}_{\mathbf{r}} H^*(\mathbf{u}) \approx \prod_{i \geqslant 0} \operatorname{End}_{\mathbf{r}} H^i(\mathbf{u}) \approx \prod_{i \geqslant 0} \prod_{\substack{w \in W^1_X \\ l(w) = i}} \text{with } \operatorname{End}_{\mathbf{r}} L(w\rho - \rho).$$

$L(w\rho - \rho)$ being irreducible, $\operatorname{End}_{\mathbf{r}} L(w\rho - \rho)$ is 1-dimensional with a canonical generator 1_w (=the identity map of $L(w\rho - \rho)$). This identifies $\operatorname{End}_{\mathbf{r}} H^*(\mathbf{u})$ with $\prod_{w \in W^1_X} \mathbb{C} 1_w$. The space $\prod_{w \in W^1_X} \mathbb{C} 1_w$ can (and will) also be thought of as $\mathbb{C}\{W^1_X\}$ (=the vector space of *all* the functions from W^1_X to $\mathbb{C}$). The product in $\mathbb{C}\{W^1_X\}$, inherited from $\operatorname{End}_{\mathbf{r}} H^*(\mathbf{u})$, is nothing but the pointwise multiplication. Let $\bar{\eta}$: $\operatorname{Ker} S \to \mathbb{C}\{W^1_X\}$ be the composite map

$$\operatorname{Ker} S \xrightarrow{\tilde{\eta}} H(\operatorname{End}_{\mathbf{r}} C(\mathbf{u}), \delta_0) \xrightarrow{\gamma_0} \operatorname{End}_{\mathbf{r}} H^*(\mathbf{u}) \approx \mathbb{C}\{W^1_X\}.$$

(2.8) *A filtration of* $C(\mathbf{g}, \mathbf{r})$ *and* $\mathbb{C}\{W^1_X\}$. Define a decreasing filtration $\mathscr{G} = (\mathscr{G}_p)_{p \in \mathbb{Z}_-}$ ($\mathbb{Z}_-$ is the set of non-positive integers) by $\mathscr{G}_p = \sum_{0 \leqslant k \leqslant -p} C^{*,k}(\mathbf{g}, \mathbf{r})$, where $C^{q,k}(\mathbf{g}, \mathbf{r}) = \operatorname{Hom}_{\mathbf{r}}(\wedge^q(\mathbf{u}) \otimes \wedge^k(\mathbf{u}^-), \mathbb{C})$. Clearly $\mathscr{G}_p$ is d', ∂'' (and hence d_0) stable. This gives rise to a filtration $\mathscr{F} = (\mathscr{F}_p)_{p \in \mathbb{Z}_-}$ of $\operatorname{End}_{\mathbf{r}} C(\mathbf{u})$ by defining $\mathscr{F}_p = \eta(\mathscr{G}_p)$. (Caution! We do not claim that $\bigcup_{p \in \mathbb{Z}_-} \mathscr{F}_p = \operatorname{End}_{\mathbf{r}} C(\mathbf{u})$.) This also gives rise to a filtration $\{\mathscr{J}_p\}_{p \in \mathbb{Z}}$ of $\operatorname{End}_{\mathbf{r}} H^*(\mathbf{u}) \approx \mathbb{C}\{W^1_X\}$, by taking the image of the induced filtration of $H(\operatorname{End}_{\mathbf{r}} C(\mathbf{u}), \delta_0)$ under the canonical map γ_0, defined in (2.6). Here again, we do not claim that $\bigcup_{p \in \mathbb{Z}_-} \mathscr{J}_p = \mathbb{C}\{W^1_X\}$. In fact, it is true if and only if $\mathbf{g}$ is a finite dimensional Lie algebra.

(2.9) *Remark.* Let $\mathscr{A}$ be any finite dimensional DGA. Then the complex $\operatorname{End}(\mathscr{A})$ comes equipped with a natural filtration $\{F_p\}_{p\in\mathbb{Z}_-}$ defined as follows: $F_0=\varepsilon(\mathscr{A})$ (recall that ε: $\mathscr{A}\to\operatorname{End}(\mathscr{A})$ is exterior multiplication). $F_{-1}=\varepsilon(\mathscr{A})+\operatorname{Der}(\mathscr{A})$, where $\operatorname{Der}(\mathscr{A})=\sum_i \operatorname{Der}^i(\mathscr{A})$ and $\operatorname{Der}^i(\mathscr{A})$ is the set of all the degree i derivations of $\mathscr{A}$. Finally (for $p\geqslant 1$) F_{-p} is the $\mathbb{C}$-span of all those elements in $\operatorname{End}(\mathscr{A})$ which can be written as products of $\leqslant p$ operators in F_{-1}. (F_{-1} can be thought of as the set of first-order differential operators on $\mathscr{A}$.)

If we take $\mathscr{A}=C(\mathbf{u})$ then, in the case when $\mathbf{g}$ is finite dimensional, the filtration $\{\mathscr{F}_p\}_{p\in\mathbb{Z}_-}$ (described above in **(2.8)**) coincides with $\{F_p\cap \operatorname{End}_{\mathbf{r}}C(\mathbf{u})\}_{p\in\mathbb{Z}_-}$, as can be easily seen. In the general case $\mathscr{F}_p$ is some appropriate "partial completion" of F_p. So, our filtration $\{\mathscr{F}_p\}$ arises as a "super" analog of the usual filtration of differential operators on a manifold.

The behavior of η under products is given by

(2.10) LEMMA. *For $\lambda\in C^{*,p}(\mathbf{g},\mathbf{r})$ and $\lambda'\in C^{*,p'}(\mathbf{g},\mathbf{r})$,*

$$\eta(\lambda\lambda')=\eta(\lambda)\,\eta(\lambda')\qquad \operatorname{mod}\mathscr{F}_{-p-p'+1}.$$

Proof. For $\alpha\in C^k(\mathbf{u})$ and $a\in\mathbf{u}$, we have

$$i(a)\,\varepsilon(\alpha)-(-1)^k\varepsilon(\alpha)\,i(a)=\varepsilon(i(a)\,\alpha)\tag{I_9}$$

as elements of $\operatorname{End} C(\mathbf{u})$.

From the above relation, by induction on p, we get (for $\boldsymbol{\beta}\in C(\mathbf{u})$, $\boldsymbol{\alpha}\in C^k(\mathbf{u})$, $\mathbf{a}\in\bigwedge^p(\mathbf{u})$, and $\mathbf{b}\in\bigwedge^{p'}(\mathbf{u})$)

$$\varepsilon(\boldsymbol{\beta})\,i(\mathbf{a})\,\varepsilon(\boldsymbol{\alpha})\,i(\mathbf{b})-(-1)^{pk}\varepsilon(\boldsymbol{\beta\alpha})\,i(\mathbf{a}\wedge\mathbf{b})\in\mathscr{F}_{-p-p'+1}\tag{I_{10}}$$

which proves the lemma. ∎

As an immediate corollary, we have

(2.11) LEMMA. *For all $p, p'\in\mathbb{Z}_-$, $\mathscr{F}_p\mathscr{F}_{p'}\subset\mathscr{F}_{p+p'}$ and $\mathscr{J}_p\cdot\mathscr{J}_{p'}\subset\mathscr{J}_{p+p'}$.*

As usual, let us define $\operatorname{Gr}\mathbb{C}\{W^1_X\}=\sum_{p\geqslant 0}\operatorname{Gr}^p$, where $\operatorname{Gr}^p=\mathscr{J}_{-p}/\mathscr{J}_{-p+1}$. In view of Lemma (2.11), $\operatorname{Gr}\mathbb{C}\{W^1_X\}$ inherits a canonical algebra structure from $\mathbb{C}\{W^1_X\}$. Denote $\operatorname{Ker} S\cap C^{p,p}(\mathbf{g},\mathbf{r})$ by $\operatorname{Ker}^{p,p}S$. By the definition of η, $\eta(\operatorname{Ker}^{p,p}S)\subset\mathscr{F}_{-p}$. Hence $\bar\eta(\operatorname{Ker}^{p,p}S)\subset\mathscr{J}_{-p}$ ($\bar\eta$ is defined in Sect. 2.7). Denote the composition $\operatorname{Ker}^{p,p}S\xrightarrow{\bar\eta}\mathscr{J}_{-p}\to\mathscr{J}_{-p}/\mathscr{J}_{-p+1}$ by $\operatorname{Gr}^p(\bar\eta)$ and let $\operatorname{Gr}(\bar\eta)$ be the map: $\sum_p\operatorname{Ker}^{p,p}S\to\operatorname{Gr}\mathbb{C}\{W^1_X\}$, such that $\operatorname{Gr}(\bar\eta)|_{\operatorname{Ker}^{p,p}S}=\operatorname{Gr}^p(\bar\eta)$. Consider the inclusion $\operatorname{Ker} S\subsetneq C(\mathbf{g},\mathbf{r})$. By [23, Theorem 3.13] it induces isomorphism (of vector spaces) $\psi_{d,S}$: $\operatorname{Ker} S\xrightarrow{\sim} H^*(\mathbf{g},\mathbf{r})$ and, moreover [23, Sect. 3], $\operatorname{Ker} S=\sum_{p\geqslant 0}\operatorname{Ker}^{p,p}S$.

Now we are in a position to state the main theorem of this section.

(2.12) THEOREM. *Let* $\mathbf{g} = \mathbf{g}(A_l)$ *be a symmetrizable Kac–Moody Lie algebra and let* $X \subset \{1, ..., l\}$ *be a subset of finite type. Let* $\mathbf{r} = \mathbf{r}_X$ *be the reductive subalgebra of* $\mathbf{g}$ *as defined in* (1.1). *Then the map* $\mathrm{Gr}(\bar{\eta}) \circ \psi_{d,S}^{-1}$: $H^*(\mathbf{g}, \mathbf{r}) \to \mathrm{Gr}\, \mathbb{C}\{W_X^1\}$ *(defined above) is an isomorphism of graded algebras. (Of course, under this isomorphism,* $H^{2p}(\mathbf{g}, \mathbf{r})$ *corresponds with* Gr^p.)

(2.13) *Remarks.* (a) By [24, Theorem 1.6], $H^*(\mathbf{g}, \mathbf{r}_X)$ is isomorphic (as a graded algebra) with $H^*(G/P_X, \mathbb{C})$ under a suitably defined integration map. This, in particular, gives a graded algebra isomorphism of $H^*(G/P_X, \mathbb{C})$ with $\mathrm{Gr}\, \mathbb{C}\{W_X^1\}$.

(b) In the case when $X = \emptyset$, we will show (Theorem 5.12) that the isomorphism of $H^*(G/B, \mathbb{C})$ with $\mathrm{Gr}\, \mathbb{C}\{W\}$ is W-equivariant, where W acts on $\mathbb{C}\{W\}$ by the left regular representation.

Proof of the Theorem (2.12). Recall the definition of the filtration $\mathscr{G}$ of $C(\mathbf{g}, \mathbf{r})$ and the filtration $\mathscr{F}$ of $\mathrm{End}_{\mathbf{r}} C(\mathbf{u})$, see (**2.8**). By Lemma (2.5), $\mathscr{F}_p$ is δ_0 stable. The corresponding spectral sequence has $E_1^{p,q}(\mathscr{F}) = H^{p+q}(\mathscr{F}_p/\mathscr{F}_{p+1}) \approx H_{d_0}^{p+q}(\mathscr{G}_p/\mathscr{G}_{p+1}) \approx H_{d'}^{p+q}(C^{*,-p}(\mathbf{g}, \mathbf{r}))$, since the differentials on $\mathscr{G}_p/\mathscr{G}_{p+1}$ induced by d_0 and d' are the same. ($H_{d'}(C^*, {}^{-p}(\mathbf{g}, \mathbf{r}))$ denotes the cohomology of the complex $C^{*,-p}(\mathbf{g}, \mathbf{r})$ with respect to the differential d'.) By (a subsequent) Lemma (2.15), $\psi_{d',S} \circ \psi_{d,S}^{-1}$: $H_d^*(\mathbf{g}, \mathbf{r}) \to H_{d'}^*(\mathbf{g}, \mathbf{r})$ is an algebra isomorphism. But, by [23, Theorem 3.15], $H_d^i(C(\mathbf{g}, \mathbf{r})) = 0$ for odd values of i and for even i, $\dim_{\mathbb{C}} H_d^i(C(\mathbf{g}, \mathbf{r})) =$ number of elements of length $i/2$ in W_X^1. In particular, $E_1^{p,q}(\mathscr{F}) = 0$ unless $p+q$ is even. But then all the differentials d_r (for $r \geqslant 1$), of the spectral sequence $E_r(\mathscr{F})$, are zero. So, we have $E_1^{p,q}(\mathscr{F}) \approx E_\infty^{p,q}(\mathscr{F})$ and (of course) this isomorphism is an algebra isomorphism.

Further, the map $\bar{\eta}$: $\mathrm{Ker}\, S \to \mathbb{C}\{W_X^1\}$ is injective. To prove this, let $0 \neq s = \sum_{w \in W_X^1} z^w s^w \in \mathrm{Ker}\, S$ be such that $\bar{\eta}(\sum z^w s^w) = 0$, for some constants $z^w \in \mathbb{C}$. ($\{s^w\}_{w \in W_X^1}$ is a $\mathbb{C}$-basis of $\mathrm{Ker}\, S$ and is defined in the next section.) Let w_0 be an element of least possible length such that $z^{w_0} \neq 0$. In particular, evaluating at w_0, we have $\bar{\eta}(\sum z^w s^w)(w_0) = 0$. But by using (a subsequent) Proposition 5.2(a) and (b), we get $\bar{\eta}(z^{w_0} s^{w_0})(w_0) = 0$ and $\bar{\eta}(s^{w_0})(w_0) \neq 0$. A contradiction! This proves that $\bar{\eta}$ is injective.

Furthermore, the canonical map γ_0: $H(\mathrm{End}_{\mathbf{r}} C(\mathbf{u}), \delta_0) \to \mathrm{End}_{\mathbf{r}} H^*(\mathbf{u})$ is an isomorphism. To prove this, denote $\bar{C}(\mathbf{u}) = \mathrm{Hom}_{\mathbb{C}}(\bigwedge(\mathbf{u}), \mathbb{C})$. We decompose $\bigwedge(\mathbf{u}) = \sum_{\theta \in \hat{\mathbf{r}}} V_\theta$, where V_θ is the isotypical component corresponding to the irreducible representation θ of $\mathbf{r}$. Of course, for any fixed θ, V_θ is finite dimensional (use the fact that root spaces are finite dimensional). So, we have

$$\bar{C}(\mathbf{u}) = \prod_{\theta \in \hat{\mathbf{r}}} V_\theta^* \ (V_\theta^* = \mathrm{Hom}_{\mathbb{C}}(V_\theta, \mathbb{C})) \quad \text{and} \quad \mathrm{End}_{\mathbf{r}} \bar{C}(\mathbf{u}) = \prod_{\theta \in \hat{\mathbf{r}}} \mathrm{End}_{\mathbf{r}}(V_\theta^*).$$

From the finite dimensionality of V_θ's, it is easy to see that $\mathrm{End}_{\mathbf{r}} C(\mathbf{u}) = \mathrm{End}_{\mathbf{r}} \bar{C}(\mathbf{u})$. Hence

$$\begin{aligned} H(\mathrm{End}_{\mathbf{r}} C(\mathbf{u})) &= H(\mathrm{End}_{\mathbf{r}} \bar{C}(\mathbf{u})) \\ &= \prod_{\theta \in \hat{\mathbf{r}}} H(\mathrm{End}_{\mathbf{r}}(V_\theta^*)) \\ &\approx \prod_{\theta \in \hat{\mathbf{r}}} \mathrm{End}_{\mathbf{r}} H(V_\theta^*) \qquad \text{(from the complete reducibility} \\ &\qquad \text{of the } \mathbf{r}\text{-module End } V_\theta^* \text{ and Proposition (2.2))} \\ &= \mathrm{End}_{\mathbf{r}} \left(\prod_{\theta \in \hat{\mathbf{r}}} H(V_\theta^*) \right) \\ &= \mathrm{End}_{\mathbf{r}} H(\bar{C}(\mathbf{u})). \\ &= \mathrm{End}_{\mathbf{r}} H(C(\mathbf{u})) = \mathrm{End}_{\mathbf{r}} H^*(\mathbf{u}). \end{aligned}$$

Now, consider the commutative diagram

$$\begin{array}{ccc} H(C(\mathbf{g}, \mathbf{r}), d_0) & \xrightarrow{H(\eta)} & H(\mathrm{End}_{\mathbf{r}} C(\mathbf{u}), \delta_0) \\ \uparrow{\scriptstyle i^*} & & \wr\downarrow{\scriptstyle \gamma_0} \\ \mathrm{Ker}\, S & \xrightarrow[\bar{\eta}]{} & \mathrm{End}_{\mathbf{r}} H^*(\mathbf{u}) \approx \mathbb{C}\{W_X^1\} \end{array}$$

where i^* is induced by the inclusion $\mathrm{Ker}\, S \subsetneq C(\mathbf{g}, \mathbf{r})$ $(d_0|_{\mathrm{Ker} S} \equiv 0)$. Defining a filtration $\tilde{\mathscr{G}} = (\tilde{\mathscr{G}}_p)_{p \in \mathbb{Z}_-}$, where $\tilde{\mathscr{G}}_p = \sum_{0 \leqslant k \leqslant -p} \mathrm{Ker}^{k,k}(S)$, of Ker S, we get that $E_1^{p,p}(\tilde{\mathscr{G}}) \approx \mathrm{Ker}^{p,p}(S)$ and $E_1^{p,q}(\tilde{\mathscr{G}}) = 0$ for $p \neq q$. So $E_1^{p,q}(\tilde{\mathscr{G}}) \to^{\sim} E_1^{p,q}(\mathscr{G})$ for all p and q. Hence i^* is an isomorphism. This, using the injectivity of $\bar{\eta}$, proves that $H(\eta)$ is injective.

Finally, using [4, Chap. XV, Sect. 1] and Lemma (2.10), we get that $\mathrm{Gr}\, \mathbb{C}\{W_X^1\}$ is isomorphic with $E_\infty(\mathscr{G})$ as graded algebras. Now (a subsequent) Lemma (2.15) completes the proof. ∎

As an immediate consequence of the theorem, we get

(2.14) Corollary. *Recall the definition of the filtration* $\{\mathscr{J}_p\}_{p \in \mathbb{Z}_+}$ *of* $\mathbb{C}\{W_X^1\}$ *from* **(2.8)**. *For any* $p \geqslant 0$, *we have* $\mathscr{J}_{-p} = \sum_{k=0}^{p} \Gamma_k$, *where* Γ_k *is the image of* $\mathrm{Ker}^{k,k}(S)$ *in* $\mathbb{C}\{W_X^1\}$ *under* $\bar{\eta}$.

We will give another "combinatorial" description of $\mathscr{J}_{-p}$ in Section 5.

The inclusion of $\mathrm{Ker}\, S \subsetneq C(\mathbf{g}, \mathbf{r})$ induces the map $\psi_{d,S}\colon \mathrm{Ker}\, S \to H_d^*(\mathbf{g}, \mathbf{r})$ and $\psi_{d',S}\colon \mathrm{Ker}\, S \to H_{d'}^*(\mathbf{g}, \mathbf{r})$. By [23, Theorem 3.13 and Remark 3.14] $\psi_{d,S}$ and $\psi_{d',S}$ are both vector space isomorphisms. We have

(2.15) LEMMA. *The map $\psi_{d',S} \circ \psi_{d,S}^{-1}\colon H_d^*(\mathbf{g}, \mathbf{r}) \to H_{d'}^*(\mathbf{g}, \mathbf{r})$ is an algebra isomorphism.*

Proof. Let $s_1, s_2 \in \operatorname{Ker} S$. There exists $s_3 \in \operatorname{Ker} S$ such that $s_1 \cdot s_2 - s_3 \in$ Image d. By [23, Lemma 3.8 and Theorem 3.13], $s_1 \cdot s_2 - s_3 \in \operatorname{Im} S$. Defining $S' = d'\partial' + \partial' d'$, we have $\frac{1}{2}S = S'$ by [23, Lemma 3.5] and hence there exists $\lambda \in C(\mathbf{g}, \mathbf{r})$ such that

$$s_1 \cdot s_2 - s_3 = (d'\partial' + \partial' d')\,\lambda.$$

Thus, on taking d', we get $d'(s_1 \cdot s_2) - d's_3 = d'\partial' d'\lambda$, i.e., $(d's_1)\,s_2 + s_1 \cdot d's_2 - d's_3 = d'\partial' d'\lambda$. Since $s_1, s_2, s_3 \in \operatorname{Ker} S \subset \operatorname{Ker} d'$, we get $d'\partial' d'\lambda = 0$. By the disjointness of d' and ∂' [23, Proposition 3.7] $\partial' d'\lambda = 0$. Hence $s_1 \cdot s_2 - s_3 \in \operatorname{Im} d'$. This proves the lemma. ∎

3. DETERMINATION OF THE INTEGRAL

In this section $\mathbf{g} = \mathbf{g}(A_l)$ will denote a symmetrizable Kac–Moody Lie algebra.

Let $X \subset \{1, \ldots, l\}$ be a subset of finite type. In [17] and recently in the general case in [23], we constructed "d, ∂ harmonic" forms $\{s^w\}_{w \in W_X^1} \subset C(\mathbf{g}, \mathbf{r}_X)$ which are dual (up to a nonzero scalar multiple) to the Schubert cells $\{V_w = Bw^{-1}P_X/P_X\}_{w \in W_X^1}$. More precisely [23, Theorem 4.5], we have, for $w, w' \in W_X^1$ of equal length,

(a) $\int_{V_w} s^{w'} = 0$, unless $w = w'$ and

(b) $\int_{V_w} s^w = (-1)^{p(p-1)/2} 2^{2p} \int_{U_w} \exp(2(w\rho - \rho)\,h(g))\,dg$, if $l(w) = p$.

(Where $U_w = wUw^{-1} \cap U$ and, for any $g \in G$, $h(g)$ denotes the projection of g on the H^+ factor under the inverse of the Iwasawa decomposition: $K \times H^+ \times U \to G$, defined by $(k, h, u) \to khu$. See (1.2), for various notations.) The additional sign factor $(-1)^{p(p-1)/2}$, in the expression of $\int_{V_w} s^w$, is due to the fact that we have taken a different orientation on complex manifolds (see (1.4)) than the one used in [23, Theorem 4.5].

In this section, we explicitly compute this integral. This, in particular, would give expression for the d, ∂ harmonic forms $\{s_0^w = s^w / \int_{V_w} s^w\}_{w \in W_X^1}$, which are exactly dual to the Schubert cells. The main result of this section is

(3.1) THEOREM. *With the notations as above, for any $w \in W$, we have*

$$2^{2l(w)} \int_{U_w} \exp(2(w\rho - \rho)\,h(g))\,dg = (4\pi)^{l(w)} \prod_{v \in w^{-1}\Delta_- \cap \Delta_+} \sigma(\rho, v)^{-1}.$$

Proof. We prove the theorem by induction on $l(w)$. In the case when $l(w)=1$, the integral can be explicitly computed. The details are given in [24, Sect. 6.16].

So, we come to the general case. Write $w=r_i v$, where r_i is a simple reflection such that $v<w$. From the Hodge-type decomposition of $C(\mathbf{g}, \mathbf{h})$ [23, Theorem 3.13], we can express

$$s^v \cdot s^{r_i} = \sum_{l(w')=l(w)} z^{w'} s^{w'} + \lambda, \tag{I_{11}}$$

for some d-exact form $\lambda \in C(\mathbf{g}, \mathbf{h})$ and some (uniquely determined) constants $z^{w'} \in \mathbb{C}$.

The next few lemmas are devoted to finding the value of z^w. Define an operator $\bar{R} = -(S-L)\tilde{M}$: $C(\mathbf{g}, \mathbf{h}) \to C(\mathbf{g}, \mathbf{h})$, where L, S and $\tilde{M}$ are defined in [23, Sect. 3]. As in the proof of [23, Lemma 3.8], one can see that $\sum_{n\geqslant 0} \bar{R}^n \mu$ converges in $C(\mathbf{g}, \mathbf{h})$, for all $\mu \in C(\mathbf{g}, \mathbf{h})$. (Topology on $C(\mathbf{g}, \mathbf{h})$ is the one, described in (2.3).) We have

(3.2) LEMMA. *Let $s_1, s_2 \in \operatorname{Ker} S$. Write $P \sum_{n=0}^{\infty} \bar{R}^n(s_1 \cdot s_2) = \sum_{w \in W} z_0^w h^w$, for some (unique) constants z_0^w. (h^w is defined in (I_{15}) and P: $C(\mathbf{g}, \mathbf{h}) \to C(\mathbf{g}, \mathbf{h}) = \operatorname{Ker} L \oplus \operatorname{Im} L$ is projection onto $\operatorname{Ker} L$). Then*

$$s_1 \cdot s_2 - \sum_{w \in W} z_0^w s^w \in \operatorname{Im} d.$$

Proof. First of all, $s_1 \cdot s_2 - d\partial\tilde{M}\ (\sum_{n\geqslant 0} \bar{R}^n)(s_1 \cdot s_2) \in \operatorname{Ker} S$. To prove this:

$$\begin{aligned}
&S(s_1 \cdot s_2) - S\, d\partial\, \tilde{M}\left(\sum_{n\geqslant 0} \bar{R}^n\right)(s_1 \cdot s_2) \\
&\quad = S(s_1 \cdot s_2) - d\partial(S-L)\,\tilde{M}\left(\sum_{n\geqslant 0} \bar{R}^n\right)(s_1 \cdot s_2) - d\partial L\tilde{M}\left(\sum_{n\geqslant 0} \bar{R}^n\right)(s_1 \cdot s_2) \\
&\qquad \text{(since } S \text{ commutes with } d \text{ and } \partial\text{)}, \\
&\quad = S(s_1 \cdot s_2) + d\partial \sum_{n\geqslant 0} \bar{R}^{n+1}(s_1 \cdot s_2) - d\partial\left(\sum_{n\geqslant 0} \bar{R}^n\right)(s_1 \cdot s_2) \\
&\qquad \text{(since, by definition, } L\tilde{M} = \mathrm{Id} \text{ on } \operatorname{Im} L \text{ and } \partial|_{\operatorname{Ker} L} = L\tilde{M}|_{\operatorname{Ker} L} = 0\text{)}, \\
&\quad = S(s_1 \cdot s_2) - d\partial(s_1 \cdot s_2) \\
&\quad = 0 \qquad \text{(since } d(s_1 \cdot s_2) = 0\text{)}.
\end{aligned}$$

Furthermore,

$$\begin{aligned}-d\partial\tilde{M}\left(\sum_{n\geqslant 0}\bar{R}^n\right)(s_1\cdot s_2) &= (-S+\partial d)\,\tilde{M}\left(\sum_{n\geqslant 0}\bar{R}^n\right)(s_1\cdot s_2)\\ &= (-L-(S-L)+\partial d)\,\tilde{M}\left(\sum_{n\geqslant 0}\bar{R}^n\right)(s_1\cdot s_2)\\ &= \sum_{n\geqslant 0}\bar{R}^{n+1}(s_1\cdot s_2)+L(\mu)\end{aligned}$$

for some $\mu\in C(\mathbf{g},\mathbf{h})$ (since $\operatorname{Im}\partial\subset\operatorname{Im}L$). So

$$s_1\cdot s_2-d\partial\tilde{M}\left(\sum_{n\geqslant 0}\bar{R}^n\right)(s_1\cdot s_2)=\left(\sum_{n\geqslant 0}\bar{R}^n\right)(s_1\cdot s_2)\bmod \operatorname{Im}L. \qquad (\mathrm{I}_{12})$$

But, since the left side of (I_{12}) belongs to Ker S, and $s^w-h^w\in\operatorname{Im}L$ (by [23, Proposition (3.17)]) the lemma follows.

(3.3) *Remark.* The above lemma is essentially due to Koch. See [20, Theorem 4.3].

(3.4.) LEMMA. *Let $w\in W$ be expressed as $w=r_iv$, with r_i a simple reflection and $v<w$. Then $\langle\Phi_w\rangle=\langle\Phi_v\rangle+k_0\alpha_i$, for some $k_0>0$ and, moreover, k_0 is equal to $\rho(v^{-1}h_i)$ (where $\langle\Phi_w\rangle$ denotes $\sum_{\alpha\in\Phi_w}\alpha$ and $\Phi_w=\Delta_+\cap w\Delta_-$).*

Proof. By [26, Sect. 2], $\langle\Phi_w\rangle=r_i\langle\Phi_v\rangle+\alpha_i=\langle\Phi_v\rangle-\langle\Phi_v\rangle(h_i)\alpha_i+\alpha_i=\langle\Phi_v\rangle+[1-(\rho-v\rho)h_i]\alpha_i$. Furthermore, $(\rho-v\rho)(h_i)=1-2\sigma(\rho,v^{-1}\alpha_i)/\sigma(\alpha_i,\alpha_i)$. Also, $\rho-v\rho$ being sum of roots, $(\rho-v\rho)(h_i)\in\mathbb{Z}$ and $\sigma(\rho,v^{-1}\alpha_i)>0$, since $v^{-1}\alpha_i\in\Delta_+$ (otherwise $\alpha_i\in\Phi_v$). Hence $(\rho-v\rho)(h_i)\leqslant 0$. This proves the lemma. ∎

(3.5) Let $\tilde{\Delta}_+$ be an index set for an orthonormal basis of root vectors of $\mathbf{n}^-$ and for each $\Phi\in\tilde{\Delta}_+$ let b_Φ be the corresponding root vector and let $-\Phi'\in\Delta_-$ be the corresponding root. Since real roots have multiplicity 1 we may regard $\Delta_+^{\mathrm{Re}}\subseteq\tilde{\Delta}_+$ and $\Phi'=\Phi$ for $\Phi\in\Delta_+^{\mathrm{Re}}$. Define $a_\Phi=-\omega_0 b_\Phi$. Clearly $\{a_\Phi\}_{\Phi\in\tilde{\Delta}_+}$ is an orthonormal basis of $\mathbf{n}$.

For a sequence $\mathbf{\Phi}=(\Phi_1,\ldots,\Phi_p)$ with $\Phi_k\in\tilde{\Delta}_+$, we define the operator $R_{\mathbf{\Phi}}$, acting on $C(\mathbf{g},\mathbf{h})$, by

$$R_{\mathbf{\Phi}}(e(\mathbf{b}\otimes\mathbf{a}))=e(\operatorname{ad}b_{\Phi_1}\circ\cdots\circ\operatorname{ad}b_{\Phi_p}\mathbf{b}\otimes\operatorname{ad}a_{\Phi_1}\circ\cdots\circ\operatorname{ad}a_{\Phi_p}\mathbf{a}) \qquad (\mathrm{I}_{13})$$

for $\mathbf{b}\otimes\mathbf{a}\in[\wedge(\mathbf{n}^-)\otimes\wedge(\mathbf{n})]^{\mathbf{h}}$. (Recall that e is defined in (2.4).)

It is easy to see that the operator $R_{\mathbf{\Phi}}$, which is defined by (I_{13}) on a dense subspace $e([\wedge(\mathbf{n}^-)\otimes\wedge(\mathbf{n})]^h)$ of $C(\mathbf{g},\mathbf{h})$, extends (uniquely) to a continuous operator (again denoted by) $R_{\mathbf{\Phi}}\colon C(\mathbf{g},\mathbf{h})\to C(\mathbf{g},\mathbf{h})$.

By [23, I_{24}, I_{34}, and I_{35}], we have

$$S - L = -2 \sum_{\Phi \in \tilde{\Delta}_+} R_\Phi \qquad \text{as operators on } C(\mathbf{g}, \mathbf{h}). \tag{I_{14}}$$

Fix $v \in W$. Let $\Phi_v = \{\beta_1, ..., \beta_p\}$ $(p = l(v))$. The element $b_{\beta_i} \in \mathbf{g}_{-\beta_i}$ (β_i being real, $\mathbf{g}_{-\beta_i}$ is 1-dimensional) is of unit norm. Define the element

$$h^v = e[(2\sqrt{-1})^p (b_{\beta_1} \wedge \cdots \wedge b_{\beta_p} \otimes a_{\beta_1} \wedge \cdots \wedge a_{\beta_p})]. \tag{I_{15}}$$

By [23, Sect. 4.2], for the special case $X = \emptyset$, $h^v \in \operatorname{Ker} L$ and further, by [23, Proposition 3.17],

$$s^v = \sum_{j \geq 0} R^j(h^v) \qquad (R \text{ is defined to be } -\tilde{M}(S - L)). \tag{I_{16}}$$

Specializing $v = r_i$, we get

$$s^{r_i} = 2\sqrt{-1} \sum_{j \geq 0} R^j(e(b_{\alpha_i} \otimes a_{\alpha_i})). \tag{I'_{16}}$$

Let $w \in W$ be written as $w = r_i v$, with $v < w$. Recall the identity (I_{11}). We are interested in finding the value of z^w. By Lemma (3.2), (I_{16}), and (I'_{16}), we have z^w = coefficient of h^w in $(2\sqrt{-1}) \sum_{n, n_1, n_2 \geq 0} \bar{R}^n(R^{n_1}(h^v) \cdot (R^{n_2}(e(b_{\alpha_i} \otimes a_{\alpha_i})))$. We have

(3.6) Lemma. *With the notations as above,*

$$z^w = (k_0 c \sqrt{-1})\, 2^{k_0} \left(\prod_{k=1}^{k_0 - 1} [\sigma(v\rho - k\alpha_i, v\rho - k\alpha_i) - \sigma(\rho, \rho)]^{-1} \right)$$

where $k_0 = \rho(v^{-1} h_i)$ and c is the coefficient of h^w in $R_{\alpha_i}^{k_0 - 1}(h^v) \cdot e(b_{\alpha_i} \otimes a_{\alpha_i})$. (By Lemma (3.4), k_0 is a positive integer.)

Proof. By Lemma (3.4) and (I_{14}), the only terms contributing to the coefficient z^w (of h^w in $2\sqrt{-1} \sum_{n, n_1, n_2 \geq 0} \bar{R}^n(R^{n_1}(h^v) \cdot R^{n_2}(e(b_{\alpha_i} \otimes a_{\alpha_i})))$ are

$$(2\sqrt{-1})\, 2^{k_0 - 1} \sum_{n_1 = 0}^{k_0 - 1} (R_{\alpha_i} \tilde{M})^{k_0 - 1 - n_1} [(\tilde{M} R_{\alpha_i})^{n_1}(h^v) \cdot e(b_{\alpha_i} \otimes a_{\alpha_i})]. \tag{E}$$

By [23, (I_{32}), (I_{36}), (I_{37})], we have $L(e(\mathbf{b} \otimes \mathbf{a})) = [\sigma(\rho - \beta, \rho - \beta) - \sigma(\rho, \rho)]\, e(\mathbf{b} \otimes \mathbf{a})$, for a weight vector $\mathbf{b} \in \bigwedge(n^-)$ of weight $-\beta$ and $\mathbf{a} \in \bigwedge(\mathbf{n})$ such that $\mathbf{b} \otimes \mathbf{a} \in [\bigwedge(\mathbf{n}^-) \otimes \bigwedge(\mathbf{n})]^{\mathbf{h}}$.

Further $\sigma(\rho, \rho) \neq \sigma(\rho - \langle \Phi_v \rangle - n_1\alpha_i, \rho - \langle \Phi_v \rangle - n_1\alpha_i)$, for any $1 \leqslant n_1 \leqslant k_0 - 1$, since

$$\begin{aligned}\sigma(v\rho - n_1\alpha_i, v\rho - n_1\alpha_i) &= \sigma(\rho, \rho) + n_1^2\sigma(\alpha_i, \alpha_i) - 2n_1\sigma(v\rho, \alpha_i)\\ &= \sigma(\rho, \rho) + n_1^2\sigma(\alpha_i, \alpha_i) - n_1\sigma(\alpha_i, \alpha_i)(v\rho(h_i))\\ &= \sigma(\rho, \rho) + (n_1^2 - k_0 n_1)\,\sigma(\alpha_i, \alpha_i)\\ &\neq \sigma(\rho, \rho).\end{aligned}$$

So the expression (E) reduces to the sum

$$(2\sqrt{-1})\, 2^{k_0-1} \sum_{n_1=0}^{k_0-1} \left(\prod_{k=1}^{k_0-1} [\sigma(v\rho - k\alpha_i, v\rho - k\alpha_i) - \sigma(\rho, \rho)]^{-1} \right)$$
$$R_{\alpha_i}^{k_0-1-n_1}[R_{\alpha_i}^{n_1}(h^v) \cdot e(b_{\alpha_i} \otimes a_{\alpha_i})].$$

But $R_{\alpha_i}^{k_0-1-n_1}[R_{\alpha_i}^{n_1}(h^v) \cdot e(b_{\alpha_i} \otimes a_{\alpha_i})] = R_{\alpha_i}^{k_0-1}(h^v) \cdot e(b_{\alpha_i} \otimes a_{\alpha_i})$. Hence the expression (E) equals

$$2\sqrt{-1}\, 2^{k_0-1}(k_0) \left(\prod_{k=1}^{k_0-1} [\sigma(v\rho - k\alpha_i, v\rho - k\alpha_i) - \sigma(\rho, \rho)]^{-1} \right)$$
$$R_{\alpha_i}^{k_0-1}(h^v) \cdot e(b_{\alpha_i} \otimes a_{\alpha_i}). \quad \blacksquare$$

The following lemma gives the coefficient of h^w in $R_{\alpha_i}^{k_0-1}(h^v) \cdot e(b_{\alpha_i} \otimes a_{\alpha_i})$.

(3.7) Lemma. *Let $w = r_i v$ be such that $v < w$. Then*

$$\begin{aligned}&(\operatorname{ad} b_{\alpha_i})^{k_0-1}\mathbf{b}_{\Phi_v} \otimes (\operatorname{ad} a_{\alpha_i})^{k_0-1}\mathbf{a}_{\Phi_v}\\ &\quad= (-1)^{k_0-1}(k_0-1!)^2 \left(\frac{\sigma(\alpha_i, \alpha_i)}{2}\right)^{k_0-1} \mathbf{b}_\Phi \otimes \mathbf{a}_\Phi\end{aligned}$$

in $\bigwedge(\mathfrak{g})$, where $\Phi = r_i(\Phi_v) \subset \Delta_+$; $k_0 > 0$ is the same as in Lemma (3.4) *and the notation $\mathbf{b}_\Phi \otimes \mathbf{a}_\Phi$, for any $\Phi = \{\Phi_1, ..., \Phi_p\} \subset \Delta_+$ consisting of real roots, means $b_{\Phi_1} \wedge \cdots \wedge b_{\Phi_p} \otimes a_{\Phi_1} \wedge \cdots \wedge a_{\Phi_p} \in \bigwedge(\mathfrak{g})$.*

Proof. Denote $l(v) = p$. Fix any positive roots $\{\bar{\Phi}_1, ..., \bar{\Phi}_p)\}$ (not necessarily real and not necessarily distinct) such that $\sum_n \bar{\Phi}_n = \langle \Phi \rangle$. If there exist $b_n \in \mathfrak{g}_{-\bar{\Phi}_n}$ satisfying $b_1 \wedge \cdots \wedge b_p \neq 0$ in $\bigwedge^p(\mathbf{n}^-)$ and none of the $\bar{\Phi}_n$ equals α_i, then $\{\bar{\Phi}_1, ..., \bar{\Phi}_p\} = \Phi$. To prove this; consider the element $0 \neq b_1 \wedge \cdots \wedge b_p \wedge b_{\alpha_i} \in \bigwedge^{p+1}(\mathbf{n}^-)$. Since $\bar{\Phi}_1 + \cdots + \bar{\Phi}_p + \alpha_i = \langle \Phi \rangle + \alpha_i = \langle \Phi_w \rangle$ (see the proof of Lemma (3.4)), by [9, Theorem 8.5], $b_1 \wedge \cdots \wedge b_p \wedge b_{\alpha_i} = \mathbf{b}_{\Phi_w}$ (up to a scalar multiple). Hence the assertion, that $\{\bar{\Phi}_1, ..., \bar{\Phi}_p\} = \Phi$, follows.

Since $\alpha_i \notin \Phi_v$, we get

$$(\operatorname{ad} b_{\alpha_i})^{k_0-1}\mathbf{b}_{\Phi_v} \otimes (\operatorname{ad} a_{\alpha_i})^{k_0-1}\mathbf{a}_{\Phi_v} = z\mathbf{b}_{\Phi} \otimes \mathbf{a}_{\Phi} \qquad (\mathrm{I}_{17})$$

for some constant z (since, by Lemma (3.4), $\langle \Phi \rangle = \langle \Phi_w \rangle - \alpha_i = \langle \Phi_v \rangle + (k_0 - 1)\,\alpha_i$).

Consider the $sl(2)$ spanned by $\{e_i, f_i, h_i\}$. Let this $sl(2)$ act on $\bigwedge(\mathbf{g})$ by the adjoint representation. We claim that

$$(\operatorname{ad} e_i)\,\mathbf{b}_{\Phi_v} = 0 \qquad (\mathrm{I}_{18})$$

$$(\operatorname{ad} h_i)\,\mathbf{b}_{\Phi_v} = (k_0 - 1)\,b_{\Phi_v}. \qquad (\mathrm{I}_{19})$$

To prove (I_{18}), it suffices (since $\alpha_i \notin \Phi_v$) to observe that, for $\beta \in \Phi_v$, if $\beta - \alpha_i$ is a root then $\beta - \alpha_i \in \Phi_v$. (If $\alpha, \beta \in v\Delta_+ \cap \Delta_+$ such that $\alpha + \beta$ is again a root, then $\alpha + \beta \in v\Delta_+ \cap \Delta_+$.) Of course, (I_{19}) follows from the definition of k_0.

(I_{18}) and (I_{19}) in conjunction with [28, Theorem 1, p. IV-4] yield

$$(\operatorname{ad} e_i)^{k_0-1}(\operatorname{ad} f_i)^{k_0-1}(\mathbf{b}_{\Phi_v}) = (k_0 - 1!)^2\mathbf{b}_{\Phi_v}. \qquad (\mathrm{I}_{20})$$

Consider the pairing $\langle\ ,\ \rangle\colon \bigwedge^p(\mathbf{n}^-) \otimes \bigwedge^p(\mathbf{n}) \to \mathbb{C}$ (defined by the Killing form) $\mathbf{b} \otimes \mathbf{a} \mapsto \langle \mathbf{b}, \mathbf{a} \rangle$. By (I_{17}), we get

$$\langle (\operatorname{ad} b_{\alpha_i})^{k_0-1}\mathbf{b}_{\Phi_v}, (\operatorname{ad} a_{\alpha_i})^{k_0-1}\mathbf{a}_{\Phi_v} \rangle = z\langle \mathbf{b}_{\Phi}, \mathbf{a}_{\Phi} \rangle.$$

Using the invariance of $\langle\ ,\ \rangle$, this yields

$$(-1)^{k_0-1}\langle (\operatorname{ad} a_{\alpha_i})^{k_0-1}(\operatorname{ad} b_{\alpha_i})^{k_0-1}\mathbf{b}_{\Phi_v}, \mathbf{a}_{\Phi_v} \rangle = z$$

$$\text{(since } \langle \mathbf{b}_{\Phi}, \mathbf{a}_{\Phi} \rangle = 1). \qquad (\mathrm{I}_{21})$$

Further, $\{e_i, e_i\} = \{f_i, f_i\} = \langle e_i, f_i \rangle = 2/\sigma(\alpha_i, \alpha_i)$ and hence a_{α_i} (resp. b_{α_i}) can be taken to be $(\sigma(\alpha_i, \alpha_i)/2)\,e_i$ (resp. $(\sigma(\alpha_i, \alpha_i)/2)\,f_i$). Now (I_{20}) and (I_{21}) give $(-1)^{k_0-1}(k_0 - 1!)^2(\sigma(\alpha_i, \alpha_i)/2)^{k_0-1} = z$ (since $\langle \mathbf{b}_{\Phi_v}, \mathbf{a}_{\Phi_v} \rangle = 1$).

Putting together Lemmas (3.6) and (3.7), we get

(3.8.) COROLLARY. $z^w = (-1)^{l(w)-1}k_0$, *where* k_0 *is as defined in Lemma* (3.4).

Proof. Use the relation $\sigma(v\rho - k\alpha_i, v\rho - k\alpha_i) - \sigma(\rho, \rho) = k(k - k_0)$ $\sigma(\alpha_i, \alpha_i)$ (see the proof of Lemma (3.4)).

(3.9) *Proof of Theorem* (3.1). Denote by d^w the integral

$2^{2l(w)} \int_{U_w} \exp(2(w\rho - \rho)\, h(g))\, dg$. Write $w = r_i v$ such that $v < w$. By (a subsequent) Corollary (3.13), we have (for some $n^{w'}$)

$$\frac{s^v}{d^v} \cdot \frac{s^{r_i}}{d^{r_i}} - (-1)^{l(v)} \frac{s^w}{d^w} + \sum_{\substack{w' \neq w \text{ and} \\ l(w') = l(w)}} n^{w'} s^{w'} \in \operatorname{Im} d.$$

Assume, by induction, that $d^v = (4\pi)^{l(v)} \prod_{\gamma \in v^{-1}\Delta_- \cap \Delta_+} \sigma(\rho, \gamma)^{-1}$. Of course, $d^{r_i} = (4\pi)\, \sigma(\rho, \alpha_i)^{-1}$. By ($\mathrm{I}_{11}$) and Corollary (3.8), $d^v d^{r_i} = (-1)^{l(v)} d^w z^w = d^w \cdot k_0$. So,

$$\begin{aligned} d^w &= \frac{(4\pi)^{l(v)+1}}{k_0} \left(\prod_{\gamma \in v^{-1}\Delta_- \cap \Delta_+} \sigma(\rho, \gamma)^{-1} \right) \cdot \sigma(\rho, \alpha_i)^{-1} \\ &= \frac{(4\pi)^{l(v)+1}}{\sigma(\rho, v^{-1}\alpha_i)} \left(\prod_{\gamma \in v^{-1}\Delta_- \cap \Delta_+} \sigma(\rho, \gamma \right)^{-1} \\ &\qquad \left(\text{since } k_0 = 2\frac{\sigma(\rho, v^{-1}\alpha_i)}{\sigma(\alpha_i, \alpha_i)} \text{ and } \sigma(\rho, \alpha_i) = \frac{1}{2}\sigma(\alpha_i, \alpha_i)\right), \\ &= (4\pi)^{l(v)+1} \left(\prod_{\gamma \in w^{-1}\Delta_- \cap \Delta_+} \sigma(\rho, \gamma)^{-1} \right) \\ &\qquad (\text{since } w^{-1}\Delta_- \cap \Delta_+ = (v^{-1}\Delta_- \cap \Delta_+) \cup \{v^{-1}\alpha_i\}). \end{aligned}$$

Hence the proof of the theorem is complete modulo the following proposition due to Bernstein, Gelfand, and Gelfand [3, Theorem (3.17)]. (Though their proof is in the finite dimensional situation, the proof goes through in the infinite dim case without any change.)

Let ε^v denote the cohomology class of G/B (where G is the group associated to any, not necessarily symmetrizable, Kac–Moody algebra $\mathbf{g}(A_l)$ and B is standard Borel subgroup of G as defined in (1.2)) which is dual to the closure of the Schubert cell $Bv^{-1}\, B/B$. (Of course, the cell, being complex, is oriented. See (1.4).)

(3.10) Proposition. *Recall, from* (1.4), *that* χ_i $(1 \leqslant i \leqslant l)$ *is any element of* $\mathbf{h}^*$ *satisfying* $\chi_i(h_j) = \delta_{i,j}$. *For any simple reflection* r_i *and any element* $v \in W$,

$$\varepsilon^{r_i} \varepsilon^v = \sum_{v \to^{\gamma} w} \chi_i(\gamma^v)\, \varepsilon^w.$$

(*As in* [3], *the notation* $v \to^{\gamma} w$ *means that* $\gamma \in \Delta_+^{\text{re}}$ *with* $\sigma_\gamma v = w$, *and* $l(w) = l(v) + 1$, *where* $\sigma_\gamma \in W$ *denotes the reflection* $\sigma_\gamma(\chi) = \chi - \chi(\gamma^v)\, \gamma$, *for all* $\chi \in \mathbf{h}^*$. *Of course*, $\gamma^v \in \mathbf{h}$ *denotes the co-root defined explicitly by* $\gamma^v = uh_j$, *if* $u\alpha_j = \gamma$ *for any* $u \in W$.)

(3.11) *Remark.* In the notation of [3], our ε^w is the same as $P_{w^{-1}}$ and that is why we have v^v instead of $v^{-1}v^v$ as in [3].

As an immediate consequence of the proposition, we have

(3.12) COROLLARY. *Let $r_i v = w \in W$ be such that $l(w) = l(v) + 1$, then the coefficient of ε^w in $\varepsilon^{r_i}\varepsilon^v$ is* 1.

We get the following as a corollary of the above corollary.

(3.13) COROLLARY. *With $r_i v = w$ as above* (i.e., $l(w) = l(v) + 1$), *the coefficient of s^w/d^w in $(s^v/d^v)(s^{r_i}/d^{r_i})$ is $(-1)^{l(v)}$.*

Proof. By [23, Theorem 4.5] and [24, Theorem 1.6], there is an algebra isomorphism $[\int]$: $H^*(\mathbf{g}, \mathbf{h}) \to H^*(G/B, \mathbb{C})$, such that the cohomology class $(s^w/d^w) \in H^*(\mathbf{g}, \mathbf{h})$ maps onto $(-1)^{p(p-1)/2}\varepsilon^w$, where $p = l(w)$. (The sign $(-1)^{p(p-1)/2}$ appears because of a different orientation convention, on complex manifolds, in this paper.)

4. THE NIL HECKE RING R AND ITS "DUAL" Λ

Throughout this section $\mathbf{g} = \mathbf{g}(A_l)$ denotes an arbitrary (not necessarily symmetrizable) Kac–Moody Lie algebra, associated to a $l \times l$ generalized Cartan matrix A, with its Cartan subalgebra $\mathbf{h}$ and Weyl group W (1.1). Let $Q = Q(\mathbf{h}^*)$ denote the quotient field of the polynomial algebra $S = S(\mathbf{h}^*)$, i.e., Q is the field of all the rational functions on $\mathbf{h}$.

(4.1) *Ring structure on Q_W.* The group W operates as a group of automorphisms on the field Q. Let Q_W be the smash product of Q with the group algebra $\mathbb{C}[W]$. More specifically, Q_W is a right Q-module (under right multiplication) with a (free) basis $\{\delta_w\}_{w \in W}$ and the multiplicative structure is given by

$$(\delta_v q_v) \cdot (\delta_w q_w) = \delta_{vw}(w^{-1}q_v)\, q_w \qquad \text{for} \quad v, w \in W \text{ and } q_v, q_w \in Q. \quad (\mathrm{I}_{22})$$

Observe that though Q_W is an associative ring (with unity δ_e), it is not an algebra over Q, since $\delta_e Q = Q\delta_e$ is not central in Q_W.

Let $S_W \subset Q_W$ be defined in the same way as Q_W with S replacing Q. The ring Q_W (and S_W) admits an involutory anti-automorphism t, defined by

$$(\delta_w q)^t = \delta_{w^{-1}}(wq) \qquad \text{for} \quad w \in W \quad \text{and} \quad q \in Q. \qquad (\mathrm{I}_{23})$$

Of course, t is not right Q-linear. Clearly the left action of W on Q_W (given as the left multiplication by δ_w, for $w \in W$) is (right) Q-linear but the right action is not.

Now, for $i=1,\ldots,l$, consider the elements

$$x_i = x_{r_i} = -(\delta_{r_i}+\delta_e)\frac{1}{\alpha_i} = \frac{1}{\alpha_i}(\delta_{r_i}-\delta_e) \in Q_W \qquad (\mathrm{I}_{24})$$

where $r_i \in W$ is a simple reflection and α_i is the corresponding simple root. Inspired by [3], we have

(4.2) PROPOSITION. *Let $w \in W$ and let $w = r_{i_1}\cdots r_{i_n}$ be a reduced expression. Then the element $x_{i_1}\cdots x_{i_n} \in Q_W$ does not depend upon the choice of reduced expression of w. We define x_w as $x_{i_1}\cdots x_{i_n}$. We denote by $\bar{x}_w = (x_{w^{-1}})'$.*

Proof. For any $\chi \in \mathbf{h}^*$, we have

$$\chi x_i - \chi(\delta_{r_i}+\delta_e)\frac{1}{\alpha_i}$$

$$= -[\delta_{r_i}(r_i\chi)+\delta_e(\chi)]\frac{1}{\alpha_i}$$

$$= -(\delta_{r_i}+\delta_e)\frac{r_i\chi}{\alpha_i} + \delta_e\frac{r_i\chi-\chi}{\alpha_i}$$

$$\chi x_i = x_i(r_i\chi) - \delta_e\chi(h_i). \qquad (\mathrm{I}_{25})$$

We assume, by induction on n, that for any v with $l(v)<n$ and any two reduced expressions $v = r_{i_1}\cdots r_{i_p} = r_{j_1}\cdots r_{j_p}$ $(p = l(v))$, we have $x_{i_1}\cdots x_{i_p} = x_{j_1}\cdots x_{j_p}$ (which we denote by x_v) and

$$\chi x_v = x_v(v^{-1}\chi) - \sum_{u \xrightarrow{\nu} v} x_u\chi(v^v) \qquad \text{for any} \quad \chi \in \mathbf{h}^*. \qquad (\mathrm{I}_{26})$$

(See Proposition (3.10), for the notations $u \xrightarrow{\nu} v$ and v^v.)
Fix $w = vr_i$ with $v < w$. Now

$$\chi x_v x_i = \left[x_v(v^{-1}\chi) - \sum_{u' \xrightarrow{\nu'} v} x_{u'}\chi(v'^v)\right]x_i \qquad (\text{by } \mathrm{I}_{26}))$$

$$\chi x_v x_i = x_v x_i(r_i v^{-1}\chi) - x_v \cdot v^{-1}\chi(h_i) - \sum_{u' \xrightarrow{\nu'} v} x_{u'}x_i\chi(v'^v) \qquad (\text{by } \mathrm{I}_{25})). \qquad (\mathrm{I}_{27})$$

Further, by [6, Theorem 1.1],

$$\{u' \colon u' \xrightarrow{\nu'} v \text{ and } u'r_i > u'\} \xrightarrow{\sim} \{u \colon u \xrightarrow{\nu} w \text{ and } u \neq v\},$$

under the map $u' \mapsto u'r_i$ and, under this correspondence, $v' \mapsto v'$. Hence we get (by I_{27}), exchange condition [13, Lemma 3.11(c)] and the fact that $x_i^2 = 0$)

$$\chi x_v x_i = x_v x_i(w^{-1}\chi) - \sum_{u \xrightarrow{\nu} w} x_u \chi(\nu^{\vee}). \tag{I_{28}}$$

In particular, if $w = vr_i = v'r_j$ with $l(v) = l(v') < l(w)$, we have

$$\chi x_v x_i - x_v x_i(w^{-1}\chi) = \chi x_{v'} x_j - x_{v'} x_j(w^{-1}\chi). \tag{I_{29}}$$

By (a subsequent) Proposition (4.3)(c), we can write

$$x_v x_i = \sum_{w' < w} \delta_{w'} q_{w'} + \delta_w \prod_{\nu \in w\Delta_+ \cap \Delta} \nu^{-1} \tag{I_{30}}$$

and

$$x_{v'} x_j = \sum_{w' < w} \delta_{w'} \bar{q}_{w'} + \delta_w \prod_{\nu \in w\Delta_+ \cap \Delta_-} \nu^{-1} \tag{I_{31}}$$

for some $q_{w'}$ and $\bar{q}_{w'} \in Q$. Substituting (I_{30}) and (I_{31}) in (I_{29}), we get

$$\sum_{w' < w} \delta_{w'}(w'^{-1}\chi)(q_{w'} - \bar{q}_{w'}) = \sum_{w' < w} \delta_{w'}(w^{-1}\chi)(q_{w'} - \bar{q}_{w'}).$$

In particular, for any $w' < w$ and any $\chi \in \mathbf{h}^*$, we have $(w'^{-1}\chi - w^{-1}\chi)(q_{w'} - \bar{q}_{w'}) = 0$. But $W \to \mathrm{Aut}(\mathbf{h}^*)$ being a faithful representation, we get $q_{w'} = \bar{q}_{w'}$ for all $w' < w$. This proves, by (I_{30}) and (I_{31}), that $x_v x_i = x_{v'} x_j$ and hence, by (I_{28}), the induction is complete. ∎

As a corollary, we get

(4.3) PROPOSITION. (a) $x_v \cdot x_w = x_{vw}$, *if* $l(vw) = l(v) + l(w)$ *and* $x_v \cdot x_w = 0$ *otherwise.*

(b) $\chi \cdot x_w = x_w(w^{-1}\chi) - \sum_{u \xrightarrow{\nu} w} x_u \chi(\nu^{\vee})$, *for any* $\chi \in \mathbf{h}^*$ *and* $w \in W$.

(c) *Write*

$$x_{v^{-1}} = \sum_w c_{v,w} \delta_{w^{-1}} \qquad \textit{for some (unique) } c_{v,w} \in Q. \tag{I_{32}}$$

Then (c_1) $c_{v,w} = 0$, *unless* $w \leqslant v$, *and* (c_2) $c_{v,v} = \prod_{\nu \in v^{-1}\Delta_- \cap \Delta_+} \nu^{-1}$. *In particular,* $c_{v,v} \neq 0$.

Proof. (a) Follows from Proposition (4.2) and the exchange condition [13, Lemma 3.11(c)] together with the fact that $x_i^2 = 0$.

(b) is nothing but (I_{26}).

(c_1) follows from [6, Theorem 1.1]. Let $v = r_{i_1} \cdots r_{i_n}$ be a reduced expression. To prove (c_2); observe that by the definition of $x_{v^{-1}}$, $c_{v,v} = [\alpha_{i_n}(r_{i_n}\alpha_{i_{n-1}}) \cdots (r_{i_n} \cdots r_{i_2}\alpha_{i_1})]^{-1}$. But then, by [26, Sect. 2], (c_2) follows. ∎

(4.4) *Remark.* Observe that (c_2) does not depend upon Proposition (4.2), in fact we use it to prove Proposition (4.2).

The elements $\{\delta_w\}_{w \in W}$ are a right (as well as left) Q-basis of Q_W. But also

(4.5) COROLLARY. *Define the matrix $C = (c_{v,w})_{v,w \in W}$, where $c_{v,w}$ is defined in (I_{32}). By Proposition (4.3)(c), C is a "lower triangular" matrix with non-zero diagonal entries and hence, in particular, $\{x_w\}_{w \in W}$ is a right (as well as left) Q-basis of Q_W.*

Now, clearly, Q has the structure of a left Q_W-module, defined explicitly by

$$(\delta_w q)\, q' = w(qq') \qquad \text{for} \quad w \in W \quad \text{and} \quad q, q' \in Q. \tag{I_{33}}$$

Our main result centers around the subring $R \subset Q_W$, defined by

$$R = \{x \in Q_W : x \cdot S \subset S\}.$$

Obviously $S_W \subset R$. Furthermore, one can easily see that x_i (and hence x_w, for any $w \in W$) belong to R. By applying the involution t, one gets another subring R^t of Q_W. One has the following crucial structure theorem for R. The proof of (a) below can be simplified in the finite case using the theory of "harmonics."

(4.6) THEOREM. (a) *R is free as a right (or left) S-module. In fact the elements $\{x_w\}_{w \in W}$ form a right (or left) S-basis of R. In particular, any $x \in R$ can be uniquely written as*

$$x = \sum x_w p_w \qquad \text{with some } p_w \in S.$$

(b) *Furthermore, one has $R \cap R^t = S_W$.*

(4.7) *Remark.* Note that R is a finitely generated ring over $\mathbb{C}$, since it is generated by $\{x_i\}_{1 \leqslant i \leqslant l}$ and S.

We need a few lemmas to prove the above theorem.

(4.8) LEMMA. *Let $e \neq w \in W$ be such that w fixes pointwise a hyperplane Π in* **h**. *Then $w = v r_i v^{-1}$, for some $v \in W$ and for some simple reflection r_i. In particular, Π is the real-root plane* $\operatorname{Ker}(v\alpha_i)$.

Proof. Step I. We first prove that, as an element of Aut $\mathbf{h}$, w is semi-simple. Since w fixes a hyperplane in $\mathbf{h}$, the only other possibility is that $w \in \operatorname{Aut} \mathbf{h}$ is unipotent and, in fact, $(w-1)^2=0$, i.e., $w^2+1=2w$. Multiplying by w^{-1}, we get $w-1=1-w^{-1}$. Considering the (dual) representation of W in h^* and evaluating at ρ, we get $-(\rho - w\rho) = \rho - w^{-1}\rho$. But, by [9, Proposition 2.5], $\rho - w^{-1}\rho$ (and $\rho - w\rho$) are both sums of $l(w)$ positive roots. Since, by assumption, $w \neq e$, we get a contradiction. So w is semi-simple.

Step II. We want to show that Π is a real-root plane, i.e., $\Pi = \operatorname{Ker}(v\alpha_i)$, for some $v \in W$ and simple root α_i.

We need some notations. Let $\mathbf{h}_{\mathbb{R}}$ be the real points of $\mathbf{h}$. (In the Definition (1.1), one can take any field k of characteristic zero in place of $\mathbb{C}$ and define $\mathbf{g}_k$, $\mathbf{h}_k$, etc.) As in [13, Sect. 3.12], define

$$C = \{h \in \mathbf{h}_{\mathbb{R}} : \alpha_i(h) \geqslant 0, \text{ for all } 1 \leqslant i \leqslant l\}$$

$$C^0 = \{h \in \mathbf{h}_{\mathbb{R}} : \alpha_i(h) > 0 \text{ for all } 1 \leqslant i \leqslant l\}.$$

Set

$$X = \bigcup_{v \in W} v \cdot C \qquad \text{and} \qquad X^0 = \bigcup_{v \in W} v \cdot C^0.$$

The Tits cone X is a convex cone by [13, Proposition 3.12]. Let $\Pi_{\mathbb{R}} = \Pi \cap \mathbf{h}_{\mathbb{R}}$, Π being a hyperplane, $\mathbf{h}_{\mathbb{R}} \backslash \Pi_{\mathbb{R}}$ has exactly two connected components $\Pi_{\mathbb{R}}^+$ and $\Pi_{\mathbb{R}}^-$. Since w is semi-simple and W leaves an integral lattice in $\mathbf{h}_{\mathbb{R}}$ stable, the only eigenvalue of $w \in \operatorname{Aut}(\mathbf{h}_{\mathbb{R}})$, different from 1, is -1. Since X^0 is W-stable, it is easy to see that $\Pi_{\mathbb{R}}^+ \cap X^0$ and $\Pi_{\mathbb{R}}^- \cap X^0$ are both non-empty. Choose a point h^+ (resp. h^-) $\in \Pi_{\mathbb{R}}^+ \cap X^0$ (resp. $\Pi_{\mathbb{R}}^- \cap X^0$). The line joining h^+ and h^- intersects the plane $\Pi_{\mathbb{R}}$ in a point (say) h^0. Since $\Pi_{\mathbb{R}}^+ \cap X^0$ and $\Pi_{\mathbb{R}}^- \cap X^0$ are both open in $\mathbf{h}_{\mathbb{R}}$ (in the Hausdorff topology) and X is convex, there is an open subset N (containing h^0) of $\Pi_{\mathbb{R}}$ contained in X. In particular, X being a cone, $\Pi_{\mathbb{R}}$ itself is contained in X. Further, since any point of X^0 has no isotropy with respect to the W-action [13, Proposition 3.12(a)], we get that

$$\Pi_{\mathbb{R}} \subset X \backslash X^0 = \bigcup_{\substack{v \in W, \\ 1 \leqslant i \leqslant l}} v(\operatorname{Ker} \alpha_{i|\mathbf{h}_{\mathbb{R}}}).$$

Since Π is a hyperplane, we get that $\Pi = \operatorname{Ker}(v\alpha_i)$, for some $v \in W$ and some simple root α_i, as required.

Step III. Finally, we want to show that $w = v r_i v^{-1}$. The element $v^{-1} w v$ has fixed plane $v^{-1}\Pi$, which is the same as $\operatorname{Ker} \alpha_i$ by Step II. Choose a point $h \in \operatorname{Ker} \alpha_{i|\mathbf{h}_{\mathbb{R}}}$, such that $\alpha_j(h) > 0$ for all $j \neq i$. By [13, Proposition 3.12(a)], the only non-trivial isotropy element (with respect to the W-action) at h is r_i. This shows that $v^{-1}wv = r_i$, proving the lemma.

(4.9) LEMMA. *Let p be an irreducible polynomial $\in S$ and let*

$$\left(\sum_{l(w)\leqslant k} p_w x_w\right)\cdot S \subset pS \tag{R}$$

where $p_w \in S$ (for all w), any p_w is either 0 or co-prime to p and $p_w \neq 0$ for some w of length k. Then p is a real root.

Proof. Rewrite $x = \sum_{l(w)\leqslant k} p_w x_w = \sum_{l(w)\leqslant k} q_w \delta_w$, where $q_w \in Q$. In fact, q_w has in its denominator only products of real roots. By Proposition (4.3)(c), for $l(w) = k$, we have $q_w = p_w \cdot c_{w^{-1},w^{-1}}$, where $c_{w^{-1},w^{-1}} = \prod_{v\in w\Delta_-\cap\Delta_+} v^{-1}$.

Define $V = \bigcup_{v\in W, v\neq e} \operatorname{Ker}(v-1)$ ($v-1$ acting on $\mathbf{h}$). We first prove that $Z(p) \subset V$, where $Z(p)$ denotes the zero set of p. If not, pick any $h_0 \in Z(p)\backslash V$. In particular h_0 has no W-isotropy. Pick w_0 of maximal length k such that $p_{w_0} \neq 0$. There exists a polynomial $p_0 \in S$ such that $p_0(w_0^{-1}h_0) = 1$ and $p_0(w^{-1}h_0) = 0$ for all those (finite in number) $w \neq w_0$ satisfying $q_w \neq 0$. Evaluating $x \cdot p_0$ at h_0, we get from (R), $p_{w_0}(h_0)\, c_{w_0^{-1},w_0^{-1}}(h_0) = 0$. (Since h_0 has no W-isotropy, $c_{w_0^{-1},w_0^{-1}}(h_0)$ makes sense and, of course, is non-zero.) Hence $p_{w_0}(h_0) = 0$, i.e., p divides p_{w_0}, which is a contradiction to the assumption. So $Z(p) \subset V$.

Since p is irreducible, we get that $Z(p) \subset \operatorname{Ker}(v-1)$, for some $v \in W$ and moreover $Z(p)$ being a hyper surface in h, $\operatorname{Ker}(v-1)$ is a hyperplane. The lemma follows now by Lemma (4.8).

(4.10) LEMMA. *Let $\{p_w\}_{l(w)\leqslant k}$ be polynomials $\in S$ such that $(\sum_{l(w)\leqslant k} p_w x_w)\cdot S \subset \alpha_i S$, for some simple root α_i. Then α_i divides all the p_w's.*

Proof. Denote by $x = (1/\alpha_i) \sum p_w x_w \in Q_W$. Write $x = x^+ + x^-$, where x^+ (resp. x^-) $= \frac{1}{2}(x + \delta_{r_i}x)$ (resp. $\frac{1}{2}(x - \delta_{r_i}x)$). Since x^+ again satisfies $x^+ \cdot S \subset S$ and, by Proposition (4.3), x^+ is of the form $(1/\alpha_i) \sum p'_v x_v$, for some $p'_v \in S$ (a similar statement holds good for x^-), we can assume that either $\delta_{r_i}x = x$ or $\delta_{r_i}x = -x$. Write

$$\alpha_i x = \sum_{l(w)\leqslant k} p_w x_w = \sum_{l(w)\leqslant k} q_w \delta_w, \tag{I$_{34}$}$$

where, as in the proof of the previous lemma, we have

$$q_w = p_w \cdot c_{w^{-1},w^{-1}} \qquad \text{if} \quad l(w) = k. \tag{I$_{35}$}$$

Fix w_0 of length k such that $p_{w_0} \neq 0$ and rewrite (I_{34})

$$\alpha_i x = x_0 + q_{w_0}\delta_{w_0} + q_{r_i w_0}\delta_{r_i w_0},$$

where

$$x_0 = \sum_{w \notin \{r_i w_0, w_0\}} q_w \delta_w. \tag{I_{36}}$$

Also fix a point $h_0 \in \operatorname{Ker} \alpha_i \cap C$ (C is defined in the proof of Lemma (4.8)) such that h_0 does not lie on any other real-root plane and choose a polynomial $p_0 \in S$ such that $p_0(w_0^{-1} h_0) = 1$ and p_0 at $w^{-1} h_0$ has a "deep" zero for any (finite in number) w such that $q_w \neq 0$ and $w \neq w_0, r_i w_0$. (Since W-isotropy at h_0 is precisely $\{1, r_i\}$, this is possible.)

Case I. $\delta_{r_i} x = x$. In this case, by (I_{36}), we have $-(r_i q_{w_0}) = q_{r_i w_0}$. In particular, $r_i w_0 < w_0$. Choose a reduced expression $w_0 = r_i \cdot r_{i_1} \cdots r_{i_{k-1}}$ (since $r_i w_0 < w_0$, this is possible). By (I_{35}), (I_{36}), and Proposition (4.3)(c), we have

$$\alpha_i x = x_0 + \frac{p_{w_0}}{\alpha_i(r_i \beta)} \delta_{w_0} + \frac{r_i \cdot p_{w_0}}{\alpha_i \cdot \beta} \delta_{r_i w_0}, \text{ where } \beta = \alpha_{i_1} \cdots (r_{i_1} \cdots r_{i_{k-2}} \alpha_{i_{k-1}}). \tag{I_{37}}$$

Evaluating $(\alpha_i^2 x) \cdot p_0$ at h_0 (since $\alpha_i(h_0) = 0$, $p_0(w_0^{-1} h_0) = 1$, and p_0 has "deep" zero at points other than $w_0^{-1} h_0$), we get, by (I_{37}), $p_{w_0}(h_0) = 0$. Hence α_i divides p_{w_0}.

Case II. $\delta_{r_i} x = -x$: In this case, we have $r_i q_{w_0} = q_{r_i w_0}$. In particular, again we have $r_i w_0 < w_0$. Analogous to (I_{37}), we get

$$\alpha_i x = x_0 + \frac{p_{w_0}}{\alpha_i(r_i \beta)} \delta_{w_0} - \frac{r_i p_{w_0}}{\alpha_i \cdot \beta} \delta_{r_i w_0},$$

where β is the same as in (I_{37}).

Considering $\alpha_i x(w_0^{-1} \alpha_i \cdot p_0)$ and evaluating at h_0, we get again $p_{w_0}(h_0) = 0$, i.e., α_i divides p_{w_0} in this case as well. This proves the lemma. ∎

(4.11) *Proof of Theorem* (4.6)(a). Let $x \in R$. By Corollary (4.5), we can write $x = (1/p) \sum_{l(w) \leqslant k} p_w x_w$, for some p, $\{p_w\}_w \in S$. We want to prove that p divides p_w, for every w. We can assume, of course, that p is irreducible. By Lemma (4.9), if p does not divide some p_w then p has to be a real root (say) $v\alpha_i$, for some $v \in W$ and simple root α_i. Since $\delta_{v^{-1}} R = R$ and $\delta_v(\sum_w S x_w) = \sum_w S x_w$ (as is easy to see), we can assume that $p = \alpha_i$. But then Lemma (4.10) proves that $x \in \sum_w Sx_w$. The rest is clear from Corollary (4.5) together with Proposition (4.3).

(b) Fix $x \in R \cap R'$. Since $x \in R$, we can write, by (a), $x = \sum p_w x_w$, for some $p_w \in S$. Express $x = \sum q_w \delta_w$, where $q_w \in Q$. Upon multiplying by a suitable polynomial, we can assume, without loss of generality, that all the q_w's have only one fixed real root (say) $v\alpha_i$ in their denominators. Further since $\delta_{v^{-1}} R = R$, $\delta_{v^{-1}} \cdot R' = R'$, and $\delta_{v^{-1}} \cdot S_W = S_W$, we can further assume that all the q_w's have only α_i in their denominators, i.e.,

$$x = \frac{1}{\alpha_i} \sum \bar{p}_w \delta_w \qquad \text{for some } \bar{p}_w \in S. \tag{I_{38}}$$

We want to prove that all the $\bar{p}_w$'s are divisible by α_i. Analogous to the proof of Lemma (4.10), considering $x = \frac{1}{2}(x + \delta_{r_i} x) + \frac{1}{2}(x - \delta_{r_i} x)$, we can assume that either $\delta_{r_i} x = x$ or $\delta_{r_i} x = -x$.

Case I. $\delta_{r_i} x = x$. In this case, by (I_{38}), we have

$$\bar{p}_e = -r_i(\bar{p}_{r_i}). \tag{I_{39}}$$

By (I_{38}), we get (taking t)

$$x^t = \frac{\bar{p}_e}{\alpha_i} \delta_e - \frac{r_i \bar{p}_{r_i}}{\alpha_i} \delta_{r_i} + \sum_{w \neq e, r_i} w^{-1} \left(\frac{\bar{p}_w}{\alpha_i} \right) \delta_{w^{-1}}. \tag{I_{40}}$$

Fix $h_0 \in \operatorname{Ker} \alpha_i$, such that h_0 does not lie on any other real-root plane. Choose a function $p_0 \in S$ such that $p_0(h_0) = 1$ and $p_0(wh_0) = 0$ if $w \neq e, r_i$ and $\bar{p}_w \neq 0$. (This is possible because, by the choice, isotropy at h_0 is precisely $\{e, r_i\}$.) Considering $\alpha_i(x^t \cdot p_0)$ and evaluating at h_0 we get (by (I_{39}) and (I_{40})) $\bar{p}_e(h_0) = 0$, i.e., α_i divides $\bar{p}_e$. To prove that α_i divides $\bar{p}_w$ for general w, we can consider $x^t \delta_w$ and argue as before.

Case II. $\delta_{r_i} x = -x$. In this case, by (I_{38}), we have

$$\bar{p}_e = r_i \bar{p}_{r_i}. \tag{I_{41}}$$

Fix h_0 and p_0 as in the previous case. Considering $\alpha_i(x \cdot p_0)$ and evaluating at h_0 we get (by (I_{38}) and (I_{41})) $\bar{p}_e(h_0) = 0$. So again α_i divides $\bar{p}_e$. This completes the proof of (b) part as well. ∎

(4.12) Definition. The elements $\{x_w\}$ have much in common with the standard basis of a Hecke ring. However, $x_{r_i}^2 = x_i^2 = 0$. This and a further nilpotence condition, in its action on Λ (Sect. 4.19), persuade us to refer to R as a *nil Hecke ring*. A departure from usual conditions is that S is not central in R.

We describe the left action of the Weyl group W in terms of $\bar{x}_w$ basis ($\bar{x}_w$ is defined in Proposition (4.2)).

(4.13) PROPOSITION. *Fix a simple reflection $r_i \in W$. Then*

$$\delta_{r_i} \bar{x}_w = -\bar{x}_w \qquad \textit{if} \quad r_i w < w \textit{ and}$$

$$= \bar{x}_{r_i w}(w^{-1}\alpha_i) - \bar{x}_w + \sum_{v \to^v r_i w} \bar{x}_v \alpha_i(v^v) \qquad \textit{otherwise.}$$

Proof.

$$\delta_{r_i} \bar{x}_w = (\delta_{r_i} - \delta_e) \frac{1}{\alpha_i} (\alpha_i \bar{x}_w) + \bar{x}_w$$

$$= \bar{x}_{r_i} \left[\bar{x}_w(w^{-1}\alpha_i) - \sum_{v' \to^{v'} w} \bar{x}_{v'} \alpha_i(v'^{v'}) \right] + \bar{x}_w \qquad \text{by} \quad (\mathrm{I}_{26}).$$

Case I. $r_i w < w$. In this case only v' with the property that $v' \to w$ and $v' \to r_i v'$ is $r_i w$. Hence, by Proposition (4.3), the above sum reduces to $-2\bar{x}_w + \bar{x}_w = -\bar{x}_w$.

Case II. $r_i w > w$. In this case,

$$\delta_{r_i} \bar{x}_w = \bar{x}_{r_i w}(w^{-1}\alpha_i) + \bar{x}_w - \sum_{\substack{v' \text{ with} \\ v' \to^{v'} w \text{ and} \\ v' \to r_i v'}} \bar{x}_{r_i v'} \alpha_i(v'^{v'}). \qquad (\mathrm{I}_{42})$$

Further, as in the proof of Proposition (4.2), it is easy to see that $\{v' : v' \to^{v'} w \text{ and } v' \to r_i v'\} \xrightarrow{\sim} \{v : v \to^v r_i w \text{ and } v \neq w\}$, under the map $v' \mapsto r_i v'$. Moreover, under this map, v' corresponds with $r_i v'$. (It is clear that $r_i(v')$ is a positive root.) Hence, by (I_{42}), we get the required result. ∎

(4.14) *Co-product structure on Q_W.* Let $Q_W \otimes_Q Q_W$ be the tensor product, considering both the copies of Q_W as right Q-modules. Define the diagonal map $\Delta: Q_W \to Q_W \otimes_Q Q_W$, by

$$\Delta(\delta_w q) = \delta_w q \otimes \delta_w = \delta_w \otimes \delta_w q \qquad \text{for} \quad w \in W \text{ and } q \in Q. \qquad (\mathrm{I}_{43})$$

Δ is clearly right Q-linear. Moreover, it is easy to see that the co-product Δ is associative and commutative with a co-unit $\varepsilon: Q_W \to Q$, defined by $\varepsilon(\delta_w q) = q$.

We introduce an *associative product structure*, denoted by $\odot$, in $Q_W \otimes_Q Q_W$, so that Δ is a ring homomorphism. Define

$$(\delta_v q_v \otimes \delta_w q_w) \odot (\delta_{v'} q_{v'} \otimes \delta_{w'} q_{w'}) = \delta_{v'(w')^{-1} vw'} q_{v'}(w'^{-1} q_v) \otimes \delta_{ww'}(w'^{-1} q_w)\, q_{w'}.$$

(I_{44})

In the next proposition, we describe the diagonal map Δ in terms of $\{\bar{x}_w\}$ basis.

(4.15) PROPOSITION. *For any $w \in W$, we have*

$$\Delta(\bar{x}_w) = \sum_{u,v \leqslant w} \bar{x}_u \otimes \bar{x}_v(p^w_{u,v})$$

for some homogeneous polynomials $p^w_{u,v} \in S$ of degree $l(u)+l(v)-l(w)$. In particular $p^w_{u,v}=0$ unless $l(u)+l(v) \geqslant l(w)$.

Proof. We prove the proposition by induction on $l(w)$. For $w=r_i$, we have

$$\Delta(\bar{x}_{r_i}) = \delta_{r_i}\left(\frac{1}{\alpha_i}\right) \otimes \delta_{r_i} - \delta_e\left(\frac{1}{\alpha_i}\right) \otimes \delta_e$$

$$= (\delta_{r_i}-\delta_e)\frac{1}{\alpha_i} \otimes (\delta_{r_i}-\delta_e) + \delta_e \otimes (\delta_{r_i}-\delta_e)\frac{1}{\alpha_i}$$

$$+ (\delta_{r_i}-\delta_e)\frac{1}{\alpha_i} \otimes \delta_e$$

$$\Delta(\bar{x}_{r_i}) = \bar{x}_{r_i} \otimes \bar{x}_{r_i}(\alpha_i) + \delta_e \otimes \bar{x}_{r_i} + \bar{x}_{r_i} \otimes \delta_e. \tag{I$_{45}$}$$

Now take arbitrary w and write $w=w'r_i$, for some simple reflection r_i so that $w'<w$. Since Δ is multiplicative, we have $\Delta(\bar{x}_w)=\Delta(\bar{x}_{w'}) \odot \Delta(\bar{x}_{r_i})$,

$$\Delta(\bar{x}_w) = \left[\sum_{u',v' \leqslant w'} \bar{x}_{u'} \otimes \bar{x}_{v'}\, p^{w'}_{u',v'}\right] \odot \left[\delta_{r_i} \otimes \delta_{r_i}\left(\frac{1}{\alpha_i}\right) - \delta_e \otimes \delta_e\left(\frac{1}{\alpha_i}\right)\right] \tag{I$_{46}$}$$

(by induction hypothesis).

It is easily seen, from (I$_{44}$), that

$$(x \otimes y) \odot (\delta_{w_0} \otimes \delta_{w_0} q) = x \cdot \delta_{w_0} \otimes y \cdot \delta_{w_0} q$$

for any $x, y \in Q_W$, $w_0 \in W$, and $q \in Q$.

Hence from (I$_{46}$), we get

$$\Delta(\bar{x}_w) = \sum_{u',v' \leqslant w'} \bar{x}_{u'}\delta_{r_i} \otimes \bar{x}_{v'}\delta_{r_i}(r_i p^{w'}_{u',v'})\frac{1}{\alpha_i} - \sum_{u',v' \leqslant w'} \bar{x}_{u'} \otimes \bar{x}_{v'}\, p^{w'}_{u',v'}\left(\frac{1}{\alpha_i}\right)$$

$$= \sum_{u',v' \leqslant w'} \bar{x}_{u'}(\delta_{r_i}-\delta_e)\frac{1}{\alpha_i} \otimes \bar{x}_{v'}(\delta_{r_i}-\delta_e)\frac{1}{\alpha_i}(r_i p^{w'}_{u',v'})\,\alpha_i + \sum_{u',v' \leqslant w'} \bar{x}_{u'}$$

$$\otimes \bar{x}_{v'}(\delta_{r_i}-\delta_e)\frac{1}{\alpha_i}(r_i p^{w'}_{u',v'}) + \sum_{u',v' \leqslant w'} \bar{x}_{u'}(\delta_{r_i}-\delta_e)\frac{1}{\alpha_i}$$

$$\otimes \bar{x}_{v'}(r_i p^{w'}_{u',v'}) + \sum_{u',v' \leqslant w'} \bar{x}_{u'} \otimes \bar{x}_{v'}\left(\frac{r_i p^{w'}_{u',v'} - p^{w'}_{u',v'}}{\alpha_i}\right)$$

$$\Delta(\bar{x}_w) = \sum_{\substack{u',v' \leqslant w' \text{ with} \\ u' < u'r_i \text{ and} \\ v' < v'r_i}} (\bar{x}_{u'r_i} \otimes \bar{x}_{v'r_i}(r_i p^{w'}_{u',v'})\, \alpha_i$$

$$+ \sum_{\substack{u',v' \leqslant w' \text{ with} \\ v' < v'r_i}} \bar{x}_{u'} \otimes \bar{x}_{v'r_i}(r_i p^{w'}_{u',v'}) + \sum_{\substack{u',v' \leqslant w' \text{ with} \\ u' < u'r_i}} \bar{x}_{u'r_i} \otimes \bar{x}_{v'}(r_i p^{w'}_{u',v'})$$

$$- \sum_{u',v' \leqslant w'} \bar{x}_{u'} \otimes \bar{x}_{v'} E_{\alpha_i}(p^{w'}_{u',v'}) \qquad \text{(by Proposition (4.3))} \qquad (\mathrm{I}_{47})$$

where E_{α_i} is the classical Bernstein–Gel'fand–Gel'fand operator [3, Sect. 1] acting on S and defined by $E_{\alpha_i}(p) = (p - r_i p)/\alpha_i$, for any $p \in S$.

The proposition follows now by observing that E_{α_i}: $S \to S$ decreases the degree by 1 and further using [6, Theorem 1.1].

(4.16) *Remark.* We will determine $\{p^w_{u,v}\}$ more specifically later in (4.31) and (4.32). The fact that Proposition (4.13) (resp. Proposition (4.15)) gives the Weyl group action (resp. the cup product) on the cohomology ring $H^*(G/B)$ would be clear in the next section. ▌

Now we dualize the concepts (and results) introduced (proved) so far in this section. These dual objects will play an important role in determining various structures on the cohomology of infinite dimensional flag varieties.

(4.17) *The algebra Ω.* Regarding Q_W as a right Q-module, let $\Omega = \mathrm{Hom}_Q(Q_W, Q)$. Since any $\psi \in \Omega$ is determined by its restriction to the (right) Q-basis $\{\delta_w\}_{w \in W}$ and conversely, we can (and often will) regard Ω as the Q-module of *all* the functions: $W \to Q$ with pointwise addition and scalar multiplication, i.e.,

$$(q\psi)\, w = q \cdot \psi(w) \qquad \text{for} \quad q \in Q,\ \psi \in \Omega, \text{ and } w \in W. \qquad (\mathrm{I}_{48})$$

Furthermore, Ω inherits a (commutative) Q-algebra structure with the product as pointwise multiplication of functions on W. In fact, this multiplication is precisely the one obtained by dualizing the Q-linear co-multiplication Δ in Q_W (see (I_{43})).

More subtly, Ω also has the structure of a left Q_W-module defined by

$$(x \cdot \psi)\, y = \psi(x^t \cdot y) \qquad \text{for} \quad x, y \in Q_W \text{ and } \psi \in \Omega. \qquad (\mathrm{I}_{49})$$

Moreover the action is Q-linear. In particular, we have the *Weyl group action* as well as the *Hecke-type operators* A_w ($w \in W$) on Ω, defined by

$$w\psi = \delta_w \cdot \psi \qquad (\mathrm{I}_{50})$$

$$A_w \psi = x_w \cdot \psi \qquad \text{for} \quad w \in W \text{ and } \psi \in \Omega. \qquad (\mathrm{I}_{51})$$

(Recall that x_w is defined in Proposition (4.2)). Taking w to be one of the simple reflections r_i, from (I_{51}), we get

$$(A_{r_i}\psi)\,v = \frac{\psi(r_i v) - \psi(v)}{v^{-1}\alpha_i} \quad \text{for} \quad \psi \in \Omega \text{ and } v \in W. \tag{I_{52}}$$

We would generally abbreviate A_{r_i} by A_i.

(4.18) *Remark.* Observe that Ω is a Q-module (under (I_{48})) as well as a left Q_W-module (under (I_{49})). Further, Q injects into Q_W by $q \to \delta_e q$ (for $q \in Q$); in particular, Ω inherits a Q-module structure (from the restriction of Q_W-module structure). But these two Q-structures are different, in general. Whenever, we refer to Ω as a Q-module, we would mean the first Q-action. ▮

We define the following important subring Λ of Ω.

(4.19) Definition. $\Lambda = \{\psi \in \Omega \colon \psi(R') \subset S$ and $\psi(\bar{x}_w) = 0$ for all but a finite number of $w \in W\}$. (R is defined just before Theorem (4.6) and $\bar{x}_w$ is defined in Proposition (4.2).)

One has the following

(4.20) Proposition. (a) *Λ is a S-subalgebra of Ω.*

(b) *Λ is a free S-module. In fact $\{\xi^w\}_w$ is an S-basis of Λ, where $\xi^w \in \Omega$ is defined (uniquely) by $\xi^w(\bar{x}_v) = \delta_{v,w}$, for $w, v \in W$.*

(c) *Λ is a stable under the left action of $R \subset Q_W$.*

Proof. To prove (a), use Proposition (4.15). Since, by Theorem (4.6)(a), $\{\bar{x}_w\}_w$ is a right S-basis of R', we get (b). Again using Theorem (4.6)(a) and Proposition(4.3)(b), (c) follows. ▮

The matrix D (defined below) is very basic to our paper.

(4.21) *The matrix D.* Define the matrix $D = (d_{v,w})_{v,w \in W}$ by $d_{v,w} = \xi^v(w)$. The relevance of D to the cup product and Weyl group action on the cohomology of infinite dimensional flag varieties will be clear in the next section.

(4.22) *Remark.* In the finite case, the matrix D can be extracted from [3, Theorem 5.9] (see Sect. 6 of this paper).

(4.23) Definition. Let $\mathscr{B} = \mathscr{B}_W$ be the space of all the functions B: $W \times W \to Q$, with the property that there exists a $d_B \geqslant 0$ such that $B(v, w) = 0$ whenever $l(v) - l(w) > d_B$. $\mathscr{B}$ is an associative algebra over Q

under pointwise addition and convolution as multiplication (i.e., $B_1 \cdot B_2(v, w) = \sum_u B_1(v, u)\, B_2(u, w)$, for $B_1, B_2 \in \mathscr{B}$).

We can think of $\mathscr{B}$ as an appropriate subspace of all the $W \times W$ matrices over Q. Under this identification, the multiplication in $\mathscr{B}$ is nothing but the matrix multiplication.

We collect various properties of $\{\xi^w\}$ in

(4.24) PROPOSITION. *For any $v, w \in W$, we have*

(a) $\xi^v(w) = d_{v,w} = 0$, *unless* $v \leqslant w$ *and* $\xi^w(w) = d_{w,w} = \prod_{\gamma \in w^{-1}\Delta_- \cap \Delta_+} \gamma$. *In particular,* $d_{w,w} \neq 0$ *and the matrix D (defined in* **(4.21)** *belongs to* $\mathscr{B}$. *In fact, it is "upper triangular" with non-zero diagonal entries and hence invertible.*

(b) $A_i \xi^w = \xi^{r_i w}$ *if* $r_i w < w$, $=0$ *otherwise.*

(c) $\xi^{r_i}(w) = \chi_i - w^{-1}\chi_i$ (χ_i *is defined in* (1.4)).

(d) $\xi^v(w)$ *is a homogeneous polynomial of degree* $l(v)$.

(e) $C^t = D^{-1}$, *where the matrix C is defined in Corollary* (4.5), C^t *denotes the transposed matrix, and* D^{-1} *denotes the inverse of the element* $D \in \mathscr{B}_W$.

(f) $\xi^u \xi^v = \sum_{u,v \leqslant w} p^w_{u,v} \xi^w$, *where* $p^w_{u,v}$ *is defined in Proposition* (4.15). *We recall that* $p^w_{u,v}$ *is a homogeneous polynomial of degree* $l(u) + l(v) - l(w)$.

(g) $r_i \xi^w = \xi^w$ *if* $r_i w > w$, $= (-w^{-1}\alpha_i)\, \xi^{r_i w} + \xi^w - \sum_{r_i w \to' v} \alpha_i(v^\vee)\, \xi^v$ *otherwise.*

(4.25) *Remark.* We will give a characterization of the matrix D in Proposition (5.5).

Proof. (a) Assume that $v \nleqslant w$ and assume further, by induction, that for any $u < w$, we have $\xi^v(u) = 0$. By Proposition (4.3)(c), we can write

$$\delta_{w^{-1}} = \Bigg[\prod_{\gamma \in w^{-1}\Delta_- \cap \Delta_+} \gamma \Bigg] x_{w^{-1}} + \sum_{u < w} q_u \delta_{u^{-1}} \qquad \text{for some} \quad q_u \in Q. \quad (\mathrm{I}_{53})$$

Taking t, we get

$$\delta_w = x^t_{w^{-1}} \Bigg[\prod_{\gamma \in w^{-1}\Delta_- \cap \Delta_+} \gamma \Bigg] + \sum_{u < w} \delta_u q_u. \qquad (\mathrm{I}_{54})$$

This proves (a).

(b) Follows by dualizing Proposition (4.3)(a).

(c) If $w=e$, (c) follows from (a). Otherwise write $w=r_j v$, for some r_j so that $v<w$ and assume, by induction, that $\xi^{r_i}(v)=\chi_i-v^{-1}\chi_i$. Now

$$(A_j\xi^{r_i})\,v=\frac{\xi^{r_i}(r_jv)-\xi^{r_i}(v)}{v^{-1}\alpha_j} \qquad \text{by } (\mathrm{I}_{52})$$

$$=[\xi^{r_i}(w)-(\chi_i-v^{-1}\chi_i)]\,\frac{1}{v^{-1}\alpha_j} \qquad \text{(by induction)}.$$

But $A_j\xi^{r_i}=\delta_{i,j}$ by (b). Hence we have

$$\xi^{r_i}(w)=\delta_{i,j}(v^{-1}\alpha_j)+\chi_i-v^{-1}\chi_i=\chi_i-w^{-1}\chi_i.$$

This proves (c).

(d) If $w=e$, there is nothing to prove. So, assume that $w\neq e$ and write $w=r_jw'$, for some r_j so that $w'<w$. Now, by (b) and (I_{52}), we have

$$(A_j\xi^{v})\,w'=\frac{\xi^{v}(w)-\xi^{v}(w')}{(w'^{-1}\alpha_j)}=0 \qquad \text{if} \quad r_jv>v,$$

$$=\xi^{r_jv}(w') \qquad \text{if} \quad r_jv<v.$$

But, by induction, we can assume that, in the case $r_jv<v$, $\xi^{r_jv}(w')$ is a homogeneous polynomial of degree $l(r_jv)$, and also $\xi^{v}(w')$ is a homogeneous polynomial of degree $l(v)$. This proves (d).

(e) Since D is invertible, it suffices to show that $D\cdot C'=\mathrm{Id}$. Fix $v, w\in W$. By definition, $\xi^{w}(\bar{x}_v)=\delta_{w,v}$. But by ($\mathrm{I}_{32}$), $x_{v^{-1}}=\sum c_{v,u}\,\delta_{u^{-1}}$, i.e., $\bar{x}_v=\sum \delta_u c_{v,u}$. Hence $\xi^{w}(\bar{x}_v)=\delta_{w,v}=\sum_u d_{w,u}c_{v,u}$. This proves (e).

(f) Follows easily by dualizing Proposition (4.15).

(g) Observe that, by Proposition (4.13), we have

$$r_i\xi^{w}=\sum \xi^{w}(\delta_{r_i}\cdot\bar{x}_u)\,\xi^{u}$$

$$=\sum_{r_iu<u} -\xi^{w}(\bar{x}_u)\,\xi^{u}$$

$$+\sum_{r_iu>u}\left[\xi^{w}(\bar{x}_{r_iu})(u^{-1}\alpha_i)-\xi^{w}(\bar{x}_u)+\sum_{v\to^{\gamma} r_iu}\xi^{w}(\bar{x}_v)\,\alpha_i(v^{\gamma})\right]\xi^{u}.$$

The rest of the proof is along the lines of the proof of Proposition (4.13). ∎

The following lemma is trivial to verify.

(4.26) LEMMA. *Let $\psi_1, \psi_2 \in \Omega$, then*

$$A_i(\psi_1 \cdot \psi_2) = (A_i\psi_1)\,\psi_2 + (r_i\psi_1)(A_i\psi_2).$$

More generally, by induction on $l(w)$, we get

(4.27) LEMMA. *Let $w = r_{i_1} \cdots r_{i_n}$ be a reduced expression. Then for any $\psi_1, \psi_2 \in \Omega$, we have*

$$A_w(\psi_1 \cdot \psi_2) = \sum_{\substack{0 \leqslant p \leqslant n \\ 1 \leqslant j_1 < \cdots < j_p \leqslant n}} A_{i_1} \circ \cdots \circ \hat{\hat{A}}_{i_{j_1}} \circ \cdots \circ \hat{\hat{A}}_{i_{j_p}} \circ \cdots \circ A_{i_n}(\psi_1)$$
$$\cdot (A_{i_{j_1}} \circ \cdots \circ A_{i_{j_p}} \psi_2)$$

where the notation $\hat{\hat{A}}_i$ means that the operator A_i is replaced by the Weyl group action r_i.

(4.28) COROLLARY. *For any $w \in W$ and $\psi \in \Omega$, we have*

$$A_w(\xi^{r_i} \cdot \psi) = \sum_{v^{-1} \xrightarrow{\gamma} w^{-1}} \chi_i(v^\gamma)(A_v\psi) + (w\xi^{r_i}) \cdot A_w\psi$$

Proof. By Proposition (4.24)(b), (g), and Lemma (4.27), we get

$$A_w(\xi^{r_i} \cdot \psi) = (w\xi^{r_i}) \cdot (A_w\xi) + \sum_{1 \leqslant j \leqslant n} (r_{i_1} \cdots A_{i_j} \cdots r_{i_n} \xi^{r_i})$$
$$\cdot (A_{i_1} \circ \cdots \circ \hat{A}_{i_j} \circ \cdots \circ A_{i_n}\psi)$$

where $w = r_{i_1} \cdots r_{i_n}$ is a reduced expression. Thus

$$A_w(\xi^{r_i} \cdot \psi) = (w\xi^{r_i}) \cdot (A_w\psi) + \sum_{\substack{\text{those } 1 \leqslant j \leqslant n \text{ such that} \\ w_j = r_{i_1} \cdots \hat{r}_{i_j} \cdots r_{i_n} \text{ is} \\ \text{reduced}}} [A_{i_j}(r_{i_{j+1}} \cdots r_{i_n} \xi^{r_i})] \cdot (A_{w_j}\psi). \tag{I_{55}}$$

By Proposition (4.24)(c), it can be easily seen that

$$A_{i_j}(r_{i_{j+1}} \cdots r_{i_n} \xi^{r_i}) = \chi_i(v^\gamma) \qquad \text{where} \quad w_j^{-1} \xrightarrow{\gamma} w^{-1}. \tag{I_{56}}$$

Substituting (I_{56}) in (I_{55}) and using [3, Proposition 2.8], we get the corollary. ▌

As an immediate consequence of Proposition (4.24)(a) and (b), we get

(4.29) LEMMA. *Let $\psi \in \Omega$. Express $\psi = \sum_w q^w \xi^w$, for some (unique) $q^w \in Q$ (infinitely many of q^w's could be non-zero). Then $q^w = (A_{w^{-1}} \psi)\, e$.*

The following proposition follows easily from Corollary (4.28) and Lemma (4.29).

(4.30) PROPOSITION. *For any $v \in W$ and simple reflection r_i, we have*

$$\xi^{r_i} \xi^v = \sum_{v \to^r w} \chi_i(v^r)\, \xi^w + \xi^{r_i}(v)\, \xi^v.$$

More generally, we have

(4.31) PROPOSITION. (generalized cup product formula). *For any $v, w \in W$, we have (by Proposition* (4.24))(f) $\xi^v \xi^w = \sum_{v,w \leqslant u} p^u_{v,w} \xi^u$, *where $p^u_{v,w}$ is a homogeneous polynomial of degree $l(v) + l(w) - l(u)$. Write $u^{-1} = r_{i_1} \cdots r_{i_n}$ as a reduced expression, then*

$$p^u_{v,w} = \sum_{\substack{\text{Those } j_1 < \cdots < j_m \text{ such} \\ \text{that } r_{i_{j_1}} \cdots r_{i_{j_m}} = w^{-1}}} A_{i_1} \circ \cdots \circ \hat{A}_{i_{j_1}} \circ \cdots \circ \hat{A}_{i_{j_m}} \circ \cdots \circ A_{i_n}(\xi^v)(e)$$

where $m = l(w)$.

Proof. Follows easily by combining Lemmas (4.27) and (4.29) together with Proposition (4.24)(a) and (b). ∎

Recall the definition of the matrix $D = (d_{v,w})_{v,w \in W}$ from **(4.21)**.

(4.32) PROPOSITION. (a) *Fix $w \in W$. Define two matrices P_w and $D_w \in \mathscr{B}_W$ by $D_w(u, v) = \delta_{u,v} d_{w,v}$ and $P_w(u, v) = p^v_{w,u}$,* for *$u, v \in W$. Of course, by definition, D_w is a diagonal matrix. Then explicitly,*

$$P_w = D \cdot D_w \cdot D^{-1}.$$

(b) *For $u,v \in W$ write*

$$u \cdot \xi^v = \sum_{l(v) - l(u) \leqslant l(w) \leqslant l(v)} g^u_{v,w} \xi^w \qquad (\mathrm{I}_{57})$$

where $g^u_{v,w}$ is a homogeneous polynomial of degree $l(v) - l(w)$. (This is possible by Proposition (4.24)(g).) *For a fixed $w \in W$, define G_w and $S_w \in \mathscr{B}_W$ by $G_w(u, v) = g^w_{u,v}$ and $S_w(u, v) = \delta_{wu,v}$. Then*

$$D \cdot S_w \cdot D^{-1} = G_w.$$

Proof. Since $p^v_{w,u}=0$ unless $u \leqslant v$, we get $P_w \in \mathscr{B}_W$:

$$\begin{aligned}(P_w D)(u,v) &= \sum_{w'} P_w(u, w')\, D(w', v)\\ &= \sum_{w'} p^{w'}_{w,u}\, d_{w',v}\\ &= d_{w,v} d_{u,v} \qquad \text{by Proposition (4.24)(f)}\\ &= (D \cdot D_w)(u, v) \qquad \text{proving (a).}\end{aligned}$$

By (I_{57}) we have $d_{u,w^{-1}v} = \sum_{w'} g^{w'}_{u,w'} d_{w',v}$, i.e., $(DS_w)(u, v) = (G_w D)(u, v)$. ∎

To conclude the section, we make the following

(4.33) DEFINITION. Recall the definition of the S-algebra Λ from (4.19). For any subset $X \subset \{1,..., l\}$ and let $W_X \subset W$ be the subgroup of W as defined in (1.1). *We define Λ^X to be the S-subalgebra of Λ, consisting of W_X invariant elements in Λ.* (Of course the W-module structure, in particular a W_X-module structure, on Λ is the one given by (I_{50}).)

The following lemma describes the structure of Λ^X.

(4.34) LEMMA. $\Lambda^X = \sum_{w \in W^1_X} S\xi^w$ *(W^1_X is defined in* (1.1). *In particular, Λ^X is a free module over S.*

Proof. Recall that W^1_X is characterized as the set of all those $w \in W$ such that w is (the unique) element of minimal length in its coset $W_X w$. In particular, for any $i \in X$, $r_i w > w$ for $w \in W^1_X$. Hence, by Proposition (4.24)(g), $\xi^w \in \Lambda^X$, for any $w \in W^1_X$.

Conversely, take $\xi \in \Lambda^X$ and write (Proposition (4.20)) $\xi = \sum_w p^w \xi^w$, where all but finitely many p^w's are zero. Fix $i \in X$. Since $r_i \xi = \xi$, we have $A_i \xi = 0$. This in particular (by Proposition (4.24)(g)), gives that $p^w = 0$, unless $r_i w > w$. So any w, with $p^w \neq 0$, belongs to W^1_X. ∎

(4.35) *Remarks.* (a) In the next section, we will see that $\mathbb{C}_0 \otimes_S \Lambda^X$ is isomorphic with the cohomology algebra $H^*(G/P_X, \mathbb{C})$, where G is the group associated to the Kac–Moody Lie algebra $\mathbf{g}$ and P_X is a parabolic (corresponding to the subset X) in G. Further, when $X = \varnothing$, the isomorphism is W-equivariant.

(b) Many concepts and results in this chapter can be extended (with suitable and easy modifications) to an arbitrary finitely generated Coxeter group W with a specified set of Coxeter generators and also equipped with a representation satisfying root-system condition, as given in [7, Sect. 2].

5. Marriage of the Second and the Fourth Chapter

Throughout this chapter, $\mathbf{g} = \mathbf{g}(A_l)$ would denote an arbitrary symmetrizable Kac–Moody Lie algebra with Cartan sub-algebra $\mathbf{h}$ and Weyl group W.

(5.1) Recall the definition of d, ∂ harmonic forms $\{s_0^w\}_{w \in W} \subset C(\mathbf{g}, \mathbf{h})$ from Section 3 (case $X = \varnothing$). (s_0^w is the unique d, ∂ harmonic form such that $\int_{Bv^{-1}B/B} s_0^w = \delta_{v,w}$, for all $v \in W$ with $l(v) = l(w)$.) Also recall the definition of the map $\bar{\eta}$: $\operatorname{Ker} S \to \mathbb{C}\{W\}$, from Theorem (2.7).

Consider the function $\tilde{D}$: $W \times W \to \mathbb{C}$, defined by $\tilde{D}(v, w) = \bar{\eta}(s_0^v)(w)$. We want to relate the function $\tilde{D}$ with the function D introduced in **(4.21)**.

(5.2) Proposition. *For any $v, w \in W$, $\tilde{D}$ satisfies the following*:

(a) $\tilde{D}(v, w) = 0$, *if $l(w) \leqslant l(v)$ and $v \neq w$.*

(b) $\tilde{D}(w, w) = \prod_{\gamma \in w^{-1}\Delta_- \cap \Delta_+} \gamma(h(\rho))$, *where $h(\rho)$ is the unique element of $\mathbf{h}$ satisfying $\chi(h(\rho)) = \sigma(\chi, \rho)$*, for all $\chi \in \mathbf{h}^*$.

(c) *For any simple reflection r_i,*

$$\tilde{D}(r_i, w) = (\chi_i - w^{-1}\chi_i)\, h(\rho) \qquad (\chi_i \textit{ is defined in } (1.4))$$

(d) *Cup-product formula*

$$\tilde{D}^{r_i}\tilde{D}^{v} = \sum_{v \xrightarrow{\gamma} w} \chi_i(v^\gamma)\, \tilde{D}^{w} + \tilde{D}(r_i, v)\, \tilde{D}^{v} \qquad \textit{as elements of } \mathbb{C}\{W\}$$

where $\tilde{D}^v \in \mathbb{C}\{W\}$ *is defined by $\tilde{D}^v(w) = \tilde{D}(v, w)$, for any $v, w \in W$ and $\tilde{D}^{r_i} \cdot \tilde{D}^v$: $W \to \mathbb{C}$ denotes the function, obtained by the pointwise multiplication of $\tilde{D}^{r_i}$ and $\tilde{D}^v$.*

Proof. (a) and (b) follows easily from the definition of $\tilde{D}$, by observing that $\sum_{j \geqslant 1} R^j(h^w) \in \operatorname{Im} L$ (see (3.5)).

(c) We show that

$$s_0^{r_i} = \frac{1}{2\pi\sqrt{-1}}\, d_{\mathbf{g}}(\tilde{\chi}_i) \qquad (\mathrm{I}_{58})$$

where $\tilde{\chi}_i \in \mathbf{g}^*$ is the element satisfying $\tilde{\chi}_{i|\mathbf{h}} = \chi_i$ and $\chi_{i|\mathbf{g}_\alpha} = 0$, for any root α. ($d_{\mathbf{g}}$ denotes the differential (of degree $+1$) of the co-chain complex $C(\mathbf{g}, \mathbb{C})$.)

We clearly have $i(h)(d_{\mathbf{g}}(\tilde{\chi}_i))$ and $\operatorname{ad}(h)(d_{\mathbf{g}}(\tilde{\chi}_i))$ equal to 0, for all $h \in \mathbf{h}$.

Hence $d_{\mathbf{g}}(\tilde{\chi}_i) \in C^2(\mathbf{g}, \mathbf{h})$. Further $\partial d_{\mathbf{g}}(\tilde{\chi}_i) \in C^1(\mathbf{g}, \mathbf{h})$ and $C^1(\mathbf{g}, \mathbf{h})$ can be easily seen to be 0. So $d_{\mathbf{g}}(\tilde{\chi}_i)$ is d, ∂ harmonic. Now

$$d_{\mathbf{g}}(\tilde{\chi}_i)(e_j, f_j) = -\chi_i(h_j) = -\delta_{i,j} \qquad \text{for all} \quad 1 \leqslant i, j \leqslant l. \tag{I_{59}}$$

Moreover, by Theorem (3.1) and the expression of s^{r_i} as given in (I'_{16}), we have

$$s_0^{r_i}(e_j, f_j) = \frac{\sigma(\alpha_i, \alpha_i)\sqrt{-1}}{4\pi} e(b_{\alpha_i} \otimes a_{\alpha_i})(e_j, f_j), \text{ i.e.,}$$

$$s_0^{r_i}(e_j, f_j) = \frac{\sqrt{-1}}{2\pi} \delta_{i,j} \qquad \text{since} \quad \|e_j\|^2 = \|f_j\|^2 = \frac{2}{\sigma(\alpha_j, \alpha_j)}. \tag{I_{60}}$$

Combining (I_{59}) and (I_{60}), (I_{58}) follows. Further, it is easy to see that

$$d_{\mathbf{g}}(\tilde{\chi}_i) = \sum_{\Phi \in \tilde{\Delta}_+} -\sigma(\chi_i, \Phi)\, e(b_\Phi \otimes a_\Phi) \tag{I_{61}}$$

($\tilde{\Delta}_+$, b_Φ, and a_Φ are defined in **(3.5)**).

Let $\Phi_w = \Delta_+ \cap w\Delta_- = \{\beta_1, ..., \beta_p\}$ $(p = l(w))$. Recall the definition of η from Lemma (2.4),

$$\eta\left(\frac{1}{2\pi\sqrt{-1}} d_{\mathbf{g}}(\tilde{\chi}_i)\right)(e(b_{\beta_1} \wedge \cdots \wedge b_{\beta_p}))$$

$$= \frac{(-1)}{2\pi\sqrt{-1}} \eta\left(\sum_{\Phi \in \tilde{\Delta}_+} \sigma(\chi_i, \Phi)\, e(b_\Phi \otimes a_\Phi)\right)(e(b_{\beta_1} \wedge \cdots \wedge b_{\beta_p}))$$

by (I_{61}).

$$= \sum_{\Phi \in \tilde{\Delta}_+} \sigma(\chi_i, \Phi)\, e(b_\Phi) \cdot (i(a_\Phi)\, e(b_{\beta_1} \wedge \cdots \wedge b_{\beta_p}))$$

(by the definition of η)

$$= \sum_{j=1}^{p} \sigma(\chi_i, \beta_j)\, e(b_{\beta_1} \wedge \cdots \wedge b_{\beta_p})$$

$$= \sigma(\chi_i, \rho - w\rho)\, e(b_{\beta_1} \wedge \cdots \wedge b_{\beta_p}) \qquad \text{since } \textstyle\sum \beta_j = \rho - w\rho.$$

Hence $\bar{\eta}(s_0^{r_i})(w) = (\chi_i - w^{-1}\chi_i)\, h(\rho)$, by the definition of $\bar{\eta}$ as in (2.7). This proves (c).

(d) Since $\tilde{D}^{r_i}\tilde{D}^v \in \mathscr{J}_{l(v)+1}$ (by Lemma (2.11) and Corollary (2.14)), we have

$$\tilde{D}^{r_i}\tilde{D}^v = \sum_{l(w) = l(v)+1} z^w \tilde{D}^w + \sum_{l(w) \leqslant l(v)} z^w \tilde{D}^w \tag{I_{62}}$$

for some constants z^w. Let p_0 be the minimum integer, such that $z^{w_0} \neq 0$, for some w_0 of length p_0. Then $p_0 \geqslant l(v)$. For otherwise, by part (a) of this proposition, we would get, by evaluating (I_{62}) at w_0, $z^{w_0}\tilde{D}^{w_0}(w_0) = 0$, which is a contradiction by part (b). Exactly the same argument shows that $z^w = 0$ for all $w \in W$ with $l(w) = l(v)$ and $w \neq v$. To determine z^v, evaluate (I_{62}) at v to get $\tilde{D}^{r_i}(v) = z^v$. Finally, by Theorem (2.12), Gr $\mathbb{C}\{W\} \approx H^*(\mathbf{g}, \mathbf{h})$ as an algebra. In view of Proposition (3.10) and [23, Theorem 4.5] along with [24, Theorem 1.6], the (d) part follows.

(5.3) LEMMA. *Let $v \neq w \in W$. Then $(v\chi_i - w\chi_i)\, h(\rho) \neq 0$, for some $1 \leqslant i \leqslant l$.*

Proof. Define $\mathbf{h}_0^* = \{\chi \in \mathbf{h}^*: \chi(h_i) = 0$, for all $1 \leqslant i \leqslant l\}$. Clearly $\mathbf{h}_0^* \cap \sum_{i=1}^l \mathbb{C}\chi_i = (0)$. Hence, by dimension count, $\mathbf{h}_0^* + \sum_{i=1}^l \mathbb{C}\chi_i = \mathbf{h}^*$. Assume that the lemma is false, i.e., $\chi_i(v^{-1}h(\rho) - w^{-1}h(\rho)) = 0$, for all i. Further $\chi(v^{-1}h(\rho) - w^{-1}h(\rho)) = 0$, for any $\chi \in \mathbf{h}_0^*$ (since $v^{-1}h(\rho) - w^{-1}h(\rho)$ lies in the span of $\{h_i\}_{1 \leqslant i \leqslant l}$). Hence $v^{-1}h(\rho) = w^{-1}h(\rho)$, which gives $v^{-1}\rho = w^{-1}\rho$. This is possible only if $v = w$, by [9, Corollary 2.6]. ∎

(5.4) DEFINITION. Recall the definition of the matrix D from **(4.21)**, which can also be viewed as a function: $W \times W \to S$. Fix any $h \in \mathbf{h}$. There is an evaluation map ev_h: $S \to \mathbb{C}$, defined by $\mathrm{ev}_h(p) = p(h)$. *We define the function $D_{h(\rho)}$: $W \times W \to \mathbb{C}$ by*

$$D_h(v, w) = \mathrm{ev}_h(D(v, w)).$$

We prove the following characterization of $D_{h(\rho)}$.

(5.5) PROPOSITION. *Let E: $W \times W \to \mathbb{C}$ be any function satisfying*

(1) *$E(v, w) = 0$ if $l(w) \leqslant l(v)$ and $w \neq v$.*

(2) *$E(w, w) = D_{h(\rho)}(w, w)$ for all $w \in W$*

(3) *$E(r_i, w) = D_{h(\rho)}(r_i, w)$ for all simple reflections r_i and all $w \in W$, and*

(4) *the cup-product formula holds for E, i.e.,*

$$E^{r_i}E^v = \sum_{v \to^v w} \chi_i(v^v)\, E^w + E(r_i, v)\, E^v \qquad \text{for all} \quad r_i \text{ and } v \in W.$$

(The notation E^v is similar to the one in Proposition (5.2)(d).) *Then $E = D_{h(\rho)}$.*

Proof. We prove the proposition by induction on $l(v, w) = l(w) - l(v)$. For (v, w) with $l(v, w) \leqslant 0$, by (1) and (2) and Proposition (4.24)(a), we have $E(v, w) = D_{h(\rho)}(v, w)$. So assume that $E(v, w) = D_{h(\rho)}(v, w)$, for v, w with $l(v, w) \leqslant n$ and let (v_0, w_0) be such that $l(v_0, w_0) = n+1$. We have, by (4), $[E^{r_i}(w_0) - E^{r_i}(v_0)]\ E^{v_0}(w_0) = \sum_{v_0 \to^{v} w} \chi_i(v^{v})\ E^{w}(w_0)$. By induction, $E^{w}(w_0) = D_{h(\rho)}(w, w_0)$ (since $l(w, w_0) = n$) and, by (3), $E^{r_i}(w_0) - E^{r_i}(v_0) = D_{h(\rho)}(r_i, w_0) - D_{h(\rho)}(r_i, v_0)$. By Lemma (5.3) and Proposition (4.24)(c), there is a $1 \leqslant i \leqslant l$ such that $D_{h(\rho)}(r_i, w_0) - D_{h(\rho)}(r_i, v_0) \neq 0$. So, by Proposition (4.30), $E(v_0, w_0) = D_{h(\rho)}(v_0, w_0)$. ∎

Recall the definition of $\tilde{D}$: $W \times W \to \mathbb{C}$ from **(5.1)**. The following result provides a bridge between Sections 2 and 4. Combining Propositions (4.24), (5.2), and (5.5), we get

(5.6) COROLLARY. *$\tilde{D} = D_{h(\rho)}$ as functions: $W \times W \to \mathbb{C}$.*

(5.7) *Remark.* There is nothing very special about $h(\rho)$ in Proposition (5.5). Any $h \in \mathbf{h}$, such that Lemma (5.3) holds for $h(\rho)$ replaced by h will do.

We recall the following.

(5.8) DEFINITION. Let K be the standard real form of a Kac–Moody group G and let T denote the "maximal torus" of K (see (1.2)). There is an action of the Weyl group $W \approx N_K(T)/T$ ($N_K(T)$ denotes the normalizer of T in K) on K/T defined as

$$n \cdot (k \bmod T) = (kn^{-1}) \bmod T, \qquad \text{for} \quad n \in N_K(T) \text{ and } k \in K.$$

In particular, W acts on the cohomology $H^*(K/T, \mathbb{Z})$ as well as on the homology $H_*(K/T, \mathbb{Z})$. Also recall, from (3.9), that $\{\varepsilon^{v}\}_{v \in W}$ denotes the $\mathbb{Z}$-basis of $H^*(K/T, \mathbb{Z})$ dual to the closures of the Schubert cells. The following lemma, in the finite case, is due to [3, Theorem 3.14(iii)]. An easy proof of the lemma (in the general case) can be given by using Proposition (3.10).

(5.9) LEMMA.

$$\begin{aligned} r_i \varepsilon^{v} &= \varepsilon^{v} && \text{if} \quad r_i v > v, \\ &= \varepsilon^{v} - \sum_{r_i v \to^{v} w} \alpha_i(v^{v})\, \varepsilon^{w} && \text{if} \quad r_i v < v. \end{aligned}$$

(5.10) *Remarks.* (a) This lemma does not require the symmetrizability assumption on G.

(b) The formula for $\sigma_\alpha \cdot P_w$ given in [3, Theorem 3.14(iii)], in the case when $l(w\sigma_\alpha) = l(w) - 1$, is incorrect. The correct formula (in their notation) is

$$\sigma_\alpha \cdot P_w = P_w + \sum_{w\sigma_\alpha \to^\gamma w'} w\alpha(H_\gamma)\, P_{w'}.$$

(5.11) Recall the definition of Λ from (4.19). Let $\mathbb{C}_0 = S/S^+$ be the 1-dimensional (over $\mathbb{C}$) S-module ($S = S(\mathbf{h}^*)$), where S^+ is the augmentation ideal (evaluation at $0 \in \mathbf{h}$) in S. By Proposition (4.20), $\mathbb{C}_0 \otimes_S \Lambda$ is an algebra and the action of R on Λ descends to give an action of R on $\mathbb{C}_0 \otimes_S \Lambda$. Also, from Proposition (4.20)(b), the elements

$$\sigma^w = 1 \otimes \xi^w \in \mathbb{C}_0 \otimes_S \Lambda \tag{I_{63}}$$

provide a $\mathbb{C}$-basis. Moreover by Proposition (4.24)(f), the algebra Λ is filtered by $\{\Lambda_p\}_{p \in \mathbb{Z}_+}$, where $\Lambda_p = \sum_{l(v) \leqslant p} S\xi^v$. Again using Proposition (4.24)(f), it is easy to see that this filtered algebra structure gives rise to an (obvious) graded (commutative) algebra structure on $\mathbb{C}_0 \otimes_S \Lambda$.

Further, besides having a ring structure and being a module for W (described in (5.8)), $H^*(K/T)$ also admits a ring of operators $\mathscr{A}$ (with $\mathbb{C}$-basis $\{A_w\}_{w \in W}$), where A_{r_i} $(1 \leqslant i \leqslant l)$ corresponds topologically to the integration on fiber for the fibration $G/B \to G/P_i$ (P_i is the minimal parabolic containing r_i). The ring of operators $\mathscr{A}$ on $H^*(K/T)$ was introduced by Bernstein, Gel'fand, and Gel'fand [3] in the finite case. The definition in the general case is carried out by Kac and Peterson.

We come to one of the main theorems of this chapter.

(5.12) THEOREM. *Let K be the standard real form of the group G associated to a symmetrizable Kac–Moody Lie algebra $\mathbf{g}$ and let T denote the "maximal torus" of K (see (1.2)). Then the map*

$$\theta\colon H^*(K/T, \mathbb{C}) \to \mathbb{C}_0 \otimes_S \Lambda$$

defined by $\theta(\varepsilon^w) = \sigma^w$, for any $w \in W$ is a graded algebra isomorphism. Moreover, the action of $w \in W$ and A_w on $H^*(K/T)$ corresponds (under θ) respectively to that of $\delta_w, x_w \in R$ on $\mathbb{C}_0 \otimes_S \Lambda$ (see (I_{49})).

Proof. We give a "geometrical" proof of this theorem. It also admits a more "algebraic" proof (see Remark (5.17)(a)).

Consider the $\mathbb{C}$-linear map $f\colon \mathbb{C}_0 \otimes_S \Lambda \to \operatorname{Gr} \mathbb{C}\{W\}$, defined by $f(\sigma^w) = \tilde{D}^w \bmod \mathscr{J}_{l(w)+1} \in \operatorname{Gr}^{l(w)} \mathbb{C}\{W\}$, where $\tilde{D}^w$, $\mathscr{J}_p$, and $\operatorname{Gr} \mathbb{C}\{W\}$ are defined respectively in Proposition (5.2), **(2.8)**, and Lemma (2.11). By Corollary (2.14), f is a $\mathbb{C}$-vector space isomorphism. Further, we recall from Proposition (4.24)(f) that $\xi^u \xi^v = \sum_{u,v \leqslant w} p^w_{u,v} \xi^w$, where $p^w_{u,v}$ is a homogeneous polynomial of degree $l(u)+l(v)-l(w)$. In particular, $\sigma^u\sigma^v = \sum_{u,v \leqslant w \text{ and } l(u)+l(v)=l(w)} p^w_{u,v}\sigma^w$. Now the fact, that f is an algebra homomorphism, follows from Corollary (5.6). By [24, Theorem 1.6], there is a natural graded algebra isomorphism $[\int]$: $H^*(\mathbf{g}, \mathbf{h}) \to H^*(K/T, \mathbb{C})$, given by an "integration" map. We claim that $f \circ \theta \circ [\int] = \mathrm{Gr}(\bar{\eta}) \circ \psi^{-1}_{d,S}$ as maps: $H^*(\mathbf{g}, \mathbf{h}) \to \mathrm{Gr}\, \mathbb{C}\{W\}$ (see Theorem (2.12)). The claim is easy to establish if we keep track of the definitions of various maps (f, θ, $\mathrm{Gr}(\bar{\eta})$, and $\psi_{d,S}$) involved and observe further (see the proof of [24, Theorem 1.6]) that $[\int](s^w_0) = \varepsilon^w$, for all $w \in W$, where s^w_0 is as defined in **(5.1)**. Now since $\mathrm{Gr}(\bar{\eta}) \circ \psi^{-1}_{d,S}$ is a graded algebra isomorphism, by Theorem (2.12), we get that θ is a graded algebra isomorphism as well.

The assertion, that θ commutes with Weyl group action, follows by combining Proposition (4.24)(g) with Lemma (5.9). Finally the claim, that the action of A_w on $H^*(K/T)$ corresponds (under θ) to that of $x_w \in R$, follows from Propositions (4.3)(a), (4.24)(b), and the analog of [3, Theorem 3.14(i)] proved by Kac and Peterson.

(5.13) COROLLARIES. (a) *We can use either of Propositions* (4.31) *or* (4.32)(a) *to determine the cup product* $\varepsilon^u\varepsilon^v$ (*for arbitrary* u *and* $v \in W$) *in terms of the Schubert basis* $\{\varepsilon^w\}$ *of* $H^*(K/T, \mathbb{Z})$. *Similarly, we can also write an expression for* $u \cdot \varepsilon^v$ (*in terms of the Schubert basis*) *as given in* (4.32)(b). *Observe that the Proposition* (4.32)(a)–(b) *gives cup product as well as the Weyl group action purely in terms of the matrix D.*

(b) *A result of Chevalley* (*that, in the finite case,* $H^*(K/T, \mathbb{C})$ *is W-isomorphic with the left regular representation of* W) *can be trivially deduced from the above theorem.*

(c) *Let* $X \subset \{1,..., l\}$ *be arbitrary. There is a standard parabolic subgroup* P_X (*as defined in* (1.2) *of the group G and let* $K_X = K \cap P_X$. *Also recall the definition of* A^X, *from* (4.33).

There is a (*unique*) *graded algebra isomorphism* θ^X: $H^*(K/K_X, \mathbb{C}) \to \mathbb{C}_0 \otimes_S A^X$, *making the following diagram commutative:*

$$\begin{tikzcd} H^*(K/K_X, \mathbb{C}) \arrow[r, "\theta^X"] \arrow[d, "p_X^*"] & \mathbb{C}_0 \otimes_S A^X \arrow[d, "\mathrm{Id.} \otimes i^X"] \\ H^*(K/T, \mathbb{C}) \arrow[r, "\theta"] & \mathbb{C}_0 \otimes_S A \end{tikzcd}$$

where p_X^* *is induced from the projection* p_X: $K/T \to K/K_X$ *and* i^X: $A^X \hookrightarrow A$ *is the canonical inclusion.*

Proof. It is well known (and easy to prove) that the map p_X^* is injective and the image of p_X^* is precisely the $\mathbb{C}$-span of $\{\varepsilon^w\}_{w \in W_X^1}$. Further, by

Lemma (4.34), the map $\mathrm{Id.} \otimes i^X$ is injective with image precisely equal to the span of $\{\sigma^w\}_{w \in W_X^1}$. This proves the corollary. ∎

The following is actually a corollary of the (c) part above. It (the following corollary) has earlier been obtained by Kac and Peterson.

(d) *With the notations as in* (c), $p_X^*(H^*(K/K_X)) = H^*(K/T)^{W_X}$, *where* W_X *is as defined in* (1.1) *and* $H^*(K/T)^{W_X}$ *denotes the* W_X*-invariants in* $H^*(K/T)$. *In particular* $H^*(K/T)^W = H^0(K/T)$.

Proof. Since the map θ (of Theorem (5.12)) commutes with W-actions, it suffices to show that, for $\xi \in \Lambda$, if $r_i(1 \otimes \xi) = 1 \otimes \xi$, for all $i \in X$ then there exists a $\xi' \in \Lambda^X$ such that $1 \otimes \xi = 1 \otimes \xi'$. Write $\xi = \sum p^w \xi^w$, for some $p^w \in S$. Fix $i \in X$. By Proposition (4.24)(g),

$$r_i \xi = \sum_{w < r_i w} p^w \xi^w + \sum_{r_i w < w} p^w \left[\xi^w - \sum_{r_i w \to^v v} \alpha_i(v^v)\, \xi^v - (w^{-1}\alpha_i)\, \xi^{r_i w} \right]. \tag{I$_{64}$}$$

Since $r_i(1 \otimes \xi) = 1 \otimes r_i \xi = 1 \otimes \xi$ (by assumption), we get by (I$_{64}$),

$$1 \otimes \sum_{r_i w < w} p^w \sum_{r_i w \to^v v} \alpha_i(v^v)\, \xi^v = 0. \tag{I$_{65}$}$$

We can rewrite (I$_{65}$) as

$$1 \otimes \sum_{r_i w < w} 2p^w \xi^w + 1 \otimes \sum_{r_i w < w} p^w \sum_{r_i w \to^v_{v \neq w} v} \alpha_i(v^v)\, \xi^v = 0. \tag{I$_{66}$}$$

For a w with $r_i w < w$ and any $v \in W$ such that $r_i w \to v$ and $v \neq w$, we have $r_i v > v$. In particular, from (I$_{66}$), we get that $p^w \in S^+$, if there exists $i \in X$ such that $r_i w < w$, i.e., $1 \otimes \xi = 1 \otimes \sum_{w \in W_X^1} p^w \xi^w$. ∎

We further generalize Corollary (5.13)(c) to Schubert varieties in G/P_X. We need some preliminaries.

(5.14) Let Ⓗ $=$ Ⓗ$_X$ be a subset of W with the following properties:

(P_1) Ⓗ is left W_X-stable.

(P_2) Whenever $w \in$ Ⓗ and $w' \leqslant w$ then $w' \in$ Ⓗ.

To any such Ⓗ, we can associate a (left) B-stable variety $V_{Ⓗ} \subset G/P_X$, defined by

$$V_{Ⓗ} = \bigcup_{w \in Ⓗ} B w^{-1} P_X / P_X. \tag{I$_{67}$}$$

By property (P_2) of $\textcircled{H}$, $V_{\textcircled{H}}$ is closed in G/P_X (see [29]). Conversely, any (left) B-stable closed subset of G/P_X is $V_{\textcircled{H}}$, for some appropriate choice of $\textcircled{H} = \textcircled{H}_X$.

Let $\Omega_{\textcircled{H}}$ denote the Q-algebra of all the maps: $\textcircled{H} \to Q$. There is a restriction map $r_{\textcircled{H}}\colon \Omega \to \Omega_{\textcircled{H}}$. Define $\Lambda^X_{\textcircled{H}} = r_{\textcircled{H}}(\Lambda^X)$. From Proposition (4.24)(a) and Lemma (4.34), we easily get

(5.15) Lemma. *$\Lambda^X_{\textcircled{H}}$ is a free S-module with a basis*

$$\{\xi^w_{|\textcircled{H}}\}_{w \in W^1_X \cap \textcircled{H}}.$$

We have the following generalization of Corollary (5.13)(c).

(5.16) Theorem. *Let G be the group associated to a Kac–Moody Lie algebra $\mathbf{g} = \mathbf{g}(A_l)$ and let X be any subset (including $X = \varnothing$) of $\{1, \ldots, l\}$. Fix $\textcircled{H} = \textcircled{H}_X \subset W$ satisfying (P_1) and (P_2) as above and let $V_{\textcircled{H}} \subset G/P_X$ be the subvariety, defined by (I_{67}). Then there is a unique graded algebra isomorphism $\theta^X_{\textcircled{H}}\colon H^*(V_{\textcircled{H}}, \mathbb{C}) \to \mathbb{C}_0 \otimes_S \Lambda^X_{\textcircled{H}}$, making the following diagram commutative:*

$$\begin{tikzcd} H^*(K/K_X, \mathbb{C}) \arrow[r, "\theta^X"] \arrow[d, "i^*_{\textcircled{H}}"] & \mathbb{C}_0 \otimes_S \Lambda^X \arrow[d, "\tilde{r}"] \\ H^*(V_{\textcircled{H}}, \mathbb{C}) \arrow[r, "\theta^X_{\textcircled{H}}"] & \mathbb{C}_0 \otimes_S \Lambda^X_{\textcircled{H}} \end{tikzcd}$$

where $i^*_{\textcircled{H}}$ is induced by the inclusion $i_{\textcircled{H}}\colon V_{\textcircled{H}} \hookrightarrow G/P_X$ and $\tilde{r} = \tilde{r}_{\textcircled{H}}$ is induced by the restriction map $r_{\textcircled{H}}$. (The grading on $\mathbb{C}_0 \otimes_S \Lambda^X_{\textcircled{H}}$ comes from the grading of $\mathbb{C}_0 \otimes_S \Lambda^X$ via $\tilde{r}$.)

Proof. Of course the map $\tilde{r}$ is, by definition, surjective. It is easy to see that the kernel of $i^*_{\textcircled{H}}$ is precisely the $\mathbb{C}$-span of $\{\varepsilon^w\}_{w \in W^1_X \setminus \textcircled{H}}$. Moreover, for any $w \in W^1_X \setminus \textcircled{H}$, $\xi^w_{|\textcircled{H}} = 0$ by Proposition (4.24)(a). Hence the $\mathbb{C}$-span of $\{\sigma^w\}_{w \in W^1_X \setminus \textcircled{H}}$ is in the kernel of $\tilde{r}$. But then due to Lemma (5.15), by dimensional considerations, it is precisely the kernel of $\tilde{r}$. This proves the theorem. ∎

(5.17) *Remarks.* (a) Theorem (5.12) admits a more "algebraic" proof using an unpublished result of Kac and Peterson (which Peterson kindly told to the second author), asserting that $A_{r_i}\varepsilon^w = \varepsilon^{r_i w}$ if $r_i w < w$ and $=0$ otherwise, together with the "twisted derivation property." For this to be valid, they do not need the symmetrizability assumption on $\mathbf{g}$. In particular, Theorem (5.12), the Corollaries (5.13)(a), (c), (d), and Theorem (5.16) are true in the general (not necessarily symmetrizable) situation.

(b) In the finite case, Theorem (2.12) was proved by the first author (unpublished) and also independently by Carrell and Lieberman [5]. Recently (and quite independently) Akyildiz, Carrell, and Lieberman [1] have proved an analog of Theorem (5.16) in the special case when G is finite dimensional and $X=\varnothing$. Their methods are very different from ours and it is not clear if their proofs can be extended to the infinite case.

6. Connection between the D-Matrix and Harmonic Polynomials

(6.1) Let G be the group associated to an arbitrary Kac–Moody Lie algebra $\mathbf{g}=\mathbf{g}(A_l)$ and let K be the standard real form of G with "maximal torus" T (see (1.2)). Denote by β: $S(\mathbf{h}^{1*})\to H^*(K/T,\mathbb{C})$ the characteristic homomorphism, given by the classifying map: $K/T\to B(T)$ corresponding to the principal T-bundle $K\to K/T$, where $\mathbf{h}^1=\mathbf{h}\cap[\mathbf{g},\mathbf{g}]$ is the $\mathbb{C}$-span of $\{h_i\}_{1\leqslant i\leqslant l}$.

Recall the definition of the algebra Λ from (4.19). Let $\tilde{\beta}$: $S=S(\mathbf{h}^*)\to\Lambda$ be the $\mathbb{C}$-algebra homomorphism defined by

$$\tilde{\beta}(p)(w)=(-1)^{\deg p}w^{-1}p \qquad \text{for} \quad p\in S \text{ and } w\in W. \tag{I_{68}}$$

The fact that $\tilde{\beta}(p)\in\Lambda$, follows from Proposition (4.24)(c). It can be easily seen that the composite map $S\to^{\tilde{\beta}}\Lambda\to^{\mathrm{ev}_0}\mathbb{C}_0\otimes_S\Lambda$ (where "evaluation at 0" ev_0: $\Lambda\to\mathbb{C}_0\otimes_S\Lambda$ is given by $\xi\mapsto 1\otimes\xi$) factors through $S(\mathbf{h}^{1*})$. (See the proof of Lemma (5.3).) We denote the map, thus obtained, by $\bar{\beta}$: $S(\mathbf{h}^{1*})\to\mathbb{C}_0\otimes_S\Lambda$. We have

(6.2) Lemma. *With the notations as above,*

(a) *The following diagram is commutative:*

$$\begin{tikzcd} S(\mathbf{h}^{1*}) \arrow[rr, "\beta"] \arrow[dr, "\bar{\beta}"'] & & H^*(K/T) \arrow[dl, "\theta"] \\ & \mathbb{C}_0\otimes_S\Lambda & \end{tikzcd}$$

where θ is defined in the Theorem (5.12).

(b) *For any $p\in S(\mathbf{h}^{1*})$, $\bar{\beta}(E_{\alpha_i}p)=x_{r_i}\cdot\bar{\beta}(p)$, where E_{α_i} is the classical Bernstein–Gel'fand–Gel'fand operator defined in the proof of Proposition* (4.15) *and the element x_{r_i} is defined by* (I_{24}).

Proof. (a) Since all the maps in the triangle are algebra homomorphisms, if suffices to show that $\theta\beta(\chi)=\bar{\beta}(\chi)$, for all $\chi\in\mathbf{h}^1$. Of course, the span of $\{\chi_{i|\mathbf{h}^1}\}_{1\leqslant i\leqslant l}$ equals $\mathbf{h}^1$, where χ_i is as in (1.4). Now $\beta(\chi_{i|\mathbf{h}^1})=\varepsilon^{r_i}$ (as is known) and $\theta(\varepsilon^{r_i})=\sigma^{r_i}$ (by the definition of θ) and hence (a) follows by Proposition (4.24)(c).

(b) The map $\bar{\beta}$ clearly commutes with W-actions. So by Lemma (4.26) and the analogous property for E_{α_i} acting on $S(\mathbf{h}^{1*})$ (which can be proved similarly), we again only need to prove that $\bar{\beta}(E_{\alpha_i}(\chi))=x_{r_i}\cdot\bar{\beta}(\chi)$, for $\chi\in\mathbf{h}^{1*}$. Further, for $1\leqslant i,j\leqslant l$, $E_{\alpha_i}(\chi_{j|\mathbf{h}^1})=\delta_{i,j}$ and $x_{r_i}\cdot\bar{\beta}(\chi_{j|\mathbf{h}^1})=x_{r_i}\cdot\sigma^{r_j}=\delta_{i,j}$ by Proposition (4.24)(b). ∎

In the next theorem we show, in the finite case, how the D-matrix can be obtained from the harmonic polynomials. Recall from (3.9) that $\{\varepsilon^w\}_{w\in W}$ is a $\mathbb{Z}$-basis of $H^*(K/T,\mathbb{Z})$.

(6.3) THEOREM. *Let* $\mathbf{g}$ *be finite dimensional. For any* $w\in W$, *choose* $f^w\in S^{l(w)}(\mathbf{h}^*)$ *such that* $\beta(f^w)=\varepsilon^w$ *and consider the matrix* $F=(f_{v,w})_{v,w\in W}$, *where* $f_{v,w}=(-1)^{l(v)}w^{-1}(f^v)$. *Then the matrix* F *(over* $S=S(\mathbf{h}^*)$*) can be decomposed as*

$$F=E\cdot D$$

where D *is as defined in* (4.21) *and* $E=(e_{v,w})_{v,w\in W}$ *is a lower triangular matrix over* S *with diagonal entries* 1 *and, in fact,* $e_{v,w}$ *is a homogeneous polynomial of degree* $l(v)-l(w)$.

(6.4) *Remarks.* (a) Choice of $\{f^w\}$, as in Theorem (6.3), is always possible in the finite case since β is surjective (in this case). In fact we can choose for $\{f^w\}$, G-harmonic polynomials on $\mathbf{g}$ [18].

(b) Of course such a decomposition, as in the above theorem, is unique. We call D (resp. E) the upper (resp. lower) triangular part of F. In particular, the upper triangular part of F does not depend upon the choice of $\{f^w\}$.

(c) A less precise (but illuminating) way to describe the theorem is that "G-harmonic polynomials on $\mathbf{g}$ determine the D-matrix."

Proof of Theorem (6.3). Recall the definition of the map $\tilde{\beta}$: $S(\mathbf{h}^*)\to\Lambda$ from (6.1). By Proposition (4.20)(b), we can write

$$\tilde{\beta}(f^v)=\sum e_{v,w}\xi^w, \tag{I_{69}}$$

for some unique $e_{v,w}\in S$. Since, for any $w\in W$, $\tilde{\beta}(f^v)(w')$ is a homogeneous polynomial of degree $l(v)$ and $\xi^w(w')$ is a homogeneous polynomial of degree $l(w)$ by Proposition (4.24)(d), by the uniqueness of decomposition in (I_{69}), we get that $e_{v,w}$ is a homogeneous polynomial of degree $l(v)-l(w)$.

Now, from Lemma (6.2), it is easy to see that for any w with $l(w)=l(v)$, we have $e_{v,w}=\delta_{v,w}$, i.e., (I_{69}) reduces to

$$\tilde{\beta}(f^v)=\sum_{l(w)<l(v)} e_{v,w}\xi^w+\xi^v. \tag{I_{70}}$$

Evaluating (I_{70}) at w', by (I_{68}), we get $(-1)^{l(v)}(w'^{-1}f^v)=\sum_{l(w)<l(v)} e_{v,w}\xi^w(w')+\xi^v(w')$. ∎

(6.6) *Remark.* Since, in the finite case, the characteristic homomorphism β: $S(\mathbf{h}^*)\to H^*(K/T)$ is surjective, by complete reducibility of W-modules, we can choose a W-equivariant splitting s of β (e.g., G-harmonic polynomials on $\mathbf{g}$ provide one such splitting). By composing s with the W-equivariant map $\tilde{\beta}$: $S\to\Lambda$ (defined in (6.1)), we get a W-equivariant map $\tilde{\beta}\circ s$: $H^*(K/T)\to\Lambda$ which splits the surjective map $\theta^{-1}\circ \mathrm{ev}_0$: $\Lambda\to H^*(K/T)$, where ev_0 is defined in (6.1) and θ is defined in Theorem (5.12). Now, the W-equivariant map $\theta^{-1}\circ\mathrm{ev}_0$ is always surjective (i.e., even in the infinite dim case), and, we just have seen that, it admits a W-equivariant splitting in the finite case. But, in general, it can be seen that it does not admit any W-equivariant splitting. The counter example exists, e.g., in any affine case.

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