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A Homology Vanishing Theorem for Kac–Moody Algebras with Coefficients in the Category $\mathcal{O}$

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INTRODUCTION

In this paper, we study the homology of a Kac–Moody Lie-algebra, associated to a symmetrizable generalized Cartan matrix, with coefficients in a module M in the category $\mathcal{O}$. We show that all the homology groups vanish if the Casimir operator acts as an automorphism on M. In fact we prove a slightly more general theorem (Theorem (1.2)). This is one of the main theorems of this paper.

We further study the homology of $\mathfrak{g}$ with coefficients in arbitrary Verma modules and also integrable highest weight modules. We prove (Proposition (1.5)) that $H_i(\mathfrak{g}, M(\lambda)) = 0$ for all $i \geqslant 0$ and for all the Verma modules $M(\lambda)$ with highest weight $\lambda \in \mathfrak{h}^*$ such that λ is not equal to $w\rho - \rho$ for any $w \in W$, whereas $H_i(\mathfrak{g}, M(w\rho - \rho)) \approx \Lambda^{i-l(w)}(\mathfrak{h})$. This result in the case when $\mathfrak{g}$ is a finite dimensional semi-simple Lie-algebra is due to Williams.

If $L(\lambda)$ is an integrable highest weight module with $\lambda \neq 0$ (λ is automatically dominant integral) then we have, using BGG resolution and Proposition (1.5), that $H_i(\mathfrak{g}, L(\lambda)) = 0$ for all $i \geqslant 0$. This is the content of our Theorem (1.7). In the case when $\mathfrak{g}$ is finite dimensional, the integrable highest weight modules are precisely the finite dimensional irreducible modules. In this case, this theorem is well known and is due to Whitehead. As in the finite dimensional case, the situation when $\lambda = 0$ (so that $L(\lambda)$ is the trivial one dimensional module) is drastically different. We have recently proved [Ku; Theorem 1.6] that the homology of the commutator

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subalgebra $[\mathfrak{g}, \mathfrak{g}]$ with trivial coefficients is isomorphic with the singular homology of the associated algebraic group G. Kac–Peterson also claim to prove this result [K6]. A formula for the Poincaré series of $H_*(G)$ is given in [K6] in terms of the Weyl group. Further, $H_*(\mathfrak{g}, k)$ can be easily computed from $H_*([\mathfrak{g}, \mathfrak{g}], k)$. See Proposition (1.9).

Finally, it seems like a reasonable conjecture to make that $H_i(\mathfrak{g}, V(\lambda)) = 0$ for all the highest weight modules $V(\lambda)$ with highest weight $\lambda \in \mathfrak{h}^*$ such that λ is not equal to $w\rho - \rho$ for any $w \in W$.

The main difficulty in carrying over the proof of Theorem (1.2) from the finite dimensional semi-simple Lie-algebras to the Kac–Moody Lie-algebras lies in the fact that in the case of infinite dimensional Kac–Moody Lie-algebras, the Casimir operator does not operate on all the $\mathfrak{g}$-modules (e.g., it does not operate on $U(\mathfrak{g})$-free modules). In particular the Casimir operator is not represented by a central element in $U(\mathfrak{g})$. So the usual (almost trivial) homological algebra argument, which works for finite dimensional semi-simple Lie-algebras $\mathfrak{g}$ due to the presence of Casimir as a central element in $U(\mathfrak{g})$, does not carry over (without a change) to the general Kac–Moody Lie-algebras. At this point, it may be of some interest to note that Chari and Ilangovan have shown [CI] that the centre of $U(\mathfrak{g})$, in general, is too small to be of much interest.

To remedy this, we make use of an associative algebra $\hat{U}(\mathfrak{g})$, given by Chari and Ilangovan [CI], which contains $U(\mathfrak{g})$ as a subalgebra. The Casimir lies in $\hat{U}(\mathfrak{g})$ as a central element and moreover all the left $U(\mathfrak{g})$-modules M in the category $\mathcal{O}$ admit an $\hat{U}(\mathfrak{g})$-module structure extending the $U(\mathfrak{g})$-module structure on it. We prove the homology vanishing theorem by showing that (Lemma (2.3)) $H_i(\mathfrak{g}, M)$ $(= \mathrm{Tor}_i^{U(\mathfrak{g})}(k, M)) \approx \mathrm{Tor}_i^{\hat{U}(\mathfrak{g})}(k, M)$ for all left $\hat{U}(\mathfrak{g})$-modules M. The main ingredient in the proof of this isomorphism is the following proposition (Proposition (2.1)):

$$H_i(\mathfrak{g}, \hat{U}(\mathfrak{g})) = 0 \qquad \text{for all } i \geqslant 1.$$

The proof of this proposition occupies Section (2). Once we have this proposition at our disposal, the usual homological algebra (see, e.g., [G]) and a lemma on the structure of $\hat{U}(\mathfrak{g})$ (Lemma (2.2)) take care of the rest. I feel that the vanishing of $H_i(\mathfrak{g}, \hat{U}(\mathfrak{g}))$ may be of interest elsewhere.

Section (0) is devoted to preliminaries and notations to be used throughout. In Section 1 we state our results. The main results are Theorems (1.2), and (1.7). Proof of Theorem (1.2) is postponed to Section 2, but Theorem (1.7) is proved in this section itself. Section 2 is fully devoted to the proof of Theorem (1.2).

0. Preliminaries and Notations

(0.1) k, throughout, will denote a field of characteristic 0. All the vector spaces are considered over k. For a vector space V, V^* will denote $\mathrm{Hom}_k(V, k)$. For a Lie-algebra $\mathfrak{g}$, its universal enveloping algebra, as usual, will be denoted by $U(\mathfrak{g})$. Let M be a left $U(\mathfrak{g})$-module; by M^t we would mean the right $U(\mathfrak{g})$-module with the underlying space being the same as M and the action being $m \cdot a = T(a) \cdot m$, for $m \in M$ and $a \in U(\mathfrak{g})$, where T is the unique anti-automorphism of $U(\mathfrak{g})$ which is -1 on $\mathfrak{g}$. Modules will be left unless explicitly stated. $H_n(\mathfrak{g}, M)$ would mean $\mathrm{Tor}_n^{U(\mathfrak{g})}(k, M) \approx \mathrm{Tor}_n^{U(\mathfrak{g})}(M^t, k)$, for any $n \geqslant 0$.

(0.2) *A symmetrizable generalized Cartan matrix* $A = (a_{ij})_{1 \leqslant i,j \leqslant l}$ is a matrix of integers satisfying $a_{ii} = 2$ for all i, $a_{ij} \leqslant 0$ if $i \neq j$ and DA is symmetric for some diagonal matrix $D = \mathrm{diag.}\,(q_1, \ldots, q_l)$ with $q_i > 0 \in \mathbb{Q}$.

Choose a triple $(\mathfrak{h}, \pi, \pi^v)$, unique up to isomorphism, where $\mathfrak{h}$ is a vector space over k of dimension $l +$ co-rank A, $\pi = \{\alpha_i\}_{1 \leqslant i \leqslant l} \subset \mathfrak{h}^*$ and $\pi^v = \{h_i\}_{1 \leqslant i \leqslant l} \subset \mathfrak{h}$ are linearly independent indexed sets satisfying $\alpha_j(h_i) = a_{ij}$. The *Kac–Moody Algebra* $\mathfrak{g} = \mathfrak{g}(A)$ is the Lie-algebra over k, generated by $\mathfrak{h}$ and the symbols e_i and f_i $(1 \leqslant i \leqslant l)$ with the defining relations $[\mathfrak{h}, \mathfrak{h}] = 0$; $[h, e_i] = \alpha_i(h)\, e_i$, $[h, f_i] = -\alpha_i(h)\, f_i$ for $h \in \mathfrak{h}$ and all $1 \leqslant i \leqslant l$; $[e_i, f_j] = \delta_{ij} h_j$ for all $1 \leqslant i, j \leqslant l$; $(\mathrm{ad}\, e_i)^{1 - a_{ij}}(e_j) = 0 = (\mathrm{ad}\, f_i)^{1 - a_{ij}}(f_j)$ for all $1 \leqslant i \neq j \leqslant l$.

$\mathfrak{h}$ is canonically embedded in $\mathfrak{g}$. Also, any ideal of $\mathfrak{g}$ that intersects $\mathfrak{h}$ trivially is zero itself by [GK].

(0.3) *Root space decomposition* [K4]. There is available the root space decomposition $\mathfrak{g} = \mathfrak{h} \oplus \sum_{\alpha \in \Delta \subset \mathfrak{h}^*} \mathfrak{g}_\alpha$, where $\mathfrak{g}_\alpha = \{x \in \mathfrak{g} : [h, x] = \alpha(h)\, x$, for all $h \in \mathfrak{h}\}$ and $\Delta = \{\alpha \in \mathfrak{h}^* \backslash \{0\}$ such that $\mathfrak{g}_\alpha \neq 0\}$. Moreover, $\Delta = \Delta_+ \cup \Delta_-$, where $\Delta_+ \subset \{\sum_{i=1}^l n_i \alpha_i : n_i \in \mathbb{Z}_+$ (= the non-negative integers) for all $i\}$ and $\Delta_- = -\Delta_+$. Elements of Δ_+ (resp. Δ_-) are called *positive* (resp. *negative*) *roots*.

Define the following Lie-subalgebras:

$$\mathfrak{n} = \sum_{\alpha \in \Delta_+} \mathfrak{g}_\alpha; \qquad \mathfrak{n}^- = \sum_{\alpha \in \Delta_-} \mathfrak{g}_\alpha; \qquad \mathfrak{b} = \mathfrak{h} \oplus \mathfrak{n}; \qquad \mathfrak{b}^- = \mathfrak{h} \oplus \mathfrak{n}^-.$$

The Lie-algebra $\mathfrak{n}$ (resp. $\mathfrak{n}^-$) is graded with respect to the grading $|x| = \Sigma n_i$ for $x \in \mathfrak{g}_\alpha$ (resp. $x \in \mathfrak{g}_{-\alpha}$), where $\alpha = \sum_{i=1}^l n_i \alpha_i$. This gives rise to a grading on $U(\mathfrak{n})$ (resp. $U(\mathfrak{n}^-)$). We denote by $U_d(\mathfrak{n})$ (resp. $U_d(\mathfrak{n}^-)$) the homogeneous elements of $U(\mathfrak{n})$ (resp. $U(\mathfrak{n}^-)$) of grade degree d.

(0.4) *Symmetric bilinear form on* $\mathfrak{h}^*$ [K4]. Define $\sigma(\alpha_i, \alpha_j) = q_i a_{ij}$, for

$1 \leqslant i, j \leqslant l$. Set $h_{\alpha_i} = q_i h_i$. It is possible to extend σ to a nondegenerate symmetric bilinear form, again denoted by σ, on $\mathfrak{h}^*$ satisfying

$$\sigma(\lambda, \alpha_i) = \lambda(h_{\alpha_i}), \qquad \text{for all } 1 \leqslant i \leqslant l \text{ and all } \lambda \in \mathfrak{h}^*.$$

We fix one such σ. Identifying $\mathfrak{h}$ with $\mathfrak{h}^*$, this form extends to a nondegenerate invariant form on $\mathfrak{g}$.

(0.5) The category $\mathcal{O}$ denotes the full category of all the left $\mathfrak{g}$-modules M, which are $\mathfrak{h}$-diagonalizable with finite dimensional weight spaces and whose set of weights lies in a finite union of sets of the form

$$D(\nu)\left(= \left\{ \nu - \sum_{i=1}^{l} n_i \alpha_i, n_i \in Z_+ \right\} \right), \qquad \text{for} \quad \nu \in \mathfrak{h}^*.$$

(0.6) *The Casimir operator* [K2, K4]. For each positive root α, choose a basis $\{e^i_\alpha\}$ of the space $\mathfrak{g}_\alpha$ and let $\{e^i_{-\alpha}\}$ be the dual basis of $\mathfrak{g}_{-\alpha}$. Define an operator, called the *Casimir operator*,

$$\Omega = \sum_k u^k u_k + 2 \sum_{\alpha \in \Delta_+} \sum_i e^i_{-\alpha} e^i_\alpha + 2\nu^{-1}(\rho),$$

where $\nu\colon \mathfrak{h} \to \mathfrak{h}^*$ is the canonical identification, $\{u^k\}$ and $\{u_k\}$ are any dual bases of $\mathfrak{h}$, and ρ is any fixed element of $\mathfrak{h}^*$ satisfying $\rho(h_i) = 1$ for all $1 \leqslant i \leqslant l$.

Ω acts on every module of the category $\mathcal{O}$. In fact, Ω is a natural transformation of the category $\mathcal{O}$. Further, Ω acts as scalar multiplication by $[\sigma(\lambda+\rho, \lambda+\rho) - \sigma(\rho, \rho)]$ on any highest weight module $V(\lambda)$ with highest weight λ.

1. Statement of the Main Results

(1.1) *Description of the algebra* $\hat{U}(\mathfrak{g})$ [CI]. Let $\mathfrak{g} = \mathfrak{g}(A)$ be the Kac–Moody Lie-algebra, associated to a symmetrizable generalized Cartan matrix A. We briefly recall the definition of an associative algebra $\hat{U}(\mathfrak{g})$, which is in some sense a completion of $U(\mathfrak{g})$, introduced by Vyjayanthi Chari and S. Ilangovan [CI].

Define $\hat{U}(\mathfrak{g}) = \prod_{d \geqslant 0} (U(\mathfrak{b}^-) \otimes_k U_d(\mathfrak{n}))$. For the description of the product structure in $\hat{U}(\mathfrak{g})$, see [CI]. The algebra $U(\mathfrak{g})$ sits inside $\hat{U}(\mathfrak{g})$ as $\sum_{d \geqslant 0} U(\mathfrak{b}^-) \otimes_k U_d(\mathfrak{n})$. (For various notations, see the previous section.)

Any left $U(\mathfrak{g})$-module M in the category $\mathcal{O}$ admits a left $\hat{U}(\mathfrak{g})$-module structure, extending the $U(\mathfrak{g})$-module structure on M. Further the Casimir operator Ω (described in Section (0.6)) can (and will) be viewed as an element (again denoted by) Ω of $\hat{U}(\mathfrak{g})$ in the sense that the action of the Casimir operator Ω on any module M in the category $\mathcal{O}$ is the same as multiplication by the element Ω ($\in \hat{U}(\mathfrak{g})$) in the extended $\hat{U}(\mathfrak{g})$-module M. Moreover, it can be seen that Ω is central in $\hat{U}(\mathfrak{g})$.

Now we are in position to state one of our main theorems.

(1.2) THEOREM. *Let $\mathfrak{g} = \mathfrak{g}(A)$ be the Kac–Moody Lie-algebra associated to a symmetrizable generalized Cartan matrix A and let $\hat{U}(\mathfrak{g})$ be the algebra described above. Let M be a left $\hat{U}(\mathfrak{g})$-module such that the element $\Omega \in \hat{U}(\mathfrak{g})$ acts as an automorphism on M then*

$$H_i(\mathfrak{g}, M) = 0 \qquad \textit{for all } i \geqslant 0.$$

(*M is viewed as a $\mathfrak{g}$-module by restriction.*)

We prove this theorem in the next section. Also, we have the following simple proposition.

(1.3) PROPOSITION. *Let $\mathfrak{g}$ be as in Theorem (1.2) and let $V(\lambda)$ be any highest weight $\mathfrak{g}$-module with highest weight λ. Further assume that $\lambda \notin \sum_{i=1}^{l} \mathbb{Z}\alpha_i$, then again we have*

$$H_i(\mathfrak{g}, V(\lambda)) = 0 \qquad \textit{for all } i \geqslant 0.$$

Proof. Consider the standard chain complex $\{\Lambda_i = \Lambda_i(\mathfrak{g}, V(\lambda))\}_{i \geqslant 0}$ (to compute the homology $H(\mathfrak{g}, V(\lambda))$), where $\Lambda_i(\mathfrak{g}, V(\lambda)) = \Lambda^i(\mathfrak{g}) \otimes_k V(\lambda)$ and $d\colon \Lambda_i \to \Lambda_{i-1}$ is given by

$$\begin{aligned} d(v_1 \wedge \cdots \wedge v_i \otimes a) &= \sum_{1 \leqslant p < q \leqslant i} (-1)^{p+q} [v_p, v_q] \wedge v_1 \wedge \cdots \wedge \hat{v}_p \wedge \cdots \wedge \hat{v}_q \wedge \cdots \wedge v_i \otimes a \\ &\quad + \sum_{p=1}^{i} (-1)^p v_1 \wedge \cdots \wedge \hat{v}_p \wedge \cdots \wedge v_i \otimes (v_p \cdot a), \end{aligned}$$

for $v_1, \ldots, v_i \in \mathfrak{g}$ and $a \in V(\lambda)$.

Clearly d is a $\mathfrak{g}$-module (in particular an $\mathfrak{h}$-module) map under the canonical $\mathfrak{g}$-module structure on Λ_i's. It is well known (and easy to prove) that $\mathfrak{g}$ (and hence $\mathfrak{h}$) acts trivially on $H_i(\mathfrak{g}, V(\lambda))$. Further, Λ_i being weight modules, we have $H_*(\mathfrak{g}, V(\lambda)) \cong H_*([\Lambda]^{\mathfrak{h}})$ where $[\Lambda]^{\mathfrak{h}}$ denotes the sub-complex $\{\Lambda_i^{\mathfrak{h}}\}_{i \geqslant 0}$ consisting of $\mathfrak{h}$-invariant elements in Λ_i.

Weights of Λ_i are of the form $\lambda + \sum_{i=1}^{l} \mathbb{Z}\alpha_i$. Since, by assumption, $\lambda \notin \sum_{i=1}^{l} \mathbb{Z}\alpha_i$, the zero weight space of Λ_i is equal to 0. This proves the proposition.

Combining Theorem (1.2) and the above proposition, we get the following.

(1.4) COROLLARY. *Let M be any $\mathfrak{g}$-module in the category $\mathcal{O}$. As is known [GL; Lemma 4.4], there exists a $\mathfrak{g}$-module filtration $0 = M_0 \subset M_1 \subset M_2 \subset \cdots$, such that $M = \bigcup_j M_j$ and each $\mathfrak{g}$-module M_{j+1}/M_j $(j \geq 0)$ is a highest weight module with highest weight λ_j. Assume that, for any $j \geq 0$, either $\sigma(\lambda_j + \rho, \lambda_j + \rho) - \sigma(\rho, \rho) \neq 0$ or $\lambda_j \notin \sum_{i=1}^{l} \mathbb{Z}\alpha_i$. Then $H_i(\mathfrak{g}, M) = 0$ for all $i \geq 0$.*

Proof. Since the Casimir Ω acts on M_{j+1}/M_j as scalar multiplication by $\sigma(\lambda_j + \rho, \lambda_j + \rho) - \sigma(\rho, \rho)$, successively using the long exact sequence (see, e.g., [H; Sect. 2]), corresponding to the coefficient sequence

$$0 \to M_j \to M_{j+1} \to M_{j+1}/M_j \to 0,$$

we get that $H_*(\mathfrak{g}, M_j) = 0$ for all j. Since the functor Tor commutes with direct limits, we get the corollary.

We compute the homology of $\mathfrak{g}$ with coefficients in arbitrary Verma modules and also the integrable highest weight modules. We have the following.

(1.5) PROPOSITION. *Let $M(\lambda)$ be the Verma module corresponding to any highest weight $\lambda \in \mathfrak{h}^*$. Then:*

(a) *If $\lambda \neq w\rho - \rho$ for any $w \in W$ (the Weyl group associated to $\mathfrak{g}$), we have*

$$H_i(\mathfrak{g}, M(\lambda)) = 0 \qquad \text{for all } i \geq 0.$$

(b) *If $\lambda = w_0\rho - \rho$ for some $w_0 \in W$, then*

$$H_i(\mathfrak{g}, M(\lambda)) \cong \Lambda^{i - l(w_0)}(\mathfrak{h})$$

($l(w_0)$ denotes the length of w_0). In particular, $H_i(\mathfrak{g}, M(\lambda)) = 0$ if $i < l(w_0)$.

(1.6) *Remark.* This result (stated in cohomology) in the case when $\mathfrak{g}$ is a finite dimensional semi-simple Lie-algebra is due to Williams [W; Theorem 4.15]. Our proof is very similar to his.

Proof. Let $k_\lambda = k$ be the left $\mathfrak{b}$-module so that $\mathfrak{n}$ acts trivially on k_λ and $\mathfrak{h}$ acts by the weight λ. By the definition, $M(\lambda) = U(\mathfrak{g}) \otimes_{U(\mathfrak{b})} k_\lambda$. Now by [CE; Proposition 4.2, p. 275], we have (observe that there is a slight difference between our notation and the notation used in [CE]. We are denoting $\mathrm{Tor}_n^{U(\mathfrak{g})}(k, M)$ by $H_n(\mathfrak{g}, M)$, whereas it is denoted by $H_n(\mathfrak{g}, M')$ in [CE].)

$$H_n(\mathfrak{b}, k_\lambda) \approx H_n(\mathfrak{g}, [k_\lambda^t \otimes_{U(\mathfrak{b})} U(\mathfrak{g})]^t) \qquad (*)$$

(Recall that for a left module M, M' is defined in Section (0.1) and similarly we can define a left module M' if we start with a right module M.)

But $[k_\lambda^t \otimes_{U(\mathfrak{b})} U(\mathfrak{g})]'$ can be easily seen to be isomorphic with the Verma module $M(\lambda)$. So, from (*), we have

$$H_n(\mathfrak{g}, M(\lambda)) \approx H_n(\mathfrak{b}, k_\lambda).$$

The Hochschild–Serre spectral sequence for homology (see [CE; p. 351]), applied to the pair $(\mathfrak{b}, \mathfrak{n})$, has $E_{p,q}^2 = H_p(\mathfrak{h}, H_q(\mathfrak{n}, k_\lambda))$ and it converges to $H_*(\mathfrak{b}, k_\lambda)$. Since $\mathfrak{n}$ acts trivially on k_λ, we get that $H_q(\mathfrak{n}, k_\lambda)$ is isomorphic with $H_q(\mathfrak{n}, k) \otimes k_\lambda$ as $\mathfrak{h}$-modules. By [GL; Theorem 8.6], we have

$$H_q(\mathfrak{n}, k) \approx \sum_{\substack{w \in W \text{ with} \\ l(w) = q}} k_{\rho - w\rho}, \qquad \text{as } \mathfrak{h}\text{-modules.}$$

Further, since $\mathfrak{h}$ acts reductively on $H_q(\mathfrak{n}, k) \otimes_k k_\lambda$, we have $H_p(\mathfrak{h}, H_q(\mathfrak{n}, k) \otimes_k k_\lambda) \approx H_p(\mathfrak{h}, [H_q(\mathfrak{n}, k) \otimes_k k_\lambda]^{\mathfrak{h}})$ ($[H_q(\mathfrak{n}, k) \otimes_k k_\lambda]^{\mathfrak{h}}$ denotes $\mathfrak{h}$-invariants). This, in particular, gives that in the case (a), $E_{p,q}^2 = 0$ for all p and q, and in the case (b), $[H_q(\mathfrak{n}, k) \otimes_k k_\lambda]^{\mathfrak{h}} = 0$ for $q \neq l(w_0)$ and $[H_{l(w_0)}(\mathfrak{n}, k) \otimes_k k_\lambda]^{\mathfrak{h}}$ is one dimensional. So $E_{p,q}^2 = 0$ for $q \neq l(w_0)$ and $E_{p,l(w_0)}^2 \approx H_p(\mathfrak{h}, k) = \Lambda^p(\mathfrak{h})$.

In particular, the spectral sequence degenerates at the E^2 term itself, i.e., $E^2 \approx E^\infty$. This proves the proposition.

Finally, we have the following generalization, of a theorem of Whitehead [J; Theorem 14, p. 96], to arbitrary symmetrizable Kac–Moody Lie-algebras.

(1.7) THEOREM.[1] *Let $L(\lambda)$ be an integrable highest weight module (also called standard or quasi-simple module) with highest weight $\lambda \neq 0$ (λ is automatically dominant integral). Then, we have*

$$H_i(\mathfrak{g}, L(\lambda)) = 0 \qquad \textit{for all } i \geq 0.$$

Proof. We recall the BGG resolution given in the general case by Garland–Lepowsky [GL; Theorem 8.7]. There is an exact sequence of $U(\mathfrak{g})$-modules:

$$\cdots \to E_2 \to E_1 \to E_0 \to L(\lambda) \to 0,$$

[1] This result, as well as the vanishing of $H^*(\mathfrak{g}, L(\lambda))$, has been obtained earlier by M. Duflo (unpublished, private communication to Victor Kac) by using J. Lepowsky's Theorem (6.6) in Generalized Verma modules, loop space cohomology and Macdonald-type identities, *Ann. Sci. École Norm. Sup.* **12** (1979), 169–234.

where each E_j has a filtration

$$0 = E_j^0 \subset E_j^1 \subset \cdots \subset E_j^{n(j)} = E_j$$

such that, for all $0 \leqslant i \leqslant n(j) - 1$, E_j^{i+1}/E_j^i is a Verma module $M(\lambda_j^i)$ with highest weight λ_j^i. Moreover, up to a rearrangement, the set $\{\lambda_j^i\}_{0 \leqslant i \leqslant n(j)-1}$ coincides with $\{w(\lambda+\rho)-\rho\}_{w \in W \text{ with } l(w)=j}$. By Proposition (1.5), $H_*(\mathfrak{g}, M(w(\lambda+\rho)-\rho)) = 0$ for any $w \in W$, since $w(\lambda+\rho)-\rho$ is not of the form $w_0\rho - \rho$ for any $w_0 \in W$. (Otherwise $0 \neq \lambda = w^{-1}w_0\rho - \rho$. But $w^{-1}w_0\rho - \rho = -\sum_{i=1}^{l} n_i\alpha_i$ with $n_i \in \mathbb{Z}_+$, which is not possible in view of [GL, Proposition 2.12].)

By the same argument used in the proof of Corollary (1.4), we get $H_*(\mathfrak{g}, E_j) = 0$ for all $j \geqslant 0$. But then $H_*(\mathfrak{g}, L(\lambda)) \approx \operatorname{Tor}_*^{U(\mathfrak{g})}(k, L(\lambda))$ is the homology of the complex (see the proof of Lemma 2.3))

$$\cdots \to k \otimes_{U(\mathfrak{g})} E_j \to k \otimes_{U(\mathfrak{g})} E_{j-1} \to \cdots \to k \otimes_{U(\mathfrak{g})} E_0 \to 0.$$

But $k \otimes_{U(\mathfrak{g})} E_j = E_j/U^+(\mathfrak{g}) \cdot E_j = 0$ (since clearly $M(\lambda)/U^+(\mathfrak{g}) \cdot M(\lambda) = 0$ for $\lambda \neq 0$).

Hence the theorem follows.

(1.8) *Remarks.* (a) The above theorem can also be deduced immediately from our Theorem (1.2) for those $L(\lambda)$ such that $\sigma(\lambda+\rho, \lambda+\rho) - \sigma(\rho, \rho) \neq 0$. In the case when $\mathfrak{g}$ is finite dimensional, affine, or hyperbolic, it is indeed true that $\sigma(\lambda+\rho, \lambda+\rho) - \sigma(\rho, \rho) \neq 0$ for all dominant integral elements $\lambda \neq 0$. (See [K3; Sect. 1].) But in general (as Victor Kac has pointed out) it is not true. The following example is due to him. Take

$$A = \begin{pmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -6 & 0 \\ 0 & -6 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{pmatrix},$$

then $3\rho = \alpha_1 - \alpha_2 - \alpha_3 + \alpha_4$ and $\sigma(\rho, \rho) = 0$. So, if we take $\mu = k\rho$ $(k > 0)$, then $\sigma(\mu+\rho, \mu+\rho) - \sigma(\rho, \rho) = 0$.

(b) We prove [Ku; Theorem 1.6] that $H_*([\mathfrak{g}, \mathfrak{g}], k)$ ($[\mathfrak{g}, \mathfrak{g}]$ denotes the commutator subalgebra of $\mathfrak{g}$) is isomorphic with the singular homology of the corresponding algebraic group G (or its subgroup K). See also [K5]. (G and K are defined in [KP].)

We come to the computation of $H_*(\mathfrak{g}, k)$. Consider the canonical surjection $\pi: \mathfrak{g} \to \mathfrak{g}/[\mathfrak{g}, \mathfrak{g}]$. From the root space decomposition, $\pi(\mathfrak{h}) = \mathfrak{g}/[\mathfrak{g}, \mathfrak{g}]$.

Hence there is a Lie-algebra splitting s: $\mathfrak{g}/[\mathfrak{g}, \mathfrak{g}] \to \mathfrak{g}$ (i.e., s is a Lie-algebra homomorphism and $\pi \circ s = \mathrm{Id}$.). Fix any $\lambda \in \mathfrak{h}^*$ such that $\lambda|_{\mathfrak{h} \cap [\mathfrak{g}, \mathfrak{g}]} = 0$. We define a 1-dimensional $\mathfrak{g}$-module k_λ by demanding that $[\mathfrak{g}, \mathfrak{g}]$ acts trivially on k_λ and $\mathfrak{h}$ acts by the weight λ on k_λ. From an analogue (though proved in cohomology and under the assumption that $\mathfrak{g}$ is finite dimensional, it is not difficult to see that their proof can be easily adopted to our situation) of [HS; Theorem 13], we get

$$H_*(\mathfrak{g}, k_\lambda) \approx H_*(\mathfrak{g}/[\mathfrak{g}, \mathfrak{g}], k) \otimes [H_*([\mathfrak{g}, \mathfrak{g}], k_\lambda)]^{\mathfrak{g}},$$

i.e., (since $\mathfrak{g}/[\mathfrak{g}, \mathfrak{g}]$ is abelian)

$$H_*(\mathfrak{g}, k_\lambda) \approx \Lambda(\mathfrak{g}/[\mathfrak{g}, \mathfrak{g}]) \otimes [H_*([\mathfrak{g}, \mathfrak{g}], k_\lambda)]^{\mathfrak{g}}. \tag{$**$}$$

Next we prove that $H_*([\mathfrak{g}, \mathfrak{g}], k)$ is trivial as a $\mathfrak{g}$-module. For, if not, let k_λ $(\lambda \neq 0)$ be a direct summand of $H_*([\mathfrak{g}, \mathfrak{g}], k)$ as an $\mathfrak{h}$-module. (Since $H_*([\mathfrak{g}, \mathfrak{g}], k)$ is trivial as a $[\mathfrak{g}, \mathfrak{g}]$-module, λ satisfies that $\lambda|_{\mathfrak{h} \cap [\mathfrak{g}, \mathfrak{g}]} = 0$.) From (**), we get

$$\begin{aligned} H_*(\mathfrak{g}, k_{-\lambda}) &\approx \Lambda(\mathfrak{g}/[\mathfrak{g}, \mathfrak{g}]) \otimes [H_*([\mathfrak{g}, \mathfrak{g}], k_{-\lambda})]^{\mathfrak{g}} \\ &\approx \Lambda(\mathfrak{g}/[\mathfrak{g}, \mathfrak{g}]) \otimes [H_*([\mathfrak{g}, \mathfrak{g}], k) \otimes k_{-\lambda}]^{\mathfrak{g}} \quad (\text{since } H_*([\mathfrak{g}, \mathfrak{g}], k_{-\lambda}) \\ &\qquad\qquad \approx H_*([\mathfrak{g}, \mathfrak{g}], k) \otimes k_{-\lambda} \text{ as } \mathfrak{g}\text{-modules}) \\ &\neq 0 \qquad (\text{since, by assumption, } k_\lambda \text{ is a direct summand of } \\ &\qquad\qquad H_*([\mathfrak{g}, \mathfrak{g}], k)). \end{aligned}$$

But by Theorem (1.7), $H_*(\mathfrak{g}, k_{-\lambda}) = 0$ ($h_i \in [\mathfrak{g}, \mathfrak{g}]$, so $\lambda(h_i) = 0$ and hence $-\lambda$ is dominant integral). This contradiction proves that $H_*([\mathfrak{g}, \mathfrak{g}], k)$ is trivial as a $\mathfrak{g}$-module as well. From (**) (taking $\lambda = 0$), we get the following

(1.9) PROPOSITION. $H_*(\mathfrak{g}, k) \approx \Lambda(\mathfrak{g}/[\mathfrak{g}, \mathfrak{g}]) \otimes H_*([\mathfrak{g}, \mathfrak{g}], k)$.

(1.10) *Conjecture.* Let $\lambda \in \mathfrak{h}^*$ be an element such that $\lambda \neq w\rho - \rho$ for any $w \in W$. It seems reasonable to conjecture that $H_i(\mathfrak{g}, V(\lambda)) = 0$ for all $i \geqslant 0$ and for any highest weight module $V(\lambda)$ with highest weight λ.

This conjecture is trivially true in the case when $\mathfrak{g}$ is finite dimensional, because the centre of $U(\mathfrak{g})$ separates the Weyl group orbits.

2. PROOF OF THEOREM (1.2)

The main ingredient in the proof of this theorem is the following proposition.

(2.1) PROPOSITION. $H_p(\mathfrak{g}, \hat{U}(\mathfrak{g}))=0$ *for all* $p \geq 1$, *where* $\mathfrak{g}$ *acts on* $\hat{U}(\mathfrak{g})$ *as the left multiplication.*

We postpone the proof of this proposition and first use it to derive our main theorem (1.2). We have the following lemmas.

(2.2) LEMMA. *The canonical map*

$$k \otimes_{U(\mathfrak{g})} \hat{U}(\mathfrak{g}) \to k \otimes_{\hat{U}(\mathfrak{g})} \hat{U}(\mathfrak{g})$$

is an isomorphism.

Proof. Let $\hat{U}(\mathfrak{g})^+ = (U(\mathfrak{b}^-))^+ \oplus \prod_{d \geq 1} [U(\mathfrak{b}^-) \otimes_k U_d(\mathfrak{n})]$, where $(U(\mathfrak{b}^-))^+$ denotes the two-sided ideal generated by $\mathfrak{b}^-$ in $U(\mathfrak{b}^-)$. It suffices to prove that any element a in $\hat{U}(\mathfrak{g})^+$ can be written as $a=\sum_{i=1}^m X_i a_i$, for some $X_i \in \mathfrak{g}$ and $a_i \in \hat{U}(\mathfrak{g})$. This follows essentially from the fact that the algebras $U(\mathfrak{b}^-)$ and $U(\mathfrak{n})$ are both finitely generated.

Write $a \in \hat{U}(\mathfrak{g})^+$ as $\sum_{d \geq 0} a_d$ with $a_d \in U(\mathfrak{b}^-) \otimes_k U_d(\mathfrak{n})$. Further, any a_d can be written as

$$a_d = \sum_{i=1}^{l} f_i b_d^i + \sum_{i=1}^{l} e_i c_d^i + \sum_{j=1}^{\dim \mathfrak{h}} \bar{h}_j e_d^j,$$

with b_d^i and $e_d^j \in U(\mathfrak{b}^-) \otimes U_d(\mathfrak{n})$ for all i and j and $c_d^i \in U_{d-1}(\mathfrak{n})$ (e_i, f_i are defined in section (0.2) and $\bar{h}_j$ is any k-basis of $\mathfrak{h}$).

Summing over d, this gives the lemma.

(2.3) LEMMA. $\operatorname{Tor}_i^{U(\mathfrak{g})}(k, M) \approx \operatorname{Tor}_i^{\hat{U}(\mathfrak{g})}(k, M)$, *for any left* $\hat{U}(\mathfrak{g})$*-module* M *and for all* i.

Proof. Let us take a $\hat{U}(\mathfrak{g})$-free resolution of the $\hat{U}(\mathfrak{g})$-module M

$$\cdots \to F_2 \to F_1 \to F_0 \to M \to 0.$$

From Proposition (2.1), $\operatorname{Tor}_i^{U(\mathfrak{g})}(k, F_j)=0$ for all $i>0$ and all $j \geq 0$, since $H_i(\mathfrak{g}, \hat{U}(\mathfrak{g}))$ is defined to be $\operatorname{Tor}_i^{U(\mathfrak{g})}(k, \hat{U}(\mathfrak{g}))$. So $\operatorname{Tor}^{U(\mathfrak{g})}(k, M)$ is given by (see [G]) the homology of the complex

$$\cdots \to k \otimes_{U(\mathfrak{g})} F_i \to k \otimes_{U(\mathfrak{g})} F_{i-1} \to \cdots \to k \otimes_{U(\mathfrak{g})} F_0 \to 0.$$

Of course, $\operatorname{Tor}^{\hat{U}(\mathfrak{g})}(k, M)$ is given by the homology of the complex

$$\cdots \to k \otimes_{\hat{U}(\mathfrak{g})} F_i \to k \otimes_{\hat{U}(\mathfrak{g})} F_{i-1} \to \cdots \to k \otimes_{\hat{U}(\mathfrak{g})} F_0 \to 0.$$

We have the following commutative diagram:

$$\begin{array}{ccccccc}
\cdots \to k\otimes_{U(\mathfrak{g})} F_i & \to & k\otimes_{U(\mathfrak{g})} F_{i-1} & \to \cdots\cdots\cdots \to & k\otimes_{U(\mathfrak{g})} F_0 & \to 0 \\
\downarrow & & \downarrow & & \downarrow & \\
\cdots \to k\otimes_{\hat{U}(\mathfrak{g})} F_i & \to & k\otimes_{\hat{U}(\mathfrak{g})} F_{i-1} & \to \cdots\cdots\cdots \to & k\otimes_{\hat{U}(\mathfrak{g})} F_0 & \to 0.
\end{array}$$

From Lemma (2.2), all the vertical maps are isomorphisms. This proves the lemma.

(2.4) *Proof of Theorem* (1.2). Since $H_i(\mathfrak{g}, M)$ is defined to be $\mathrm{Tor}_i^{U(\mathfrak{g})}(k, M)$, in view of Lemma (2.3), it suffices to show that $\mathrm{Tor}_i^{\hat{U}(\mathfrak{g})}(k, M)=0$. But the Casimir Ω is a central element in the algebra $\hat{U}(\mathfrak{g})$, which acts as 0 on k and as an automorphism on M (by assumption). From this, the vanishing of $\mathrm{Tor}_i^{\hat{U}(\mathfrak{g})}(k, M)$ follows easily. ∎

We come to the proof of Proposition (2.1). Let us recall the following well known facts, which will be used often.

(1) $H_i(\mathfrak{a}, \mathfrak{a}', M)^* \approx H^i(\mathfrak{a}, \mathfrak{a}', M^*)$, for any Lie-algebra $\mathfrak{a}$ and a subalgebra $\mathfrak{a}'$ with M being a left $\mathfrak{a}$-module.

(2) The map $e\colon \mathrm{Hom}_k(V, W^*) \to [V \otimes_k W]^*$ defined by $e(f)(v\otimes w)=f(v)(w)$, for $f\in \mathrm{Hom}_k(V, W^*)$, $v\in V$ and $w\in W$, is an $\mathfrak{a}$-module isomorphism for any left $\mathfrak{a}$-modules V and W.

(2.5) LEMMA. $H_p(\mathfrak{g}, \hat{U}(\mathfrak{g})) \approx H_p(\mathfrak{g}, \mathfrak{b}^-, \hat{U}(\mathfrak{g}))$, *for all* $p\geqslant 0$.

Proof. The Hochschild–Serre spectral sequence [HS; Sect. 2], applied to the Lie-algebra pair $(\mathfrak{g}, \mathfrak{b}^-)$ with coefficients in $\hat{U}(\mathfrak{g})^*$, is a convergent spectral sequence with

$$E_1^{p,q}=H^q(\mathfrak{b}^-, \mathrm{Hom}_k(\Lambda^p(\mathfrak{g}/\mathfrak{b}^-), \hat{U}(\mathfrak{g})^*)) \approx H^q(\mathfrak{b}^-, [\Lambda^p(\mathfrak{n}) \otimes_k \hat{U}(\mathfrak{g})]^*)$$

and converging to $H^*(\mathfrak{g}, \hat{U}(\mathfrak{g})^*)$.

In a subsequent lemma (Lemma (2.7)), we prove that $H^q(\mathfrak{b}^-, [\Lambda^p(\mathfrak{n}) \otimes_k \hat{U}(\mathfrak{g})]^*)=0$ for all $q>0$ and all $p\geqslant 0$. This gives that $E_r^{p,q}=0$ for all $r\geqslant 1$ and $q>0$. Hence $E_2^{p,q}\approx E_\infty^{p,q}$ for all p and q. Further $E_2^{p,0}=H^p(\mathfrak{g}, \mathfrak{b}^-, \hat{U}(\mathfrak{g})^*)$. (See [HS; Sect. 2].) This gives that the canonical restriction map $H^p(\mathfrak{g}, \mathfrak{b}^-, \hat{U}(\mathfrak{g})^*) \to H^p(\mathfrak{g}, \hat{U}(\mathfrak{g})^*)$ is an isomorphism for all p. But then $H_p(\mathfrak{g}, \hat{U}(\mathfrak{g})) \to H_p(\mathfrak{g}, \mathfrak{b}^-, \hat{U}(\mathfrak{g}))$ itself is an isomorphism. (Use the fact that if V and W are two k-vector spaces with a linear map $f\colon V\to W$ such that the induced dual map $f^*\colon W^*\to V^*$ is an isomorphism, then f itself is an isomorphism.)

(2.6) LEMMA. $H^q(\mathfrak{n}^-, [\Lambda^p(\mathfrak{n}) \otimes_k \hat{U}(\mathfrak{g})]^*)=0$ *for all* $q>0$ *and for all* $p\geqslant 0$, *where* $\mathfrak{n}=\mathfrak{g}/\mathfrak{b}^-$ *is a* $U(\mathfrak{n}^-)$*-module under the adjoint action and* $\hat{U}(\mathfrak{g})$ *is a* $U(\mathfrak{n}^-)$*-module under the left multiplication.*

Proof. There exists a $U(\mathfrak{n}^-)$-resolution of the trivial module k

$$\cdots \to F_q \xrightarrow{d_q} F_{q-1} \to \cdots \to F_1 \xrightarrow{d_1} F_0 \xrightarrow{d_0} k \to 0,$$

such that all the F_q's are free and finitely generated left $U(\mathfrak{n}^-)$-modules. In fact, by [GL; Section 7], F_q can be further chosen to be a graded submodule of $U(\mathfrak{n}^-) \otimes_k \Lambda^q(\mathfrak{n}^-)$, such that the map d_q is the restriction of the differential in the canonical resolution of the trivial module k:

$$\cdots \to U(\mathfrak{n}^-) \otimes_k \Lambda^q(\mathfrak{n}^-) \to \cdots \to U(\mathfrak{n}^-) \otimes_k \Lambda^0(\mathfrak{n}^-) \to k.$$

(We take the product grading on $U(\mathfrak{n}^-) \otimes_k \Lambda^q(\mathfrak{n}^-)$.) The latter fact will be used only in a subsequent lemma (Lemma (2.9)).

Any $v \in \Lambda^p(\mathfrak{n})$ is contained in a finite-dimensional (over k) $U(\mathfrak{n}^-)$–invariant submodule M_v of $\Lambda^p(\mathfrak{n})$. (This is easy to see in view of the fact that the root spaces $\mathfrak{g}_\alpha$ are finite dimensional.) Consider the finitely generated and free resolution of the $U(\mathfrak{n}^-)$–module M_v:

$$\cdots \to F_q \otimes_k M_v \xrightarrow{d_q \otimes 1} F_{q-1} \otimes_k M_v \to \cdots \to F_1 \otimes_k M_v$$
$$\xrightarrow{d_1 \otimes 1} F_0 \otimes_k M_v \xrightarrow{d_0 \otimes 1} M_v.$$

(The fact that $F_q \otimes_k M_v$'s are $U(\mathfrak{n}^-)$-free, follows from the Hopf principle [GL; Proposition 1.7].)

Let $K_q = \ker(d_q \otimes 1 = \tilde{d}_q)$. We have the exact sequence

$$F_{q+2} \otimes_k M_v \xrightarrow{\tilde{d}_{q+2}} F_{q+1} \otimes_k M_v \xrightarrow{\tilde{d}_{q+1}} K_q \to 0,$$

and hence K_q is finitely presented as a $U(\mathfrak{n}^-)$-module. We prove that

$$\hat{U}(\mathfrak{g})^t \otimes_{U(\mathfrak{n}^-)} K_q \to \hat{U}(\mathfrak{g})^t \otimes_{U(\mathfrak{n}^-)} (F_q \otimes_k M_v)$$

is injective for all $q \geqslant 0$. Consider the following commutative diagram:

$$\begin{array}{ccc}
\left[\prod_{d \geqslant 0} [U(\mathfrak{b}^-) \otimes_k U_d(\mathfrak{n})]\right]^t \otimes_{U(\mathfrak{n}^-)} K_q & \xrightarrow{\sim} & \prod_{d \geqslant 0} [[U(\mathfrak{b}^-) \otimes_k U_d(\mathfrak{n})]^t \otimes_{U(\mathfrak{n}^-)} K_q] \\
\downarrow & & \downarrow \\
\hat{U}(\mathfrak{g})^t \otimes_{U(\mathfrak{n}^-)} (F_q \otimes_k M_v) & \to & \prod_{d \geqslant 0} [[U(\mathfrak{b}^-) \otimes_k U_d(\mathfrak{n})]^t \otimes_{U(\mathfrak{n}^-)} (F_q \otimes_k M_v)].
\end{array}$$

The top horizontal map is an isomorphism. (See [B; Sect. 3, No. 7].) The right vertical map is injective, as $[U(\mathfrak{b}^-) \otimes_k U_d(\mathfrak{n})]^t$ is free right $U(\mathfrak{n}^-)$-module. So the left vertical map is also injective. This gives (on using the right exactness of $\otimes$) that $\mathrm{Tor}_q^{U(\mathfrak{n}^-)}(\hat{U}(\mathfrak{g})^t, M_v) = 0$ for all $q > 0$. Which, in turn (since $\Lambda^p(\mathfrak{n})$ is a direct limit of M_v's) implies that

$$\begin{aligned}\mathrm{Tor}_q^{U(\mathfrak{n}^-)}(\hat{U}(\mathfrak{g})^t, \Lambda^p(\mathfrak{n})) &\approx \mathrm{Tor}_q^{U(\mathfrak{n}^-)}(k, \hat{U}(\mathfrak{g}) \otimes_k \Lambda^p(\mathfrak{n}))\\ &= H_q(\mathfrak{n}^-, \hat{U}(\mathfrak{g}) \otimes_k \Lambda^p(\mathfrak{n})) = 0\end{aligned}$$

for all $q > 0$.

This proves the lemma.

Now we come to the following

(2.7) Lemma. *$H^q(\mathfrak{b}^-, [\Lambda^p(\mathfrak{n}) \otimes_k \hat{U}(\mathfrak{g})]^*) = 0$ for all $q > 0$ and all $p \geq 0$.*

Proof. Consider the Hochschild–Serre spectral sequence relative to an ideal, corresponding to the pair $(\mathfrak{b}^-, \mathfrak{n}^-)$ (coefficients being $[\Lambda^p(\mathfrak{n}) \otimes_k \hat{U}(\mathfrak{g})]^*$), with

$$E_2^{q,s} = H^q(\mathfrak{h}, H^s(\mathfrak{n}^-, [\Lambda^p(\mathfrak{n}) \otimes_k \hat{U}(\mathfrak{g})]^*))$$

and converging to $H^{q+s}(\mathfrak{b}^-, [\Lambda^p(\mathfrak{n}) \otimes_k \hat{U}(\mathfrak{g})]^*)$. From Lemma (2.6), $H^s(\mathfrak{n}^-, [\Lambda^p(\mathfrak{n}) \otimes_k \hat{U}(\mathfrak{g})]^*) = 0$ for all $s > 0$ and hence the spectral sequence degenerates at the E_2 term itself. So $H^q(\mathfrak{b}^-, [\Lambda^p(\mathfrak{n}) \otimes_k \hat{U}(\mathfrak{g})]^*) \approx H^q(\mathfrak{h}, H^0(\mathfrak{n}^-, [\Lambda^p(\mathfrak{n}) \otimes_k \hat{U}(\mathfrak{g})]^*))$.But $H^0(\mathfrak{n}^-, [\Lambda^p(\mathfrak{n}) \otimes_k \hat{U}(\mathfrak{g})]^*)$ is the space of invariants in $[\Lambda^p(\mathfrak{n}) \otimes_k \hat{U}(\mathfrak{g})]^*$ which is equal to $[\Lambda^p(\mathfrak{n}) \otimes_k \hat{U}(\mathfrak{g})/\mathfrak{n}^- \cdot [\Lambda^p(\mathfrak{n}) \otimes_k \hat{U}(\mathfrak{g})]]^*$. Now from an argument completely analogous to the one used in the proof of the next Lemma (2.8), we get that this is isomorphic (as an $\mathfrak{h}$-module) with

$$\left[\Lambda^p(\mathfrak{n}) \otimes_k \prod_{d \geq 0} [U(\mathfrak{h}) \otimes_k U_d(\mathfrak{n})]\right]^*.$$

Hence

$$\begin{aligned}H^q(\mathfrak{b}^-, [\Lambda^p(\mathfrak{n}) \otimes_k \hat{U}(\mathfrak{g})]^*) &\approx H^q\left(\mathfrak{h}, \left[\Lambda^p(\mathfrak{n}) \otimes_k \prod_d [U(\mathfrak{h}) \otimes_k U_d(\mathfrak{n})]\right]^*\right)\\ &\approx H_q\left(\mathfrak{h}, \Lambda^p(\mathfrak{n}) \otimes_k \prod_d [U(\mathfrak{h}) \otimes_k U_d(\mathfrak{n})]\right)^*\\ &\approx \mathrm{Tor}_q^{U(\mathfrak{h})}\left(\Lambda^p(\mathfrak{n})^t, \prod_{d \geq 0} [U(\mathfrak{h}) \otimes_k U_d(\mathfrak{n})]\right)^*.\end{aligned}$$

But $U(\mathfrak{h})$ being a commutative Noetherian ring, $\prod_{d \geqslant 0} [U(\mathfrak{h}) \otimes_k U_d(\mathfrak{n})]$ is a flat $U(\mathfrak{h})$-module [C; Theorem 2.1] and hence the lemma follows.

(2.8) LEMMA. *$H_p(\mathfrak{g}, \mathfrak{b}^-, \hat{U}(\mathfrak{g})) \approx H_p(\mathfrak{n}, \prod_{d \geqslant 0} U_d(\mathfrak{n}))$ for all p, where $\prod_{d \geqslant 0} U_d(\mathfrak{n})$ is considered as a left $U(\mathfrak{n})$-module by left multiplication.*

Proof. Let $\Lambda(\mathfrak{g}, \mathfrak{b}^-, \hat{U}(\mathfrak{g})) = (\cdots \to \Lambda_p \to^{d_p} \Lambda_{p-1} \to \cdots \to \Lambda_0 \to 0)$ be the standard complex for computing the homology $H(\mathfrak{g}, \mathfrak{b}^-, \hat{U}(\mathfrak{g}))$. We recall that

$$\Lambda_p = [\Lambda^p(\mathfrak{n}) \otimes_k \hat{U}(\mathfrak{g})]/\mathfrak{b}^- \cdot [\Lambda^p(\mathfrak{n}) \otimes_k \hat{U}(\mathfrak{g})]$$

and

$$\begin{aligned} &d_p(\overline{v_1 \wedge \cdots \wedge v_p \otimes a}) \\ &\quad = \sum_{1 \leqslant i < j \leqslant p} (-1)^{i+j} \overline{[v_i, v_j] \wedge v_1 \wedge \cdots \wedge \hat{v}_i \wedge \cdots \wedge \hat{v}_j \wedge \cdots \wedge v_p \otimes a} \\ &\quad\quad + \sum_{i=1}^{p} (-1)^i \overline{v_1 \wedge \cdots \wedge \hat{v}_i \wedge \cdots \wedge v_p \otimes (v_i \cdot a)}, \end{aligned}$$

for $v_1, \ldots, v_p \in \mathfrak{n}$ and $a \in \hat{U}(\mathfrak{g})$,

where $\overline{v_1 \wedge \cdots \wedge v_p \otimes a}$ denotes the projection of $v_1 \wedge \cdots \wedge v_p \otimes a \in \Lambda^p(\mathfrak{n}) \otimes_k \hat{U}(\mathfrak{g})$ onto Λ_p.

We define a *chain* map $\beta = (\beta_p)$ with $\beta_p \colon \Lambda_p(\mathfrak{n}, \prod_d U_d(\mathfrak{n})) \to \Lambda_p(\mathfrak{g}, \mathfrak{b}^-, \hat{U}(\mathfrak{g}))$ as follows:

$$\beta_p\left(v \otimes \sum_d x_d\right) = \overline{v \otimes \sum_d 1 \otimes x_d}, \qquad \text{for } v \in \Lambda^p(\mathfrak{n}) \text{ and } x_d \in U_d(\mathfrak{n}).$$

Further we define a map $\alpha_p(d)$: $\Lambda^p(\mathfrak{n}) \otimes_k U(\mathfrak{b}^-) \otimes_k U_d(\mathfrak{n}) \to \Lambda^p(\mathfrak{n}) \otimes_k U_d(\mathfrak{n})$ by $\alpha_p(d)(v \otimes b \otimes x_d) = T(b) \cdot v \otimes x_d$ for $v \in \Lambda^p(\mathfrak{n})$, $b \in U(\mathfrak{b}^-)$, and $x_d \in U_d(\mathfrak{n})$. (Recall from Section (0.1) that T is the unique anti-automorphism of $U(\mathfrak{b}^-)$ such that $T(X) = -X$ for $X \in \mathfrak{b}^-$.) This gives rise to a map

$$\alpha = \alpha_p \colon \Lambda^p(\mathfrak{n}) \otimes_k \prod_{d \geqslant 0} [U(\mathfrak{b}^-) \otimes_k U_d(\mathfrak{n})] \to \Lambda^p(\mathfrak{n}) \otimes_k \prod_{d \geqslant 0} U_d(\mathfrak{n}),$$

defined by $v \otimes (\sum_{d \geqslant 0} b_d \otimes x_d) \mapsto \sum_d T(b_d) \cdot v \otimes x_d$ for $v \in \Lambda^p(\mathfrak{n})$, $b_d \in U(\mathfrak{b}^-)$, and $x_d \in U_d(\mathfrak{n})$. This map is well defined, since $U(\mathfrak{b}^-) \cdot v$ is finite-dimensional subspace of $\Lambda^p(\mathfrak{n})$ for any $v \in \Lambda^p(\mathfrak{n})$ (see the proof of Lemma (2.6)).

The map α can be easily seen to factor through $\mathfrak{b}^- \cdot [\Lambda^p(\mathfrak{n}) \otimes_k \hat{U}(\mathfrak{g})]$. We denote by $\tilde{\alpha}$ the map α defined on the factor space

$$\Lambda^p(\mathfrak{n}) \otimes_k \hat{U}(\mathfrak{g})/\mathfrak{b}^- \cdot [\Lambda^p(\mathfrak{n}) \otimes_k \hat{U}(\mathfrak{g})].$$

Clearly $\tilde{\alpha} \circ \beta = \mathrm{Id}$. Further, to prove that $(\beta \circ \alpha)\, x - x \in \mathfrak{b}^- \cdot [\Lambda^p(\mathfrak{n}) \otimes_k \hat{U}(\mathfrak{g})]$ for all $x \in \Lambda^p(\mathfrak{n}) \otimes_k \hat{U}(\mathfrak{g})$, observe the following:

(1) $v \otimes Xa \equiv -(Xv) \otimes a \bmod \mathfrak{b}^- \cdot [U(\mathfrak{b}^-) \cdot v \otimes_k U(\mathfrak{b}^-)]$ for $v \in \Lambda^p(\mathfrak{n})$, $X \in \mathfrak{b}^-$, and $a \in U(\mathfrak{b}^-)$,

(2) $U(\mathfrak{b}^-) \cdot v$ is finite dimensional $/k$, for all $v \in \Lambda^p(\mathfrak{n})$,

(3) $U(\mathfrak{b}^-)$ is a finitely generated algebra $/k$.

This proves that the map β is an isomorphism of the chain complexes and hence the lemma follows.

Finally, we have the following

(2.9) LEMMA.[2] *$H_p(\mathfrak{n}, \prod_{d \geqslant 0} U_d(\mathfrak{n})) = 0$ for all $p \geqslant 1$.*

Proof. Choose a $U(\mathfrak{n})$-resolution of the trivial module k (as stated in the proof of Lemma (2.6)):

$$\cdots \to F_p \xrightarrow{d_p} F_{p-1} \to \cdots \to F_1 \xrightarrow{d_1} F_0 \xrightarrow{d_0} k \to 0,$$

with F_p a graded submodule of $U(\mathfrak{n}) \otimes_k \Lambda^p(\mathfrak{n})$. Further F_p is free and finitely generated left $U(\mathfrak{n})$-module and all the differentials d_p are graded morphisms.

Let $K_p = \ker d_p$, then K_p is finitely generated graded $U(\mathfrak{n})$-module with homogeneous generators (say) $\{v_1, ..., v_r\}$. For a $U(\mathfrak{n})$-module M, let $\hat{M}$ denote its $\mathfrak{m}$-adic completion with respect to the ideal $\mathfrak{m} = U(\mathfrak{n})^+$ of $U(\mathfrak{n})$. We prove that $\hat{K}_p \to \hat{F}_p$ is injective for all $p \geqslant 0$. For this, it suffices to show that for any n, there exists n' such that

$$[\mathfrak{m}^{n'} \cdot F_p] \cap K_p \subset \mathfrak{m}^n \cdot K_p.$$

Let $x \in [\mathfrak{m}^{n'} \cdot F_p] \cap K_p$ be a homogeneous element of degree d, so that $d \geqslant n'$. Of course x can be written as $x = \sum_{i=1}^r a_i v_i$ with $a_i \in U(\mathfrak{n})$. Further, we can choose a_i to be homogeneous of degree $(d - \deg. v_i)$. So, if we take

[2] M. V. Nori has pointed out the following very short proof of this lemma: The sequence $\cdots \to F_p(d) \to F_{p-1}(d) \to \cdots \to F_0(d) \to k(d) \to 0$ is exact, where $F_p(d)$ denotes the elements of grade degree d in F_p, and hence $\cdots \to \prod_{d \geqslant 0} F_p(d) \to \prod_d F_{p-1}(d) \to \cdots \to \prod_{d \geqslant 0} F_0(d) \to k \to 0$ is also exact. But $\prod_{d \geqslant 0} F_p(d)$ can be canonically identified with $[\prod_{d \geqslant 0} U_d(\mathfrak{n})]' \otimes_{U(\mathfrak{n})} F_p$.

$n' = \max_{1 \leqslant i \leqslant r}\{\deg v_i + n\}$ then $\deg a_i \geqslant n$. Hence $a_i \in \mathfrak{m}^n$, as $\mathfrak{m}^n = \sum_{i \geqslant n} U_i(\mathfrak{n})$. (This follows easily from the fact that $U(\mathfrak{n})$ is generated, as an algebra, by its elements of degree 1.) This shows that $x \in \mathfrak{m}^n \cdot K_p$, so the assertion that $\hat{K}_p$ injects into $\hat{F}_p$ is established.

Now we want to show that $[\prod_{d \geqslant 0} U_d(\mathfrak{n})]' \otimes_{U(\mathfrak{n})} K_p$ injects into $[\prod_{d \geqslant 0} U_d(\mathfrak{n})]' \otimes_{U(\mathfrak{n})} F_p$ for all $p \geqslant 0$, which will prove the lemma.

The k-algebra $\prod_{d \geqslant 0} U_d(\mathfrak{n})$ can be identified with the k-algebra $\hat{U}(\mathfrak{n})$, where $\hat{U}(\mathfrak{n})$ is the $\mathfrak{m}$-adic completion of the k-algebra $U(\mathfrak{n})$ with respect to the two-sided ideal $\mathfrak{m}$. There exists a canonical map $\hat{U}(\mathfrak{n})' \otimes_{U(\mathfrak{n})} K_p \to \hat{K}_p$, induced from the map $[U(\mathfrak{n})/\mathfrak{m}^d]' \otimes_{U(\mathfrak{n})} K_p \to K_p/\mathfrak{m}^d \cdot K_p$ defined by $\bar{a} \otimes x \mapsto \overline{T(a) \cdot x}$ for $a \in U(\mathfrak{n})$ and $x \in K_p$. We have the following commutative diagram:

$$\begin{array}{ccccc}
\hat{U}(\mathfrak{n})' \otimes_{U(\mathfrak{n})} F_{p+2} & \to & \hat{U}(\mathfrak{n})' \otimes_{U(\mathfrak{n})} F_{p+1} & \to & \hat{U}(\mathfrak{n})' \otimes_{U(\mathfrak{n})} K_p \to 0 \\
\downarrow & & \downarrow & & \downarrow \\
\hat{F}_{p+2} & \longrightarrow & \hat{F}_{p+1} & \longrightarrow & \hat{K}_p \to 0.
\end{array}$$

The top horizontal map is exact due to right exactness of $\otimes$. The first two left side vertical maps are isomorphisms, because F_{p+2} and F_{p+1} are free and finitely generated. To prove the exactness of the bottom sequence, we have the exact sequence:

$$0 \to K_{p+1} \to F_{p+1} \xrightarrow{d_{p+1}} K_p \to 0$$

and hence the following sequence is also exact (see [M; Proposition on p. 167]).

$$0 \to \hat{K}_{p+1} \to \hat{F}_{p+1} \to \hat{K}_p \to 0. \tag{S$_1$}$$

Similarly we have the exact sequence:

$$0 \to \hat{K}_{p+2} \to \hat{F}_{p+2} \to \hat{K}_{p+1} \to 0. \tag{S$_2$}$$

By (S_1) and (S_2), we get that

$$\hat{F}_{p+2} \to \hat{F}_{p+1} \to \hat{K}_p \to 0$$

is exact. So, using the five lemma, we get that $\hat{U}(\mathfrak{n})' \otimes_{U(\mathfrak{n})} K_p \xrightarrow{\sim} \hat{K}_p$. This establishes the lemma.

(2.10) *Proof of Proposition* (2.1). Lemmas (2.5), (2.8), and (2.9) put together immediately give Proposition (2.1).

(2.11) *Remark.* Suppose we take a larger completion $\hat{U}_{\mathscr{F}}(\mathfrak{g}) = \prod_{d \geq 0}[U(\mathfrak{n}^-) \otimes \mathscr{F} \otimes U_d(\mathfrak{n})]$ instead of $\hat{U}(\mathfrak{g})$, where $\mathscr{F} \supset S(\mathfrak{h})$ is an 'appropriate' class of functions on a subspace of $\mathfrak{h}^*$, and ask if our "crucial" (to the proof of Theorem (1.2)) Proposition (2.1) and Lemma (2.2) are true with $\hat{U}(\mathfrak{g})$ replaced by $\hat{U}_{\mathscr{F}}(\mathfrak{g})$. This would enable one, with an appropriate choice of $\mathscr{F}$, to get many more central operators (other than the Casimir). (See [K5].) This, in turn, could be used to prove our vanishing theorem (1.2) for a larger class of modules M. See also conjecture (1.10).

But we briefly sketch an argument to show that if $\mathscr{F} \supset S(\mathfrak{h})$ satisfies the following:

(a) $\mathscr{F}$ is a commutative ring with an augmentation $\varepsilon: \mathscr{F} \to k$ (i.e., ε is a ring homomorphism such that $\varepsilon \circ i = \mathrm{Id}.$, where $i: k \hookrightarrow S(\mathfrak{h})$ is the canonical inclusion) such that $\varepsilon(\mathfrak{h}) = 0$;

(b) $\mathscr{F}$ is free as a $S(\mathfrak{h})$-module;

(c) $\mathfrak{h}$ generates $\mathscr{F}^+ = \ker \varepsilon$ as $\mathscr{F}$-module.

Then $\mathscr{F} = S(\mathfrak{h})$.

(Observe that our proof of Proposition (2.1) and Lemma (2.2) uses the properties (a), (b), and (c) above.)

Take a $S(\mathfrak{h})$-free basis $\{e_n\}_{n \in I}$ of $\mathscr{F}$. By assumption

$$e_n^+ = e_n - i\varepsilon(e_n) = \sum_{m \in I} \sum_j h_j P_{j,m}^n e_m, \qquad \text{for some } P_{j,m}^n \in S(\mathfrak{h}), \qquad (*)$$

where $\{h_j\}$ is a k-basis of $\mathfrak{h}$ and $P_{j,m}^n = 0$ for all but finitely many m. Write $1 = \sum_{m \in I} f_m e_m$, with $f_m \in S(\mathfrak{h})$ and all but finitely many $f_m = 0$. Hence, by $(*)$,

$$e_n = \sum_{m \in I} \left\{ \sum_j h_j P_{j,m}^n + \varepsilon(e_n) f_m \right\} e_m.$$

So, $\sum_j h_j P_{j,m}^n + \varepsilon(e_n) f_m = \delta_{n,m}$ for all $n, m \in I$. This, in particular, shows that I is finite and hence $\mathscr{F}$ is a Noetherian ring. (Otherwise we can choose $n_0 \in I$ such that $f_{n_0} = 0$. So $\sum_j h_j P_{j,n_0}^{n_0} = 1$ in $S(\mathfrak{h})$, a contradiction!)

By induction, it is easy to see that $\mathscr{F} = S^+(\mathfrak{h})^p \cdot \mathscr{F} + S(\mathfrak{h})$ for all $p \geq 1$ (use $\mathscr{F}^+ \subset (\mathscr{F}^+)^2 + S^+(\mathfrak{h})$). In particular the $S(\mathfrak{h})$-module $\mathscr{F}/S(\mathfrak{h})$ satisfies $S^+(\mathfrak{h})^p \cdot (\mathscr{F}/S(\mathfrak{h})) = \mathscr{F}/S(\mathfrak{h})$, so the completion $\widehat{\mathscr{F}/S(\mathfrak{h})}$ of the $S(\mathfrak{h})$-module $\mathscr{F}/S(\mathfrak{h})$ along $S^+(\mathfrak{h})$ is 0. This gives that $\widehat{S(\mathfrak{h})} \xrightarrow{\sim} \hat{\mathscr{F}}$. But $\mathscr{F}$ being $S(\mathfrak{h})$-free, rank $\mathscr{F} = 1$. Now, it is easy to see that $\mathscr{F} = S(\mathfrak{h})$.

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Note added in proof.

(1) This paper was earlier circulated as MSRI preprint No. 033-84-7 with the title "Homology of Kac–Moody Lie-algebras with arbitrary coefficients".

(2) We prove a fairly general vanishing theorem for the Ext bi-functor (which, in particular, gives our theorem (1.2)), by a different method, in "Extension of the category $\mathcal{O}^{g}$ and a vanishing theorem for the Ext bi-functor for Kac–Moody algebras" (to appear in *J. Algebra*).

(3) In the above paper, we also settle conjecture (1.10) in affirmative for the affine (including twisted affine) algebras. For general symmetrizable Kac–Moody algebras, we show that if $\lambda \in K^{w.g.}$ (it is defined in the above mentioned paper and it contains the Tits cone), the conjecture is again true.

(4) We learned of a paper by Chiu Sen "The homology of Kac–Moody Lie algebras with coefficients in a generalized Verma module," *J. Algebra* **90** (1984), where Proposition (1.5) is proved.

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