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Mathematics

# The nil Hecke ring and cohomology of $G/P$ for a Kac–Moody group $G$

**(Kac–Moody algebra/Borel subalgebra/cohomology algebra/Schubert cells/smash product)**

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**ABSTRACT** **Let $G$ be the group with Borel subgroup $B$, associated to a Kac–Moody Lie algebra $\mathfrak{g}$ (with Weyl group $W$ and Cartan subalgebra $\mathfrak{h}$). Then $H^*(G/B)$ has, among others, four distinguished structures (*i*) an algebra structure, (*ii*) a distinguished basis, given by the Schubert cells, (*iii*) a module for $W$, and (*iv*) a module for Hecke-type operators $A_w$, for $w \in W$. We construct a ring $R$, which we refer to as the nil Hecke ring, which is very simply and explicitly defined as a functor of $W$ together with the $W$-module $\mathfrak{h}$ alone and such that all these four structures on $H^*(G/B)$ arise naturally from the ring $R$.**

## Section 1

To any (not necessarily symmetrizable) generalized $l \times l$ Cartan matrix $A$, one associates a Kac–Moody (1, 2) algebra $\mathfrak{g} = \mathfrak{g}(A)$ over $\mathbb{C}$ and group $G = G(A)$. If $A$ is a classical Cartan matrix, then $G$ is a finite dimensional semisimple algebraic group over $\mathbb{C}$. We refer to this as the finite case. In general, one has subalgebras of $\mathfrak{g}$; $\mathfrak{h} \subset \mathfrak{b} \subset \mathfrak{p}$, the Cartan subalgebra, Borel subalgebra, and a parabolic subalgebra, respectively. One also has the corresponding subgroups $H \subset B \subset P$. Let $W$ be the Weyl group associated to $(\mathfrak{g}, \mathfrak{h})$ and let $\{r_i\}_{1 \le i \le l}$ denote the set of simple reflections. The group $W$ operates on $\mathfrak{h}$, its dual space $\mathfrak{h}^*$, the symmetric algebra $S = S(\mathfrak{h}^*)$, and the quotient field $Q = Q(\mathfrak{h}^*)$ of $S$.

$W$ parameterizes the Schubert cell decomposition of the generalized flag variety $G/B = \cup_{w \in W} V_w$ $( = Bw^{-1}B/B)$. A suitable subset $W^1 \subset W$ does the same for $G/P$.

Our principal concern is the cohomology ring $H(G/P)$. For notational convenience, we restrict our attention to the case when $P = B$, although many of our results extend to $G/P$ and in fact to *arbitrary Schubert varieties* $\subset G/P$.

Now, besides having a ring structure, $H(G/B)$ is also a module for $W$. In addition, in the finite case, a ring of operators $\mathcal{A}$ (with $\mathbb{C}$-basis $\{A_w\}_{w \in W}$) on $H(G/B)$ was introduced in ref. 3, where $A_{r_i}$ $(1 \le i \le l)$, although defined algebraically, correspond topologically to the integration on fiber for the fibration $G/B \to G/P_i$ ($P_i$ is minimal parabolic containing $r_i$). The definition of the ring of operators $\mathcal{A}$ on $H(G/B)$ has been explicitly carried out in the general case by V. Kac and D. Peterson (unpublished work).

The problems, we wish to deal with, are to describe $H(G/B)$ (*i*) as a ring, in particular the cup product of arbitrary two Schubert classes, and (*ii*) as a module for $W$ and $\mathcal{A}$. Our main result is that all these structures arise very naturally from a single ring $R$, which in itself admits a simple and concrete definition, using only the Weyl group $W$ and its representation on $\mathfrak{h}^*$ but which has some rather remarkable properties. We refer to $R$ as the *nil Hecke ring*, corresponding to the pair $(W, \mathfrak{h})$ for reasons that will be clear later.

We would like to remark that there are a number of serious obstacles in trying to directly pass from the finite to the general infinite case, and as a consequence we have sought a new approach. Among the obstacles are (*i*) the characteristic homomorphism: $S \to H(G/B)$ fails to be surjective in general, (*ii*) the failure of complete reducibility of $W$-modules $S$ and $H(G/B)$, (*iii*) the absence of "harmonics," and (*iv*) the absence of the fundamental (top) cohomology class and failure of Poincaré duality. An approach, which remains valid in the general case, was motivated from theorem 5.9 of ref. 3, proved by B.K. This theorem arises from the correspondence of the Lie algebra cohomology $H(\mathfrak{n})$ ($\mathfrak{n}$ is the nil-radical of $\mathfrak{b}$) and $H(G/B)$, as proved by Kostant (4) in the finite case, and was established, in the general case, by Kumar (5).

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## Section 2

The group $W$ operates as a group of automorphisms on the field $Q = Q(\mathfrak{h}^*)$. Let $Q_W$ be the smash product of $Q$ with the group algebra $\mathbb{C}[W]$. More specifically, $Q_W$ is a right $Q$-module (under right multiplication by $Q$) with a (free) basis $\{\delta_w\}_{w \in W}$ and the multiplicative structure is given by

$$(\delta_{w_1} q_1)(\delta_{w_2} q_2) = \delta_{w_1 w_2}(w_2^{-1} q_1) q_2,$$

for $q_1, q_2 \in Q$ and $w_1, w_2 \in W$.

Observe that $\delta_e Q = Q \delta_e$ is not central in $Q_W$.

The ring $Q_W$ has an involutary anti-automorphism, defined by

$$(\delta_w q)^t = \delta_{w^{-1}}(wq), \text{ for } q \in Q \text{ and } w \in W.$$

Let $S_W \subset Q_W$ be defined in the same way as $Q_W$ with $S = S(\mathfrak{h}^*)$ replacing $Q$.

Now, for $i = 1, \ldots, l$, let

$$x_i = -(\delta_{r_i} + \delta_e)\left[\frac{1}{\alpha_i}\right] = \frac{1}{\alpha_i}(\delta_{r_i} - \delta_e),$$

where $\alpha_i$ is the simple root corresponding to $r_i$. Also let $\mathrm{l} : W \to Z_+$ be the length function. Inspired by ref. 3, we have the following.

PROPOSITION 2.1. *Let* $\mathrm{w} \in \mathrm{W}$ *and let* $\mathrm{w} = \mathrm{r}_{\mathrm{i}_1} \ldots \mathrm{r}_{\mathrm{i}_n}$ *be a reduced expression. Then the element* $\mathrm{x_w} \in \mathrm{Q_W}$ *defined by* $\mathrm{x_w} = \mathrm{x}_{\mathrm{i}_1} \ldots \mathrm{x}_{\mathrm{i}_n}$ *is independent of the reduced expression. Furthermore, for* v, w $\in$ W,

$$\mathrm{x_v} \cdot \mathrm{x_w} = \mathrm{x_{vw}} \textit{ if } \mathrm{l(vw)} = \mathrm{l(v)} + \mathrm{l(w)}$$
$$= 0 \textit{ otherwise}. \qquad \square$$

Let $\Delta_+$ (resp $\Delta_-$) denote the set of positive (resp negative) roots and let $\le$ denote the Bruhat partial ordering on $W$. The elements $\{\delta_w\}_{w \in W}$ are a right (as well as left) $Q$-basis of $Q_W$. But also

PROPOSITION 2.2. *The elements* $\{\mathrm{x_w}\}_{\mathrm{w} \in \mathrm{W}}$ *form a right (as* *well as left)* Q*-basis of* $Q_W$. *Write*

$$\delta_{w^{-1}} = \sum_v d_{v,w} x_v - 1, \textit{ for some (unique) } d_{v,w} \in Q.$$

*Then* (a) $d_{v,w} \in S$, *in fact* $d_{v,w}$ *is a homogeneous polynomial of degree* $l(v)$.

(b) $d_{v,w} = 0$, *unless* $v \leq w$.

(c) $d_{w,w} = \prod_{\nu \in w^{-1}\Delta_- \cap \Delta_+} \nu.$ □

The upper triangular and invertible (over $Q$) matrix $D = (d_{v,w})_{v,w\in W}$ relating the two bases $\{\delta_w\}$ and $\{x_w\}$ will play an important role in this paper.

*Remark 2.3*: In the finite case, the matrix $D$ may be extracted from theorem 5.9 of ref. 3. □

Now clearly, $Q$ has the structure of a left $Q_W$-module, defined explicitly by

$$(\delta_w q)q' = w(qq'), \text{ for } w \in W \text{ and } q, q' \in Q.$$

Our main result centers around the subring $R \subset Q_W$, defined by

$$R = \{a \in Q_W : a \cdot S \subset S\}.$$

Obviously $S_W \subset R$. Furthermore, one can easily see that $x_w \in R$, for any $w \in W$.

By applying the involution $t$, one gets another subring $R^t$ of $Q_W$. One has the following crucial structure theorem for $R$. The proof of part a of *Theorem 2.4* can be simplified in the finite case using the theory of "harmonics."

THEOREM 2.4. (a) R *is free as a right (as well as left)* S*-module. In fact the elements* $\{x_w\}_{w\in W}$ *form a right (as well as left)* S*-basis of* R. *In particular, any* $a \in R$ *can be uniquely written as*

$$a = \sum_w x_w p_w, \textit{ with some } p_w \in S.$$

(b) *Furthermore, one has* $R \cap R^t = S_W$. □

*Remark 2.5*: Note that $R$ is a finitely generated ring over $\mathbb{C}$, since it is generated by $\{x_i\}_{1\leq i\leq l}$ and $S$. □

The elements $\{x_w\}$ have much in common with the standard basis of a Hecke ring. However, $x_{r_i}^2 = x_i^2 = 0$. This and a further nilpotence condition, in its action on $\Lambda$ (*Definition 2.6*), persuade us to refer to $R$ as a *nil Hecke ring*. A departure from usual conditions is that $S$ is not central in $R$.

*Definition 2.6*: Regarding $Q_W$ as a right $Q$-module, let $\Omega = \mathrm{Hom}_Q(Q_W, Q)$. Since any $\psi \in \Omega$ is determined by its restriction to the base $\{\delta_w\}$ (and conversely), we can regard $\Omega$ as the $Q$-module of all the functions: $W \to Q$ with pointwise addition and scalar multiplication. Furthermore, $\Omega$ inherits a (commutative) algebra (over $Q$) structure, with the product as pointwise multiplication of functions on $W$. On the other hand, more subtly $\Omega$ also has the structure of a left $Q_W$-module defined by

$$(a\cdot\psi)w = \psi(a^t\cdot\delta_w), \text{ for } a \in Q_W,\ \psi \in \Omega, \text{ and } w \in W,$$

and the action is $Q$-linear.

It may be remarked that $\Omega$ is a $Q$-module, as well as a left $Q_W$-module, and since $Q$ injects into $Q_W$ under $q \to \delta_e q$ in particular a $Q$-module, but these two actions of $Q$ do not coincide. Whenever we refer to $\Omega$ as a $Q$-module, we would mean the first $Q$-action. Now let

$$\Lambda = \{\psi \in \Omega : \psi(R^t) \subseteq S \text{ and } \psi(x_w^t) = 0$$

for all but a finite number of $w \in W\}$.

One has the following:

PROPOSITION 2.7. (a) $\Lambda$ *is an* S*-subalgebra of* $\Omega$.

(b) $\Lambda$ *is a free* S*-module. In fact* $\{\xi^w\}_{w\in W}$ *is an* S*-basis of* $\Lambda$, *where* $\xi^w \in \Omega$ *is defined (uniquely) by* $\xi^w(x_{v^{-1}}^t) = \delta_{v,w}$, *for* $v, w \in W$.

*Observe that* $\xi^w(\delta_v) = d_{w,v}$ ($d_{w,v}$ *is defined in* Proposition 2.2).

(c) $\Lambda$ *is stable under the left action of* $R \subset Q_W$. □

Let $\mathbb{C}_0 = S/S^+$ be the one-dimensional (over $\mathbb{C}$) $S$-module, where $S^+$ is the augmentation ideal (evaluation at $0 \in \mathfrak{h}$) in $S$. By *Proposition 2.7*, $\mathbb{C}_0 \otimes_S \Lambda$ is clearly an algebra and the action of $R$ on $\Lambda$ descends to give an action of $R$ on $\mathbb{C}_0 \otimes_S \Lambda$. Also, from *Proposition 2.7*, the elements $\sigma^w = 1 \otimes \xi^w \in \mathbb{C}_0 \otimes_S \Lambda$ provide a $\mathbb{C}$-basis. Furthermore, the filtered structure on $\Lambda$ (given by the length of $w$) gives rise to a graded commutative algebra structure on $\mathbb{C}_0 \otimes_S \Lambda$.

But now $H(G/B, \mathbb{C})$ has a $\mathbb{C}$-basis $\{\varepsilon^w\}_{w\in W}$, which is dual to the homology basis defined by the closure $\bar{V}_w$ of the Schubert cells $V_w$. Furthermore $H(G/B)$ is a module for $W$ and $\mathcal{A}$ (*Section 1*).

Our next result is the following:

THEOREM 2.8. *The map* $\theta : H(G/B, \mathbb{C}) \to \mathbb{C}_0 \otimes_S \Lambda$, *defined by* $\theta(\varepsilon^w) = \sigma^w$ *for all* $w \in W$, *is a graded algebra isomorphism. Moreover, the action of* $w \in W$ *and* $A_w$ *on* $H(G/B)$ *corresponds (under* $\theta$*) respectively to that of* $\delta_w$, $x_w \in R$ *on* $\mathbb{C}_0 \otimes_S \Lambda$. □

This theorem enables us to write down the cup product of any two elements $\varepsilon^v\varepsilon^w$ in terms of $\{\varepsilon^u\}$ basis, because of the following:

PROPOSITION 2.9. (a) *For any* $v, w \in W$

$$\xi^v\xi^w = \sum_{v,w\leq u} p_{v,w}^u \xi^u,$$

*where* $p_{v,w}^u$ *is a (unique) homogeneous polynomial of degree* $l(v) + l(w) - l(u)$. *In particular,* $p_{v,w}^u = 0$ *if* $l(v) + l(w) < l(u)$.

(b) *Fix* $v \in W$ *and define the two matrices* $D_v$ *and* $P_v$ *by* $D_v(w,u) = \delta_{u,w} d_{v,u}$ *and* $P_v(w,u) = p_{v,w}^u$.

*Then explicitly,* $P_v = D \cdot D_v \cdot D^{-1}$, *where* D *is defined in* Proposition 2.2. □

A similar result holds for the action of arbitrary $w \in W$ on $H(G/B)$.

In the symmetrizable case *Theorem 2.8* admits a proof and an interpretation in the geometrical setting, which we feel worthwhile to elaborate upon.

## Section 3

In this section, we put symmetrizability assumption on $\mathfrak{g}$. Let $\mathfrak{n}$ be the nil-radical of $\mathfrak{b}$.

Let $C(\mathfrak{n})$ (resp $C(\mathfrak{g}, \mathfrak{h})$) denote the co-chain complex, with co-chain map $d$, associated to the Lie algebra $\mathfrak{n}$ (resp Lie algebra pair $(\mathfrak{g}, \mathfrak{h})$) and let End $C(\mathfrak{n})$ denote the algebra of *all* the $\mathbb{C}$-linear maps: $C(\mathfrak{n}) \to C(\mathfrak{n})$ (product coming from composition of maps). Let $\mathrm{End}_{\mathfrak{h}} C(\mathfrak{n})$ be the subspace of $\mathfrak{h}$ invariants in End $C(\mathfrak{n})$. End $C(\mathfrak{n})$ inherits, from $C(\mathfrak{n})$, a derivation $\delta$, such that $\delta^2 = 0$.

The map $\eta$ (defined below) is basic for this section.

LEMMA 3.1. *With suitable topologies on* $C(\mathfrak{g}, \mathfrak{h})$ *and* $\mathrm{End}_{\mathfrak{h}} C(\mathfrak{n})$, *there exists a (unique) continuous map* $\eta : C(\mathfrak{g}, \mathfrak{h}) \to \mathrm{End}_{\mathfrak{h}} C(\mathfrak{n})$ *such that*

$$\eta(\alpha \otimes e(a)) = \left[\frac{2\pi}{\sqrt{-1}}\right]^p \varepsilon(\alpha) i(a),$$

$$\textit{for } \alpha \in C(\mathfrak{n}) \textit{ and } a \in \Lambda^p(\mathfrak{n}),$$

*where* $\varepsilon$ *and* i *are the usual exterior and interior multiplications on the Grassmann algebra* $C(\mathfrak{n})$ *and* $e : \Lambda(\mathfrak{n}) \to C(\mathfrak{n}^-)$ *is induced from the Killing form.*

*Moreover $\eta$ is injective.* □

Recall the operator $S = d\partial + \partial d$, acting on $C(\mathfrak{g}, \mathfrak{h})$, from section 3.4 of ref. 5. Although $\eta$ is not a co-chain map, we have the following:

PROPOSITION 3.2. $\delta(\eta(s)) = 0$, *for* $s \in Ker\, S$. *In particular, we get a map* $\hat{\eta} : Ker\, S \rightarrow H(End_{\mathfrak{h}}C(\mathfrak{n}), \delta)$. □

Let $\mathbb{C}\{W\}$ denote the ring of *all* the functions: $W \rightarrow \mathbb{C}$, the product being pointwise multiplication. Using a result of Garland and Lepowsky (ref. 6, theorem 8.6), on the structure of $H^*(\mathfrak{n})$ as an $\mathfrak{h}$-module, and the Kunneth theorem, we can identify the ring $H(\mathrm{End}_{\mathfrak{h}}C(\mathfrak{n}), \delta)$ with the ring $\mathbb{C}\{W\}$. The map $\hat{\eta}$, under this identification, gives rise to the map $\bar{\eta} :$ Ker $S \rightarrow \mathbb{C}\{W\}$.

Section 3.3. *A filtration of* $C(\mathfrak{g}, \mathfrak{h})$ and $End_{\mathfrak{h}}C(\mathfrak{n})$. The complex $C(\mathfrak{g}, \mathfrak{h})$ has a decreasing filtration $\{\mathcal{G}_n\}_{n\in Z^-}$ defined by

$$\mathcal{G}_n = \sum_{0\le k\le -n} C^{*,k}(\mathfrak{g}, \mathfrak{h}).$$

By taking the image of $\mathcal{G}_n$ under $\eta$, we get a filtration $\{\mathcal{F}_n\}_{n\in Z^-}$ of $\mathrm{End}_{\mathfrak{h}}C(\mathfrak{n})$. This filtration is an "appropriate completion" of a filtration on $\mathrm{End}_{\mathfrak{h}}C(\mathfrak{n})$ arising as a "super" analogue of the usual filtration of differential operators on a manifold.

The filtration $\{\mathcal{F}_n\}$ gives rise to a decreasing filtration $\mathcal{J}_n$ ($n \in Z^-$) of $\mathbb{C}\{W\}$. Explicitly $\mathcal{J}_n = \bar{\eta}(\mathrm{Ker}\, S \cap \mathcal{G}_n)$.

LEMMA 3.4. $\mathcal{J}_m \cdot \mathcal{J}_n \subset \mathcal{J}_{m+n}$. □

In particular, we can speak of the corresponding graded algebra Gr $\mathbb{C}\{W\} = \Sigma_{n\ge 0}\, \mathrm{Gr}^n$, where $\mathrm{Gr}^n = \mathcal{J}_{-n}/\mathcal{J}_{-n+1}$.

Let $\{s_0^w\}_{w\in W}$ be the $d - \partial$ harmonic forms, which are exactly dual to the Schubert homology classes (see theorem 4.5 of ref. 5). We define the elements $\underline{\xi}^w \in \mathbb{C}\{W\}$ by $\underline{\xi}^w = \bar{\eta}(s_0^w)$. Now we are ready to state the following:

THEOREM 3.5. *The map* $\beta : H^*(\mathfrak{g}, \mathfrak{h}) \rightarrow Gr\, \mathbb{C}\{W\}$, *defined by* $\beta[s_0^w] = \underline{\xi}^w \bmod \mathcal{J}_{-l(w)+1}$, *is a graded algebra isomorphism.* ($[s_0^w]$ *denotes the cohomology class of* $s_0^w$.) □

The following theorem links *Section 2* with $\underline{\xi}^w$.

THEOREM 3.6. *For any* v, w $\in$ W, $\underline{\xi}^w(v)(\bar{h}_\rho) = \xi^w(v)$, *where* $h_\rho$ *is an element of* $\mathfrak{h}$ *satisfying* $\sigma(\chi, \rho) = \chi(h_\rho)$ *and* $\xi^w$ *is as defined in* Proposition 2.7. □

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