Geometry of Schubert varieties and Demazure character formula

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1 Notation

The base field in this note is taken to be the field of complex numbers $\mathbb{C}$. The varieties are, by definition, quasi-projective, reduced (but not necessarily irreducible) schemes.

Let G be a semisimple, simply-connected, complex algebraic group. A Borel subgroup B is any maximal connected, solvable subgroup; any two of which are conjugate to each other. We will also fix a maximal torus $H \subset B$. The Lie algebras of G, B, and H are given by $\mathfrak{g}$, $\mathfrak{b}$, and $\mathfrak{h}$, respectively. For a fixed B, any subgroup $P \subset G$ containing B is called a *standard parabolic*.

2 Representations of G

Let $R \subset \mathfrak{h}^*$ denote the set of roots of $\mathfrak{g}$. Recall,

$$\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha \in R} \mathfrak{g}_\alpha, \text{ where } \mathfrak{g}_\alpha := \{x \in \mathfrak{g} : [h, x] = \alpha(h)x \text{ for all } h \in \mathfrak{h}\}.$$

Our choice of B gives rise to R^+, the set of positive roots, such that

$$\mathfrak{b} = \mathfrak{h} \oplus \bigoplus_{\alpha \in R^+} \mathfrak{g}_\alpha.$$

We let $\{\alpha_1, \dots, \alpha_\ell\} \subset \mathfrak{h}^*$ be the simple roots and let $\{\alpha_1^\vee, \dots, \alpha_\ell^\vee\} \subset \mathfrak{h}$ be the simple coroots, where $\ell := \dim \mathfrak{h}$ (called the *rank* of $\mathfrak{g}$).

Elements of $X(H) := \mathrm{Hom}(H, \mathbb{C}^*)$ are called *integral weights*, and can be identified with

$$\mathfrak{h}^*_{\mathbb{Z}} = \{\lambda \in \mathfrak{h}^* : \lambda(\alpha_i^\vee) \in \mathbb{Z},\, \forall\, i\},$$

by taking derivatives. The dominant integral weights $X(H)_+$ are those integral weights $\lambda \in X(H)$ such that $\lambda(\alpha_i^\vee) \geq 0$, for all i.

We let $V(\lambda)$ denote the irreducible G-module with highest weight $\lambda \in X(H)_+$. Then, $V(\lambda)$ has a unique B-stable line such that H acts on this line by λ. This gives a one-to-one correspondence between the set of isomorphism classes of irreducible finite dimensional algebraic representations of G and $X(H)_+$.

3 Tits system

Let $N = N_G(H)$ be the normalizer of H in G, and let $W = N/H$ be the Weyl group, which acts on H by conjugation. For each $i = 1, \ldots, \ell$, consider the subalgebra

$$\mathfrak{sl}_2(i) := \mathfrak{g}_{\alpha_i} \oplus \mathfrak{g}_{-\alpha_i} \oplus \mathbb{C}\,\alpha_i^\vee \subset \mathfrak{g}.$$

There is an isomorphism of Lie algebras $\mathfrak{sl}_2 \to \mathfrak{sl}_2(i)$, taking $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ to $\mathfrak{g}_{\alpha_i}$, $\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$ to $\mathfrak{g}_{-\alpha_i}$, and $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ to $\alpha_i^\vee$. This isomorphism gives rise to a homomorphism $SL_2 \to G$. Let $\overline{s_i}$ denote the image of $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ in G. Then, $\overline{s_i} \in N$ and $S = \{s_i\}_{i=1}^{\ell}$ generates W as a group, where s_i denotes the image of $\overline{s_i}$ under $N \to N/H$. These $\{s_i\}$ are called *simple reflections*. For details about the Weyl group, see [3, §24,27].

The conjugation action of W on H gives rise to an action on $\mathfrak{h}$ via taking derivatives and also on $\mathfrak{h}^*$ by taking duals. Below are explicit formulae for these induced actions:

$$\begin{aligned} s_j : \mathfrak{h} \to \mathfrak{h} \quad &: \quad h \mapsto h - \alpha_j(h)\alpha_j^\vee \\ s_j : \mathfrak{h}^* \to \mathfrak{h}^* \quad &: \quad \beta \mapsto \beta - \beta(\alpha_j^\vee)\alpha_j. \end{aligned}$$

Theorem 1. *The quadruple (G, B, N, S) forms a Tits system (also called a BN-pair), i.e., the following are true:*

(a) $H = B \cap N$ and S generates W as a group;

(b) B *and* N *generate* G *as a group;*

(c) For every i, $s_iBs_i \not\subseteq B$*;*

(d) For every $1 \leq i \leq \ell$ *and* $w \in W$, $(Bs_iB)(BwB) \subset (Bs_iwB) \cup (BwB)$.

There are many consequences of this theorem. For example, (W, S) is a Coxeter group. In particular, there is a length function on W, denoted by $\ell : W \to \mathbb{Z}_+$. For any $w \in W$, $\ell(w)$ is defined to be the minimal $k \in \mathbb{Z}_+$ such that $w = s_{i_1} \dots s_{i_k}$ with each $s_{i_j} \in S$. A decomposition $w = s_{i_1} \dots s_{i_k}$ is called a *reduced decomposition* if $\ell(w) = k$.

We also have the *Bruhat-Chevalley ordering*: $v \leq w$ if v can be obtained by deleting some simple reflections from a reduced decomposition of w.

Axiom (d) above can be refined:

$$(Bs_iB)(BwB) \subset Bs_iwB \text{ if } s_iw > w. \tag{d'}$$

Thus, if we have a reduced decomposition $w = s_{i_1} \dots s_{i_k}$, then

$$BwB = (Bs_{i_1}B) \dots (Bs_{i_k}B), \tag{1}$$

which can be obtained from (d') by inducting on $k = \ell(w)$.

We also have the Bruhat decomposition:

$$G = \bigsqcup_{w \in W} BwB.$$

Theorem 2. *The set of standard parabolics are in one-to-one correspondence with subsets of the set* $[\ell] = \{1, \dots, \ell\}$*. Specifically, if* $I \subset [\ell]$*, let*

$$P_I = \bigsqcup_{w \in \langle s_i : i \in I \rangle} BwB,$$

where $\langle s_i : i \in I \rangle$ *denotes the subgroup of* W *generated by the enclosed elements. Then,* $I \mapsto P_I$ *is the bijection.*

Sketch of the proof. By (1) and (d), P_I is clearly a subgroup containing B. Conversely, if $P \supset B$, then, by the Bruhat decomposition,

$$P = \bigsqcup_{w \in S_P} BwB,$$

for some subset $S_P \subset W$. Let I be the following set:

$$\{i \in [\ell] : s_i \text{ occurs in a reduced decomposition of some } w \in S_P\}.$$

From the above (specifically Axiom (d) and (d')), one can prove $P_I = P$. □

4 A fibration

We begin with a technical theorem.

Theorem 3. *Let F be a closed, algebraic subgroup of G and X be an F-variety. Then, $E = G \times_F X$ is a G-variety, where*

$$G \times_F X := G \times X/\sim \qquad \text{with} \qquad (gf, x) \sim (g, fx)$$

for all $g \in G$, $f \in F$, and $x \in X$. The equivalence class of (g, x) is denoted by $[g, x]$. Then, G acts on E by:

$$g' \cdot [g, x] = [g'g, x].$$

In particular, $G \times_F \{pt\} = G/F$ is a variety. Furthermore, the map $\pi : E \to G/F$ given by $[g, x] \mapsto gF$ is a G-equivariant isotrivial fibration with fiber X.

The variety structure on G/F can be characterized by the following universal property: if Y is any variety, then $G/F \to Y$ is a morphism if and only if the composition $G \to G/F \to Y$ is a morphism.

Now, B is a closed subgroup. To see this, we only need to show that $\overline{B}$ is solvable (B being a maximal solvable subgroup, it will follow that $B = \overline{B}$). Since the commutator $G \times G \to G$ is a continuous map, we have that $[\overline{F}, \overline{F}] \subset \overline{[F, F]}$, for any $F \subset G$. Using this fact and induction, $D_n(\overline{F}) \subset \overline{D_n(F)}$ for all n, where $D_n(F)$ denotes the n–th term in the derived series of F. Since $D_n(B)$ is trivial for large n, $D_n(\overline{B})$ becomes trivial for large n, and $\overline{B}$ is solvable. Thus, G/B is a variety. We wish to give an explicit realization of this variety structure. In the process, we will show that G/B is a projective variety.

Take any regular $\lambda \in X(H)_+$, so that $\lambda(\alpha_i^\vee) > 0$ for all i. The representation $G \to \mathrm{Aut}(V(\lambda))$ gives rise to a map

$$\pi : G/B \to \mathbb{P}V(\lambda), \quad g \mapsto [g \cdot v],$$

since $[v]$ is fixed by B, where v is a highest weight vector of $V(\lambda)$.

Claim. *π is a morphism and injective.*

Proof. π is a morphism since the composition $G \to G/B \to \mathbb{P}V(\lambda)$ is a morphism. To prove injectivity, it suffices to show that the stabilizer of $[v]$ is exactly B. Let P be the stabilizer. Now, $B \subset P$, so P is parabolic and hence $P = P_I$ for some $I \subset [\ell]$. If $I = \varnothing$, then $P = B$. Towards a contradiction, assume $s_i \in P$. Then, s_i stabilizes λ, but

$$s_i(\lambda) = \lambda - \lambda(\alpha_i^\vee)\alpha_i \neq \lambda,$$

since λ is regular. □

We claim $X = \pi(G/B)$ is closed. We will need the following theorem:

Theorem 4 (Borel fixed–point theorem, see §21 in [3])**.** *Let Z be a projective variety with an action of a solvable group. Then, Z has a fixed point.*

Clearly, $\overline{X}$ is G-stable as a subspace of $\mathbb{P}V(\lambda)$. It follows that $\overline{X} \setminus X$ is G-stable. Thus, $\overline{X} \setminus X$ has a B-fixed point which contradicts the existence of a unique highest weight vector. Thus, $\overline{X} \setminus X = \varnothing$ and X is closed.

Lastly, to show X and G/B are isomorphic varieties, we use the following proposition from algebraic geometry:

Proposition 5 (Theorem A.11 in [1])**.** *If $f : Y \to Z$ is a bijective morphism between irreducible varieties and Z is normal, then f is an isomorphism.*

Observe that X is smooth because it is a G-orbit (G takes smooth points to smooth points and any variety has at least one smooth point). In particular, X is normal and $\pi : G/B \to X$ is an isomorphism.

5 Line bundles on G/B

For any $\lambda \in X(H)$, we define a line bundle $\mathcal{L}(\lambda)$ on G/B. Recall that $B = H \ltimes U$, where $U = [B, B]$ is the unipotent radical. Extend $\lambda : H \to \mathbb{C}^*$ to $\lambda : B \to \mathbb{C}^*$ by letting λ map U to 1. Consider $\mathbb{C} = \mathbb{C}_\lambda$ as a B-module, where $b \cdot z = \lambda(b)z$. Then, $\mathcal{L}(\lambda)$ is the line bundle: $\pi : G \times_B \mathbb{C}_{-\lambda} \to G/B$. Note that λ is made negative in the definition of $\mathcal{L}(\lambda)$.

The space of global sections

$$H^0(G/B, \mathcal{L}(\lambda)) := \{\sigma : G/B \to G \times_B \mathbb{C}_{-\lambda} : \pi \circ \sigma = \mathrm{id}\}$$

is a G-module, where the G-action is given by

$$(g \cdot \sigma)(g'B) = g\sigma(g^{-1}g'B).$$

Also, this module is finite dimensional since G/B is projective and any cohomology of coherent sheaves on projective varieties is finite dimensional.

6 Borel–Weil theorem

Theorem 6 (Borel–Weil theorem). *If $\lambda \in X(H)_+$, then there is a G-module isomorphism*

$$H^0(G/B, \mathcal{L}(\lambda)) \simeq V(\lambda)^*.$$

Proof. If we pull back the line bundle $\mathcal{L} = \mathcal{L}(\lambda)$ (given by $\pi : G \times_B \mathbb{C}_{-\lambda} \to G/B$) under $G \to G/B$, we get the bundle $\hat{\mathcal{L}}$, which is $\hat{\pi} : G \times \mathbb{C}_{-\lambda} \to G$. We wish to compare sections of these two bundles.

Sections of $\hat{\mathcal{L}}$ are of the form $\sigma(g) = (g, f(g))$, for some map $f : G \to \mathbb{C}_{-\lambda}$, so we can identify $H^0(G, \hat{\mathcal{L}})$ with $k[G] \otimes \mathbb{C}_{-\lambda}$. There is a B-action on $k[G]$ given by $(b \cdot f)(g) = f(gb)$. Acting diagonally, we get an action on $k[G] \otimes \mathbb{C}_{-\lambda}$. Since $k[G] \otimes \mathbb{C}_{-\lambda}$ is naturally isomorphic to $k[G]$ (make the second coordinate 1), we get a new B-action on $k[G]$ given by

$$(b \cdot f)(g) = \lambda(b)^{-1} f(gb). \tag{2}$$

Use this action to make $H^0(G, \hat{\mathcal{L}})$ a B-module.

Sections of $\mathcal{L}$ are of the form $\sigma(gB) = [g, f(g)]$, for some map $f : G \to \mathbb{C}_{-\lambda}$. In order to insure that σ is well-defined, we require that for any $b \in B$:

$$[g, f(g)] = [gb, f(gb)] = [g, b \cdot f(gb)] = [g, \lambda(b)^{-1} f(gb)].$$

Therefore, f must have the property that $f(g) = \lambda(b)^{-1} f(gb)$ for all $b \in B$. It follows that

$$\left[H^0(G, \hat{\mathcal{L}})\right]^B = H^0(G/B, \mathcal{L}).$$

Now, it suffices to show $\left[H^0(G, \hat{\mathcal{L}})\right]^B \simeq V(\lambda)^*$.

Consider the following two $(G \times G)$-modules. First, $k[G]$ has a $(G \times G)$-action given by $((g_1, g_2) \cdot f)(g) = f(g_1^{-1} g g_2)$. Second, acting coordinate-wise, we have:

$$\mathcal{M} := \bigoplus_{\mu \in X(H)_+} V(\mu)^* \otimes V(\mu).$$

It follows from the Peter–Weyl theorem and Tanaka–Krein duality that these are isomorphic as $(G\times G)$-modules. The explicit isomorphism is $\Phi = \sum_\mu \Phi_\mu : \mathcal{M} \to k[G]$, where $\Phi_\mu : V(\mu)^* \otimes V(\mu) \to k[G]$ is given by

$$\Phi_\mu(f \otimes v)(g) = f(gv).$$

Furthermore, $k[G] \otimes \mathbb{C}_{-\lambda}$ has a $(G \times B)$-action given diagonally, where G is forgotten when $G \times B$ acts on the second coordinate $\mathbb{C}_{-\lambda}$, and the action of $G \times B$ on $k[G]$ is the restriction of the $G \times G$ action given above. Since $H^0(G, \hat{\mathcal{L}}) \simeq k[G] \otimes \mathbb{C}_{-\lambda}$ as (left) G-modules, where G acts on $k[G]$ via $(g \cdot f)(x) = f(g^{-1}x)$, for $g, x \in G$ and $f \in k[G]$. Since the action of G on $k[G] \otimes \mathbb{C}_{-\lambda}$ commutes with the B-action given by equation (2), we get an induced G-action on the space of B-invariants:

$$\begin{aligned} \left[H^0(G, \hat{\mathcal{L}})\right]^B &\simeq [k[G] \otimes \mathbb{C}_{-\lambda}]^B \\ &\simeq \bigoplus_{\mu \in X(H)_+} [V(\mu)^* \otimes V(\mu) \otimes \mathbb{C}_{-\lambda}]^B \\ &\simeq \bigoplus_{\mu \in X(H)_+} V(\mu)^* \otimes [V(\mu) \otimes \mathbb{C}_{-\lambda}]^B \\ &\simeq \bigoplus_{\mu \in X(H)_+} V(\mu)^* \otimes [\mathbb{C}_\mu \otimes \mathbb{C}_{-\lambda}]^H \\ &\simeq V(\lambda)^*, \end{aligned}$$

since $\mathbb{C}_\mu \otimes \mathbb{C}_{-\lambda}$ will only have H-invariants if $\mu = \lambda$. □

It follows from the next section that the higher cohomology vanishes; that is, for $\lambda \in X(H)_+$ and $i \geq 1$, $H^i(G/B, \mathcal{L}(\lambda)) = 0$.

7 Borel–Weil–Bott theorem

Let ρ be half the sum of the positive roots. Since G is simply-connected, $\rho \in X(H)_+$. Also, ρ has the property that $\rho(\alpha_i^\vee) = 1$ for all i. We will need a shifted action of the Weyl group on $\mathfrak{h}^*$ given by:

$$w \star \lambda = w(\lambda + \rho) - \rho.$$

Theorem 7 (Borel–Weil–Bott)**.** *If $\lambda \in X(H)_+$ and $w \in W$, then*

$$H^p(G/B, \mathcal{L}(w \star \lambda)) = \begin{cases} V(\lambda)^* & \text{if } p = \ell(w) \\ 0 & \text{if } p \neq \ell(w) \end{cases}.$$

Before we prove this theorem, we need to establish a number of results. For any i, let P_i denote the minimal parabolic subgroup $P_i = B \sqcup Bs_iB$. In what follows, if M is a B-module, the notation $H^p(G/B, M)$ is the p-th sheaf cohomology for the sheaf of sections of the bundle $G \times_B M \to G/B$.

Lemma 8. *If M is a P_i-module, then $H^p(G/B, M \otimes \mathbb{C}_\mu) = 0$, for all $p \geq 0$ and any $\mu \in X(H)$ such that $\mu(\alpha_i^\vee) = 1$.*

Proof. Apply the Leray–Serre spectral sequence to the fibration $G/B \to G/P_i$ with fiber P_i/B and the vector bundle on G/B corresponding to the B-module $M \otimes \mathbb{C}_\mu$. Thus,

$$E_2^{p,q} = H^p(G/P_i, H^q\,(P_i/B, M \otimes \mathbb{C}_\mu)) \Longrightarrow H^*(G/B, M \otimes \mathbb{C}_\mu).$$

If we can show $E_2^{p,q} = 0$, then we are done.

It suffices to show $H^q(P_i/B, M \otimes \mathbb{C}_\mu)$ vanishes for all $q \geq 0$. By the next exercise, we have

$$H^q(P_i/B, M \otimes \mathbb{C}_\mu) \simeq M \otimes H^q(P_i/B, \mathbb{C}_\mu),$$

since M is a P_i-module by assumption. Since $P_i/B \simeq SL_2(i)/B(i) \simeq \mathbb{P}^1$, where $SL_2(i)$ is the subgroup of P_i with Lie algebra $\mathfrak{sl}_2(i)$ and $B(i)$ is the standard Borel subgroup of $SL_2(i)$, we have that

$$H^q(P_i/B, \mathbb{C}_\mu) \simeq H^q(\mathbb{P}^1, \mathcal{O}(-\mu(\alpha_i^\vee))) = H^q(\mathbb{P}^1, \mathcal{O}(-1)),$$

which is known to be zero (for example, [2, Ch. III, Theorem 5.1]). □

Exercise 9. For any closed subgroup $F \subset G$, if M is a G-module, then $G \times_F M \to G/F$ is a trivial vector bundle.

Proposition 10. *If for some i, $\mu \in X(H)$ has the property that $\mu(\alpha_i^\vee) \geq -1$, then for all $p \geq 0$,*

$$H^p(G/B, \mathcal{L}(\mu)) \simeq H^{p+1}(G/B, \mathcal{L}(s_i \star \mu)).$$

Proof. First, consider the case where $\mu(\alpha_i^\vee) \geq 0$. Let $X_i := P_i/B \simeq \mathbb{P}^1$ and $\mathcal{H} := H^0(X_i, \mathcal{L}(\mu+\rho))$. It can easily be seen (by using the definition of the action of P_i on $\mathcal{H}$) that the action of the unipotent radical U_i of P_i is trivial on $\mathcal{H}$. Moreover, P_i/U_i is isomorphic with the subgroup $\widehat{SL_2(i)}$ of G generated by $SL_2(i)$ and H. Thus, by the Borel-Weil theorem for $G = \widehat{SL_2(i)}$, we get $\mathcal{H} \simeq V_i(\mu+\rho)^*$, as $\widehat{SL_2(i)}$-modules, where $V_i(\mu+\rho)$ is the irreducible $\widehat{SL_2(i)}$-module with highest weight $\mu+\rho$. (Even though we stated the Borel-Weil theorem for semisimple, simply-connected groups, the same proof gives the result for any connected, reductive group.) Thus, we have the weight space decomposition (as H-modules):

$$\mathcal{H} \simeq V_i(\mu+\rho)^* = \bigoplus_{j=0}^{(\mu+\rho)(\alpha_i^\vee)} \mathbb{C}_{-(\mu+\rho)+j\alpha_i}.$$

There is a short exact sequence of B-modules:

$$0 \longrightarrow K \longrightarrow \mathcal{H} \longrightarrow \mathbb{C}_{-(\mu+\rho)} \longrightarrow 0,$$

where K, by definition, is the kernel of the projection. Tensoring with $\mathbb{C}_\rho$, we get the following exact sequence of B-modules:

$$0 \longrightarrow K \otimes \mathbb{C}_\rho \longrightarrow \mathcal{H} \otimes \mathbb{C}_\rho \longrightarrow \mathbb{C}_{-\mu} \longrightarrow 0.$$

Passing to the long exact cohomology sequence, we get:

$$\cdots \to H^p(G/B, \mathcal{H} \otimes \mathbb{C}_\rho) \to H^p(G/B, \mathbb{C}_{-\mu}) \to \\ H^{p+1}(G/B, K \otimes \mathbb{C}_\rho) \to H^{p+1}(G/B, \mathcal{H} \otimes \mathbb{C}_\rho) \to \cdots.$$

By the previous lemma, $H^p(G/B, \mathcal{H} \otimes \mathbb{C}_\rho) = 0$ for all p. Thus,

$$H^p(G/B, \mathcal{L}(\mu)) = H^p(G/B, \mathbb{C}_{-\mu}) \simeq H^{p+1}(G/B, K \otimes \mathbb{C}_\rho). \tag{3}$$

Consider another short exact sequence of B-modules:

$$0 \longrightarrow \mathbb{C}_{-s_i(\mu+\rho)} \longrightarrow K \longrightarrow M \longrightarrow 0,$$

where M is just the cokernal of the inclusion. In particular, as H-modules,

$$M = \bigoplus_{j=1}^{(\mu+\rho)(\alpha_i^\vee)-1} \mathbb{C}_{-(\mu+\rho)+j\alpha_i},$$

so it may be regarded as a P_i-module. Then, as B-modules, we can tensor with $\mathbb{C}_\rho$ to arrive at the following exact sequence:

$$0 \longrightarrow \mathbb{C}_{-s_i \star \mu} \longrightarrow K \otimes \mathbb{C}_\rho \longrightarrow M \otimes \mathbb{C}_\rho \longrightarrow 0.$$

Again, passing to the long exact sequence, we see:

$$\cdots \to H^p(G/B, M \otimes \mathbb{C}_\rho) \to H^{p+1}(G/B, \mathbb{C}_{-s_i \star \mu}) \to \\ H^{p+1}(G/B, K \otimes \mathbb{C}_\rho) \to H^{p+1}(G/B, M \otimes \mathbb{C}_\rho) \to \cdots .$$

By the previous lemma, $H^p(G/B, M \otimes \mathbb{C}_\rho) = 0$ for all p. Thus,

$$H^{p+1}(G/B, \mathcal{L}(s_i \star \mu)) = H^{p+1}(G/B, \mathbb{C}_{-s_i \star \mu}) \simeq H^{p+1}(G/B, K \otimes \mathbb{C}_\rho). \quad (4)$$

Combining equations (3) and (4), we get the proposition in the case where $\mu(\alpha_i^\vee) \geq 0$.

For the case that $\mu(\alpha_i^\vee) = -1$, we have that $s_i \star \mu = \mu$, so the statement reduces to proving that $H^p(G/B, \mathcal{L}(\mu)) = 0$, for all p. In this case, $K = 0$. From the isomorphism $\mathcal{H} \otimes \mathbb{C}_\rho \simeq \mathbb{C}_{-\mu}$, we conclude $H^p(G/B, \mathcal{L}(\mu)) \simeq H^p(G/B, \mathcal{H} \otimes \mathbb{C}_\rho)$ which vanishes by the previous lemma. □

Corollary 11. *If $\mu \in X(H)_+$ and $w \in W$, then for all $p \in \mathbb{Z}$, as G-modules:*

$$H^p(G/B, \mathcal{L}(\mu)) \simeq H^{p+\ell(w)}(G/B, \mathcal{L}(w \star \mu)).$$

Proof. We induct on $\ell(w)$. Assume the above for all $v \in W$ such that $\ell(v) < \ell(w)$, and write $w = s_i v$ for some $v < w$. Then,

$$H^p(G/B, \mathcal{L}(\mu)) \simeq H^{p+\ell(v)}(G/B, \mathcal{L}(v \star \mu)).$$

Now $(v \star \mu)(\alpha_i^\vee) = (\mu + \rho)(v^{-1}\alpha_i^\vee) - 1 \geq -1$, since $v^{-1}\alpha_i^\vee$ is a positive coroot and $\mu + \rho$ is dominant. So, applying Proposition 10, we get:

$$H^p(G/B, \mathcal{L}(\mu)) \simeq H^{p+\ell(v)+1}(G/B, \mathcal{L}(s_i \star (v \star \mu))) = H^{p+\ell(w)}(G/B, \mathcal{L}(w \star \mu)),$$

which is our desired result. □

We are now ready to prove the Borel–Weil–Bott theorem.

Proof of the Borel–Weil–Bott theorem. From the above corollary,

$$H^p(G/B, \mathcal{L}(w \star \lambda)) \simeq H^{p-\ell(w)}(G/B, \mathcal{L}(\lambda)).$$

We claim that $H^j(G/B, \mathcal{L}(\lambda)) = 0$ if $j \neq 0$. Indeed, if $j < 0$, this is true. Let w_0 denote the unique longest word in the Weyl group, so that $\ell(w_0) = \dim(G/B)$. If $j > 0$, then by Corollary 11,

$$H^j(G/B, \mathcal{L}(\lambda)) \simeq H^{j+\dim(G/B)}(G/B, \mathcal{L}(w_0 \star \lambda)) = 0.$$

This implies

$$H^p(G/B, \mathcal{L}(w \star \lambda)) = \begin{cases} H^0(G/B, \mathcal{L}(\lambda)) & \text{if } p = \ell(w) \\ 0 & \text{if } p \neq \ell(w) \end{cases},$$

which is our desired result, by the Borel–Weil theorem. □

Exercise 12. Show that for any μ not contained in $W\star(X(H)_+)$, $H^p(G/B, \mathcal{L}(\mu)) = 0$, for all $p \geq 0$.

8 Schubert varieties

For any $w \in W$, let $X_w := \overline{BwB/B} \subset G/B$ denote the corresponding *Schubert variety.* This variety is projective and irreducible of dimension $\ell(w)$. By the Bruhat decomposition, we have the following decomposition of X_w:

$$X_w = \bigsqcup_{v \leq w} BvB/B.$$

9 Bott–Samelson–Demazure–Hansen variety

Let $\mathfrak{W}$ be the set of all ordered sequences $\mathfrak{w} = (s_{i_1}, \ldots, s_{i_n})$, $n \geq 0$, of simple reflections, called *words.* For any such word, define the *Bott–Samelson–Demazure–Hansen variety* as follows: if $n = 0$ (thus, $\mathfrak{w}$ is the empty sequence), $Z_{\mathfrak{w}}$ is a point. For $\mathfrak{w} = (s_{i_1}, \ldots, s_{i_n})$, with $n \geq 1$, define

$$Z_{\mathfrak{w}} = P_{i_1} \times \cdots \times P_{i_n}/B^n,$$

where the product group B^n acts on $P_{\mathfrak{w}} := P_{i_1} \times \cdots \times P_{i_n}$ from the right via:

$$(p_1, \ldots, p_n) \cdot (b_1, \ldots, b_n) = (p_1 b_1, b_1^{-1} p_2 b_2, \ldots, b_{n-1}^{-1} p_n b_n).$$

This action is free and proper. The group P_{i_1} (in particular, B) acts on $Z_{\mathfrak{w}}$ via its left multiplication on the first factor.

Lemma 13. *$Z_{\mathfrak{w}}$ is a smooth projective variety.*

Sketch of the proof. Induct on the length of $\mathfrak{w}$, where length refers to the number of terms in the sequence. Let $\mathfrak{v}$ be the last $n-1$ terms in the sequence $\mathfrak{w}$, so that $\mathfrak{w} = (s_{i_1}) \cup \mathfrak{v}$, where order is preserved when taking the union.

Let

$$\pi : Z_{\mathfrak{w}} \simeq P_{i_1} \times_B Z_{\mathfrak{v}} \longrightarrow Z_{(s_{i_1})} = P_{i_1}/B \simeq \mathbb{P}^1$$

be the map $[p_1, \ldots, p_n] \mapsto p_1 B$. This map has fiber $Z_{\mathfrak{v}}$ and since it is a fibration, we get that $Z_{\mathfrak{w}}$ is smooth. Furthermore, $Z_{\mathfrak{w}}$ is complete since $\mathbb{P}^1$ is complete and the fibers of π are complete by induction.

Furthermore, it is a trivial fibration restricted to $\mathbb{P}^1 \setminus \{x\}$, for any $x \in \mathbb{P}^1$. Hence, projectivity follows from the Chevalley–Kleiman criterion asserting that a smooth complete variety is projective if and only if any finite set of points is contained in an affine open subset. □

There is a map $\xi : \mathfrak{W} \to W$ given by $\mathfrak{w} = (s_{i_1}, \ldots, s_{i_n}) \mapsto s_{i_1} \cdots s_{i_n}$. For any $\mathfrak{w} \in \mathfrak{W}$, we say $\mathfrak{w}$ is reduced if $s_{i_1} \cdots s_{i_n}$ is a reduced decomposition of $\xi(w)$.

For $\mathfrak{w} \in \mathfrak{W}$, consider the map $\theta_{\mathfrak{w}} : Z_{\mathfrak{w}} \to G/B$ given by $[p_1, \ldots, p_n] \mapsto p_1 \cdots p_n B$.

Lemma 14. *If $\mathfrak{w}$ is reduced, then $\theta_{\mathfrak{w}}(Z_{\mathfrak{w}}) = X_{\xi(\mathfrak{w})}$. Moreover, $\theta_{\mathfrak{w}}$ is a desingularization of $X_{\xi(\mathfrak{w})}$; that is, it is birational and proper.*

If $\mathfrak{w}$ is not reduced, then $\theta_{\mathfrak{w}}(Z_{\mathfrak{w}})$ is NOT equal to $X_{\xi(\mathfrak{w})}$ in general.

Sketch of the proof. The open subset of $Z_{\mathfrak{w}}$ given by

$$(B s_{i_1} B) \times \cdots \times (B s_{i_n} B)/B^n$$

maps isomorphically to the open cell BwB/B by (1) of Section (3). □

10 A fundamental cohomology vanishing theorem

Let $\mathfrak{w} = (s_{i_1}, \ldots, s_{i_n})$ be an arbitrary word. For any j, $1 \leq j \leq n$, define $\mathfrak{w}(j) = (s_{i_1}, \ldots, \widehat{s_{i_j}}, \ldots, s_{i_n})$. The variety

$$Z_{\mathfrak{w}(j)} = P_{i_1} \times \cdots \times \widehat{P_{i_j}} \times \cdots \times P_{i_n}/B^{n-1}$$

embeds into $Z_{\mathfrak{w}}$ by:

$$[p_1, \ldots, p_{j-1}, p_{j+1}, \ldots, p_n] \mapsto [p_1, \ldots, p_{j-1}, 1, p_{j+1}, \ldots, p_n].$$

Denote also by $Z_{\mathfrak{w}(j)}$ the images of these maps. These are divisors in $Z_{\mathfrak{w}}$.

For $\lambda \in X(H)_+$, let $\mathcal{L}_{\mathfrak{w}}(\lambda) = \theta_{\mathfrak{w}}^*(\mathcal{L}(\lambda))$ be the pull back of $\mathcal{L}(\lambda)$ under the map $\theta_{\mathfrak{w}} : Z_{\mathfrak{w}} \to G/B$. We state the following fundamental theorem without proof.

Theorem 15 (Theorem 8.1.8 in [1])**.** *Let $\mathfrak{w} = (s_{i_1}, \ldots, s_{i_n})$ be a word and let $1 \leq p \leq q \leq n$ be such that $(s_{i_p}, \ldots, s_{i_q})$ is reduced. Then, for any $\lambda \in X(H)_+$ and $r > 0$,*

$$H^r\left(Z_{\mathfrak{w}}, \mathcal{O}_{Z_{\mathfrak{w}}}\left(-\sum_{j=p}^{q} Z_{\mathfrak{w}(j)}\right) \otimes \mathcal{L}_{\mathfrak{w}}(\lambda)\right) = 0.$$

Also, $H^r(Z_{\mathfrak{w}}, \mathcal{L}_{\mathfrak{w}}(\lambda)) = 0$.

Corollary 16. *For any word $\mathfrak{w} = (s_{i_1}, \ldots, s_{i_n})$, $\lambda \in X(H)_+$, and j such that $1 \leq j \leq n$, the map*

$$H^0(Z_{\mathfrak{w}}, \mathcal{L}_{\mathfrak{w}}(\lambda)) \to H^0(Z_{\mathfrak{w}(j)}, \mathcal{L}_{\mathfrak{w}(j)}(\lambda))$$

is surjective.

Proof. Consider the short exact sequence:

$$0 \longrightarrow \mathcal{O}_{Z_{\mathfrak{w}}}(-Z_{\mathfrak{w}(j)}) \longrightarrow \mathcal{O}_{Z_{\mathfrak{w}}} \longrightarrow \mathcal{O}_{Z_{\mathfrak{w}(j)}} \longrightarrow 0,$$

where $\mathcal{O}_{Z_{\mathfrak{w}}}(-Z_{\mathfrak{w}(j)})$ is identified with the ideal sheaf of $Z_{\mathfrak{w}(j)}$ inside $Z_{\mathfrak{w}}$. Since $\mathcal{L}_{\mathfrak{w}}(\lambda)$ is locally free, we may tensor the above sequence to get the exact sequence:

$$0 \longrightarrow \mathcal{O}_{Z_{\mathfrak{w}}}(-Z_{\mathfrak{w}(j)}) \otimes \mathcal{L}_{\mathfrak{w}}(\lambda) \longrightarrow \mathcal{L}_{\mathfrak{w}}(\lambda) \longrightarrow \mathcal{L}_{\mathfrak{w}(j)}(\lambda) \longrightarrow 0.$$

Passing to the long exact sequence and applying Theorem 15 gives us our desired result. $\square$

11 Geometry of Schubert varieties

In this section we show that Schubert varieties are normal, have rational singularities, and are Cohen-Macaulay.

Theorem 17 (Zariski's Main Theorem, see [2], Chap. III, Corollary 11.4 and its proof)**.** *If $f : X \to Y$ is a birational projective morphism between irreducible varieties and X is smooth, then Y is normal if and only if $f_*\mathcal{O}_X = \mathcal{O}_Y$.*

Lemma 18 (Lemma A.32 in [1])**.** *If $f : X \to Y$ is a surjective morphism between projective varieties and $\mathcal{L}$ is an ample line bundle on Y such that $H^0(Y, \mathcal{L}^{\otimes d}) \to H^0(X, (f^*\mathcal{L})^{\otimes d})$ is an isomorphism for all large d, then $f_*\mathcal{O}_X = \mathcal{O}_Y$.*

For any $w \in W$, choose a reduced decomposition $w = s_{i_1} \cdots s_{i_n}$, with each $s_{i_j} \in S$, and take $\mathfrak{w} = (s_{i_1}, \ldots, s_{i_n})$. Then, $\theta_{\mathfrak{w}} : Z_{\mathfrak{w}} \to X_w$ is a desingularization. By the previous two results, to prove the normality of X_w, it suffices to find an ample line bundle $\mathcal{L}(\lambda)$ such that for all large d,

$$H^0(X_w, \mathcal{L}(\lambda)^{\otimes d}) \simeq H^0(Z_{\mathfrak{w}}, \mathcal{L}_{\mathfrak{w}}(\lambda)^{\otimes d}).$$

In fact, $\mathcal{L}(\lambda)$ is ample on G/B if and only if λ is a dominant, regular weight (this claim is easy to prove from the results in Section 4). Since the restriction of ample line bundles are ample, in order to show that X_w is normal, it suffices to prove the following theorem:

Theorem 19. *If $\lambda \in X(H)_+$ and $w \in W$, then $H^0(X_w, \mathcal{L}(\lambda)) \to H^0(Z_{\mathfrak{w}}, \mathcal{L}_{\mathfrak{w}}(\lambda))$ is an isomorphism.*

Before we give the proof, we recall the following useful lemma:

Lemma 20 (Projection formula, Exercise 8.3 of Chap. III in [1])**.** *If $f : X \to Y$ is any morphism, η is a vector bundle on Y, $\mathcal{S}$ is a quasi-coherent sheaf on X, then for all i:*

$$R^i f_*(\mathcal{S} \otimes f^*\eta) \simeq (R^i f_*\mathcal{S}) \otimes \eta.$$

Proof of the theorem. This map is clearly injective since $Z_{\mathfrak{w}} \twoheadrightarrow X_w$. Choose a reduced decomposition of the longest element $w_0 \in W$, $w_0 = s_{i_1} \cdots s_{i_N}$, each $s_{i_j} \in S$, $N = \dim(G/B) = |R^+|$, and let $\mathfrak{w} = (s_{i_1}, \ldots, s_{i_N})$. Introduce the following notation: for $0 \leq j \leq N$, let $w_j = s_{i_1} \cdots s_{i_j}$ and $\mathfrak{w}_j = (s_{i_1}, \ldots, s_{i_j})$. Consider the following diagram:

$$
\begin{array}{ccc}
Z_{\mathfrak{w}_N} & \xrightarrow{\theta_{\mathfrak{w}_N}} & X_{w_N} = G/B \\
\uparrow & & \uparrow \\
Z_{\mathfrak{w}_{N-1}} & \xrightarrow{\theta_{\mathfrak{w}_{N-1}}} & X_{w_{N-1}} \\
\uparrow & & \uparrow \\
Z_{\mathfrak{w}_{N-2}} & \xrightarrow{\theta_{\mathfrak{w}_{N-2}}} & X_{w_{N-2}} \\
\uparrow & & \uparrow \\
\vdots & & \vdots
\end{array}
$$

In this diagram, the horizontal arrows are surjective and the vertical arrows (which are the canonical inclusions) are injective. Passing to global sections, we get:

$$
\begin{array}{ccc}
H^0(Z_{\mathfrak{w}_N}, \mathcal{L}_{\mathfrak{w}_N}(\lambda)) & \longleftarrow & H^0(X_{w_N}, \mathcal{L}(\lambda)) \\
\downarrow & & \downarrow \\
H^0(Z_{\mathfrak{w}_{N-1}}, \mathcal{L}_{\mathfrak{w}_{N-1}}(\lambda)) & \longleftarrow & H^0(X_{w_{N-1}}, \mathcal{L}(\lambda)) \\
\downarrow & & \downarrow \\
H^0(Z_{\mathfrak{w}_{N-2}}, \mathcal{L}_{\mathfrak{w}_{N-2}}(\lambda)) & \longleftarrow & H^0(X_{w_{N-2}}, \mathcal{L}(\lambda)) \\
\downarrow & & \downarrow \\
\vdots & & \vdots
\end{array}
$$

In this diagram, the horizontal arrows are of course injective and the vertical arrows on the left are surjective by Corollary 16. Furthermore, by Lemma 20 (with $\mathcal{S} = \mathcal{O}_{Z_{\mathfrak{w}_N}}$ and $\eta = \mathcal{L}(\lambda)$) and Theorem 17, the top horizontal arrow is an isomorphism. By a standard diagram chase, all of the horizontal arrows are isomorphisms.

Since $w_0 = w(w^{-1}w_0)$ and $\ell(w^{-1}w_0) = \ell(w_0) - \ell(w)$, a reduced decomposition of w_0 can always be obtained so that the first $\ell(w)$ terms of the decomposition give the word $\mathfrak{w}$. This completes the proof. $\square$

Thus, using Theorem 17 and Lemma 18, we get the following:

Corollary 21. *Any Schubert variety X_w is normal.*

Corollary 22. *For any $v \le w$ and $\lambda \in X(H)_+$, the restriction map*

$$H^0(X_w, \mathcal{L}(\lambda)) \to H^0(X_v, \mathcal{L}(\lambda))$$

is surjective.

Proof. By the above proof, $H^0(G/B, \mathcal{L}(\lambda)) \to H^0(X_v, \mathcal{L}(\lambda))$ is surjective and hence so is $H^0(X_w, \mathcal{L}(\lambda)) \to H^0(X_v, \mathcal{L}(\lambda))$. □

An irreducible projective variety Y has *rational singularities* if for some desingularization $f : X \to Y$ we have that $f_*\mathcal{O}_X = \mathcal{O}_Y$ and $R^i f_*\mathcal{O}_X = 0$ for all $i > 0$. This definition does not depend on a choice of desingularization. (In characteristic $p > 0$, we also need to assume that $R^i f_*\kappa_X = 0$, for the canonical bundle κ_X.) To prove that X_w has rational singularities, we use the following theorem of Kempf:

Theorem 23 (Lemma A.31 in [1])**.** *Let $f : X \to Y$ be a morphism of projective varieties such that $f_*\mathcal{O}_X = \mathcal{O}_Y$. Assume there exists an ample line bundle $\mathcal{L}$ on Y such that $H^i(X, (f^*\mathcal{L})^{\otimes d}) = 0$ for all $i > 0$ and all large d. Then, $R^i f_*\mathcal{O}_X = 0$ for $i > 0$.*

Corollary 24. *Any Schubert variety X_w has rational singularities.*

Proof. It suffices to prove $H^i(Z_{\mathfrak{w}}, \mathcal{L}_{\mathfrak{w}}(d\lambda)) = 0$ for all large d, for all $i > 0$, and some regular $\lambda \in X(H)_+$, which follows from Theorem 15. □

We recall the following general theorem:

Theorem 25 (Lemma A.38 in [1])**.** *Any projective variety which has rational singularities is Cohen-Macaulay.*

Thus, we get:

Corollary 26. *X_w is Cohen-Macaulay.*

Another consequence of having rational singularities (which we will use later) is given in the following two results.

Proposition 27. *Let Y be a projective variety with rational singularities. Then, for any desingularization $f : X \to Y$ and any vector bundle η on Y, $H^i(Y, \eta) \to H^i(X, f^*\eta)$ is an isomorphism for $i \ge 0$.*

Proof. Applying the Leray-Serre spectral sequence, we have

$$E_2^{p,q} = H^p(Y, R^q f_* f^* \eta) \Longrightarrow H^*(X, f^*\eta).$$

By the projection formula (with $\mathcal{S} = \mathcal{O}_X$),

$$R^q f_*(\mathcal{O}_X \otimes f^*\eta) \simeq \eta \otimes (R^q f_* \mathcal{O}_X).$$

Since Y has rational singularities, $R^q f_* \mathcal{O}_X = 0$ for $q > 0$. Therefore, $E_2^{p,q} = 0$ for $q > 0$, and hence $H^p(Y, \eta) \simeq E_2^{p,0}$ for all p, and the result follows. □

Corollary 28. *For any $\lambda \in X(H)$ and $i \geq 0$,*

$$H^i(X_w, \mathcal{L}(\lambda)) \simeq H^i(Z_{\mathfrak{w}}, \mathcal{L}_{\mathfrak{w}}(\lambda)),$$

for any reduced word $\mathfrak{w}$ with $\xi(\mathfrak{w}) = w$. In particular, for any $\lambda \in X(H)_+$, $H^i(X_w, \mathcal{L}(\lambda)) = 0$ if $i > 0$.

Proof. By Corollary 24 and Proposition 27, $H^i(X_w, \mathcal{L}(\lambda)) \simeq H^i(Z_{\mathfrak{w}}, \mathcal{L}_{\mathfrak{w}}(\lambda))$, which vanishes by Theorem 15 for $\lambda \in X(H)_+$. □

12 Demazure modules

Let $w \in W$ and $\lambda \in X(H)_+$. The Demazure module $V_w(\lambda) \subset V(\lambda)$ is the B-submodule defined by $V_w(\lambda) = \mathcal{U}(\mathfrak{b}) \cdot V(\lambda)_{w\lambda}$, where $\mathcal{U}(\mathfrak{b})$ is the enveloping algebra of $\mathfrak{b}$ and $V(\lambda)_{w\lambda}$ is the weight space of $V(\lambda)$ with weight $w\lambda$. Observe that $V(\lambda)_{w\lambda}$ is one-dimensional. The formal character of $V_w(\lambda)$ is defined by

$$\text{ch } V_w(\lambda) = \sum_{\mu \in X(H)} \dim(V_w(\lambda)_\mu)\ e^\mu.$$

If $w = w_0$, then $V_w(\lambda) = V(\lambda)$. Therefore, ch $V_{w_0}(\lambda)$ is given by the Weyl character formula.

For an arbitrary $\mathfrak{w} \in \mathfrak{W}$, we need to introduce the Demazure operators $D_{\mathfrak{w}}$. For each simple reflection s_i, let $D_{s_i} : \mathbb{Z}[X(H)] \to \mathbb{Z}[X(H)]$ be the $\mathbb{Z}$-linear map given by:

$$D_{s_i}(e^\mu) = \frac{e^\mu - e^{s_i\mu - \alpha_i}}{1 - e^{-\alpha_i}}.$$

Given $\mathfrak{w} = (s_{i_1}, \ldots, s_{i_n}) \in \mathfrak{W}$, define $D_{\mathfrak{w}} : \mathbb{Z}[X(H)] \to \mathbb{Z}[X(H)]$ by

$$D_{\mathfrak{w}} = D_{s_{i_1}} \circ \cdots \circ D_{s_{i_n}}.$$

In what follows, we will also need $* : \mathbb{Z}[X(H)] \to \mathbb{Z}[X(H)]$ given by

$$* e^{\mu} = e^{-\mu},$$

and extended $\mathbb{Z}$-linearly.

Theorem 29. *For any reduced word* $\mathfrak{w}$*,*

$$\operatorname{ch} V_{\xi(\mathfrak{w})}(\lambda) = D_{\mathfrak{w}}(e^{\lambda}).$$

Proof. The first step is to show $V_w(\lambda)^* \simeq H^0(X_w, \mathcal{L}(\lambda))$.

By the Borel–Weil theorem, $V(\lambda)^* \simeq H^0(G/B, \mathcal{L}(\lambda))$. The isomorphism $\phi : V(\lambda)^* \to H^0(G/B, \mathcal{L}(\lambda))$ is explicitly given by $\phi(f)(gB) = [g, f(gv_\lambda)]$, where v_λ is a highest weight vector in $V(\lambda)$.

By Corollary 22, the restriction $H^0(G/B, \mathcal{L}(\lambda)) \to H^0(X_w, \mathcal{L}(\lambda))$ is surjective. Let ϕ_w denote the composition

$$V(\lambda)^* \to H^0(G/B, \mathcal{L}(\lambda)) \to H^0(X_w, \mathcal{L}(\lambda)).$$

We compute the kernal of ϕ_w; i.e., find all $f \in V(\lambda)^*$ such that $\phi_w(f)$ is the zero section. It suffices to check that $\phi_w(f) = 0$ on BwB/B, since BwB/B is a dense open subset of X_w. For $f \in V(\lambda)^*$,

$$\begin{aligned} \phi_w(f) = 0 &\iff f(BwB \cdot v_\lambda) = 0 \\ &\iff f(B \cdot v_{w\lambda}) = 0 \\ &\iff f \text{ vanishes on } V_w(\lambda). \end{aligned}$$

Thus, $\ker \phi_w = \{f \in V(\lambda)^* : f|_{V_w(\lambda)} = 0\}$; that is, we have the following exact sequence:

$$0 \longrightarrow \left(\frac{V(\lambda)}{V_w(\lambda)}\right)^* \longrightarrow V(\lambda)^* \longrightarrow H^0(X_w, \mathcal{L}(\lambda)) \longrightarrow 0.$$

Therefore, $H^0(X_w, \mathcal{L}(\lambda))^* \simeq V_w(\lambda)$, which completes the first step.

Now, take a reduced decomposition of $w = s_{i_1} \cdots s_{i_n}$ and let $\mathfrak{w} = (s_{i_1}, \ldots, s_{i_n})$. The map $Z_{\mathfrak{w}} \to X_w$ is B-equivariant and by Corollary 28, $H^i(Z_{\mathfrak{w}}, \mathcal{L}_{\mathfrak{w}}(\lambda)) \simeq$

$H^i(X_w, \mathcal{L}(\lambda))$ for all i as B-modules (for any $\lambda \in X(H)$). Therefore, their characters coincide; that is,

$$\text{ch } H^i(Z_{\mathfrak{w}}, \mathcal{L}_{\mathfrak{w}}(\lambda)) = \text{ch } H^i(X_w, \mathcal{L}(\lambda)).$$

Consider the Euler–Poincaré characteristic:

$$\chi_H(Z_{\mathfrak{w}}, \mathcal{L}_{\mathfrak{w}}(\lambda)) := \sum_i (-1)^i \text{ ch} \, H^i(Z_{\mathfrak{w}}, \mathcal{L}_{\mathfrak{w}}(\lambda)) \in \mathbb{Z}[X(H)].$$

Since $\text{ch} \, H^0(X_w, \mathcal{L}(\lambda)) = \chi_H(Z_{\mathfrak{w}}, \mathcal{L}_{\mathfrak{w}}(\lambda))$ for $\lambda \in X(H)_+$, it suffices to show:

$$\chi_H(Z_{\mathfrak{w}}, \mathcal{L}_{\mathfrak{w}}(\lambda)) = * \, D_{\mathfrak{w}}(e^{\lambda}).$$

In fact, we will prove a stronger result which is given as the next theorem. □

Theorem 30. *For a B-module M, let $G \times_B M \to G/B$ be the associated vector bundle. Denote its pull back to $Z_{\mathfrak{w}}$ (for any word $\mathfrak{w}$) under the morphism $\theta_{\mathfrak{w}} : Z_{\mathfrak{w}} \to G/B$ by $\theta_{\mathfrak{w}}^* M$. Then,*

$$\chi_H(Z_{\mathfrak{w}}, \theta_{\mathfrak{w}}^* M) = * \, D_{\mathfrak{w}}(* \, \text{ch } M).$$

Proof. We induct on the length n of $\mathfrak{w} = (s_{i_1}, \ldots, s_{i_n})$. The Leray spectral sequence for the fibration $Z_{\mathfrak{w}} \to Z_{\mathfrak{w}(n)}$ which maps $[p_1, \ldots, p_n] \mapsto [p_1, \ldots, p_{n-1}]$, with fibers $\mathbb{P}^1 \simeq P_{i_n}/B$, takes the form

$$E_2^{p,q} = H^p \left(Z_{\mathfrak{w}(n)}, \theta_{\mathfrak{w}(n)}^* (H^q(P_{i_n}/B, \theta_{s_{i_n}}^* M)) \right),$$

and converges to $H^{p+q}(Z_{\mathfrak{w}}, \theta_{\mathfrak{w}}^* M)$. From this we see that

$$\chi_H(Z_{\mathfrak{w}}, \theta_{\mathfrak{w}}^* M) = \chi_H(Z_{\mathfrak{w}(n)}, \theta_{\mathfrak{w}(n)}^* (\chi_H(P_{i_n}/B, \theta_{s_{i_n}}^* M))).$$

By induction,

$$\begin{aligned} \chi_H(Z_{\mathfrak{w}}, \theta_{\mathfrak{w}}^* M) &= * \, D_{\mathfrak{w}(n)} \left(* \, \chi_H(P_{i_n}/B, \theta_{s_{i_n}}^* M) \right) \\ &= * \, D_{\mathfrak{w}(n)}(* \; * \; D_{s_{i_n}}(* \, \text{ch} \, M)), \text{ by the next exercise} \\ &= * \, D_{\mathfrak{w}}(* \, \text{ch} \, M). \end{aligned}$$

□

Combining Theorem 30 for $M = \mathbb{C}_\lambda$ and Corollary 28, we get the following:

Corollary 31. *For any reduced word $\mathfrak{w}$, the operator $D_{\mathfrak{w}}$ depends only upon $\xi(\mathfrak{w})$.*

Exercise 32. Show $\chi_H(P_{i_n}/B, \mathbb{C}_\mu) = * \, D_{s_{i_n}}(e^{-\mu})$ and conclude that

$$\chi_H(P_{i_n}/B, \theta_{s_{i_n}}^* M) = * \, D_{s_{i_n}}(* \, \text{ch} \, M).$$

13 Verma modules

For any $\lambda \in \mathfrak{h}^*$, define the *Verma module*

$$M(\lambda) := \mathcal{U}(\mathfrak{g}) \otimes_{\mathcal{U}(\mathfrak{b})} \mathbb{C}_\lambda,$$

where $\mathfrak{h}$ acts on $\mathbb{C}_\lambda$ by the action of $\mathfrak{h}$ via the weight λ and the nil-radical $\mathfrak{n} := [\mathfrak{b}, \mathfrak{b}]$ acts trivially. Then, $M(\lambda)$ is a $\mathfrak{g}$-module under left multiplication by elements of $\mathfrak{g}$ on the $\mathcal{U}(\mathfrak{g})$ factor. If $\lambda \in \mathfrak{h}^* \setminus X(H)$, then $\mathbb{C}_\lambda$ is only a representation of $\mathfrak{b}$ and not of B.

Exercise 33. Show

$$\text{ch } M(\lambda) = e^\lambda \prod_{\beta \in R^+} \left(1 - e^{-\beta}\right)^{-1}.$$

14 BGG resolution

Let $\lambda \in X(H)_+$ and $N = |R^+|$. We define a resolution of the form:

$$0 \to \mathcal{F}_N \to \cdots \to \mathcal{F}_2 \to \mathcal{F}_1 \to \mathcal{F}_0 = M(\lambda) \to V(\lambda) \to 0, \tag{5}$$

where

$$\mathcal{F}_p := \bigoplus_{\substack{v \in W \\ \ell(v) = p}} M(v \star \lambda).$$

Fix one non-zero vector $1_\lambda \in \mathbb{C}_\lambda$, then $1 \otimes 1_\lambda \in M(\lambda)$ maps to v_λ, a highest weight vector in $V(\lambda)$. This map extends to

$$x \otimes 1_\lambda \mapsto x \cdot v_\lambda,$$

for each $x \in \mathcal{U}(\mathfrak{g})$. Up to a scalar, this is the unique $\mathfrak{g}$-module map $M(\lambda) \to V(\lambda)$. To define the maps $\delta_j : \mathcal{F}_{j+1} \to \mathcal{F}_j$, we first recall the following theorem:

Theorem 34 (Theorem 9.2.3 in [1])**.** *Let $\lambda \in X(H)_+$ and $v, w \in W$. If $w \not\geq v$, then*

$$\text{Hom}_{\mathfrak{g}}(M(w \star \lambda), M(v \star \lambda)) = 0.$$

If $w \geq v$, then it is one-dimensional.

We now define $\mathfrak{g}$-module maps $\mathcal{F}_{v,w} : M(w\star\lambda) \to M(v\star\lambda)$. If $w \not\geq v$, then $\mathcal{F}_{v,w} = 0$. For any $w \in W$, take any non-zero $\mathfrak{g}$-module map $i_w : M(w\star\lambda) \to M(\lambda)$.

Exercise 35. For any $\lambda, \mu \in \mathfrak{h}^*$, a $\mathfrak{g}$-module map $M(\lambda) \to M(\mu)$ is injective if non-zero.

When $w > v$ and $\ell(w) = \ell(v) + 1$, we write $w \leftarrow v$. There exists a unique choice of non-zero $\mathfrak{g}$-maps $\mathcal{F}_{v,w} : M(w \star \lambda) \to M(v \star \lambda)$ for every $w \leftarrow v$ satisfying

$$i_v \circ \mathcal{F}_{v,w} = i_w.$$

Define

$$\delta_p = \sum_{\substack{\ell(w)=p+1 \\ w\leftarrow v}} \epsilon_{v,w}\mathcal{F}_{v,w} : \mathcal{F}_{p+1} \to \mathcal{F}_p,$$

where $\epsilon_{v,w} \in \{\pm 1\}$ are chosen satisfying the following result due to Bernstein–Gelfand–Gelfand.

Lemma 36 (Lemma 9.2.2 in [1]). *There is a choice of $\epsilon : \{v \to w\} \to \{\pm 1\}$ satisfying the following: for any square as below consisting of elements $v, w, x, y \in W$*

$$\begin{array}{ccc} v & \longrightarrow & x \\ \downarrow & & \downarrow \\ y & \longrightarrow & w \end{array},$$

we have

$$\epsilon_{v,x}\epsilon_{x,w}\epsilon_{v,y}\epsilon_{y,w} = -1.$$

The following is the celebrated Bernstein–Gelfand–Gelfand (BGG, for short) resolution.

Theorem 37. *The above sequence (5) is a resolution of $V(\lambda)$ for any $\lambda \in X(H)_+$.*

We will give two applications of this powerful result.

15 Application: Weyl character formula

Corollary 38 (Weyl character formula)**.** *If* $\lambda \in X(H)_+$*, then*

$$\mathrm{ch}\ V(\lambda) = \frac{\sum_{w \in W} (-1)^{\ell(w)} e^{w \star \lambda}}{\sum_{w \in W} (-1)^{\ell(w)} e^{w \star 0}}.$$

Proof. From the exactness of complex (5), we easily see that

$$\mathrm{ch}\ V(\lambda) = \sum_{p \geq 0} (-1)^p \,\mathrm{ch}\, \mathcal{F}_p = \sum_{w \in W} (-1)^{\ell(w)} \mathrm{ch}\, M(w \star \lambda).$$

In particular, by Exercise 33,

$$1 = \mathrm{ch}\ V(0) = \sum_{w \in W} (-1)^{\ell(w)} \mathrm{ch}\, M(w \star 0) = \left(\sum_{w \in W} (-1)^{\ell(w)} e^{w \star 0} \right) \prod_{\beta \in R^+} \left(1 - e^{-\beta}\right)^{-1},$$

which implies

$$\prod_{\beta \in R^+} \left(1 - e^{-\beta}\right) = \sum_{w \in W} (-1)^{\ell(w)} e^{w \star 0}.$$

Therefore,

$$\begin{aligned} \mathrm{ch}\ V(\lambda) &= \sum_{w \in W} (-1)^{\ell(w)} \mathrm{ch}\, M(w \star \lambda) \\ &= \left(\sum_{w \in W} (-1)^{\ell(w)}\, e^{w \star \lambda} \right) \prod_{\beta \in R^+} \left(1 - e^{-\beta}\right)^{-1} \\ &= \frac{\sum_{w \in W} (-1)^{\ell(w)} e^{w \star \lambda}}{\sum_{w \in W} (-1)^{\ell(w)} e^{w \star 0}}. \end{aligned}$$

□

16 Application: Kostant's theorem on $\mathfrak{n}$-homology

Let $\mathfrak{n} = [\mathfrak{b}, \mathfrak{b}]$ denote the nil-radical of the Borel $\mathfrak{b}$. Then, by definition,

$$H_i(\mathfrak{n}^-, V(\lambda)) := \mathrm{Tor}_i^{\mathcal{U}(\mathfrak{n}^-)}(\mathbb{C}, V(\lambda)),$$

where $\mathfrak{n}^- := [\mathfrak{b}^-, \mathfrak{b}^-]$ and $\mathfrak{b}^- = \mathfrak{h} \oplus \bigoplus_{\beta \in R^+} \mathfrak{g}_{-\beta}$. Since H normalizes $\mathfrak{n}^-$ and $V(\lambda)$ is a $\mathfrak{g}$-module, $H_i(\mathfrak{n}^-, V(\lambda))$ is canonically an H-module. Now, we prove the following theorem due to Kostant as a consequence of the BGG resolution.

Theorem 39. *As H-modules,*

$$H_i(\mathfrak{n}^-, V(\lambda)) \simeq \bigoplus_{\substack{w \in W \\ \ell(w)=i}} \mathbb{C}_{w \star \lambda}.$$

Proof. By the PBW-theorem, $M(\lambda)$ is free as a $\mathcal{U}(\mathfrak{n}^-)$-module. Specifically, as $\mathcal{U}(\mathfrak{n}^-)$-modules, we have:

$$\mathcal{U}(\mathfrak{g}) \simeq \mathcal{U}(\mathfrak{n}^-) \otimes_{\mathbb{C}} \mathcal{U}(\mathfrak{b}).$$

Thus, as $\mathcal{U}(\mathfrak{n}^-)$-modules,

$$M(\lambda) = \big(\mathcal{U}(\mathfrak{n}^-) \otimes_{\mathbb{C}} \mathcal{U}(\mathfrak{b})\big) \otimes_{\mathcal{U}(\mathfrak{b})} \mathbb{C}_\lambda = \mathcal{U}(\mathfrak{n}^-) \otimes_{\mathbb{C}} \mathbb{C}_\lambda.$$

Since any Verma module is free as a $\mathcal{U}(\mathfrak{n}^-)$-module, by the BGG resolution (5) and the definition of Tor, we get that

$$\mathrm{Tor}_i^{\mathcal{U}(\mathfrak{n}^-)}(\mathbb{C}, V(\lambda)) = H_i(\mathbb{C} \otimes_{\mathcal{U}(\mathfrak{n}^-)} \mathcal{F}_*).$$

To compute $H_i(\mathbb{C} \otimes_{\mathcal{U}(\mathfrak{n}^-)} \mathcal{F}_*)$, observe

$$\begin{aligned}
\mathbb{C} \otimes_{\mathcal{U}(\mathfrak{n}^-)} \mathcal{F}_j &= \bigoplus_{\ell(w)=j} \mathbb{C} \otimes_{\mathcal{U}(\mathfrak{n}^-)} (\mathcal{U}(\mathfrak{n}^-) \otimes_{\mathbb{C}} \mathbb{C}_{w \star \lambda}) \\
&= \bigoplus_{\ell(w)=j} \mathbb{C} \otimes_{\mathbb{C}} \mathbb{C}_{w \star \lambda} \\
&= \bigoplus_{\ell(w)=j} \mathbb{C}_{w \star \lambda}.
\end{aligned}$$

But all of the $\mathfrak{h}$-module maps

$$\mathbb{C} \otimes_{\mathcal{U}(\mathfrak{n}^-)} \mathcal{F}_j = \bigoplus_{\ell(w)=j} \mathbb{C}_{w \star \lambda} \xrightarrow{1 \otimes \delta_{j-1}} \bigoplus_{\ell(v)=j-1} \mathbb{C}_{v \star \lambda} = \mathbb{C} \otimes_{\mathcal{U}(\mathfrak{n}^-)} \mathcal{F}_{j-1}$$

are zero, since for $\lambda \in X(H)_+$ we have

$$v \star \lambda = w \star \lambda \Longleftrightarrow v = w.$$

Thus, $H_i(\mathbb{C} \otimes_{\mathcal{U}(\mathfrak{n}^-)} \mathcal{F}_*) = \bigoplus_{\ell(w)=i} \mathbb{C}_{w \star \lambda}$, proving the theorem. □

17 Grothendieck–Cousin complex

Our aim is to realize/prove the BGG-resolution geometrically.

Let X be a variety and let it be filtered by closed (but not necessarily irreducible) sub-varieties

$$X = X_0 \supset X_1 \supset X_2 \supset \cdots .$$

Let $\mathcal{S}$ be a coherent sheaf (a vector bundle is enough for our purposes) on X. Then, there exists a complex (called the *Grothendieck–Cousin complex*) as follows:

$$0 \to H^0(X, \mathcal{S}) \to H^0_{X_0/X_1}(X, \mathcal{S}) \to H^1_{X_1/X_2}(X, \mathcal{S}) \to H^2_{X_2/X_3}(X, \mathcal{S}) \to \cdots .$$

For $Z \subset Y \subset X$ closed, $H^i_{Y/Z}(X, \mathcal{S})$ is the cohomology with support (see Appendix B of [1]).

Theorem 40 (Kempf)**.** *The above complex is exact if the following properties hold:*

(1) X is a Cohen-Macaulay irreducible variety,

(2) $\mathcal{S}$ is a vector bundle,

(3) the maps $X_i \smallsetminus X_{i+1} \to X$ are affine morphisms for all i (i.e., inverse images of affine open subsets are affine) and $X_i \smallsetminus X_{i+1}$ are affine varieties,

(4) codimension of each irreducible component of X_i in X is at least i,

(5) $H^n(X, \mathcal{S}) = 0$ if $n > 0$.

In our case, take $X = G/B$ and $X_i = \bigcup_{\ell(v) \geq i} X^v$, where $X^v = \overline{B^- vB/B} = \overline{w_0 B w_0 vB/B} = w_0 X_{w_0 v}$, where B^- is the subgroup of G with Lie algebra $\mathfrak{b}^-$. Take $\mathcal{S} = \mathcal{L}(\lambda)$ for $\lambda \in X(H)_+$.

Since G/B is smooth, it is Cohen–Macaulay, and property (1) follows. Of course, (2) is given. Since X_w is of dimension $\ell(w)$, property (4) follows. Property (5) follows from the Borel–Weil–Bott theorem, Theorem 7. Finally,

$$X_i \smallsetminus X_{i+1} = \left(\bigcup_{\ell(v)=i} X^v \right) \smallsetminus \left(\bigcup_{\ell(w) \geq i+1} X^w \right) = \bigsqcup_{\ell(v)=i} B^- vB/B,$$

which is affine since $BvB/B \simeq \mathbb{A}^{\ell(v)}$. If we check that the inclusion $\varphi : B^-vB/B \to G/B$ is an affine morphism, then (3) will be verified.

Let $U_{R^+\cap vR^-}$ be the subgroup of G with the Lie algebra

$$\bigoplus_{\beta\in R^+\cap vR^-} \mathfrak{g}_\beta.$$

Then, the map $U_{R^+\cap vR^-} \to BvB/B$, $g \mapsto gvB$, is a biregular isomorphism. Now, identifying BvB/B with $U_{R^+\cap vR^-}$ as above, we get a biregular isomorphism

$$BvB/B \times B^-vB/B \simeq vB^-B/B \simeq U^-, \tag{6}$$

under $(g,x) \mapsto gx$, where $U^- := [B^-, B^-]$. For any affine open subset V of G/B, by the following exercise, $(vB^-B/B) \cap V$ is an affine open subset of vB^-B/B. But, B^-vB/B is an affine closed subset of vB^-B/B by the above isomorphism (6). Thus, $V \cap (B^-vB/B)$ is a closed subset of affine $V\cap(vB^-B/B)$ and hence $V\cap(B^-vB/B)$ is an affine open subset of B^-vB/B. This establishes (3).

Exercise 41. If U, V are affine open in any variety Y, then $U \cap V$ is affine.

Theorem 42 (Lemma 9.3.5 and Proposition 9.3.7 in [1])**.** *As $\mathfrak{g}$-modules, for any $p \geq 0$,*

$$H^p_{X_p/X_{p+1}}(G/B, \mathcal{L}(\lambda)) \simeq \bigoplus_{\ell(w)=p} M(w \star \lambda)^\vee,$$

where $^\vee$ denotes the restricted dual.

Thus, in our case the Grothendieck–Cousin complex becomes the resolution (due to Kempf)

$$0 \longrightarrow V(\lambda)^* \longrightarrow \mathcal{F}_0^\vee \longrightarrow \mathcal{F}_1^\vee \longrightarrow \cdots \longrightarrow \mathcal{F}_N^\vee \longrightarrow 0,$$

which is dual to the BGG resolution.

18 Remarks

We have not given any historical comments. The interested reader can find them in sections 8.C and 9.C of [1].

References

[1] Kumar, Shrawan. *Kac-Moody Groups, their Flag Varieties and Representation Theory*. Boston, Birkhäuser, 2002.

[2] Hartshorne, Robin. *Algebraic Geometry*. New York: Springer–Verlag, 1977.

[3] Humphreys, James. *Linear Algebraic Groups*. New York: Springer–Verlag, 1975.