Transformation Groups
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Components of $V(\rho) \otimes V(\rho)$ and Dominant Weight Polyhedra for Affine Kac–Moody Lie Algebras

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Abstract
Kostant asked the following question: Let $\mathfrak{g}$ be a simple Lie algebra over the complex numbers. Let λ be a dominant integral weight. Then, $V(\lambda)$ is a component of $V(\rho) \otimes V(\rho)$ if and only if $\lambda \leq 2\rho$ under the usual Bruhat–Chevalley order on the set of weights. In an earlier work with R. Chirivi and A. Maffei, the second author gave an affirmative answer to this question up to a saturation factor. The aim of the current work is to extend this result to untwisted affine Kac–Moody Lie algebra $\mathfrak{g}$ associated with any simple Lie algebra $\mathring{\mathfrak{g}}$ (up to a saturation factor). In fact, we prove the result for affine sl_n without any saturation factor. Our proof requires some additional techniques including the Goddard–Kent–Olive construction and study of the characteristic cone of non-compact polyhedra.

1 Introduction

Let $\mathfrak{g}$ be a symmetrizable Kac–Moody Lie algebra over $\mathbb{C}$. In particular, we could let $\mathfrak{g}$ be a (finite-dimensional) semisimple Lie algebra or an untwisted affine Kac–Moody Lie algebra. Given two dominant integral weights λ and μ of $\mathfrak{g}$, the tensor decomposition problem seeks to understand the irreducible components of $V(\lambda) \otimes V(\mu)$, where $V(\lambda)$ is the irreducible representation of $\mathfrak{g}$ with highest weight λ, and similarly for μ. This is a classic problem, with approaches and applications in representation theory, geometry, conformal field theory, and algebraic combinatorics, just to name a few.

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In his study of the exterior algebra $\bigwedge \mathfrak{g}$ of semisimple Lie algebras, Kostant [1] found that as $\mathfrak{g}$-representations, $\bigwedge \mathfrak{g}$ decomposes as

$$\bigwedge \mathfrak{g} \cong 2^{\ell} \left(V(\rho) \otimes V(\rho) \right),$$

where ℓ is the rank of $\mathfrak{g}$, and ρ is half the sum of positive roots. Because of the role $\bigwedge \mathfrak{g}$ plays in the computation of Lie algebra homology, and also in the structure of the Clifford algebra $\mathcal{C}(\mathfrak{g})$ as was Kostant's motivation, one would like to understand the decomposition of the tensor product $V(\rho) \otimes V(\rho)$ into irreducible components. In this direction, Kostant made the following conjecture, first recorded in the work of Berenstein–Zelevinsky, where they also gave a proof of this conjecture when $\mathfrak{g} = sl_{n+1}$ [2].

Conjecture 1.1 Let $\mathfrak{g}$ be a semisimple Lie algebra over $\mathbb{C}$. Let $\lambda \in \mathcal{P}^+$ be a dominant integral weight such that $\lambda \leq 2\rho$ in dominance order. Then, $V(\lambda) \subset V(\rho) \otimes V(\rho)$.

Of course, the opposite implication clearly holds. Later, Chrivì–Kumar–Maffei [3] proved a weaker version of Kostant's conjecture, which recovers Berenstein–Zelevinsky's result for sl_{n+1}, by showing that the expected components appear in the tensor product decomposition "up to saturation," in the language of the saturated tensor cone; see Sect. 4 for the precise definitions.

Theorem 1.2 *([3, Theorem 3]). Let $\mathfrak{g}$ be a (finite-dimensional) simple Lie algebra over $\mathbb{C}$. Let $\lambda \in \mathcal{P}^+$ be a dominant integral weight such that $\lambda \leq 2\rho$ in dominance order. Then, for any saturation factor $d \geq 1$ of $\mathfrak{g}$, $V(d\lambda) \subset V(d\rho) \otimes V(d\rho)$. In particular, for $\mathfrak{g} = sl_n$, $V(\lambda) \subset V(\rho) \otimes V(\rho)$.*

The proof of Theorem 1.2 in loc. cit. makes use of a set of inequalities due to Berenstein–Sjamaar [4] which controls the possible irreducible components of $V(\lambda) \otimes V(\mu)$ up to saturation. The tensor decomposition problem is a particular example of a larger class of branching problems, which consider the restriction of representations of $\mathfrak{g}$ to an embedded subalgebra $\mathfrak{g}_1 \hookrightarrow \mathfrak{g}$. In the tensor decomposition case, this corresponds to the diagonal embedding $\mathfrak{g} \hookrightarrow \mathfrak{g} \oplus \mathfrak{g}$. A possible representation-theoretic generalization of Conjecture 1.1 is to consider the branching of the representation $V(\rho)$ of $\mathfrak{g}$ to a subalgebra $\mathfrak{g}_1$. This was studied by Nadimpalli–Pattanayak [5], guided by the approach in [3], generalizing Theorem 1.2 via the corresponding inequalities of the related "branching cone" (see [6] for a survey on the cones and inequalities for the tensor product and more general branching setting).

We now propose a different representation-theoretic generalization; in particular, we consider the tensor decomposition problem for any untwisted affine Kac–Moody Lie algebra $\mathfrak{g}$. In this setting, one can again try to determine the irreducible components of the tensor product $V(\rho) \otimes V(\rho)$, where ρ now is the sum of fundamental weights (see Sect. 2 for conventions). However, as the highest weight irreducible representations in this case are infinite-dimensional, and the tensor product $V(\rho) \otimes V(\rho)$ has infinitely-many components, the problem is more subtle. Nevertheless, we prove the following theorem, which is a verbatim generalization of Theorem 1.2 (cf. Theorems 5.6 and 6.2 in the text).

Theorem 1.3 *Let $\mathfrak{g}$ be an untwisted affine Kac–Moody Lie algebra, $\lambda \in \mathcal{P}^+$ a dominant integral weight such that $\lambda \leq 2\rho$ in dominance order. Then,*

1. *for any saturation factor $d \geq 1$ of $\mathfrak{g}$, we have $V(d\lambda) \subset V(d\rho) \otimes V(d\rho)$.*
2. *if $\mathfrak{g} = A_n^{(1)} = \widehat{sl}_{n+1}$, then $V(\lambda) \subset V(\rho) \otimes V(\rho)$.*

We note that, unlike in the semisimple case, Theorem 1.3(2) does not follow from 1.3(1), as the smallest saturation factor d for $\widehat{sl}_{n+1}$ is $d = 2$. While a similar approach via inequalities as in [3] and [5] in the affine Kac–Moody setting would be effective to prove Theorem 1.3(1) (see Remark 5.3), such an approach would not address 1.3(2).

Instead, because of its utility in the $\mathfrak{g} = \widehat{sl}_{n+1}$ case, our primary tool in proving Theorem 1.3 is the action of the Virasoro algebra on $V(\rho) \otimes V(\rho)$ via the Goddard–Kent–Olive (or GKO) construction, which we recall in Sect. 4.2. The applicability of the GKO construction to the tensor decomposition problem allows us to restrict attention to certain "maximal" components $V(\nu)$ of $V(\rho) \otimes V(\rho)$, whose existence demonstrates the appearance of the components $V(\nu - k\delta)$ for all $k \geq 0$. This technique originates with Kac–Wakimoto [7] and has seen applications in the study of the affine tensor semigroup (see Sect. 4.1 for references). More recently, the GKO action was crucially used in a similar fashion in our previous work on root components for affine Kac–Moody Lie algebras [8].

To apply the GKO construction effectively, we must first understand the dominant weights $\lambda \leq 2\rho$. To do this, we introduce the notion of the *dominant weight polyhedron* D_μ for a highest weight integrable (irreducible) representation $V(\mu)$ of an affine Kac–Moody Lie algebra. In the finite case, these are compact polytopes and are completely describable by their vertices. This description was used in [3, Proposition 9] to express those $\lambda \leq 2\rho$ in a particularly useful form. However, in the affine case, these polyhedra are no longer compact, so they cannot be completely determined by their vertices. Taking this into account, we give in Proposition 3.9 an a priori larger set of points that contains the vertices of D_μ for regular dominant μ; in fact, in Corollary 3.10, we show that these points are precisely the vertices when μ is regular dominant. By a general result for the structure of polyhedra, we then give an explicit decomposition of D_μ for μ regular dominant. With this result, we recover a decomposition result for $\lambda \leq 2\rho$ in Proposition 5.1 analogous to the one in the semisimple case as in [3].

Finally, we apply the GKO construction to the specific problem of determining the components of $V(\rho) \otimes V(\rho)$ up to a saturation factor d as in Theorem 1.3. We further propose an exact analogue of Kostant's conjecture (without any saturation factor) for affine Kac–Moody Lie algebras in Conjecture 6.3.

2 Notation and Conventions for Affine Kac–Moody Lie Algebras

We fix here briefly the key notation and conventions used throughout the paper. For a full treatment, we refer to [9, Chapters 6, 7], whose conventions we adopt.

By $\mathbb{Z}_+$ (resp. $\mathbb{Q}_+$), we mean the nonnegative integers (resp., rational numbers).

Throughout, we denote by $\mathfrak{g}$ the untwisted affine Kac–Moody Lie algebra, associated with a finite-dimensional simple Lie algebra $\mathring{\mathfrak{g}}$. We fix a choice of Cartan subalgebra and Borel subalgebra $\mathfrak{h} \subset \mathfrak{b} \subset \mathfrak{g}$ based on a choice of Cartan subalgebra $\mathring{\mathfrak{h}}$

and a Borel subalgebra $\mathring{\mathfrak{b}} \supset \mathring{\mathfrak{h}}$ of $\mathring{\mathfrak{g}}$. Relative to this choice, we choose a set of simple roots $\Delta = \{\alpha_0, \alpha_1, \ldots, \alpha_\ell\} = \{\alpha_i\}_{i\in I}$, where $\ell = \operatorname{rank} \mathring{\mathfrak{g}}$ and $I := \{0, 1, \ldots, \ell\}$. We set Φ the set of roots of $\mathfrak{g}$ and denote by Φ^+ (resp., Φ^-) the set of positive roots (resp., negative roots) for this choice. The set of roots Φ can further be partitioned into the set of real roots Φ_{Re} and imaginary roots Φ_{Im}, with basic imaginary root $\delta := \alpha_0 + \theta$, where θ is the highest root of $\mathring{\mathfrak{g}}$. We denote by $d,\ K \in \mathfrak{h}$ the derivation and central elements of $\mathfrak{g}$, respectively.

The Weyl group of $\mathfrak{g}$ is denoted by W and has simple reflections $\{s_0, s_1, \ldots, s_\ell\}$ associated with the simple roots.

We let

$$\mathcal{P}^+ := \{\lambda \in \mathfrak{h}^* : \langle \lambda, \alpha_i^\vee \rangle \in \mathbb{Z}_+ \, \forall i \in I\}$$

be the set of dominant integral weights of $\mathfrak{g}$. The fundamental weights $\Lambda_i \in \mathcal{P}^+$ are defined uniquely by $\langle \Lambda_i, \alpha_j^\vee \rangle = \delta_{ij}$, the Kronecker delta, for all simple coroots $\alpha_j^\vee$ and $\langle \Lambda_i, d \rangle = 0$. Then, the corresponding Weyl vector $\rho \in \mathcal{P}^+$ is given by $\rho = \sum_{i=0}^{\ell} \Lambda_i$ (so that $\langle \rho, d \rangle = 0$). For any $\lambda \in \mathcal{P}^+$, the value of $\langle \lambda, K \rangle$ is called the level of λ; throughout, we will restrict to those $\lambda \in \mathcal{P}^+$ with **positive** level.

The weight lattice $\mathcal{P}$ and root lattice $\mathcal{Q}$ are given by

$$\mathcal{P} := \bigoplus_{i=0}^{\ell} \mathbb{Z}\Lambda_i \oplus \mathbb{C}\delta, \quad \mathcal{Q} := \bigoplus_{i=0}^{\ell} \mathbb{Z}\alpha_i.$$

Finally, the similarly defined objects for $\mathring{\mathfrak{g}}$ will be denoted by a circle, i.e., the root lattice $\mathring{\mathcal{Q}}$ of $\mathring{\mathfrak{g}}$, etc. We fix also a normalized invariant form $(\cdot|\cdot)$ on $\mathfrak{g}$ such that $(\theta|\theta) = 2$, where again $\theta \in \mathring{\Phi}^+$ is the highest root of the underlying simple Lie algebra $\mathring{\mathfrak{g}}$; we will always use this choice of normalization.

3 Irreducible Highest Weight Representations and Dominant Weight Polyhedra

In this section, we recall some basic results on the structure of irreducible highest weight representations of $\mathfrak{g}$. While many of the results mentioned hold more generally at the level of symmetrizable Kac–Moody algebras, we focus only on statements for the affine Lie algebra setting. We use [9, Chapters 11, 12] as our primary sources.

3.1 Weights of Irreducible Highest Weight Modules

Let $\lambda \in \mathcal{P}^+$ be a dominant integral weight of $\mathfrak{g}$. We denote by $V(\lambda)$ the corresponding irreducible, highest weight representation of $\mathfrak{g}$ with highest weight λ. As an $\mathfrak{h}$-module via restriction, $V(\lambda)$ has a decomposition into weight spaces

$$V(\lambda) = \bigoplus_{\mu \in \mathcal{P}} V(\lambda)_\mu,$$

where $V(\lambda)_\mu := \{v \in V(\lambda) : h.v = \langle \mu, h\rangle v \ \forall h \in \mathfrak{h}\}$. We set $\mathcal{P}(\lambda)$ to be the set of weights of the representation $V(\lambda)$, that is, $\mathcal{P}(\lambda) = \{\mu \in \mathcal{P} : V(\lambda)_\mu \neq 0\}$. This is a (typically infinite) W-invariant set of weights in $\mathcal{P}$.

For any two weights $\lambda, \mu \in \mathcal{P}$, we say that $\lambda \geq \mu$ if $\lambda - \mu \in \mathcal{Q}^+ := \bigoplus_{i=0}^{\ell} \mathbb{Z}_+\alpha_i$. This defines a partial order on the weight lattice $\mathcal{P}$, the *dominance order*. As $V(\lambda)$ is a highest weight representation, for any $\mu \in \mathcal{P}(\lambda)$, necessarily $\mu \leq \lambda$. While it is not immediately clear from an algebraic perspective for which $\mu \leq \lambda$ we have $\mu \in \mathcal{P}(\lambda)$, this question has a nice combinatorial answer. We record now the following standard proposition, taken from [9, Proposition 12.5(a)(b)], which gives a concise description of $\mathcal{P}(\lambda)$.

Proposition 3.1 *For $\lambda \in \mathcal{P}^+$, let $V(\lambda)$ be the irreducible highest weight representation with highest weight λ, and set of weights $\mathcal{P}(\lambda)$. Then,*

$$\mathcal{P}(\lambda) = W \cdot \{\mu \in \mathcal{P}^+ : \mu \leq \lambda\} = (\lambda + \mathcal{Q}) \cap \mathrm{conv}_{\mathbb{Q}}\{w\lambda : w \in W\},$$

where $\mathrm{conv}_{\mathbb{Q}}$ *denotes the rational convex hull.*

3.2 δ-Maximal Weights

Let $\lambda \in \mathcal{P}^+$ be a dominant integral weight, and consider the associated highest weight irreducible representation $V(\lambda)$ with set of weights $\mathcal{P}(\lambda)$. We recall the following definition.

Definition 3.2 A weight $\mu \in \mathcal{P}(\lambda)$ is called δ-maximal if $\mu + k\delta \notin \mathcal{P}(\lambda)$ for any $k > 0$. We denote by $\mathcal{P}_{max}(\lambda)$ the set of all δ-maximal weights.

Since δ is fixed by all Weyl group elements $w \in W$, the set of δ-maximal weights in $\mathcal{P}(\lambda)$ is W-invariant. But as any weight in $\mathcal{P}(\lambda)$ can be W-translated to a dominant weight (cf. Proposition 3.1), it suffices to understand the set of δ-maximal *dominant* weights; we denote this set by $\mathcal{P}^+_{max}(\lambda)$. The study of δ-maximal dominant weights plays a crucial role in applications of representation theory to modular forms and conformal field theory, via understanding (up to renormalization) the "string functions"

$$t \mapsto \dim(V(\lambda)_{\mu - t\delta})$$

for $\mu \in \mathcal{P}^+_{max}(\lambda)$ (see [9, Chapter 12]). As these will play an important role in what is to come, we record the following lemma on the "δ-strings" $\{\mu - k\delta\}_{k\geq 0} \subset \mathcal{P}(\lambda)$, which is a reformulation of [9, Prop. 12.5(e)].

Lemma 3.3 *Let $\lambda \in \mathcal{P}^+$ and $\mu \in \mathcal{P}(\lambda)$. Then, $\mu - k\delta \in \mathcal{P}(\lambda)$ for all $k \geq 0$.*

Combining Lemma 3.3 and Proposition 3.1, we can get the following description of δ-maximal dominant weights, taken from [11, Proposition 4.4].

Proposition 3.4 *Let $a_0, a_1, \ldots, a_\ell$ be defined by $\delta = \sum_{i\in I} a_i\alpha_i$ and let $\lambda \in \mathcal{P}^+$. Then, $\mathcal{P}^+_{max}(\lambda)$ is the set of all $\mu \in \mathcal{P}^+$ such that*

1. $\mu \leq \lambda$, and
2. if $\lambda - \mu = \sum_{i \in I} c_i \alpha_i$ then there exists some $i \in I$ such that $c_i < a_i$.

Proof Suppose for contradiction that $\mu \in \mathcal{P}^+_{max}(\lambda)$ and $\lambda - \mu = \sum_{i \in I} c_i \alpha_i$ with $c_i \geq a_i$ for all i. Set $\mu' := \mu + \delta \in \mathcal{P}^+$. Then, we have

$$\lambda - \mu' = \lambda - \mu - \delta = \sum_{i \in I} (c_i - a_i)\alpha_i.$$

Note that since $c_i - a_i \geq 0$ for all i by assumption, we get that $\lambda - \mu' \in \mathcal{Q}^+$ or that $\mu' \leq \lambda$ in dominance order. But then, by Proposition 3.1, we get that $\mu' \in \mathcal{P}(\lambda)$, which contradicts μ being δ-maximal. The converse is clear. $\square$

3.3 Dominant Weight Polyhedra

We next want to better understand the dominant weights $\mu \in \mathcal{P}^+$ such that $\mu \in \mathcal{P}(\lambda)$. By Lemma 3.3, it would suffice to understand those weights in $\mathcal{P}^+_{max}(\lambda)$. However, this is in general a subtle issue, as it is not straightforward to describe uniformly the set $\mathcal{P}^+_{max}(\lambda)$ in terms of λ. We instead consider a larger set, which we refer to as the *dominant weight polyhedron* associated with λ and denote it by D_λ. This will be a rational, (generically) unbounded polyhedron in the sense of [12], which we take as our standard reference.

We restrict for our purposes to the case when $\lambda \in \mathcal{P}^+$ is regular dominant, so that $\langle \lambda, \alpha_i^\vee \rangle > 0$ for all $i \in I := \{0, 1, \ldots, \ell\}$. *We will also assume that* $\langle \lambda, d \rangle = 0$, up to replacing λ with $\lambda - \langle \lambda, d \rangle \delta$. Let $\mathcal{C}_\mathbb{Q}$ denote the rational dominant chamber, defined by

$$\mathcal{C}_\mathbb{Q} := \{\mu \in \mathfrak{h}^*_\mathbb{Q} : \langle \mu, \alpha_i^\vee \rangle \geq 0 \; \forall i \in I\},$$

where $\mathfrak{h}^*_\mathbb{Z} := \left(\oplus_{i=0}^{\ell} \mathbb{Z}\Lambda_i\right) \oplus \mathbb{Z}\delta$ and $\mathfrak{h}^*_\mathbb{Q} := \mathbb{Q} \otimes_\mathbb{Z} \mathfrak{h}^*_\mathbb{Z}$. We now fix the following definition.

Definition 3.5 For a dominant integral weight $\lambda \in \mathcal{P}^+$, the dominant weight polyhedron D_λ is defined by

$$D_\lambda := \mathcal{C}_\mathbb{Q} \cap \mathrm{conv}_\mathbb{Q}\{w\lambda : w \in W\}.$$

Thus, by Proposition 3.1, the points $\mu \in D_\lambda$ correspond to rational dominant weights such that $N\mu \in \mathcal{P}(N\lambda) \cap \mathcal{P}^+$ for some scaling factor $N \in \mathbb{Z}_{>0}$. Again by Proposition 3.1, the points $\mu \in D_\lambda$ are determined by the set of inequalities for $\mu \in \mathfrak{h}^*_\mathbb{Q}$:

$$\begin{cases} \langle \mu, \alpha_i^\vee \rangle \geq 0 \; \forall i \in I, \\ \langle \lambda - \mu, x_i \rangle \geq 0 \; \forall i \in I, \\ \langle \lambda - \mu, K \rangle = 0, \end{cases}$$

where x_i is the fundamental coweight defined by $\langle \alpha_j, x_i \rangle = \delta_{i,j}$. The faces of D_λ are given by replacing a subset of the above inequalities by equalities. To this end, let

$J \subset I$ and consider the sets

$$\begin{aligned} A_J &:= \sum_{k\in J} \mathbb{Q}_+\Lambda_k, \\ \overline{A}_J &:= A_J + \mathbb{Q}_+(-\delta), \\ B_J &:= \sum_{k\in J} \mathbb{Q}_+\alpha_k, \\ C_J = C_J(\lambda) &:= \lambda - B_J. \end{aligned}$$

In particular, we set $\overline{A} = \overline{A}_I = \sum_{k\in I} \mathbb{Q}_+\Lambda_k + \mathbb{Q}_+(-\delta)$, $B = B_I = \sum_{k\in I} \mathbb{Q}_+\alpha_k$, and $C = C_I = \lambda - B$. Then, $\overline{A}_H \cap C_K$ are faces of D_λ for $H, K \subset I$. By construction, we have that

$$D_\lambda = \overline{A} \cap C; \tag{1}$$

note that as any $\mu \in D_\lambda$ satisfies $\langle \mu, d\rangle \leq \langle \lambda, d\rangle$ since $N\mu \leq N\lambda$ (for some positive integer N) in dominance order, the restriction to $\mathbb{Q}_+(-\delta)$ in the definition of $\overline{A}$ suffices.

Unlike when considering a finite-dimensional semisimple Lie algebra $\mathring{\mathfrak{g}}$, the polyhedra D_λ are typically unbounded. First, we want to determine the "infinite directions" of the polyhedron D_λ. We introduce the following definition from [12, Section 8.2].

Definition 3.6 For a polyhedron P, the characteristic cone $\operatorname{char.cone}(P)$ is given by

$$\operatorname{char.cone}(P) := \{y : x + y \in P \ \forall x \in P\}.$$

Lemma 3.7 *For $\lambda \in \mathcal{P}^+$, we have* $\operatorname{char.cone}(D_\lambda) = \mathbb{Q}_+(-\delta)$.

Proof As in (1), $D_\lambda = \overline{A} \cap C$. Clearly $\mathbb{Q}_+(-\delta) \subseteq \operatorname{char.cone}(\overline{A} \cap C)$. Let $y \in \mathfrak{h}^*_{\mathbb{Q}}$ such that $y \in \operatorname{char.cone}(\overline{A} \cap C)$. Since $\lambda \in \overline{A} \cap C$, we immediately conclude $y \in -\sum_{k\geq 0} \mathbb{Q}_+\alpha_k$, by the definition of C.

Then, for any $x \in \overline{A} \cap C$, we have $x + Ny \in \overline{A} \cap C$ for any $N \in \mathbb{Z}_+$. Assume that $\langle y, \alpha_j^\vee\rangle < 0$ for some j. Taking N sufficiently large, we get that

$$\langle x + Ny, \alpha_j^\vee\rangle = \langle x, \alpha_j^\vee\rangle + N\langle y, \alpha_j^\vee\rangle < 0.$$

Contradiction, since $x + Ny \in \overline{A}$, and any $z \in \overline{A}$ satisfies $\langle z, \alpha_j^\vee\rangle \geq 0$ for all j. Thus, $\langle y, \alpha_j^\vee\rangle \geq 0$ for all j.

Next, suppose that $\langle y, \alpha_j^\vee\rangle > 0$ for some j. Note that we can write the canonical central element K of $\mathfrak{g}$ as $K = \sum_{k\geq 0} a_k^\vee \alpha_k^\vee$ for the appropriate positive labels $a_k^\vee$, and as y pairs nonnegatively with all $\alpha_k^\vee$, we get

$$\langle y, K\rangle > 0;$$

contradiction, since $y \in -\sum_{k\geq 0} \mathbb{Q}_+\alpha_k$, and α_k pairs to zero with K for all $k \geq 0$.

Thus, $\langle y, \alpha_k^\vee \rangle = 0$ for all $k \geq 0$. Now, using that $\mathfrak{h}^*_{\mathbb{Q}} = \mathring{\mathfrak{h}}^*_{\mathbb{Q}} \oplus (\mathbb{Q}(\delta) + \mathbb{Q}(\Lambda_0))$, we get that

$$y \in \mathring{\mathfrak{h}}^*_{\mathbb{Q}} \oplus \mathbb{Q}(\delta)$$

and is orthogonal to all of $\mathring{\mathfrak{h}}^*_{\mathbb{Q}}$; thus, necessarily $y \in \mathbb{Q}(\delta)$. In particular, this forces $y \in \mathbb{Q}_+(-\delta)$, as desired. □

We are next interested in the minimal faces of D_λ. By the general theory of polyhedra [12, Section 8.5(22)], the minimal faces of a polyhedron P are translates of the "lineality space"

$$\text{lin. space}(P) := \text{char. cone}(P) \cap -\text{char. cone}(P).$$

Then, by Lemma 3.7, in the case of D_λ, we get that $\text{lin. space}(D_\lambda) = \{0\}$. Thus, the minimal faces of D_λ are zero-dimensional, hence vertices. To understand these vertices, we will examine the various face intersections $A_H \cap C_K$ and $\overline{A}_H \cap C_K$, for subsets $H, K \subseteq I$, and analyze those which intersect in single points. First, in the next lemma, we introduce the possible candidates for the vertices of D_λ.

Lemma 3.8 *Let $\lambda \in \mathcal{P}^+$ be a regular dominant integral weight with $\langle \lambda, d \rangle = 0$. For any subset $J \subsetneq I$, let $W_J = \langle s_j : j \in J \rangle \subset W$ be the corresponding (finite) parabolic subgroup corresponding to the simple roots $\{\alpha_j\}_{j \in J}$. Define*

$$\hat{b}_J(\lambda) := \frac{1}{|W_J|} \sum_{w \in W_J} w(\lambda).$$

Then, $\hat{b}_J(\lambda) \in D_\lambda$. In fact, for each $J \subsetneq I$, we have $\hat{b}_J(\lambda) \in \overline{A}_{I \setminus J}$, and $\hat{b}_J(\lambda) \neq \hat{b}_{J'}(\lambda)$ for $J \neq J'$.

Proof First, we show that $\hat{b}_J(\lambda) \in D_\lambda$. To this end, by its definition, $\hat{b}_J(\lambda) \in \text{conv}_{\mathbb{Q}}\{w\lambda : w \in W\}$. It remains to show that $\hat{b}_J(\lambda) \in \mathcal{C}_{\mathbb{Q}}$. Let $\alpha_i^\vee$ be a simple coroot. We consider two cases: first, suppose that $i \in J$. Then, we have that

$$|W_J| \langle \hat{b}_J(\lambda), \alpha_i^\vee \rangle = \left\langle \sum_{w \in W_J} w(\lambda), \alpha_i^\vee \right\rangle = \left\langle \lambda, \sum_{w \in W_J} w^{-1}(\alpha_i^\vee) \right\rangle = 0, \tag{2}$$

since $i \in J$ so that $\sum_{w \in W_J} w^{-1}(\alpha_i^\vee) = 0$. Else if $i \notin J$, then for all $w \in W_J$, we have $w^{-1}(\alpha_i^\vee)$ is a positive coroot, hence

$$|W_J| \langle \hat{b}_J(\lambda), \alpha_i^\vee \rangle = \left\langle \sum_{w \in W_J} w(\lambda), \alpha_i^\vee \right\rangle = \left\langle \lambda, \sum_{w \in W_J} w^{-1}(\alpha_i^\vee) \right\rangle > 0, \tag{3}$$

since λ is regular dominant. Thus, $\hat{b}_J(\lambda) \in \mathcal{C}_{\mathbb{Q}}$, as desired, and thus $\hat{b}_J(\lambda) \in D_\lambda$.

Further, since by (2) and (3), $\hat{b}_J(\lambda)$ is a rational dominant weight that vanishes precisely on $\{\alpha_j^\vee\}_{j\in J}$, we get that $\hat{b}_J(\lambda) \in \overline{A}_{I\backslash J}$ by definition, since $\langle w\lambda, d\rangle \leq \langle \lambda, d\rangle = 0$.

Finally, if $j \in J\backslash J'$ we have $\langle \hat{b}_J(\lambda), \alpha_j^\vee\rangle = 0$ while $\langle \hat{b}_{J'}(\lambda), \alpha_j^\vee\rangle > 0$, by (2) and (3), so $\hat{b}_J(\lambda) \neq \hat{b}_{J'}(\lambda)$. □

Proposition 3.9 *Let $\lambda \in \mathcal{P}^+$ be a regular dominant integral weight, and assume that $\langle \lambda, d\rangle = 0$. Then, the vertices of D_λ are contained in the set $\{\hat{b}_J(\lambda)\}_{J\subsetneq I}$.*

Proof As shown above, the minimal faces of D_λ are points, hence vertices. We show that they are contained in the set $\{\hat{b}_J(\lambda)\}_{J\subsetneq I}$. This is done by considering the following ten claims.

Claim 1 If $H \nsubseteq K$, then $\overline{A}_{I\backslash H} \cap C_K = \varnothing$.

Take $y \in \overline{A}_{I\backslash H} \cap C_K$ and let $y = \lambda - x$ for some $x \in B_K$. Write

$$x = \lambda - y = \lambda - \sum_{h\notin H} a_h \Lambda_h + b\delta \tag{4}$$

where $x \in \sum_{k\in K} \mathbb{Q}_+\alpha_k$, $a_h, b \in \mathbb{Q}_+$. Take $j \in H\backslash K$. Evaluating (4) at $\alpha_j^\vee$, we get a contradiction.

Claim 2 If $H \subsetneq K$ and $0 \notin K$, then $A_{I\backslash H} \cap C_K$ contains $\hat{b}_H(\lambda)$ and $\hat{b}_K(\lambda)$.

Clearly, $\hat{b}_H(\lambda), \hat{b}_K(\lambda) \in C_K$. By Lemma 3.8, we know that $\hat{b}_H(\lambda), \hat{b}_K(\lambda) \in \overline{A}_{I\backslash H}$. Now, since $0 \notin K$ (and thus also not in H), we get that $\langle \hat{b}_K(\lambda), d\rangle = \langle \hat{b}_H(\lambda), d\rangle = 0$, so that $\hat{b}_K(\lambda), \hat{b}_H(\lambda) \in A_{I\backslash H}$.

Claim 3 For any subsets $H, K \subset I$, $A_{I\backslash H} \cap C_K = A_{I\backslash H} \cap C_{K\backslash\{0\}}$.

This follows easily since for any $x \in A_{I\backslash H}$, $\langle x, d\rangle = 0$, whereas $\langle \alpha_0, d\rangle = 1$.

Claim 4 For $H \subsetneq K \subsetneq I$ and $0 \in H$, $\hat{b}_H(\lambda), \hat{b}_K(\lambda) \in \overline{A}_{I\backslash H} \cap C_K$ and $A_{I\backslash H} \cap C_K = \varnothing$.

Of course, $\hat{b}_H(\lambda), \hat{b}_K(\lambda) \in C_K$, and by Lemma 3.8, $\hat{b}_H(\lambda), \hat{b}_K(\lambda) \in \overline{A}_{I\backslash H}$. For the second part, by Claim 3, $A_{I\backslash H} \cap C_K = A_{I\backslash H} \cap C_{K\backslash\{0\}} = \varnothing$, by Claim 1 since $0 \in H$.

Claim 5 For $H \subsetneq K \subsetneq I$ and $0 \in K\backslash H$, $\hat{b}_H(\lambda), \hat{b}_{K\backslash\{0\}}(\lambda) \in A_{I\backslash H} \cap C_K$, even if $H = K\backslash\{0\}$.

By Claim 3, $A_{I\backslash H} \cap C_K = A_{I\backslash H} \cap C_{K\backslash\{0\}}$. Furthermore, $H \subset K\backslash\{0\}$ since $0 \notin H$. Thus, the claim follows from Claim 2. (Since $0 \notin H$, it is easy to see that $\hat{b}_H(\lambda) \in A_{I\backslash H}$.)

Claim 6 For $H \subset I\backslash\{i_1, i_2\} with\ i_1 \neq i_2$, none of $A_{I\backslash H} \cap C_I$ or $\overline{A}_{I\backslash H} \cap C_I$ contain exactly one point.

If $i_1 = 0$, $A_{I\backslash H} \cap C_I = A_{I\backslash H} \cap C_{I\backslash\{0\}}$ contains at least two points by Claim 2 and hence so does $\overline{A}_{I\backslash H} \cap C_I$. If neither of i_1, i_2 is 0 (i.e., by claim 5 taking $K = I\backslash\{i_1\}$, we can assume that $0 \in H$), then

$$A_{I\backslash H} \cap C_I = A_{I\backslash H} \cap C_{I\backslash\{0\}} = \varnothing,$$

where the equalities follow from Claims 3 and 1 respectively. Furthermore,

$$\overline{A}_{I\backslash H} \cap C_I \supset \overline{A}_{I\backslash H} \cap C_{I\backslash\{i_1\}} \supset \{\hat{b}_H(\lambda), \hat{b}_{I\backslash\{i_1\}}(\lambda)\}$$

by Claim 4.

Claim 7 For $H = I\backslash\{i_0\}$, $K = I$: if $i_0 = 0$, then $A_{I\backslash H} \cap C_I$ contains the point $\hat{b}_{I\backslash\{0\}}(\lambda)$. If $i_0 \neq 0$, then $A_{I\backslash H} \cap C_I = \varnothing$ and $\overline{A}_{I\backslash H} \cap C_I$ contains the point $\hat{b}_{I\backslash\{i_0\}}(\lambda)$.

Let $i_0 = 0$. By Lemma 3.8, we have $\hat{b}_{I\backslash\{0\}}(\lambda) \in \overline{A}_{\{0\}}$. But since $H = I\backslash\{0\}$, necessarily $\langle \hat{b}_{I\backslash\{0\}}(\lambda), d\rangle = \langle \lambda, d\rangle = 0$, so in fact $\hat{b}_{I\backslash\{0\}}(\lambda) \in A_{\{0\}}$. Of course, $\hat{b}_{I\backslash\{0\}}(\lambda) \in C_I$.

For $i_0 \neq 0$, by Claim 3, $A_{I\backslash H} \cap C_I = A_{I\backslash H} \cap C_{I\backslash\{0\}}$, and since $0 \in H$, by Claim 1 this is empty. Now, again by Lemma 3.8, $\hat{b}_{I\backslash\{i_0\}}(\lambda) \in \overline{A}_{I\backslash H} \cap C_I$.

Claim 8 For any H such that $0 \notin H$, $A_{I\backslash H} \cap C_H = \overline{A}_{I\backslash H} \cap C_H$ contains the point $\hat{b}_H(\lambda)$.

The identity $A_{I\backslash H} \cap C_H = \overline{A}_{I\backslash H} \cap C_H$ follows since, for any $y \in C_H$, $\langle y, d\rangle = 0$. Further, $\hat{b}_H(\lambda) \in C_H$ by construction and $\hat{b}_H(\lambda) \in \overline{A}_{I\backslash H}$ by Lemma 3.8.

Claim 9 For any H such that $0 \in H \subsetneq I$, $A_{I\backslash H} \cap C_H = \varnothing$ and $\overline{A}_{I\backslash H} \cap C_H$ contains the point $\hat{b}_H(\lambda)$.

$A_{I\backslash H} \cap C_H = A_{I\backslash H} \cap C_{H\backslash\{0\}}$, by Claim 3. But by Claim 1, this is empty. Finally, $\hat{b}_H(\lambda) \in C_H$ by construction, and by Lemma 3.8, $\hat{b}_H(\lambda) \in \overline{A}_{I\backslash H}$.

Claim 10 $\overline{A}_{I\backslash I} \cap C_I = \varnothing$.

We need to find $\{a_i\}_{i\in I}$, $a_i \geq 0$, such that

$$\lambda - \sum_{i\in I} a_i\alpha_i \in \mathbb{Q}_+(-\delta).$$

Now, the above implies $\langle \lambda, \alpha_j^\vee\rangle = \sum_{i\in I} a_i \langle \alpha_i, \alpha_j^\vee\rangle$ for all $\alpha_j^\vee$. But this can be rewritten as

$$\Lambda = A \cdot C, \tag{5}$$

where $\Lambda = (\lambda_0, \lambda_1, \lambda_2, \ldots, \lambda_\ell)$, for $\lambda_j := \langle \lambda, \alpha_j^\vee\rangle > 0$, $A = (a_0, a_1, \ldots, a_\ell)$, and $C = \left(\alpha_i(\alpha_j^\vee)\right)_{i,j\geq 0}$ is the affine Cartan matrix. But (5) is contradicted by [9, Theorem 4.3].

Combining all the cases covered by Claims 1–10, we get that if any of $A_{I\backslash H} \cap C_K$ or $\overline{A}_{I\backslash H} \cap C_K$ for any $H, K \subseteq I$ is a single point, then that point is one of the points from the set $\{\hat{b}_H(\lambda)\}_{H\subsetneq I}$. □

Corollary 3.10 *Following the notation and assumptions as in Proposition 3.9, the set of vertices of D_λ is equal to (not just contained in) $\{\hat{b}_H(\lambda)\}_{H \subsetneq I}$.*

Proof Take any $H \subsetneq I$ and let $\mu \in \overline{A}_{I \setminus H} \cap C_H$. Then, we can write

$$\mu = \sum_{i \notin H} m_i \Lambda_i - m\delta = \lambda - \sum_{k \in H} n_k \alpha_k, \quad \text{for some } m_i, m, n_k \in \mathbb{Q}_+.$$

Evaluating the above at $\alpha_i^\vee$, for $i \in H$, we get

$$\langle \lambda, \alpha_i^\vee \rangle - \sum_{k \in H} n_k \langle \alpha_k, \alpha_i^\vee \rangle = 0, \quad \text{for any } i \in H. \tag{6}$$

Consider the row matrices $\boldsymbol{\lambda}_H := \big(\langle \lambda, \alpha_i^\vee \rangle\big)_{i \in H}$; $\mathbf{n} := (n_k)_{k \in H}$ and the $|H| \times |H|$-matrix $C_H := \big(\langle \alpha_k, \alpha_i^\vee \rangle\big)_{k, i \in H}$. Since H is a proper subset of I, C_H is the Cartan matrix of a semisimple Lie algebra. In particular, it is invertible. Hence, the above equation (6) gives

$$\boldsymbol{\lambda}_H = \mathbf{n} \cdot C_H, \text{ i.e., } \boldsymbol{\lambda}_H \cdot C_H^{-1} = \mathbf{n}.$$

Thus, $\mathbf{n}$ is unique if it exists satisfying $n_k \in \mathbb{Q}_+$ for all $k \in H$. This shows that $\overline{A}_{I \setminus H} \cap C_H$ has at most one point. However, as shown in the Claims 8 and 9 of the proof of Proposition 3.9, it has at least the point $\hat{b}_H(\lambda)$. This proves the corollary. □

Specializing to $\lambda = 2\rho$, we can simplify the above to get the following corollary.

Corollary 3.11 *The vertices of $D_{2\rho}$ are given by the set $\{\hat{b}_J(2\rho)\}_{J \subsetneq I}$, where*

$$\hat{b}_J(2\rho) = \rho + w_0^J(\rho),$$

with $w_0^J \in W_J$ the longest element.

Proof By definition, we have

$$\hat{b}_J(2\rho) = \frac{1}{|W_J|} \sum_{w \in W_J} w(2\rho).$$

We rewrite this as

$$\hat{b}_J(2\rho) = 2\rho + \frac{1}{|W_J|} \sum_{w \in W_J} (w(2\rho) - 2\rho) = 2\rho + \frac{2}{|W_J|} \sum_{w \in W_J} w(\rho) - \rho.$$

Now, we have

$$w(\rho) - \rho = -\sum_{\beta \in \Phi_J(w)} \beta,$$

where $\Phi_J(w) := \{\beta \in \Phi_J^+ : w^{-1}(\beta) \in -\Phi_J^+\}$ and Φ_J^+ are the positive roots in the sub-root system corresponding to J. Every root $\beta \in \Phi_J^+$ appears in precisely half of the sets $\Phi_J(w)$ as w varies across W_J (since for any $\beta \in \Phi_J^+$ and $w \in W_J$, β belongs to exactly one of $\Phi_J(w)$ or $\Phi_J(ww_0^J)$); thus we get

$$\hat{b}_J(2\rho) = 2\rho - \frac{2}{|W_J|} \sum_{\beta \in \Phi_J^+} \frac{|W_J|}{2} \beta = 2\rho - \sum_{\beta \in \Phi_J^+} \beta = 2\rho + (w_0^J(\rho) - \rho) = \rho + w_0^J(\rho).$$

□

Finally, by the decomposition theorem for polyhedra in terms of minimal faces and the characteristic cone (cf. [12, Section 8.9]), we get the following crucial proposition.

Proposition 3.12 *For any $\lambda \in \mathcal{P}^+$ regular dominant with $\langle \lambda, d \rangle = 0$, the dominant weight polyhedron D_λ decomposes as*

$$D_\lambda = \mathrm{conv}_{\mathbb{Q}} \left(\{\hat{b}_J(\lambda)\}_{J \subsetneq I} \right) + \mathbb{Q}_+(-\delta).$$

Remark 3.13 In the case of a finite-dimensional semisimple Lie algebra, the vertices of the corresponding dominant weight polytope for 2ρ were worked out by Kostant, as reported in [2]. For any dominant weight, formulas for the vertices resembling those in Proposition 3.9 were given in [13]. In this case, the vertices completely determine the polytope, as it is compact. While we will not need it here, we conjecture the following stronger results.

Conjecture 3.14 Let $\lambda \in \mathcal{P}^+$ be an arbitrary (not necessarily regular) dominant integral weight with positive level. Then, the vertices of D_λ are given precisely by the set $\{\hat{b}_J(\lambda)\}_{J \subsetneq I}$ (which may have repeated elements), and D_λ decomposes as

$$D_\lambda = \mathrm{conv}_{\mathbb{Q}} \left(\{\hat{b}_J(\lambda)\}_{J \subsetneq I} \right) + \mathbb{Q}_+(-\delta).$$

4 Tensor Product of Irreducible Representations and the Goddard–Kent–Olive Construction

In this section, we will review some of the key features of the tensor decomposition problem, which seeks to understand the irreducible components of $V(\lambda) \otimes V(\mu)$ for $\lambda, \mu \in \mathcal{P}^+$. We will also briefly introduce an action of the Virasoro algebra on $V(\lambda) \otimes V(\mu)$, given by the *Goddard–Kent–Olive* (or GKO) construction, and its application to the tensor decomposition problem.

Throughout, we will take $\lambda, \mu \in \mathcal{P}^+$ with $\lambda(d) = \mu(d) = 0$, without loss of generality, as we could replace λ (and similarly μ) up to twisting by $k\delta$ via $V(\lambda) \otimes V(k\delta) = V(\lambda + k\delta)$.

4.1 The Tensor Semigroup

We return now to the tensor decomposition problem, as discussed in the introduction. To this end, consider $\lambda, \mu \in \mathcal{P}^+$ and the tensor product decomposition

$$V(\lambda) \otimes V(\mu) \cong \bigoplus_{\nu \in \mathcal{P}^+} V(\nu)^{m^{\nu}_{\lambda,\mu}},$$

where $m^{\nu}_{\lambda,\mu} \in \mathbb{Z}_+$ is the multiplicity of the subrepresentation $V(\nu)$ in $V(\lambda) \otimes V(\mu)$. While the problem of determining for which ν we have $m^{\nu}_{\lambda,\mu} \neq 0$ (or in fact determining $m^{\nu}_{\lambda,\mu}$ precisely) has a long history which has developed exact methods to give exact solutions (crystals, the Littelmann path model, among others), predicting components which appear in $V(\lambda) \otimes V(\mu)$ based only off of the pair (λ, μ) is challenging. One such example of components, known as the Parthasarathy–Ranga Rao–Varadarajan or PRV components, is given in the following theorem originally due to Kumar [14] for semisimple Lie algebras. Its extension to any symmetrizable Kac–Moody Lie algebra was proved in [15] and Mathieu [16].

Theorem 4.1 *Let $\lambda, \mu \in \mathcal{P}^+$ be two dominant integral weights and $w, v \in W$ two Weyl group elements such that $\nu := w\lambda + v\mu \in \mathcal{P}^+$. Then $V(\nu) \subset V(\lambda) \otimes V(\mu)$.*

As above, we are interested in the set of triples $\{(\lambda, \mu; \nu) \in (\mathcal{P}^+)^3 \colon m^{\nu}_{\lambda,\mu} \neq 0\}$. This set in fact forms a semigroup, which we refer to as the tensor semigroup for $\mathfrak{g}$. This follows, for example, from the analogue of the Borel–Weil theorem for symmetrizable Kac–Moody Lie algebras ([10, Corollary 8.3.12]). The tensor semigroup and its generators are difficult to understand, even in the finite-dimensional semisimple case. We consider a simpler object, known as the saturated tensor semigroup or saturated tensor cone, in the following definition.

Definition 4.2 The *saturated tensor cone* associated with $\mathfrak{g}$, denoted $\Gamma(\mathfrak{g})$, is given by

$$\Gamma(\mathfrak{g}) := \{(\lambda, \mu; \nu) \in (\mathcal{P}^+_{\mathbb{Q}})^3 : \exists N \geq 1 \text{with } m^{N\nu}_{N\lambda,N\mu} \neq 0\};$$

that is, $V(N\nu) \subset V(N\lambda) \otimes V(N\mu)$ for some $N \geq 1$, where

$$\mathcal{P}^+_{\mathbb{Q}} := \{\lambda \in \mathfrak{h}^* : \langle \lambda, \alpha_i^\vee \rangle \in \mathbb{Q}_+ \; \forall i \in I\}.$$

For a finite-dimensional semisimple Lie algebra $\mathring{\mathfrak{g}}$, the saturated tensor cone $\Gamma(\mathring{\mathfrak{g}})$ has been studied extensively using geometric methods; we refer to the survey [6]. In this case, $\Gamma(\mathring{\mathfrak{g}})$ is a closed polyhedral convex cone with inequalities determined by data coming from the cohomology of the associated flag varieties. An important notion in this topic is that of a saturation factor for $\Gamma(\mathring{\mathfrak{g}})$, which we define below.

Definition 4.3 We say that a positive integer $d \geq 1$ is a saturation factor of $\mathring{\mathfrak{g}}$ (or equivalently of $\Gamma(\mathring{\mathfrak{g}})$) if, for any $(\lambda, \mu; \nu) \in \Gamma(\mathring{\mathfrak{g}}) \cap (\mathcal{P}^+)^3$ such that $\lambda + \mu - \nu$ is in the root lattice, we have $V(d\nu) \subset V(d\lambda) \otimes V(d\mu)$.

For example, the saturation theorem of Knutson and Tao [17] says that $d = 1$ is a saturation factor for $\Gamma(sl_{n+1})$. The saturation conjecture of Kapovich and Millson [18] posits that $d = 1$ should also be a saturation factor for any simply-laced simple Lie algebra; it is known that the minimal saturation factor for the non-simply laced types satisfy $d > 1$.

When $\mathfrak{g}$ is an affine Kac–Moody Lie algebra, $\Gamma(\mathfrak{g})$ is no longer closed nor polyhedral (the semigroup is infinitely generated); however, a similar set of inequalities governing the points of $\Gamma(\mathfrak{g})$ were conjectured by Brown and Kumar [11] and later proven by Ressayre [19] in this setting. Furthermore, it is not a priori clear that a saturation factor d (analogously defined as in Definition 4.3) should exist for $\Gamma(\mathfrak{g})$. Nevertheless, in loc. cit., Ressayre provides saturation factors in this setting, again building on the work of Brown and Kumar, who computed the saturation factors for $\mathfrak{g} = A_1^{(1)}$. In the following sections, we will rely only on the existence of a saturation factor for $\Gamma(\mathfrak{g})$ and not make explicit use of their specific values apart from the case of $d = 1$ for $\Gamma(sl_{n+1})$.

Finally, we recall the definition of a δ-maximal component $V(\nu) \subset V(\lambda) \otimes V(\mu)$, taken from [11, § 6] or [8, Definition 2.2].

Definition 4.4 Let $V(\nu) \subset V(\lambda) \otimes V(\mu)$ be an irreducible component of the tensor product. We say that $V(\nu)$ is δ-maximal if $V(\nu + k\delta) \not\subset V(\lambda) \otimes V(\mu)$ for any $k > 0$.

The δ-maximal components are natural tensor product analogues of the δ-maximal dominant weights $\mathcal{P}^+_{max}(\lambda)$ of an irreducible representation $V(\lambda)$. Likewise, we can often recover arbitrary components in $V(\lambda) \otimes V(\mu)$ by focusing on just the δ-maximal components. Motivated by the similarities, we would like to understand the associated "δ-strings" in $\Gamma(\mathfrak{g})$, which should correspond to subrepresentations $V(\nu - k\delta) \subset V(\lambda) \otimes V(\mu)$. Unfortunately, the exact analogue of Lemma 3.3 does not hold in general for the appearance of components $V(\nu - k\delta)$. However, it is not too far from being correct; the goal of the next section is to give the precise analogue of Lemma 3.3.

4.2 The Goddard–Kent–Olive Construction

We next introduce the key technical tool which will allow us to study the components of $V(\rho) \otimes V(\rho)$, which is the Goddard–Kent–Olive (or GKO, for short) construction of the Virasoro algebra. The GKO construction, which is particularly applicable in the study of branching problems and the tensor decomposition problem, is a "relative" version of the earlier Sugawara coset construction. For further details, we refer to [20, Lecture 10]. First, we recall the definition of the Virasoro algebra, as follows.

Definition 4.5 The Virasoro algebra Vir is a Lie algebra over $\mathbb{C}$ with basis $\{c, L_k : k \in \mathbb{Z}\}$ with commutator relations

$$[L_k, L_j] = (k - j)L_{k+j} + \frac{1}{12}(k^3 - k)\delta_{k,-j}c, \ [Vir, c] = 0.$$

Given a finite-dimensional semisimple Lie algebra $\mathring{\mathfrak{g}}$, the Sugawara construction realizes Vir as a subalgebra of $\widehat{U}(\mathfrak{g})$, a particular completion of the universal enveloping algebra of the associated affine Kac–Moody Lie algebra. The utility of the construction is to examine highest weight irreducible representations $V(\lambda)$ of $\mathfrak{g}$ as representations of the Virasoro algebra. When we consider this realization of the Virasoro algebra, we will denote the basis by $\{L_k^{\mathfrak{g}}, c^{\mathfrak{g}}\}$.

The GKO construction considers an embedding $\mathring{\mathfrak{g}}_1 \hookrightarrow \mathring{\mathfrak{g}}_2$ of finite-dimensional semisimple (or more generally, reductive) Lie algebras and the actions of their affinizations $\mathfrak{g}_1$ and $\mathfrak{g}_2$ on $\mathfrak{g}_2$-modules. In particular, we can take the diagonal embedding $\mathring{\mathfrak{g}} \hookrightarrow \mathring{\mathfrak{g}} \oplus \mathring{\mathfrak{g}}$. Then the Virasoro algebra realization on the $\mathfrak{g} \oplus \mathfrak{g}$-module $V(\lambda) \otimes V(\mu)$, for $\lambda, \mu \in \mathcal{P}^+$ given by the GKO construction is

$$\begin{aligned} L_k^{GKO} &= L_k^{\mathfrak{g}\oplus\mathfrak{g}} - L_k^{\mathfrak{g}}, \\ c^{GKO} &= c^{\mathfrak{g}\oplus\mathfrak{g}} - c^{\mathfrak{g}}. \end{aligned}$$

The following proposition, taken from [20], summarizes the key properties of this Virasoro action on a tensor product $V(\lambda) \otimes V(\mu)$.

Proposition 4.6 *([20, Proposition 10.3]). Let $\mathfrak{g}$ be the affine Kac–Moody Lie algebra associated to a simple Lie algebra $\mathring{\mathfrak{g}}$ and $\lambda, \mu \in \mathcal{P}^+$ with levels $\langle \lambda, K\rangle = l$, $\langle \mu, K\rangle = m$. Then*

1. *$V(\lambda) \otimes V(\mu)$ is a unitarizable Virasoro representation with nonnegative central charge*
$$(\dim \mathring{\mathfrak{g}}) \left(\frac{l}{l+h^\vee} + \frac{m}{m+h^\vee} - \frac{l+m}{l+m+h^\vee} \right),$$
where $h^\vee$ is the dual Coxeter number of $\mathfrak{g}$ ([9, Section 6.1]).
2. *L_0^{GKO} acts on $V(\lambda) \otimes V(\mu)$ by*
$$\frac{1}{2} \left(\frac{(\lambda|\lambda+2\rho)}{l+h^\vee} + \frac{(\mu|\mu+2\rho)}{m+h^\vee} - \frac{\Omega}{l+m+h^\vee} \right),$$
where Ω is the Casimir operator of $\mathfrak{g}$ ([10, Section 1.5]) and $(\cdot|\cdot)$ is the normalized form on $\mathfrak{h}^*$ as in [10, Lemma 13.1.8]. Recall [10, Lemma 2.1.16] that Ω acts on a component $V(\nu) \subset V(\lambda) \otimes V(\mu)$ via the scalar $(\nu|\nu+2\rho)$.
3. *For all k, $[L_k^{GKO}, \mathfrak{g}'] = 0$; i.e., the L_k^{GKO} are intertwining operators for the representation of $\mathfrak{g}' := [\mathfrak{g}, \mathfrak{g}]$ on $V(\lambda) \otimes V(\mu)$.*

In particular, given a δ-maximal component $V(\nu) \subset V(\lambda) \otimes V(\mu)$, Proposition 4.6(3) says that L_k^{GKO} stabilizes the $\mathfrak{g}'$ isotypical component

$$\sum_{n\geq 0} V(\nu - n\delta)^{m_{\lambda,\mu}^{\nu-n\delta}} \subset V(\lambda) \otimes V(\mu),$$

with $L_k^{GKO} \cdot v_\nu$ either zero or a highest weight vector for the $\mathfrak{g}$-action, where v_ν is the highest weight vector of $V(\nu)$. We refer to Kac–Wakimoto [7] who considered the connection between the action of the GKO construction on these components and the nonvanishing of the multiplicities $m_{\lambda,\mu}^{\nu-k\delta}$. This was further used in the contexts of the affine tensor semigroup and affine tensor cone by Brown–Kumar [11] and Ressarye [19]. These methods were adapted and crucially used in our previous work [8] on the existence of root components in the tensor decomposition problem. The following proposition summarizes the technical application of this approach; we refer to [8, Section 3] for a more in-depth treatment and proofs.

Proposition 4.7 *Let $\lambda, \mu \in \mathcal{P}^+$ with positive levels l, m, respectively. Let $V(\nu) \subset V(\lambda) \otimes V(\mu)$ be a δ-maximal component. Then,*

1. *If $\frac{(\lambda|\lambda+2\rho)}{l+h^\vee} + \frac{(\mu|\mu+2\rho)}{m+h^\vee} - \frac{(\nu|\nu+2\rho)}{l+m+h^\vee} \neq 0$, then $V(\nu - k\delta) \subset V(\lambda) \otimes V(\mu)$ for all $k \geq 0$.*
2. *Else, $V(\nu - k\delta) \subset V(\lambda) \otimes V(\mu)$ for $k = 0$, $k \geq 2$, and $V(\nu - \delta) \not\subset V(\lambda) \otimes V(\mu)$.*

Thus, the key conclusion is that by computing the action of L_0^{GKO} on a potential δ-maximal component $V(\nu) \subset V(\lambda) \otimes V(\nu)$, we can show the existence of all submodules of the form $V(\nu - k\delta)$ by demonstrating the appearance of $V(\nu)$ in the tensor product. We note, however, that this does *not* give a way to construct the δ-maximal components themselves; this typically must be done independently of the Virasoro action considerations. These constructions, as well as verifying the positivity of the L_0^{GKO} action, will form the basis of our approach when restricting to $\lambda = \mu = \rho$ in the following sections.

5 Components of $V(d\rho) \otimes V(d\rho)$

In this section, $\mathfrak{g}$ is the affine Kac–Moody Lie algebra associated with a simple Lie algebra $\mathring{\mathfrak{g}}$.

With the previous details at our disposal, we can now consider the components of $V(d\rho) \otimes V(d\rho)$ for a saturation factor $d \geq 1$ of $\mathfrak{g}$. In particular, recall that by definition (as in Definition 4.3) if $\lambda \leq 2\rho$ is a dominant integral weight such that $V(N\lambda) \subset V(N\rho) \otimes V(N\rho)$ for some $N \geq 1$, then for any saturation factor $d \geq 1$, we have $V(d\lambda) \subset V(d\rho) \otimes V(d\rho)$. Our approach is to first show, using the convexity of the saturated tensor semigroup, that such an N exists for all dominant weights $\lambda \leq 2\rho$. First, we have the following proposition which is an adaptation of [3, Proposition 9] to the affine setting.

Proposition 5.1 *Let $\lambda \in \mathcal{P}^+$ with $\lambda \leq 2\rho$. Then, we can write $\lambda = \rho + \beta$ for some weight $\beta \in \mathcal{P}(\rho)$.*

Proof Since $\lambda \leq 2\rho$, we have that $\lambda - \rho \in \rho + \mathcal{Q}$. So, by Proposition 3.1, it suffices to show that $\lambda - \rho \in \mathrm{conv}_{\mathbb{Q}}\{w\rho : w \in W\}$. Now, by Proposition 3.12 and Corollary 3.11, we can write

$$\lambda = \sum_{J \subsetneq I} c_J \hat{b}_J(2\rho) + c_\delta(-\delta),$$

where as in Corollary 3.11, $\hat{b}_J(2\rho) = \rho + w_0^J(\rho)$, and $c_J, c_\delta \in \mathbb{Q}_+$ with $\sum_J c_J = 1$. Thus we have

$$\lambda - \rho = \sum_{J \subsetneq I} c_J w_0^J(\rho) + c_\delta(-\delta).$$

Clearing denominators, we can write

$$N(\lambda - \rho) = \sum_{J \subsetneq I} c'_J w_0^J(\rho) + c'_\delta(-\delta),$$

where $N \in \mathbb{Z}_{\geq 1}$, and $c'_J, c'_\delta \in \mathbb{Z}_+$ with $\sum_J c'_J = N$. Now,

$$\sum_{J \subsetneq I} c'_J w_0^J(\rho) = \sum_{J \subsetneq I} \frac{c'_J}{N} w_0^J(N\rho) \in \mathcal{P}(N\rho),$$

by Proposition 3.1 since

$$N\rho - \sum_{J \subsetneq I} c'_J w_0^J(\rho) = -\sum_{J \subsetneq I} c'_J (w_0^J(\rho) - \rho) \in \mathcal{Q}.$$

By Lemma 3.3, we have that subtracting $c'_\delta(\delta)$ will remain a weight of $\mathcal{P}(N\rho)$, so we conclude $N(\lambda - \rho) \in \mathcal{P}(N\rho)$. Since $N(\lambda - \rho)$ is an integral weight in $\mathcal{P}(N\rho)$, by Proposition 3.1 we can write

$$N(\lambda - \rho) = \sum_{w \in W} a_w w(N\rho) = N \sum_{w \in W} a_w w(\rho)$$

for some $a_w \in \mathbb{Q}_+$ with $\sum_{w \in W} a_w = 1$. Dividing by N and using Proposition 3.1 again gives the result since

$$\Big(\sum_{w \in W} a_w w(\rho)\Big) - \rho = \lambda - 2\rho \in \mathcal{Q} \text{ by assumption.}$$

□

Remark 5.2 By the same proof, since the candidates for vertices of the dominant weight polyhedron depend linearly on the highest weight, we have that for any integer $M \geq 1$, if $\lambda \in \mathcal{P}^+$ is such that $\lambda \leq 2M\rho$, then we can write $\lambda = M\rho + \beta$ for some weight $\beta \in \mathcal{P}(M\rho)$.

Remark 5.3 Using Proposition 5.1, one could alternatively adapt the proof of Theorem 3 as in [3] in a straightforward way to prove our Theorem 1.3(1) using the results of Ressayre [19] on the inequalities for the affine tensor semigroup. We do not take that approach, however, and instead highlight the use of the GKO construction, which will be key to the case $\mathfrak{g} = A_n^{(1)}$.

Now, we can use Proposition 5.1 and Remark 5.2 for $\lambda \leq 2M\rho$ to simplify the proof of the following computational proposition. This, in turn, will allow us to apply the GKO construction and reduce the proof of Theorem 1.3 to the δ-maximal components by using Proposition 4.7.

Proposition 5.4 *Let $M \geq 1$ be an integer, and let $\lambda \in \mathcal{P}^+$ with $\lambda \leq 2M\rho$. Then*

$$\frac{2(M\rho|(M+2)\rho)}{(M+1)h^\vee} - \frac{(\lambda|\lambda+2\rho)}{(2M+1)h^\vee} > 0.$$

Proof Write $\lambda = M\rho + \beta$ for some $\beta \in \mathcal{P}(M\rho)$. Substituting this for λ and by careful simplifications, we get

$$\begin{aligned}&\frac{2(M\rho|(M+2)\rho)}{(M+1)h^\vee} - \frac{(\lambda|\lambda+2\rho)}{(2M+1)h^\vee}\\ &= \frac{1}{(M+1)(2M+1)h^\vee}\big((2M+4)((M\rho|M\rho)-(M\rho|\beta))\\ &\quad +(M+1)((M\rho|M\rho)-(\beta|\beta)) + 2(M\rho|M\rho) + (2\rho|M\rho-\beta)\big).\end{aligned}$$

But by [9, Proposition 11.4], we have that $(M\rho|M\rho)-(M\rho|\beta) \geq 0$ and $(M\rho|M\rho) - (\beta|\beta) \geq 0$, since $\beta \in \mathcal{P}(M\rho)$. Finally, one can check that $2(M\rho|M\rho) = 2M^2(\rho|\rho) > 0$, and $(2\rho|M\rho-\beta) \geq 0$, since $M\rho - \beta \in \mathcal{Q}^+$. Thus the expression of the proposition is strictly positive. □

With this in hand, we can apply Proposition 4.7 to conclude the following. Observe that, by definition, $h^\vee := \langle \rho, K\rangle$.

Corollary 5.5 *Let M be a positive integer. Take $\lambda \in \mathcal{P}^+$ such that $V(\lambda) \subset V(M\rho) \otimes V(M\rho)$ is a δ-maximal component. Then, $V(\lambda - k\delta) \subset V(M\rho) \otimes V(M\rho)$ for any $k \geq 0$.*

We can now prove the following theorem, which is the first major result of this paper.

Theorem 5.6 *Take $\lambda \in \mathcal{P}^+$ such that $\lambda \leq 2\rho$. Then, $V(d\lambda) \subset V(d\rho) \otimes V(d\rho)$ for any saturation factor d of $\mathfrak{g}$.*

Proof As in the proof of Proposition 5.1, write

$$\lambda = \sum_{J \subsetneq I} c_J \hat{b}_J(2\rho) + c_\delta(-\delta), \quad \text{for } c_J, c_\delta \in \mathbb{Q}_+.$$

Again clearing denominators, we write (for some integer $N \geq 1$)

$$N\lambda = \sum_{J \subsetneq I} c'_J \hat{b}_J(2\rho) + c'_\delta(-\delta)$$

with each $c'_J, c'_\delta \in \mathbb{Z}_+$ and $\sum_J c'_J = N$. But since $c'_J\hat{b}_J(2\rho) = c'_J\rho + w_0^J(c'_J\rho)$ (cf. Corollary 3.11), via the PRV components of Theorem 4.1 we get that $V(c'_J\hat{b}_J(2\rho)) \subset V(c'_J\rho) \otimes V(c'_J\rho)$ for all J. (Observe that $\rho + w_0^J(\rho)$ is a dominant weight.) Thus, by additivity in the tensor semigroup, we get

$$V(\Lambda) \subset V(N\rho) \otimes V(N\rho),$$

where $\Lambda := \sum_{J \subsetneq I} c'_J \hat{b}_J(2\rho)$. Finally, by Corollary 5.5, we get (by replacing Λ with the corresponding δ-maximal component) that

$$V(N\lambda) = V(\Lambda - c'_\delta(\delta)) \subset V(N\rho) \otimes V(N\rho).$$

Thus, by the existence of a saturation factor for the affine tensor cone, we have that $V(d\lambda) \subset V(d\rho) \otimes V(d\rho)$ for any saturation factor d. ◻

Remark 5.7 Assume that $\lambda \in \mathcal{P}^+$, $\lambda \leq 2\rho$ is such that

$$V(d\lambda) \subset V(d\rho) \otimes V(d\rho), \quad \text{for some } d \in \mathbb{Z}_{>0}.$$

Then, for any $k \in \mathbb{Z}$ and $\lambda' \in \mathcal{P}^+$ such that $\lambda' = \lambda + 2\,k\delta$,

$$V(d\lambda') \subset V(d(\rho + k\delta)) \otimes V(d(\rho + k\delta)).$$

To prove this, observe that for any $\mu \in \mathcal{P}^+$ and $k \in \mathbb{Z}$,

$$V(\mu + k\delta) \simeq V(\mu) \otimes V(k\delta).$$

6 The Case $\mathfrak{g} = A_n^{(1)}$ and Conjecture for Other Types

In this section, we restrict our attention to the case when $\mathfrak{g} = A_n^{(1)} = \widehat{sl}_{n+1}$. By Theorem 5.6, we know that for any saturation factor d for $\mathfrak{g}$, we have that $V(d\lambda) \subset V(d\rho) \otimes V(d\rho)$ for all dominant $\lambda \leq 2\rho$. However, unlike in the corresponding finite-dimensional simple Lie algebra $\mathring{\mathfrak{g}} = sl_{n+1}$, the saturation factor d *cannot* be taken to be $d = 1$. It is known ([19, Theorem 3]) that in type $A_n^{(1)}$, any $d \geq 2$ is a saturation factor.

While the method of proof of Theorem 5.6 applied to a finite-dimensional simple Lie algebra gives a new proof of Theorem 1.2, we will assume the validity of Theorem 1.2 and use it to prove the corresponding result for $A_n^{(1)}$. By Corollary 5.5, in order to prove the following Theorem 6.2, it suffices to consider the case when $\lambda \in \mathcal{P}^+_{max}(2\rho)$. Since for $A_n^{(1)}$ we have that $\delta = \alpha_0 + \alpha_1 + \cdots + \alpha_n$, by Proposition 3.4, we know that any δ-maximal dominant weight $\lambda \leq 2\rho$ satisfies

$$\text{supp}(2\rho - \lambda) \subsetneq I = \{0, 1, \ldots, n\},$$

where, for any $\gamma \in \mathcal{Q}^+$, we define the support of γ as $\text{supp}(\gamma) = \{i \in I : c_i \neq 0 \text{ where } \gamma = \sum_{i \in I} c_i \alpha_i\}$. We first give the following lemma, for certain such λ, following a similar approach as in [8, Proof of Proposition 6.4].

Lemma 6.1 *Let $\lambda \in \mathcal{P}^+_{max}(2\rho)$ such that $2\rho - \lambda \in \mathring{\mathcal{Q}}^+$. Then $V(\lambda) \subset V(\rho) \otimes V(\rho)$.*

Proof Let $\mathring{V}(\rho)$ be the $\mathring{\mathfrak{g}}$-submodule of $V(\rho)$ generated by the highest weight vector $v_\rho \in V(\rho)$, and similarly for $\mathring{V}(\lambda)$. Then, by Theorem 1.2, we know that

$$\mathring{V}(\lambda) \subset \mathring{V}(\rho) \otimes \mathring{V}(\rho).$$

Let $v_\lambda \in V(\rho) \otimes V(\rho)$ be a corresponding highest weight vector for $\mathring{\mathfrak{g}}$ that generates $\mathring{V}(\lambda)$. We claim that this is in fact a highest weight vector for $\mathfrak{g}$. All that remains to check is that $e_0.v_\lambda = 0$, where e_0 is a root vector corresponding to the positive root α_0. But by convention, we have that $\rho(d) = 0$, so that no vector in $V(\rho) \otimes V(\rho)$ can have weight ν with $\nu(d) \geq 1$. And, since $2\rho - \lambda \in \mathring{\mathcal{Q}}^+$, we know that $\lambda(d) = 0$. Then, if $e_0.v_\lambda \neq 0$, we have that $e_0.v_\lambda$ has weight satisfying $(\lambda + \alpha_0)(d) = \alpha_0(d) = 1$, a contradiction. Thus, $e_0.v_\lambda = 0$, so that $V(\lambda) \subset V(\rho) \otimes V(\rho)$. □

By the symmetry of the Dynkin diagram of type $A_n^{(1)}$, Lemma 6.1 actually suffices to handle the case of all δ-maximal weights $\lambda \in \mathcal{P}^+_{max}(2\rho)$. Together with Corollary 5.5 for $M = 1$, we are able to prove the following theorem.

Theorem 6.2 *Let $\mathfrak{g} = A_n^{(1)}$, and let $\lambda \in \mathcal{P}^+$ with $\lambda \leq 2\rho$. Then, $V(\lambda) \subset V(\rho) \otimes V(\rho)$.*

Proof First, let $\lambda \in \mathcal{P}^+_{\max}(2\rho)$ be a δ-maximal weight. We know that $\operatorname{supp}(2\rho - \lambda) \subsetneq I$. Consider any diagram automorphism σ of the Dynkin diagram of $A_n^{(1)}$ such that $\operatorname{supp}(\sigma(2\rho - \lambda)) \subset I \backslash \{0\}$. Then, σ induces an automorphism of $\mathfrak{g}$ and also of $V(\rho)$ and $V(\rho) \otimes V(\rho)$ (since ρ is stable under the diagram automorphism). If $\lambda' := \sigma(\lambda) \in \mathcal{P}^+$ is the corresponding dominant integral weight, we thus have $\lambda' \leq 2\rho$ and $\operatorname{supp}(2\rho - \lambda') \subset I \backslash \{0\}$, so by Lemma 6.1 we get that $V(\lambda') \subset V(\rho) \otimes V(\rho)$. Applying σ^{-1}, we get that $V(\lambda) \subset V(\rho) \otimes V(\rho)$. Now, the theorem follows by applying Corollary 5.5. □

For $\mathfrak{g}$ of other types, we do not have sufficient diagram automorphisms to reduce to those λ with $2\rho - \lambda \in \mathring{\mathcal{Q}}^+$. Even if this were the case, as Conjecture 1.1 is only known to hold in types A_n and the exceptional types, the proof method used for Theorem 6.2 to leverage the result for $\mathring{\mathfrak{g}}$ is not easily adaptable.

However, direct computer-verified computations via the software Sage [21] show that Theorem 6.2 extends for low rank examples like $\mathfrak{g} = B_2^{(1)}$, $G_2^{(1)}$, $B_3^{(1)}$, and $D_4^{(1)}$. Thus, we conclude by proposing the following analogue of Kostant's conjecture for affine Kac–Moody Lie algebras.

Conjecture 6.3 Let $\mathfrak{g}$ be any untwisted affine Kac–Moody Lie algebra, and $\lambda \in \mathcal{P}^+$ such that $\lambda \leq 2\rho$. Then, $V(\lambda) \subset V(\rho) \otimes V(\rho)$.

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Declarations

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