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Subalgebra generated by ad-locally nilpotent elements of Borcherds Generalized Kac-Moody Lie algebras

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A R T I C L E I N F O

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A B S T R A C T

We determine the Lie subalgebra $\mathfrak{g}_{\text{nil}}$ of a Borcherds symmetrizable generalized Kac-Moody Lie algebra $\mathfrak{g}$ generated by ad-locally nilpotent elements and show that it is 'essentially' the same as the Levi subalgebra of $\mathfrak{g}$ with its simple roots precisely the real simple roots of $\mathfrak{g}$.

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1. Introduction

Let $\mathfrak{g} = \mathfrak{g}(A)$ be the symmetrizable Generalized Kac-Moody (GKM) algebra associated to a $\ell \times \ell$ matrix A (cf. Section 2). Let

$$\mathfrak{g}^o_{\text{nil}} := \{x \in \mathfrak{g} : \ \operatorname{ad} x \text{ acts locally nilpotently on } \mathfrak{g}\},$$

and let $\mathfrak{g}_{\text{nil}} \subset \mathfrak{g}$ be the Lie subalgebra generated by $\mathfrak{g}^o_{\text{nil}}$. Then, we prove the following theorem (cf. Theorem 3.1):

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Theorem. *Let $\mathfrak{g} = \mathfrak{g}(A)$ be as above, where $\ell \geq 2$ and A is indecomposable, i.e., the corresponding Dynkin diagram is connected. Then,*

$$\mathfrak{g}'(B) \subset \mathfrak{g}_{\text{nil}} \subset \mathfrak{g}'(B) + \mathfrak{h},$$

where $B \subset A$ is the submatrix parameterized by those i such that $a_{i,i} = 2$, $\mathfrak{h}$ is the Cartan subalgebra and $\mathfrak{g}'(B)$ is the derived subalgebra of $\mathfrak{g}(B)$.

As shown in Remark 3.2, the assumption $\ell \geq 2$ in the above theorem is necessary in general.

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2. Basic definition

In this section, we recall the definition of Borcherds Generalized Kac-Moody Lie algebras $\mathfrak{g}$ (for short GKM algebras). For a more extensive treatment of $\mathfrak{g}$ and its properties, see Chapters 1, 11 of [3] and the papers [1] and [2].

Definition 2.1. Let $A = (a_{i,j})$ be a $\ell \times \ell$ matrix (for $\ell \geq 1$) with real entries, satisfying the following properties:

(P1) either $a_{i,i} = 2$ or $a_{i,i} \leq 0$,
(P2) $a_{i,j} \leq 0$ if $i \neq j$, and $a_{i,j} \in \mathbb{Z}$ if $a_{i,i} = 2$,
(P3) $a_{i,j} = 0$ if and only if $a_{j,i} = 0$.

Fix a *realization* of A, which is a triple $(\mathfrak{h}, \Pi, \Pi^\vee)$ consisting of a complex vector space $\mathfrak{h}$, $\Pi = \{\alpha_1, \dots, \alpha_\ell\} \subset \mathfrak{h}^*$ and $\Pi^\vee = \{\alpha_1^\vee, \dots, \alpha_\ell^\vee\} \subset \mathfrak{h}$ are indexed subsets, satisfying the following three conditions:

(Q1) both sets Π and $\Pi^\vee$ are linearly independent,
(Q2) $\alpha_j(\alpha_i^\vee) = a_{i,j}$, for all i, j,
(Q3) $\ell - \operatorname{rank} A = \dim \mathfrak{h} - \ell$.

By [3], Proposition 1.1, such a realization is unique up to an isomorphism of the triple.

Now, the Borcherds Generalized Kac-Moody Lie algebra (for short GKM algebra) $\mathfrak{g}(A)$ is defined as the Lie algebra generated by $\{e_i, f_i, \mathfrak{h}\}_{1 \leq i \leq \ell}$ subject to the following relations:

(R1) $[e_i, f_j] = \delta ij \alpha_i^\vee$, for all i,
(R2) $[h, h'] = 0$, for all $h, h' \in \mathfrak{h}$,
(R3) $[h, e_i] = \alpha_i(h) e_i;\ [h, f_i] = -\alpha_i(h) f_i$, for all $1 \leq i \leq \ell$ and $h \in \mathfrak{h}$,

(R4) $(\operatorname{ad} e_i)^{1-a_{i,j}} e_j = (\operatorname{ad} f_i)^{1-a_{i,j}} f_j = 0$, if $a_{i,i} = 2$ and $i \neq j$,
(R5) $[e_i, e_j] = [f_i, f_j] = 0$, if $a_{i,j} = 0$.

The matrix A (or the Lie algebra $\mathfrak{g}(A)$) is called *symmetrizable* if there exists an invertible diagonal matrix $D = \operatorname{diag}(\epsilon_1, \ldots, \epsilon_\ell)$ such that the matrix DA is symmetric.

3. Main theorem and its proof

Theorem 3.1. *Let $\mathfrak{g} = \mathfrak{g}(A)$ be the symmetrizable GKM algebra associated to a $\ell \times \ell$ matrix A as in the last section. Assume further that $\ell \geq 2$ and A is indecomposable, i.e., the corresponding Dynkin diagram is connected. Let*

$$\mathfrak{g}^o_{\text{nil}} := \{x \in \mathfrak{g} : \ \operatorname{ad} x \text{ acts locally nilpotently on } \mathfrak{g}\},$$

and let $\mathfrak{g}_{\text{nil}} \subset \mathfrak{g}$ be the Lie subalgebra generated by $\mathfrak{g}^o_{\text{nil}}$. Then,

$$\mathfrak{g}'(B) \subset \mathfrak{g}_{\text{nil}} \subset \mathfrak{g}'(B) + \mathfrak{h},$$

where $B \subset A$ is the submatrix parameterized by those i such that $a_{ii} = 2$, i.e., α_i is a real root and $\mathfrak{g}'(B)$ is the derived subalgebra of $\mathfrak{g}(B)$.

Proof. Consider the $\mathbb{Z}$-gradation of $\mathfrak{g}$ induced from a homomorphism $\theta : Q := \oplus_i \mathbb{Z}\alpha_i \to \mathbb{Z}$. Then, for any $x \in \mathfrak{g}^o_{\text{nil}}, x_+(\theta) \in \mathfrak{g}^o_{\text{nil}}$, where $x_+(\theta)$ is the top degree component of x in the $\mathbb{Z}$-gradation of $\mathfrak{g}$ induced by θ. To prove this, observe that for any $y \in \mathfrak{g}_\alpha$ (where $\mathfrak{g}_\alpha$ is the root space corresponding to the root α or 0),

$$(\operatorname{ad} x)^n(y) = (\operatorname{ad} x_+(\theta))^n(y) + \text{ lower degree terms.}$$

Similarly, for $x \in \mathfrak{g}^o_{\text{nil}}, x_-(\theta) \in \mathfrak{g}^o_{\text{nil}}$, where $x_-(\theta)$ is the lowest degree component of x.

Further, given any nonzero $x \in \mathfrak{g}$, we can get a gradation $\theta_x : Q \to \mathbb{Z}$ as above (depending upon x) such that all the homogeneous degree components of x (under θ_x) belong to root spaces $\mathfrak{g}_\beta$. To prove this, write $x = \sum_j x_{\beta_j}$, where β_j are distinct roots or zero, $x_{\beta_j} \in \mathfrak{g}_{\beta_j}$ and each $x_{\beta_j} \neq 0$. Consider the finite collection of weights: $\{\beta_j - \beta_k\}_{j \neq k} \subset \mathfrak{h}^*$. Now, we can find a vector $\gamma = \gamma_x \in \mathbb{Q}^\ell = Q \otimes_\mathbb{Z} \mathbb{Q}$ such that for the standard dot product $(\cdot, \cdot)$ in $\mathbb{Q}^\ell$,

$$\theta_x(\beta_j - \beta_k) := (\beta_j - \beta_k, \gamma) \neq 0, \text{ for any } j \neq k. \tag{1}$$

To prove the above equation, consider the $(\ell - 1)$-dimensional subspace $V_{j,k} \subset \mathbb{Q}^\ell$ (for any $j \neq k$) perpendicular to $\beta_j - \beta_k$. Since the collection $\{\beta_j - \beta_k\}_{j \neq k}$ is finite, we can find a vector γ such that the equation (1) is satisfied. We can further take $\gamma \in Q \simeq \mathbb{Z}^\ell$ by clearing the denominators.

So, if $x \in \mathfrak{g}^o_{\rm nil}$, then either x belongs to the center $Z(\mathfrak{g})$ of $\mathfrak{g}$ or the root component $x_\beta \in \mathfrak{g}^o_{\rm nil}$ for some root β ($\beta \neq 0$). (To prove this: if x belongs to the Cartan subalgebra $\mathfrak{h}$, then it will have to lie in $Z(\mathfrak{g})$ of $\mathfrak{g}$ by [3], Proposition 1.6. But, if it does not lie in $\mathfrak{h}$, then, as observed in the beginning of the proof by making a choice of θ_x as above, $x_+(\theta_x) \in \mathfrak{g}^o_{\rm nil}$ for the top degree component $x_+(\theta_x)$ of x in $\mathbb{Z}$-gradation θ_x of $\mathfrak{g}$.) Moreover, if some nonzero root component of x belongs to the root space $\mathfrak{g}_\delta$ such that δ contains an imaginary simple root δ_p (i.e., with $a_{p,p} \leq 0$) with nonzero coefficient, we can assume that $x_\delta \in \mathfrak{g}^o_{\rm nil}$ (possibly with a different nonzero root component of x corresponding to a root containing an imaginary simple root with nonzero coefficient). This is achieved by taking γ as above but requiring $\theta_x(\alpha_p)$ to be much larger for all the imaginary simple roots α_p as compared to the values $\theta_x(\alpha_q)$ for all the real simple roots α_q (i.e., those with $a_{q,q} = 2$).

By using the Cartan involution ω of $\mathfrak{g}$ (i.e., $\omega(e_i) = -f_i, \omega(f_i) = -e_i, \omega(h) = -h \, \forall h \in \mathfrak{h}$), if needed, we can further assume that δ is a positive root. Write

$$\delta = \sum_p (m_p \alpha_p) + \sum_q (n_q \alpha_q), \text{ for } m_p, n_q \geq 0,$$

where α_p (resp. α_q) run over all the imaginary (resp. real) simple roots. In particular, some $m_p > 0$. By [3], Exercise 11.21, the support $\operatorname{supp}(\delta)$ is connected. Assume first that δ is not an imaginary simple root. Further, taking some W-translate (where W is the Weyl group of $\mathfrak{g}$, cf. [3], §11.13), we can assume that $\delta(\alpha_q^\vee) \leq 0$ for all the real simple coroots $\alpha_q^\vee$ (cf. [3], Identity 11.13.3). Now, with respect to the W-invariant symmetric bilinear form $\langle \cdot, \cdot \rangle$ on $\mathfrak{h}^*$ (cf. [3], §2.1),

$$\begin{aligned} \langle \delta, \delta \rangle &= \sum_p m_p \langle \delta, \alpha_p \rangle + \sum_q n_q \langle \delta, \alpha_q \rangle \\ &= \sum_q n_q \langle \delta, \alpha_q \rangle + \sum_{p,q} m_p n_q \langle \alpha_q, \alpha_p \rangle + \sum_{p,p'} m_p m_{p'} \langle \alpha_{p'}, \alpha_p \rangle, \end{aligned} \tag{2}$$

where $\alpha_{p'}$ also runs over imaginary simple roots. Now, by assumption,

$$\langle \delta, \alpha_q \rangle \leq 0, \quad \text{for all the real simple roots.} \tag{3}$$

For any imaginary simple root α_p and any real simple root α_q, we have

$$\langle \alpha_q, \alpha_p \rangle \leq 0, \text{ since } a_{p,q} \leq 0. \tag{4}$$

Further, for imaginary simple roots $\alpha_p, \alpha_{p'}$,

$$\langle \alpha_{p'}, \alpha_p \rangle \leq 0, \text{ by [3], Identity 2.1.6.} \tag{5}$$

Observe that we can take the normalizing factor $\epsilon_i > 0$ for each $1 \leq i \leq \ell$ as can be seen from the identity:

$$\epsilon_i a_{i,j} = \epsilon_j a_{j,i}, \text{ for all } 1 \le i, j \le \ell,$$

where the diagonal matrix $D = \mathrm{diag}(\epsilon_1, \cdots, \epsilon_\ell)$ is such that DA is a symmetric matrix. Moreover, since there exists p with $m_p \neq 0$ and since $\mathrm{supp}\,\delta$ is connected and δ is not a simple root, by [3], Identity 2.1.6,

$$\begin{aligned} &\langle \alpha_{p'}, \alpha_p \rangle < 0, \text{ for some } p' \neq p \text{ with } m_{p'} \neq 0 \text{ and } \alpha_{p'} \text{ an imaginary simple root} \\ &\text{or } \langle \alpha_q, \alpha_p \rangle < 0 \text{ for some } q \text{ with } n_q \neq 0 \text{ and } \alpha_q \text{ a real simple root.} \end{aligned} \tag{6}$$

Thus, combining the equations (2) - (6), we get:

$$\langle \delta, \delta \rangle < 0.$$

By [3], Corollary 9.12, $\oplus_{k>0}\,\mathfrak{g}_{k\delta}$ is a free Lie algebra on a basis of the form $\oplus_{k>0}\,\mathfrak{g}^o_{k\delta}$, where

$$\begin{aligned} \mathfrak{g}^o_{k\delta} := \{x \in \mathfrak{g}_{k\delta} : \langle x, y \rangle = 0 \forall y \text{ in the Lie subalgebra generated by} \\ \mathfrak{g}_{-\delta}, \mathfrak{g}_{-2\delta}, \dots, \mathfrak{g}_{-(k-1)\delta}\}. \end{aligned}$$

Observe next that $\mathfrak{g}_{k\delta} \neq 0$ for any $k > 0$ by [3], Identity 11.13.3. If $\mathfrak{g}_\delta$ is one dimensional, then so is $\mathfrak{g}_{-\delta}$ and hence $\mathfrak{g}^o_{2\delta} \neq 0$. (To prove $\dim \mathfrak{g}_{-\delta} = 1$, observe that, due to the existence of the Cartan involution, $\dim \mathfrak{g}_\beta = \dim \mathfrak{g}_{-\beta}$ for any root β, cf. [3], Identity 1.3.5 and Theorem 11.13.1. Moreover, $\mathfrak{g}_{-\delta}$ being one dimensional, the Lie subalgebra of $\mathfrak{g}$ generated by $\mathfrak{g}_{-\delta}$ is $\mathfrak{g}_{-\delta}$ itself. Thus, $\mathfrak{g}^o_{2\delta} \neq 0$ by the definition.) Thus, $\oplus_{k>0}\,\mathfrak{g}_{k\delta}$ is a free Lie algebra on at least 2 generators. If $\dim \mathfrak{g}_\delta \ge 2$, then $\oplus_{k>0}\,\mathfrak{g}_{k\delta}$ is again a free Lie algebra on at least two generators (since $\mathfrak{g}^o_\delta = \mathfrak{g}_\delta$). Thus, $\mathrm{ad}(x_\delta)$ can not act locally nilpotently on $\oplus_{k>0}\,\mathfrak{g}_{k\delta}$ and hence on $\mathfrak{g}$ (since the enveloping algebra of a free Lie algebra is the tensor algebra on the same generators and now use [4], Identity (3) of Definition 1.3.2).

Now, let $\delta = \alpha_p$ be an imaginary simple root. Then, again $x_\delta = e_p$ can not act nilpotently on any e_i, $i \neq p$ such that $a_{i,p} \neq 0$. (This is where we have used the assumption that A is indecomposable and $\ell \ge 2$.) To prove this, use [3], Identity 11.13.3 by observing that $(n\alpha_p + \alpha_i) \in K$ for all $n \ge 2$ in the notation of [3].

Thus, we conclude that any $x \in \mathfrak{g}^o_{\mathrm{nil}}$ must be of the form $x \in \mathfrak{g}(B) + \mathfrak{h}$. Hence,

$$\mathfrak{g}_{\mathrm{nil}} \subset \mathfrak{g}'(B) + \mathfrak{h}.$$

Further, by [4], Lemma 1.3.3(a) and the defining relations of $\mathfrak{g}(A)$, $e_i, f_i \in \mathfrak{g}^o_{\mathrm{nil}}$ for any real simple root α_i. Thus,

$$\mathfrak{g}'(B) \subset \mathfrak{g}_{\mathrm{nil}}.$$

This proves the theorem. □

Remark 3.2. (a) Define

$$\mathfrak{g}'^{o}_{\mathrm{nil}} := \{x \in \mathfrak{g}' : \ \mathrm{ad}\, x \text{ acts locally nilpotently on } \mathfrak{g}'\}$$

and let $\mathfrak{g}'_{\mathrm{nil}} \subset \mathfrak{g}'$ be the Lie subalgebra generated by $\mathfrak{g}'^{o}_{\mathrm{nil}}$. Then, by the same proof as above,

$$\mathfrak{g}'(B) \subset \mathfrak{g}'_{\mathrm{nil}} \subset (\mathfrak{g}'(B) + \mathfrak{h}) \cap \mathfrak{g}'.$$

(b) It is easy to see that the above theorem remains true in the case A is parameterized by $\mathbb{N} \times \mathbb{N}$.

(c) For the 1×1-matrix $A = (0)$, following [3], §2.9, $\mathfrak{g}(A) = \mathfrak{h} \oplus \mathbb{C}e_1 \oplus \mathbb{C}f_1$, where $\mathfrak{h} = \mathbb{C}\alpha_1^\vee \oplus \mathbb{C}d$ and $[e_1, f_1] = \alpha_1^\vee, [\alpha_1^\vee, \mathfrak{g}] = 0, [d, e_1] = e_1, [d, f_1] = -f_1$. Thus, in this case, $\mathfrak{g}_{\mathrm{nil}} = \mathfrak{g}'$. Hence, the assumption $\ell \geq 2$ in the above theorem is necessary in general.

(d) One interesting consequence of the above theorem is that the only connected 'reasonable' group attached to a GKM algebra $\mathfrak{g}(A)$ is the one coming from its subalgebra $\mathfrak{g}(B)$ (up to an $\mathfrak{h}$-factor).

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