Representation ring of Levi subgroups versus cohomology ring of flag varieties II

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Abstract

For any reductive group G and a parabolic subgroup P with its Levi subgroup L, the first author in [Ku2] introduced a ring homomorphism $\xi^P_\lambda : \mathrm{Rep}^{\mathbb{C}}_{\lambda-\mathrm{poly}}(L) \to H^*(G/P, \mathbb{C})$, where $\mathrm{Rep}^{\mathbb{C}}_{\lambda-\mathrm{poly}}(L)$ is a certain subring of the complexified representation ring of L (depending upon the choice of an irreducible representation $V(\lambda)$ of G with highest weight λ). In this paper we study this homomorphism for $G = \mathrm{Sp}(2n)$ and its maximal parabolic subgroups P_{n-k} for any $1 \leq k \leq n$ (with the choice of $V(\lambda)$ to be the defining representation $V(\omega_1)$ in $\mathbb{C}^{2n}$). Thus, we obtain a $\mathbb{C}$-algebra homomorphism $\xi_{n,k} : \mathrm{Rep}^{\mathbb{C}}_{\omega_1-\mathrm{poly}}(\mathrm{Sp}(2k)) \to H^*(IG(n-k, 2n), \mathbb{C})$. Our main result asserts that $\xi_{n,k}$ is injective when n tends to ∞ keeping k fixed. Similar results are obtained for the odd orthogonal groups.

1 Introduction

This is a follow-up of first author's work [Ku2].

Let G be a connected reductive group over $\mathbb{C}$ with a Borel subgroup B and maximal torus $T \subset B$. Let P be a standard parabolic subgroup with the Levi subgroup L containing T. Let $V(\lambda)$ be an irreducible almost faithful representation of G with highest weight λ (i.e., the corresponding map $\rho_\lambda : G \to \mathrm{Aut}(V(\lambda))$ has finite kernel). Then, Springer defined an adjoint-equivariant regular map with Zariski dense image $\theta_\lambda : G \to \mathfrak{g}$ (depending upon λ) (cf. Definition 1). Using this the first author defined in [Ku2] a certain subring $\mathrm{Rep}^{\mathbb{C}}_{\lambda-\mathrm{poly}}(L)$ of the complexified representation ring $\mathrm{Rep}^{\mathbb{C}}(L)$ (cf. Definition 3). For $G = \mathrm{GL}(n)$ and $V(\lambda)$ the defining representation $\mathbb{C}^n$, the ring $\mathrm{Rep}_{\lambda-\mathrm{poly}}(G) := \mathrm{Rep}^{\mathbb{C}}_{\lambda-\mathrm{poly}}(G) \cap \mathrm{Rep}(G)$ coincides with the standard notion of polynomial representation ring of $\mathrm{GL}(n)$ (cf. the equation (4)).

Coming back to the general case, the first author [Ku2] defined a surjective $\mathbb{C}$-algebra homomorphism

$$\xi^P_\lambda : \mathrm{Rep}^{\mathbb{C}}_{\lambda-\mathrm{poly}}(L) \to H^*(G/P, \mathbb{C}) \tag{1}$$

(cf. Theorem 4).

Specializing the above result to the case when $G = \mathrm{GL}(n)$, $V(\lambda)$ is the standard defining representation $\mathbb{C}^n$ and $P = P_r$ (for any $1 \leq r \leq n-1$) is the maximal parabolic subgroup so that the flag variety G/P_r is the Grassmannian $\mathrm{Gr}(r,n)$ and $L_r = \mathrm{GL}(r) \times \mathrm{GL}(n-r)$ and restricting ξ^P_λ to the component $\mathrm{GL}(r)$, one recovers the classical ring homomorphism

$$\phi_n : \mathrm{Rep}_{\mathrm{poly}}(\mathrm{GL}(r)) \to H^*(\mathrm{Gr}(r,n))$$

as shown in [Ku2, §5].

Fix $r \geq 1$ and define the stable cohomology ring

$$\mathbb{H}^*(\mathrm{Gr}_r, \mathbb{Z}) := \varprojlim H^*(\mathrm{Gr}(r,n), \mathbb{Z})$$

as the inverse limit. Then, the homomorphisms ϕ_n combine to give a ring homomorphism

$$\phi_\infty : \mathrm{Rep}_{\mathrm{poly}}(\mathrm{GL}(r)) \to \mathbb{H}^*(\mathrm{Gr}_r, \mathbb{Z}).$$

Moreover, by the explicit description of ϕ_n (cf. [F, §9.4] and also [Ku2, §5]) it is immediately seen that ϕ_∞ is a ring *isomorphism*.

The aim of this paper is to analyze the corresponding question for the Symplectic groups $\mathrm{Sp}(2k)$ *as well as the odd orthogonal groups* $\mathrm{SO}(2k+1)$.

Let us fix a positive integer k and consider the isotropic Grassmannian $\mathrm{IG}(n-k, 2n)$ consisting of $n-k$-dimensional isotropic subspaces of $V = \mathbb{C}^{2n}$ with respect to a non-degenerate symplectic form. Then, $\mathrm{IG}(n-k, 2n)$ is the quotient $\mathrm{Sp}(2n)/P^C_{n-k}$ of $\mathrm{Sp}(2n)$ by the standard maximal parabolic subgroup P^C_{n-k} corresponding to the $n-k$-th node of the Dynkin diagram of $\mathrm{Sp}(2n)$ (following the indexing convention as in [Bo]). Let L^C_{n-k} denote the Levi subgroup of P^C_{n-k}. Then,

$$L^C_{n-k} \simeq \mathrm{GL}(n-k) \times \mathrm{Sp}(2k).$$

We take the standard representation of $\mathrm{Sp}(2n)$ in $\mathbb{C}^{2n}$ and abbreviate the corresponding $\mathrm{Rep}^{\mathbb{C}}_{\lambda-\mathrm{poly}}(L^C_{n-k})$ by $\mathrm{Rep}^{\mathbb{C}}_{\mathrm{poly}}(L^C_{n-k})$. Thus, following (1), we get a ring homomorphism

$$\xi^{P^C_{n-k}} : \mathrm{Rep}^{\mathbb{C}}_{\mathrm{poly}}(L^C_{n-k}) \to H^*(\mathrm{IG}(n-k, 2n), \mathbb{C}).$$

Restricting $\xi^{P^C_{n-k}}$ to the component $\mathrm{Sp}(2k)$, we get a ring homomorphism

$$\xi_{n,k} : \mathrm{Rep}^{\mathbb{C}}_{\mathrm{poly}}(\mathrm{Sp}(2k)) \to H^*(\mathrm{IG}(n-k, 2n), \mathbb{C}).$$

Define the *stable cohomology ring* (cf. Definition 15)

$$\mathbb{H}^*(\mathrm{IG}_k, \mathbb{Z}) := \varprojlim H^*(\mathrm{IG}(n-k, 2n), \mathbb{Z})$$

as the inverse limit. Then, the homomorphisms $\xi_{n,k}$ combine to give a ring homomorphism

$$\xi_k : \mathrm{Rep}^{\mathbb{C}}_{\mathrm{poly}}(\mathrm{Sp}(2k)) \to \mathbb{H}^*(\mathrm{IG}_k, \mathbb{Z}).$$

Following is our first main result of the paper (cf. Theorem 16 for a more precise assertion).

Theorem A. *The above ring homomorphism* $\xi_k : \mathrm{Rep}^{\mathbb{C}}_{\mathrm{poly}}(\mathrm{Sp}(2k)) \to \mathbb{H}^*(\mathrm{IG}_k, \mathbb{C})$ *is injective.*

However, it is not surjective (cf. Remark 17).

There are parallel results for the odd orthogonal groups $SO(2k+1)$. Specifically, consider the isotropic Grassmannian $OG(n-k, 2n+1)$ consisting of $n-k$-dimensional isotropic subspaces of $V = \mathbb{C}^{2n+1}$ with respect to a non-degenerate symmetric form. Then, $OG(n-k, 2n+1)$ is the quotient $SO(2n)/P^B_{n-k}$ of $SO(2n+1)$ by the standard maximal parabolic subgroup P^B_{n-k} corresponding to the $n-k$-th node of the Dynkin diagram of $SO(2n+1)$. Let L^B_{n-k} denote the Levi subgroup of P^B_{n-k}. Then,

$$L^B_{n-k} \simeq GL(n-k) \times SO(2k+1).$$

We take the standard representation of $SO(2n+1)$ in $\mathbb{C}^{2n+1}$ and abbreviate the corresponding $\mathrm{Rep}^{\mathbb{C}}_{\lambda-\mathrm{poly}}(L^B_{n-k})$ by $\mathrm{Rep}^{\mathbb{C}}_{\mathrm{poly}}(L^B_{n-k})$. Thus, following (1), we get a ring homomorphism

$$\xi^{P^B_{n-k}} : \mathrm{Rep}^{\mathbb{C}}_{\mathrm{poly}}(L^B_{n-k}) \to H^*(OG(n-k, 2n+1), \mathbb{C}).$$

Restricting $\xi^{P^B_{n-k}}$ to the component $SO(2k+1)$, we get a ring homomorphism

$$\bar{\xi}_{n,k} : \mathrm{Rep}^{\mathbb{C}}_{\mathrm{poly}}(SO(2k+1)) \to H^*(OG(n-k, 2n+1), \mathbb{C}).$$

Similar to $\mathbb{H}^*(IG_k, \mathbb{Z})$, define the *stable cohomology ring* (cf. Definition 28)

$$\mathbb{H}^*(OG_k, \mathbb{Z}) := \varprojlim H^*(OG(n-k, 2n+1), \mathbb{Z})$$

as the inverse limit. Then, the homomorphisms $\bar{\xi}_{n,k}$ combine to give a ring homomorphism

$$\bar{\xi}_k : \mathrm{Rep}^{\mathbb{C}}_{\mathrm{poly}}(SO(2k+1)) \to \mathbb{H}^*(OG_k, \mathbb{Z}).$$

Following is our second main result of the paper (cf. Theorem 29 for a more precise assertion).

Theorem B. *The above ring homomorphism* $\bar{\xi}_k : \mathrm{Rep}^{\mathbb{C}}_{\mathrm{poly}}(SO(2k+1)) \to \mathbb{H}^*(OG_k, \mathbb{C})$ *is injective.*

However, it is not surjective (cf. Remark 30).

The proofs rely on some results of Buch-Kresch-Tamvakis from [BKT1] and [BKT2] and earlier results of the first author [Ku2].

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2 Prelimanaries and Notation

We recall some notation and results from [Ku2].

Let G be a connected reductive group over $\mathbb{C}$ with a Borel subgroup B and maximal torus $T \subset B$. Let P be a standard parabolic subgroup with the Levi subgroup L containing T. We denote their Lie algebras by the corresponding Gothic characters: $\mathfrak{g}, \mathfrak{b}, \mathfrak{t}, \mathfrak{p}, \mathfrak{l}$ respectively. We denote by $\Delta = \{\alpha_1, \ldots, \alpha_\ell\} \subset \mathfrak{t}^*$ the set of simple roots. The fundamental weights of $\mathfrak{g}$ are denoted by $\{\omega_1, \ldots, \omega_\ell\} \subset \mathfrak{t}^*$. Let W (resp. W_L) be the Weyl group of G (resp. L). Then, W is generated by the simple reflections $\{s_i\}_{1 \le i \le \ell}$. Let

W^P denote the set of smallest coset representatives in the cosets in W/W_L. *Throughout the paper we follow the indexing convention as in [Bo, Planche I - IX].*

Let $X(T)$ be the group of characters of T and let $D \subset X(T)$ be the set of dominant characters (with respect to the given choice of B and hence positive roots, which are the roots of $\mathfrak{b}$). Then, the isomorphism classes of finite dimensional irreducible representations of G are bijectively parameterized by D under the correspondence $\lambda \in D \rightsquigarrow V(\lambda)$, where $V(\lambda)$ is the irreducible representation of G with highest weight λ. We call $V(\lambda)$ *almost faithful* if the corresponding map $\rho_\lambda : G \to \mathrm{Aut}(V(\lambda))$ has finite kernel.

Recall the Bruhat decomposition for the flag variety:

$$G/P = \sqcup_{w \in W^P} \Lambda_w^P, \ \text{ where } \Lambda_w^P := BwP/P.$$

Let $\bar{\Lambda}_w^P$ denote the closure of Λ_w^P in G/P. We denote by $[\bar{\Lambda}_w^P] \in H_{2\ell(w)}(G/P, \mathbb{Z})$ its fundamental class. Let $\{\epsilon_w^P\}_{w \in W^P}$ denote the Kronecker dual basis of the cohomology, i.e.,

$$\epsilon_w^P([\bar{\Lambda}_v^P]) = \delta_{w,v}, \ \text{ for any } v, w \in W^P.$$

Thus, ϵ_w^P belongs to the singular cohomology:

$$\epsilon_w^P \in H^{2\ell(w)}(G/P, \mathbb{Z}).$$

We abbreviate ϵ_w^B by ϵ_w. Then, for any $w \in W^P$, $\epsilon_w^P = \pi^*(\epsilon_w)$, where $\pi : G/B \to G/P$ is the standard projection.

We will often abbreviate ϵ_w^P by ϵ_w when the reference to P is clear from the context.

Definition 1. Let $V(\lambda)$ be any almost faithful irreducible representation of G. Following Springer (cf. [BR, §9]), define the map

$$\theta_\lambda : G \to \mathfrak{g} \quad \text{(depending upon } \lambda)$$

as follows:

$$\begin{array}{ccc} G & \xrightarrow{\rho_\lambda} & \mathrm{Aut}(V(\lambda)) \subset \mathrm{End}(V(\lambda)) = \mathfrak{g} \oplus \mathfrak{g}^\perp \\ & \searrow^{\theta_\lambda} & \downarrow \pi \\ & & \mathfrak{g} \end{array}$$

where $\mathfrak{g}$ sits canonically inside $\mathrm{End}(V(\lambda))$ via the derivative $d\rho_\lambda$, the orthogonal complement $\mathfrak{g}^\perp$ is taken with respect to the standard conjugate $\mathrm{Aut}(V(\lambda))$-invariant form on $\mathrm{End}(V(\lambda))$: $\langle A, B \rangle := \mathrm{tr}(AB)$, and π is the projection to the $\mathfrak{g}$-factor. (By considering a compact form K of G, it is easy to see that $\mathfrak{g} \cap \mathfrak{g}^\perp = \{0\}$.)

Since $\pi \circ d\rho_\lambda$ is the identity map, θ_λ is a local diffeomorphism at 1 (and hence with Zariski dense image). Of course, by construction, θ_λ is an algebraic morphism. Moreover, since the decomposition $\mathrm{End}(V(\lambda)) = \mathfrak{g} \oplus \mathfrak{g}^\perp$ is G-stable, it is easy to see that θ_λ is G-equivariant under conjugation.

We recall the following lemma from [Ku2, Lemma 2].

Lemma 2. *The above morphism restricts to $\theta_{\lambda|T} : T \to \mathfrak{t}$.*

For any $\mu \in X(T)$, we have a G-equivariant line bundle $\mathcal{L}(\mu)$ on G/B associated to the principal B-bundle $G \to G/B$ via the one dimensional B-module μ^{-1}. (Any $\mu \in X(T)$ extends uniquely to a character of B.) The one dimensional B-module μ is also denoted by $\mathbb{C}_\mu$. Recall the surjective Borel homomorphism

$$\beta : S(\mathfrak{t}^*) \to H^*(G/B, \mathbb{C}),$$

which takes a character $\mu \in X(T)$ to the first Chern class of the line bundle $\mathcal{L}(\mu)$. (We realize $X(T)$ as a lattice in $\mathfrak{t}^*$ via taking derivative.) We then extend this map linearly over $\mathbb{C}$ to $\mathfrak{t}^*$ and extend further as a graded algebra homomorphism from $S(\mathfrak{t}^*)$ (doubling the degree). Under the Borel homomorphism,

$$\beta(\omega_i) = \epsilon_{s_i}, \quad \text{for any fundamental weight } \omega_i. \tag{2}$$

Fix a compact form K of G. In particular, $T_o := K \cap T$ is a (compact) maximal torus of K. Then, $W \simeq N(T_o)/T_o$, where $N(T_o)$ is the normalizer of T_o in K. Recall that β is W-equivariant under the standard action of W on $S(\mathfrak{t}^*)$ and the W-action on $H^*(G/B, \mathbb{C})$ induced from the W-action on $G/B \simeq K/T_o$ via

$$(nT_o) \cdot (kT_o) := kn^{-1}T_o, \quad \text{for } n \in N(T_o) \text{ and } k \in K.$$

Thus, for any standard parabolic subgroup P with the Levi subgroup L containing T, restricting β, we get a surjective graded algebra homorphism:

$$\beta^P : S(\mathfrak{t}^*)^{W_L} \to H^*(G/B, \mathbb{C})^{W_L} \simeq H^*(G/P, \mathbb{C}),$$

where the last isomorphism, which is induced from the projection $G/B \to G/P$, can be found, e.g., in [Ku1, Corollary 11.3.14].

Now, the Springer morphism $\theta_{\lambda|T} : T \to \mathfrak{t}$ (restricted to T) gives rise to the corresponding W-equivariant injective algebra homomorphism on the affine coordinate rings:

$$(\theta_{\lambda|T})^* : \mathbb{C}[\mathfrak{t}] = S(\mathfrak{t}^*) \to \mathbb{C}[T].$$

Thus, on restriction to W_L-invariants, we get an injective algebra homomorphism

$$\theta_\lambda(P)^* : \mathbb{C}[\mathfrak{t}]^{W_L} = S(\mathfrak{t}^*)^{W_L} \to \mathbb{C}[T]^{W_L}.$$

(Since W_L-invariants depend upon the choice of the parabolic subgroup P, we have included P in the notation of $\theta_\lambda(P)^*$.) Now, let $\mathrm{Rep}(L)$ be the representation ring of L and let $\mathrm{Rep}^{\mathbb{C}}(L) := \mathrm{Rep}(L) \otimes_{\mathbb{Z}} \mathbb{C}$ be its complexification. Then, as it is well known,

$$\mathrm{Rep}^{\mathbb{C}}(L) \simeq \mathbb{C}[T]^{W_L} \tag{3}$$

obtained from taking the character of an L-module restricted to T.

We will often identify a virtual representation of L with its character restricted to T (which is automatically W_L-invariant).

Definition 3. We call a virtual character $\chi \in \mathrm{Rep}^{\mathbb{C}}(L)$ of L a *λ-polynomial character* if the corresponding function in $\mathbb{C}[T]^{W_L}$ is in the image of $\theta_\lambda(P)^*$. The set of all

λ-polynomial characters of L, which is, by definition, a subalgebra of $\mathrm{Rep}^{\mathbb{C}}(L)$ isomorphic to the algebra $S(\mathfrak{t}^*)^{W_L}$, is denoted by $\mathrm{Rep}^{\mathbb{C}}_{\lambda-\mathrm{poly}}(L)$. Of course, the map $\theta_\lambda(P)^*$ induces an algebra isomorphism (still denoted by)

$$\theta_\lambda(P)^* : S(\mathfrak{t}^*)^{W_L} \simeq \mathrm{Rep}^{\mathbb{C}}_{\lambda-\mathrm{poly}}(L),$$

under the identification (3).

It is easy to see that

$$\mathrm{Rep}_{\omega_1-\mathrm{poly}}(\mathrm{GL}(n)) = \mathrm{Rep}_{\mathrm{poly}}(\mathrm{GL}(n)), \tag{4}$$

where $\mathrm{Rep}_{\mathrm{poly}}(\mathrm{GL}(n))$ denotes the subring of the representation ring $\mathrm{Rep}(\mathrm{GL}(n))$ spanned by the irreducible polynomial representations of $\mathrm{GL}(n)$.

We recall the following result from [Ku2, Theorem 5].

Theorem 4. *Let $V(\lambda)$ be an almost faithful irreducible G-module and let P be any standard parabolic subgroup. Then, the above maps (specifically $\beta^P \circ (\theta_\lambda(P)^*)^{-1}$) give rise to a surjective $\mathbb{C}$-algebra homomorphism*

$$\xi^P_\lambda : \mathrm{Rep}^{\mathbb{C}}_{\lambda-\mathrm{poly}}(L) \to H^*(G/P, \mathbb{C}).$$

Moreover, let Q be another standard parabolic subgroup with Levi subgroup R containing T such that $P \subset Q$ (and hence $L \subset R$). Then, we have the following commutative diagram:

$$\begin{array}{ccc} \mathrm{Rep}^{\mathbb{C}}_{\lambda-\mathrm{poly}}(R) & \xrightarrow{\xi^Q_\lambda} & H^*(G/Q, \mathbb{C}) \\ \downarrow{\scriptstyle\gamma} & & \downarrow{\scriptstyle\pi^*} \\ \mathrm{Rep}^{\mathbb{C}}_{\lambda-\mathrm{poly}}(L) & \xrightarrow{\xi^P_\lambda} & H^*(G/P, \mathbb{C}), \end{array}$$

where π^* is induced from the standard projection $\pi : G/P \to G/Q$ and γ is induced from the restriction of representations.

3 Injectivity Result for the Symplectic Group

In this section, we consider the symplectic group $G = \mathrm{Sp}(2n)$ ($n \geq 2$). We take the Springer morphism for $\mathrm{Sp}(2n)$ with respect to the first fundamental weight $\lambda = \omega_1$. We will abbreviate the Springer morphism θ_{ω_1} by θ, ξ^P_λ by ξ^P and $\mathrm{Rep}^{\mathbb{C}}_{\omega_1-\mathrm{poly}}(G)$ by $\mathrm{Rep}^{\mathbb{C}}_{\mathrm{poly}}(G)$.

Let $V = \mathbb{C}^{2n}$ be equipped with the nondegenerate symplectic form $\langle\, ,\, \rangle$ so that its matrix $(\langle e_i, e_j\rangle)_{1\leq i,j\leq 2n}$ in the standard basis $\{e_1, \ldots, e_{2n}\}$ is given by

$$E_C = \begin{pmatrix} 0 & J \\ -J & 0 \end{pmatrix},$$

where J is the anti-diagonal matrix $(1,\dots,1)$ of size n. Let

$$\mathrm{Sp}(2n) := \{g \in \mathrm{SL}(2n) : g \text{ leaves the form } \langle\,,\,\rangle \text{ invariant}\}$$

be the associated symplectic group. Clearly, $\mathrm{Sp}(2n)$ can be realized as the fixed point subgroup $\mathrm{SL}(2n)^\sigma$ under the involution $\sigma : \mathrm{SL}(2n) \to \mathrm{SL}(2n)$ defined by $\sigma(A) = E_C(A^t)^{-1}E_C^{-1}$.

The involution σ keeps both of B and T stable, where B and T are the standard Borel and maximal torus respectively of $\mathrm{SL}(2n)$. Moreover, B^σ (respectively, T^σ) is a Borel subgroup (respectively, a maximal torus) of $\mathrm{Sp}(2n)$. We denote B^σ, T^σ by $B_C = B_{C_n}, T_C = T_{C_n}$ respectively. Then, T_C is given as follows:

$$T_C = \left\{\mathbf{t} = \mathrm{diag}(t_1,\dots,t_n,t_n^{-1},\dots,t_1^{-1}) : t_i \in \mathbb{C}^*\right\}. \tag{5}$$

Its Lie algebra is given by

$$\mathfrak{t}_C = \left\{\dot{\mathbf{t}} = \mathrm{diag}(x_1,\dots,x_n,-x_n,\dots,-x_1) : x_i \in \mathbb{C}\right\}. \tag{6}$$

We recall the following lemma from [Ku2, Lemma 10].

Lemma 5. *The Springer morphism $\theta : G \to \mathfrak{g}$ for $G = \mathrm{Sp}(2n)$ is given by*

$$g \mapsto \frac{g - E_C^{-1}g^tE_C}{2}, \quad \textit{for } g \in G.$$

(Observe that this is the Cayley transform.)

From the description of the Springer morphism given above, we immediately get the following (cf. [Ku2, Corollary 11]):

Corollary 6. *Restricted to the maximal torus as above, we get the following description of the Springer map θ:*

$$\theta(\mathbf{t}) = \mathrm{diag}(\bar{t}_1,\dots,\bar{t}_n,-\bar{t}_n,\dots,-\bar{t}_1), \quad \textit{where } \bar{t}_i := \frac{t_i - t_i^{-1}}{2}.$$

The following result follows easily from Corollary 6 together with the description of the Weyl group (cf. [Ku2, Proposition 12]).

Proposition 7. *Let $f : T \to \mathbb{C}$ be a regular map. Then, $f \in \mathrm{Rep}^{\mathbb{C}}_{\mathrm{poly}}(G)$ if and only if the following is satisfied:*

There exists a symmetric polynomial $P_f(x_1,\dots,x_n)$ such that

$$f(\mathbf{t}) = P_f\left((\bar{t}_1)^2,\dots,(\bar{t}_n)^2\right), \ \textit{for } \mathbf{t} \in T_C \textit{ given by } (5).$$

We recall the following result from [Ku2, Proposition 24].

Lemma 8. *Under the homomorphism $\xi^B : \mathrm{Rep}^{\mathbb{C}}_{\mathrm{poly}}(T) \to H^*(G/B,\mathbb{C})$ of Theorem 4,*

$$\bar{t}_i \mapsto (\epsilon_{s_i} - \epsilon_{s_{i-1}}), \quad \textit{for any } 1 \le i \le n.$$

Definition 9. For $1 \leq r \leq n$, we let $\mathrm{IG}(r, 2n)$ to be the set of r-dimensional isotropic subspaces of V with respect to the form $\langle\ ,\ \rangle$, i.e.,

$$\mathrm{IG}(r, 2n) := \{M \in \mathrm{Gr}(r, 2n) : \langle v, v' \rangle = 0,\ \forall\, v, v' \in M\}.$$

Then, $\mathrm{IG}(r, 2n)$ is the quotient $\mathrm{Sp}(2n)/P_r^C$ of $\mathrm{Sp}(2n)$ by the standard maximal parabolic subgroup P_r^C with $\Delta \setminus \{\alpha_r\}$ as the set of simple roots of its Levi component L_r^C. (Again we take L_r^C to be the unique Levi subgroup of P_r^C containing T_C.) Then,

$$L_r^C \simeq \mathrm{GL}(r) \times \mathrm{Sp}(2(n-r)).$$

In this case, by the identity (4), Corollary 6 and Proposition 7,

$$\mathrm{Rep}_{\mathrm{poly}}^{\mathbb{C}}(L_r^C) \simeq \mathbb{C}_{\mathrm{sym}}[\bar{t}_1, \ldots, \bar{t}_r] \otimes_{\mathbb{C}} \mathbb{C}_{\mathrm{sym}}[(\bar{t}_{r+1})^2, \ldots, (\bar{t}_n)^2], \tag{7}$$

where $\mathbb{C}_{\mathrm{sym}}$ denotes the subalgebra of the polynomial ring consisting of symmetric polynomials.

From now on we fix $k \geq 0$ and consider $\mathrm{IG}(n-k, 2n)$*.*

Following [BKT1, Definition 1.1], a partition $\lambda : \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_d > 0$ is said to be *k-strict* if no part greater than k is repeated (i.e., $\lambda_j > k \Rightarrow \lambda_{j+1} < \lambda_j$). The Schubert varieties in $\mathrm{IG}(n-k, 2n)$ are parametrized by k-strict partitions contained in the $(n-k) \times (n+k)$ rectangle. The codimension of this variety is equal to $|\lambda| := \sum \lambda_i$. Let $\sigma_\lambda \in H^{2|\lambda|}(\mathrm{IG}(n-k, 2n), \mathbb{Z})$ denote the cohomology class Poincaré dual to the fundamental class $[X_\lambda]$ of the Schubert variety associated to λ. Let $\mathcal{P}(k, n)$ denote the set of k-strict partitions contained in the $(n-k) \times (n+k)$ rectangle. Thus, $\{\sigma_\lambda\}_{\lambda \in \mathcal{P}(k,n)}$ gives the Schubert basis of $H^*(\mathrm{IG}(n-k, 2n), \mathbb{Z})$.

We have the following short exact sequence of vector bundles over $\mathrm{IG}(n-k, 2n)$:

$$0 \to \mathcal{S} \to \bar{\mathcal{E}} \to \mathcal{Q} \to 0,$$

where $\bar{\mathcal{E}}$ is the trivial bundle of rank $2n$, $\mathcal{S}$ is the tautological subbundle of rank $n-k$ and $\mathcal{Q}$ is the quotient bundle of rank $n+k$. Let $c_i = c_i(\mathcal{Q})$ $(1 \leq i \leq n+k)$ denote the i^{th} Chern class of the quotient bundle $\mathcal{Q}$. Then, these classes are so called the *special Schubert classes*. Then, by [BKT1, §1.2],

$$c_i = \sigma_i, \tag{8}$$

where $\sigma_i := \sigma_{(i)}$ and (i) is the partition with single term i.

We have the following presentation of the cohomology ring due to [BKT1, Theorem 1.2]. In the following we follow the convention that $c_0 = 1$ and $c_p = 0$ if $p < 0$ or $p > n+k$.

Theorem 10. *The cohomology ring $H^*(\mathrm{IG}(n-k, 2n), \mathbb{Z})$ is presented as a quotient of the ploynomial ring $\mathbb{Z}[c_1, \ldots, c_{n+k}]$ modulo the relations:*

$$(R_{n,k}^p)\ (n-k+1 \leq p \leq n+k): \qquad \det(c_{1+j-i})_{1 \leq i,j \leq p} = 0,$$

and

$$(S_{n,k}^s)\ (k+1 \leq s \leq n): \qquad c_s^2 + 2\sum_{i=1}^{n+k-s} (-1)^i c_{s+i} c_{s-i} = 0.$$

Our original proof of the following result was longer. The following shorter proof is due to L. Mihalcea.

Proposition 11. *The map* $\xi^{P_{n-k}} : \mathrm{Rep}^{\mathbb{C}}_{\mathrm{poly}}(L^C_{n-k}) \to H^*(\mathrm{IG}(n-k, 2n), \mathbb{C})$ *of Theorem 4 under the decomposition* (7) *for* $r = n - k$ *takes, for* $1 \leq i \leq k$,

$$e_i\left((\bar{t}_{n-k+1})^2, \ldots, (\bar{t}_n)^2\right) \mapsto c_i^2 + 2\sum_{j=1}^{i}(-1)^j c_{i+j}c_{i-j},$$

where $c_i = c_i(Q)$ *is as defined before Theorem 10 and* e_i *is the i-th elementary symmetric function.*

Before we come to the proof of the proposition, we need the following two lemmas:

Let $\mathrm{Fl} = G/B$ be the full flag variety for $G = \mathrm{Sp}(2n)$. It consists of partial flags

$$F_\bullet : \ F_1 \subset F_2 \subset \ldots \subset F_n \subset E := \mathbb{C}^{2n}, \ \text{such that each } F_j \text{ is isotropic and } \dim F_j = j.$$

We can complete the partial flag to a full flag by taking $F_{n+j} := F^\perp_{n-j}$. The flags $F_\bullet$ give rise to a sequence of tautological vector bundles over Fl:

$$\mathcal{F}_1 \subset \mathcal{F}_2 \subset \ldots \subset \mathcal{F}_n \subset \mathcal{E}, \ \text{ with rank } \mathcal{F}_j = j,$$

where $\mathcal{E} : \mathrm{Fl} \times \mathbb{C}^{2n} \to \mathrm{Fl}$ is the trivial rank $2n$ vector bundle. For $1 \leq j \leq n$, define

$$x_j := -c_1(\mathcal{F}_j/\mathcal{F}_{j-1}),$$

where $\mathcal{F}_0$ is taken to be the vector bundle of rank 0.

Lemma 12. *For* $1 \leq j \leq n$, *the Schubert divisor* $\epsilon_{s_j} \in H^2(\mathrm{Fl}, \mathbb{Z})$ *is given by*

$$\epsilon_{s_j} = -c_1(\mathcal{F}_j) = x_1 + \cdots + x_j.$$

In particular, under ξ^B *for* $G = \mathrm{Sp}(2n)$, $\bar{t}_j \mapsto x_j$ *for any* $1 \leq j \leq n$.

Proof. The first part follows from the identity (8).
The 'In particular' statement follows from Lemma 8. □

For $1 \leq j \leq n$, let

$$Q_j := \mathcal{E}/\mathcal{F}_j.$$

Observe that the symplectic form gives an isomorphism of vector bundles:

$$Q_j \simeq (\mathcal{F}_j^\perp)^*. \tag{9}$$

Lemma 13. *For* $0 \leq j \leq n$, *the following holds:*

$$c(Q_j)c(Q_j^*) = \prod_{p=j+1}^{n}(1 - x_p)(1 + x_p),$$

where c is the total Chern class.

Proof. By definition,

$$\prod_{p=j+1}^{n}(1-x_p)(1+x_p) = \prod_{p=j+1}^{n} c(\mathcal{F}_p/\mathcal{F}_{p-1})\cdot c\left((\mathcal{F}_p/\mathcal{F}_{p-1})^*\right) = \frac{c(\mathcal{F}_n)}{c(\mathcal{F}_j)}\cdot\frac{c(\mathcal{F}_n^*)}{c(\mathcal{F}_j^*)}. \tag{10}$$

From the exact sequence $0\to\mathcal{F}_j\to\mathcal{E}\to\mathcal{Q}_j\to 0$, we get

$$c(\mathcal{Q}_j)\cdot c(\mathcal{F}_j) = 1 \text{ and } c(\mathcal{Q}_j^*)c(\mathcal{F}_j^*) = 1, \tag{11}$$

and hence taking $j=n$ in the above equation and using the equation (9), we get

$$c(\mathcal{F}_n)c(\mathcal{F}_n^*) = 1, \text{ since } \mathcal{F}_n^\perp = \mathcal{F}_n. \tag{12}$$

Combining the equations (10), (11) and (12), we get the lemma. □

Proof. (of Proposition 11) By taking terms of degree $2i$ and $j=n-k$ in Lemma 13, we obtain in $H^*(\mathrm{Fl},\mathbb{Z})$:

$$c_i(\mathcal{Q}_j)^2 + 2\sum_{p=1}^{i}(-1)^p c_{i+p}(\mathcal{Q}_j)\cdot c_{i-p}(\mathcal{Q}_j) = e_i(x_{j+1}^2,\ldots,x_n^2).$$

By the definition, the bundle $\mathcal{S}$ pulls back to the bundle $\mathcal{F}_{n-k}$ over Fl under the projection $\mathrm{Fl}\to\mathrm{IG}(n-k,2n)$. Thus, the proposition follows from Lemma 12. □

Remark 14. Even though we do not need, the map $\xi^{P_{n-k}}:\mathrm{Rep}_{\mathrm{poly}}^{\mathbb{C}}(L_{n-k}^C)\to H^*(\mathrm{IG}(n-k,2n),\mathbb{C})$ of Theorem 4 under the decomposition (7) takes for $1\le i\le n-k$,

$$e_i(\bar{t}_1,\ldots,\bar{t}_{n-k})\mapsto c_i(S) = \epsilon_{s_{n-k-i+1}\cdots s_{n-k}}.$$

This follows from [BKT1, §1.2].

Definition 15. [Inverse Limit] For any $k\ge 0$, define the *stable cohomology ring* [BKT2, §1.3] as

$$\mathbb{H}^*(\mathrm{IG}_k,\mathbb{Z}) = \varprojlim H^*(\mathrm{IG}(n-k,2n),\mathbb{Z})$$

as the inverse limit (in the category of graded rings) of the inverse system

$$\cdots\leftarrow H^*(\mathrm{IG}(n-k,2n),\mathbb{Z})\xleftarrow{\pi_n^*} H^*(\mathrm{IG}(n-k+1,2n+2),\mathbb{Z})\leftarrow\cdots,$$

where $\pi_n:\mathrm{IG}(n-k,2n)\hookrightarrow\mathrm{IG}(n-k+1,2n+2)$ is given by $V\mapsto T_n(V)\oplus\mathbb{C}e_{n+1}$ and $T_n:\mathbb{C}^{2n}\to\mathbb{C}^{2n+2}$ is the linear embedding taking $e_i\mapsto e_i$ for $1\le i\le n$ and taking $e_i\mapsto e_{i+2}$ for $n+1\le i\le 2n$.

This ring has an additive basis consisting of Schubert classes σ_λ for each $k-$strict partition λ. The natural ring homorphism $\varphi_{k,n}:\mathbb{H}^*(\mathrm{IG}_k,\mathbb{Z})\to H^*(\mathrm{IG}(n-k,2n),\mathbb{Z})$ takes σ_λ to σ_λ whenever λ fits in a $(n-k)\times(n+k)$ rectangle and to zero otherwise. In particular, $\varphi_{k,n}$ is surjective. From the definition of the Chern classes $c_j = c_j^n(Q)$, it is easy to see that under the restriction map $\pi_n^*: H^*(\mathrm{IG}(n-k+1,2n+2),\mathbb{Z})\to H^*(\mathrm{IG}(n-k,2n),\mathbb{Z}), c_j^{n+1}\mapsto c_j^n$ for $1\le j\le n+k$ and $c_{n+k+1}^{n+1}\mapsto 0$.

From the presentation of the ring $H^*(\mathrm{IG}(n-k, 2n), \mathbb{Z})$ (Theorem 10), none of the determinantal relations hold in the inverse limit. So, $\mathbb{H}^*(\mathrm{IG}_k, \mathbb{Z})$ is isomorphic to the polynomial ring $\mathbb{Z}[c_1, c_2, \ldots]$ modulo the relations:

$$(S^s)\ (s > k): \qquad c_s^2 + 2\sum_{i=1}^{s} (-1)^i c_{s+i} c_{s-i} = 0. \tag{13}$$

Take $k \geq 1$. Recall from Proposition 7 that

$$\mathrm{Rep}^{\mathbb{C}}_{\mathrm{poly}}(\mathrm{Sp}(2k)) \simeq \mathbb{C}_{\mathrm{sym}}\left[(\bar{h}_1)^2, \ldots, (\bar{h}_k)^2\right].$$

Define a ring homomorphism (for any $1 \leq k \leq n$)

$$\iota^n_k : \mathrm{Rep}^{\mathbb{C}}_{\mathrm{poly}}(\mathrm{Sp}(2k)) \to \mathrm{Rep}^{\mathbb{C}}_{\mathrm{poly}}(L^C_{n-k})$$

by taking $f(\bar{\mathbf{h}}) \mapsto 1 \otimes f(\bar{\mathbf{t}})$, where $\bar{\mathbf{h}} := (\bar{h}_1, \ldots, \bar{h}_k)$, $\bar{\mathbf{t}} := (\bar{t}_{n-k+1}, \ldots, \bar{t}_n)$ and $f(\bar{\mathbf{t}})$ is the same polynomial written in the $\bar{\mathbf{t}}$-variables uner the transformation $\bar{h}_p \mapsto \bar{t}_{n-k+p}$. This gives rise to the map $\xi_{n,k} := \xi^{P_{n-k}} \circ \iota^n_k : \mathrm{Rep}^{\mathbb{C}}_{\mathrm{poly}}(\mathrm{Sp}(2k)) \to H^*(\mathrm{IG}(n-k, 2n), \mathbb{C})$. Consider the following diagram, which is commutative because of Proposition 11.

$$\begin{array}{ccc}
 & & \uparrow{\scriptstyle \pi^*_{n-1}} \\
\mathrm{Rep}^{\mathbb{C}}_{\mathrm{poly}}(\mathrm{Sp}(2k)) & \xrightarrow{\ \xi_{n,k}\ } & H^*(\mathrm{IG}(n-k, 2n), \mathbb{C}) \\
 & \searrow^{\xi_{n+1,k}} & \uparrow{\scriptstyle \pi^*_{n}} \\
 & & H^*(\mathrm{IG}(n-k+1, 2n+2), \mathbb{C}) \\
 & & \uparrow{\scriptstyle \pi^*_{n+1}}
\end{array}$$

The compatible ring homomorphisms $\xi_{n,k} : \mathrm{Rep}^{\mathbb{C}}_{\mathrm{poly}}(\mathrm{Sp}(2k)) \to H^*(\mathrm{IG}(n-k, 2n), \mathbb{C})$ combine to give a ring homomorphism

$$\xi_k : \mathrm{Rep}^{\mathbb{C}}_{\mathrm{poly}}(\mathrm{Sp}(2k)) \to \mathbb{H}^*(\mathrm{IG}_k, \mathbb{C}).$$

The following theorem is one of our main results of the paper.

Theorem 16. *Let $k \geq 1$ be an integer. The above ring homomorphism $\xi_k : \mathrm{Rep}^{\mathbb{C}}_{\mathrm{poly}}(\mathrm{Sp}(2k)) \to \mathbb{H}^*(\mathrm{IG}_k, \mathbb{C})$ takes the generators*

$$e_i\left((\bar{h}_1)^2, \ldots, (\bar{h}_k)^2\right) \mapsto c_i^2 + 2\sum_{j=1}^{i} (-1)^j c_{i+j} c_{i-j}, \ \ \textit{for any } 1 \leq i \leq k, \tag{14}$$

where $c_i = c_i(Q)$.

In particular, ξ_k is injective.

Proof. The first part follows from Proposition 11.

We next prove the injectivity of ξ_k:

By Proposition 7, $\mathrm{Rep}^{\mathbb{C}}_{\mathrm{poly}}(\mathrm{Sp}(2k))$ is a polynomial ring over $\mathbb{C}$ generated by $\{e_1(\bar{\mathbf{h}}), \ldots, e_k(\bar{\mathbf{h}})\}$. Let $\mathrm{Rep}^{\mathbb{Z}}_{\mathrm{poly}}(\mathrm{Sp}(2k))$ be the polynomial subring over $\mathbb{Z}$ generated by $\{e_1(\bar{\mathbf{h}}), \ldots, e_k(\bar{\mathbf{h}})\}$. Then, by the equation (14),

$$\xi_k(\mathrm{Rep}^{\mathbb{Z}}_{\mathrm{poly}}(\mathrm{Sp}(2k))) \subset \mathbb{H}^*(\mathrm{IG}_k, \mathbb{Z}).$$

Thus, on restriction, we get the ring homomorphism

$$\xi^{\mathbb{Z}}_k : \mathrm{Rep}^{\mathbb{Z}}_{\mathrm{poly}}(\mathrm{Sp}(2k)) \to \mathbb{H}^*(\mathrm{IG}_k, \mathbb{Z}).$$

Observe further that $\xi^{\mathbb{Z}}_k$ is a homomorphism of graded rings if we assign degree $4i$ to each e_i (and the standard cohomological degree to $\mathbb{H}^*(\mathrm{IG}_k, \mathbb{Z})$). Let K be the Kernel of $\xi^{\mathbb{Z}}_k$. Since $\mathbb{H}^*(\mathrm{IG}_k, \mathbb{Z})$ is a free $\mathbb{Z}$-module of finite rank in each degree, the induced homomorphism

$$\mathbb{Z}/(2) \otimes_{\mathbb{Z}} K \to \mathbb{Z}/(2) \otimes_{\mathbb{Z}} \mathrm{Rep}^{\mathbb{Z}}_{\mathrm{poly}}(\mathrm{Sp}(2k)) \ \text{ is injective.} \tag{15}$$

We next observe that the induced homomorphism

$$\mathbb{Z}/(2) \otimes_{\mathbb{Z}} \mathrm{Rep}^{\mathbb{Z}}_{\mathrm{poly}}(\mathrm{Sp}(2k)) \to \mathbb{Z}/(2) \otimes_{\mathbb{Z}} \mathbb{H}^*(\mathrm{IG}_k, \mathbb{Z}) \ \text{ is injective.} \tag{16}$$

To prove this, observe that

$$\mathbb{Z}/(2) \otimes_{\mathbb{Z}} \mathrm{Rep}^{\mathbb{Z}}_{\mathrm{poly}}(\mathrm{Sp}(2k)) \simeq \mathbb{Z}/(2)[e_1, \ldots, e_k], \tag{17}$$

and, by the defining relations (S^s) $(s > k)$ of $\mathbb{H}^*(\mathrm{IG}_k, \mathbb{Z})$ as in equation (13),

$$\mathbb{Z}/(2) \otimes_{\mathbb{Z}} \mathbb{H}^*(\mathrm{IG}_k, \mathbb{Z}) \simeq \mathbb{Z}/(2)[c_1, \ldots, c_k] \otimes \frac{\mathbb{Z}/(2)[c_{k+1}, c_{k+2} \ldots]}{\langle c^2_{k+1}, c^2_{k+2} \ldots \rangle}. \tag{18}$$

Moreover, under the above identifications (17) and (18), by the first part of the theorem, the ring homomorphism $\xi^{\mathbb{Z}}_k$ modulo 2 is given by

$$e_i \mapsto c^2_i, \ \text{ for any } 1 \leq i \leq k.$$

In particular, it is injective. From this we obtain that

$$\mathbb{Z}/(2) \otimes_{\mathbb{Z}} K = 0.$$

But, since K is a finitely generated torsionfree $\mathbb{Z}$-module in each graded degree (thus free) we get that

$$K = 0.$$

Since $\mathbb{C}$ is a torsionfree $\mathbb{Z}$-module, this clearly gives the injectivity of ξ_k (cf. [Sp, Chap. 5, §2, Lemma 5]). This proves the theorem. □

Remark 17. The ring homomorphism $\xi_k : \mathrm{Rep}^{\mathbb{C}}_{\mathrm{poly}}(\mathrm{Sp}(2k)) \to \mathbb{H}^*(\mathrm{IG}_k, \mathbb{C})$ of the above Theorem 16 is *not* surjective, as can be easily seen since the domain is a finitely generated $\mathbb{C}$-algebra (by Proposition 7) whereas the range is not (for otherwise for each n, $H^*(\mathrm{IG}(n-k, 2n), \mathbb{C})$ would be generated by a fixed finite number of generators independent of n).

4 Injectivity Result for the Odd Orthogonal Group

The treatment in this section is parallel to that of the last section dealing with Sp($2n$). But, we include some details for completeness.

In this section, we consider the special orthogonal group $G = \mathrm{SO}(2n+1)$ ($n \geq 2$). We take the Springer morphism for SO($2n+1$) with respect to the first fundamental weight $\lambda = \omega_1$. We will abbreviate θ_{ω_1} by θ, ξ^P_λ by ξ^P and $\mathrm{Rep}^{\mathbb{C}}_{\omega_1-\mathrm{poly}}(G)$ by $\mathrm{Rep}^{\mathbb{C}}_{\mathrm{poly}}(G)$.

Let $V' = \mathbb{C}^{2n+1}$ be equipped with the nondegenerate symmetric form $\langle\, , \rangle$ so that its matrix $E_B = (\langle e_i, e_j\rangle)_{1\leq i,j\leq 2n+1}$ (in the standard basis $\{e_1, \dots, e_{2n+1}\}$) is the $(2n+1) \times (2n+1)$ antidiagonal matrix with $1'$s all along the antidiagonal except at the $(n+1, n+1)$-th place where the entry is 2. Note that the associated quadratic form on V' is given by

$$Q(\sum t_i e_i) = t_{n+1}^2 + \sum_{i=1}^{n} t_i t_{2n+2-i}.$$

Let

$$\mathrm{SO}(2n+1) := \{g \in \mathrm{SL}(2n+1) : g \text{ leaves the quadratic form } Q \text{ invariant}\}$$

be the associated special orthogonal group. Clearly, SO($2n+1$) can be realized as the fixed point subgroup $\mathrm{SL}(2n+1)^\delta$ under the involution $\delta : \mathrm{SL}(2n+1) \to \mathrm{SL}(2n+1)$ defined by $\delta(A) = E_B^{-1}(A^t)^{-1}E_B$. The involution δ keeps both of B and T stable, where B (resp. T) is the standard Borel (resp. maximal torus) of SL($2n+1$). Moreover, B^δ (respectively, T^δ) is a Borel subgroup (respectively, a maximal torus) of SO($2n+1$). We denote B^δ, T^δ by $B_B = B_{B_n}, T_B = T_{B_n}$ respectively. Then, T_B is given by:

$$T_B = \left\{\mathbf{t} = \mathrm{diag}(t_1, \dots, t_n, 1, t_n^{-1}, \dots, t_1^{-1}) : \; t_i \in \mathbb{C}^*\right\}. \tag{19}$$

Its Lie algebra is given by

$$\mathfrak{t}_B = \left\{\dot{\mathbf{t}} = \mathrm{diag}(x_1, \dots, x_n, 0, -x_n, \dots, -x_1) : \; x_i \in \mathbb{C}\right\}. \tag{20}$$

We recall the following lemma from [Ku2, Lemma 10].

Lemma 18. *The Springer morphism $\theta : G \to \mathfrak{g}$ for $G = \mathrm{SO}(2n+1)$ is given by*

$$g \mapsto \frac{g - E_B^{-1} g^t E_B}{2}, \;\; \textit{for } g \in G.$$

(Observe that this is the Cayley transform.)

From the description of the Springer morphism given above, we immediately get the following (cf. [Ku2, Corollary 11]:

Corollary 19. *Restricted to the maximal torus T_B as above, we get the following description of the Springer map θ:*

$$\theta(\mathbf{t}) = \mathrm{diag}(\bar{t}_1, \dots, \bar{t}_n, 0, -\bar{t}_n, \dots, -\bar{t}_1)., \;\; \textit{where } \bar{t}_i := \frac{t_i - t_i^{-1}}{2}.$$

The following result follows easily from Corollary 19 together with the description of the Weyl group (cf. [Ku2, Proposition 12]).

Proposition 20. *Let $f : T_B \to \mathbb{C}$ be a regular map. Then, $f \in \mathrm{Rep}^{\mathbb{C}}_{\mathrm{poly}}(G)$ if and only if the following is satisfied:*

There exists a symmetric polynomial $P_f(x_1, \dots, x_n)$ such that

$$f(\mathbf{t}) = P_f\left((\bar{t}_1)^2, \dots, (\bar{t}_n)^2\right), \ \text{for } \mathbf{t} \in T_B \ \text{given by (19).}$$

We recall the following result from [Ku2, Proposition 24].

Lemma 21. *Under the homomorphism $\xi^B : \mathrm{Rep}^{\mathbb{C}}_{\mathrm{poly}}(T_B) \to H^*(G/B, \mathbb{C})$ of Theorem 4 for $G = \mathrm{SO}(2n+1)$,*

$$\bar{t}_i \mapsto (\epsilon_{s_i} - \epsilon_{s_{i-1}}), \ \text{for any } 1 \le i < n,$$

and

$$\bar{t}_n \mapsto 2\epsilon_{s_n} - \epsilon_{s_{n-1}}.$$

Definition 22. For $1 \le r \le n$, let $\mathrm{OG}(r, 2n+1)$ be the set of r-dimensional isotropic subspaces of V' with respect to the quadratic form Q, i.e.,

$$\mathrm{OG}(r, 2n+1) := \{M \in \mathrm{Gr}(r, V') : Q(v) = 0, \ \forall\, v \in M\}.$$

Then, $\mathrm{OG}(r, 2n+1)$ is the quotient $\mathrm{SO}(2n+1)/P^B_r$ of $\mathrm{SO}(2n+1)$ by the standard maximal parabolic subgroup P^B_r with $\Delta \setminus \{\alpha_r\}$ as the set of simple roots of its Levi component L^B_r. (Again we take L^B_r to be the unique Levi subgroup of P^B_r containing T_B.) Then,

$$L^B_r \simeq \mathrm{GL}(r) \times \mathrm{SO}(2(n-r)+1). \tag{21}$$

In this case, by the identity (4) and Proposition 20,

$$\mathrm{Rep}^{\mathbb{C}}_{\mathrm{poly}}(L^B_r) \simeq \mathbb{C}_{\mathrm{sym}}[\bar{t}_1, \dots, \bar{t}_r] \otimes_{\mathbb{C}} \mathbb{C}_{\mathrm{sym}}[(\bar{t}_{r+1})^2, \dots, (\bar{t}_n)^2].$$

From now on we fix $k \ge 0$ and consider $\mathrm{OG}(n-k, 2n+1)$.

The Schubert varieties in $\mathrm{OG}(n-k, 2n+1)$ are again parametrized by $\mathcal{P}(k, n)$ consisting of k-strict partitions contained in the $(n-k)\times(n+k)$ rectangle. The codimension of this variety is equal to $|\lambda|$. Let $\tau_\lambda \in H^{2|\lambda|}(\mathrm{OG}(n-k, 2n+1), \mathbb{Z})$ denote the cohomology class Poincaré dual to the corresponding fundamental class $[X^B_\lambda]$ of the Schubert variety associated to λ.Thus, $\{\tau_\lambda\}_{\lambda\in\mathcal{P}(k,n)}$ gives the Schubert basis of $H^*(\mathrm{OG}(n-k, 2n+1), \mathbb{Z})$ (cf. [BKT1, §2.1]).

We have the following short exact sequence of vector bundles over $\mathrm{OG}(n-k, 2n+1)$:

$$0 \to \mathcal{S}_B \to \bar{\mathcal{E}}' \to \mathcal{Q}_B \to 0,$$

where $\bar{\mathcal{E}}'$ is the trivial bundle of rank $2n+1$, $\mathcal{S}_B$ is the tautological subbundle of rank $n-k$ and $\mathcal{Q}_B$ is the quotient bundle of rank $n+k+1$. Let $c_i = c_i(\mathcal{Q}_B)$ $(1 \le i \le n+k)$ denote the i^{th} Chern class of the quotient bundle $\mathcal{Q}$. (Observe that $c_{n+k+1} = 0$ as can be seen by pulling $\mathcal{Q}_B$ to $\mathrm{SO}(2n+1)/T_B$, where it admits a nowhere vanishing section given by the vector e_{n+1}.) Then, by [BKT1, §2.3],

$$c_i(\mathcal{Q}_B) = \begin{cases} \tau_i & if \quad 1 \le i \le k \\ 2\tau_i & if \quad k < i \le n+k, \end{cases} \tag{22}$$

where $\tau_i := \tau_{(i)}$ and (i) is the partition with single term i.

We have the following presentation of the cohomology ring due to [BKT1, Theorem 2.2(a)]. In the following we follow the convention that $\tau_0 = 1$ and $\tau_p = 0$ if $p < 0$ or $p > n + k$.

Theorem 23. *The cohomology ring $H^*(\mathrm{OG}(n-k, 2n+1), \mathbb{Z})$ is presented as a quotient of the ploynomial ring $\mathbb{Z}[\tau_1, \ldots, \tau_{n+k}]$ modulo the relations:*

$$(\bar{R}^p_{n,k})\ (n-k+1 \le p \le n): \qquad \det(\delta_{1+j-i}\tau_{1+j-i})_{1\le i,j\le p} = 0,$$

$$(\bar{R}^p_{n,k})\ (n+1 \le p \le n+k): \qquad \sum_{r=k+1}^{p} (-1)^r \tau_r \det(\delta_{1+j-i}\tau_{1+j-i})_{1\le i,j\le p-r} = 0,$$

and

$$(\bar{S}^{\,s}_{n,k})\ (k+1 \le s \le n): \qquad \tau_s^2 + \sum_{i=1}^{s} (-1)^i \delta_{s-i}\tau_{s+i}\tau_{s-i} = 0,$$

where $\delta_p = 1$ if $p \le k$ and $\delta_p = 2$ otherwise.

Proposition 24. *The map $\xi^{P^B_{n-k}} : \mathrm{Rep}^{\mathbb{C}}_{\mathrm{poly}}(L^B_{n-k}) \to H^*(\mathrm{OG}(n-k, 2n+1), \mathbb{C})$ of Theorem 4 under the decomposition* (21) *takes, for $1 \le i \le k$,*

$$e_i\left((\bar{t}_{n-k+1})^2, \ldots, (\bar{t}_n)^2\right) \mapsto c_i^2 + 2\sum_{j=1}^{i} (-1)^j c_{i+j} c_{i-j},$$

where $c_i = c_i(Q_B)$ and e_i is the i-th elementary symmetric function.

Proof. It follows by the same proof as that of the corresponding Propositionn 11 once we use the following two lemmas. □

Let $\mathrm{Fl}_B = G/B_B$ be the full flag variety for $G = \mathrm{SO}(2n+1)$. It consists of partial flags

$$\bar{F}_\bullet : \ \bar{F}_1 \subset \bar{F}_2 \subset \ldots \subset \bar{F}_n \subset E' := \mathbb{C}^{2n+1}, \text{ such that each } \bar{F}_j \text{ is istropic and } \dim F_j = j.$$

We can complete the partial flag to a full flag by taking $\bar{F}_{n+j} := \bar{F}^{\perp}_{n-j}$. The flags $\bar{F}_\bullet$ give rise to a sequence of tautological vector bundles over Fl_B:

$$\bar{\mathcal{F}}_1 \subset \bar{\mathcal{F}}_2 \subset \ldots \subset \bar{\mathcal{F}}_n \subset \mathcal{E}', \text{ with rank } \bar{\mathcal{F}}_j = j,$$

where $\mathcal{E}' : \mathrm{Fl}_B \times \mathbb{C}^{2n+1} \to \mathrm{Fl}_B$ is the trivial rank $2n+1$ vector bundle. For $1 \le j \le n$, define

$$\bar{x}_j := -c_1(\bar{\mathcal{F}}_j / \bar{\mathcal{F}}_{j-1}),$$

where $\bar{\mathcal{F}}_0$ is taken to be the vector bundle of rank 0.

The first part of the following lemma follows from equation (22). The 'In particular' statement follows from Lemma 21.

Lemma 25. *For $1 \leq j \leq n$, the Schubert divisor $\epsilon_{s_j} \in H^2(\mathrm{Fl}_B, \mathbb{Z})$ is given by*

$$\epsilon_{s_j} = -c_1(\bar{\mathcal{F}}_j) = \bar{x}_1 + \cdots + \bar{x}_j, \ \ \textit{for } j < n, \ \ \textit{and}$$

$$2\epsilon_{s_n} = -c_1(\bar{\mathcal{F}}_n) = \bar{x}_1 + \cdots + \bar{x}_n.$$

In particular, under ξ^B for $G = \mathrm{SO}(2n+1)$, $\bar{t}_j \mapsto \bar{x}_j$ for any $1 \leq j \leq n$.

For $1 \leq j \leq n$, let

$$\bar{Q}_j := \mathcal{E}'/\bar{\mathcal{F}}_j.$$

Observe that the orthogonal form gives an isomorphism of vector bundles:

$$\bar{Q}_j \simeq (\bar{\mathcal{F}}_j^{\perp})^*. \tag{23}$$

Lemma 26. *For $0 \leq j \leq n$, the following holds:*

$$c(\bar{Q}_j)c(\bar{Q}_j^*) = \prod_{p=j+1}^{n} (1 - \bar{x}_p)(1 + \bar{x}_p),$$

where c is the total Chern class.

Proof. The lemma follows by the same proof as that of the corresponding Lemma 13 once we observe that

$$c(\bar{\mathcal{F}}_n) = c(\bar{\mathcal{F}}_{n+1}),$$

which follows from the fact that $\bar{\mathcal{F}}_{n+1}/\bar{\mathcal{F}}_n$ pulled back to $\mathrm{SO}(2n+1)/T_B$ admits a nowhere vanishing section since the vector e_{n+1} is held fixed by T_B. ◻

Remark 27. Even though we do not need, the map $\xi^{P^B_{n-k}} : \mathrm{Rep}^{\mathbb{C}}_{\mathrm{poly}}(L^B_{n-k}) \to H^*(\mathrm{OG}(n-k, 2n+1), \mathbb{C})$ of Theorem 4 under the decomposition (21) takes for $1 \leq i \leq n-k$,

$$e_i\,(\bar{t}_1, \ldots, \bar{t}_{n-k}) \mapsto c_i(S_B) = \epsilon_{s_{n-k-i+1}\cdots s_{n-k}}, \ \ \text{if } k > 0,$$

$$e_i\,(\bar{t}_1, \ldots, \bar{t}_{n-k}) \mapsto c_i(S_B) = 2\epsilon_{s_{n-k-i+1}\cdots s_{n-k}}, \ \ \text{if } k = 0.$$

Definition 28. [Inverse Limit] Analogous to Definition 15, for any $k \geq 0$, define the *stable cohomology ring* [BKT2, §3.2] as

$$\mathbb{H}^*(\mathrm{OG}_k, \mathbb{Z}) = \varprojlim H^*(\mathrm{OG}(n-k, 2n+1), \mathbb{Z})$$

as the inverse limit (in the category of graded rings) of the inverse system

$$\cdots \leftarrow H^*(\mathrm{OG}(n-k, 2n+1), \mathbb{Z}) \xleftarrow{\bar{\pi}_n^*} H^*(\mathrm{OG}(n-k+1, 2n+3), \mathbb{Z}) \leftarrow \cdots,$$

where $\bar{\pi}_n : \mathrm{OG}(n-k, 2n+1) \hookrightarrow \mathrm{OG}(n-k+1, 2n+3)$ is given by $V \mapsto \bar{T}_n(V) \oplus \mathbb{C}e_{n+1}$ and $\bar{T}_n : \mathbb{C}^{2n+1} \to \mathbb{C}^{2n+3}$ is the linear embedding taking $e_i \mapsto e_i$ for $1 \leq i \leq n$, taking $e_{n+1} \mapsto e_{n+2}$ and taking $e_i \mapsto e_{i+2}$ for $n+2 \leq i \leq 2n+1$.

This ring has an additive basis consisting of Schubert classes τ_λ for each $k-$strict partition λ. The natural ring homorphism $\bar{\varphi}_{k,n} : \mathbb{H}^*(\mathrm{OG}_k, \mathbb{Z}) \to H^*(\mathrm{OG}(n-k, 2n+1), \mathbb{Z})$

takes τ_λ to τ_λ whenever λ fits in a $(n-k)\times(n+k)$ rectangle and to zero otherwise. In particular, $\bar\varphi_{k,n}$ is surjective. From the definition of the Chern classes $c_j = c_j^n(Q_B)$, it is easy to see that under the restriction map $\bar\pi_n^* : H^*(\mathrm{OG}(n-k+1, 2n+3), \mathbb{Z}) \to H^*(\mathrm{OG}(n-k, 2n+1), \mathbb{Z}), c_j^{n+1} \mapsto c_j^n$ for $1 \le j \le n+k$ and $c_{n+k+1}^{n+1} \mapsto 0$.

From the presentation of the ring $H^*(\mathrm{OG}(n-k, 2n+1), \mathbb{Z})$ (Theorem 23), $\mathbb{H}^*(\mathrm{OG}_k, \mathbb{Z})$ is isomorphic with the polynomial ring $\mathbb{Z}[\tau_1, \tau_2, \ldots]$ modulo the relations:

$$(\bar S^s)\ (s > k): \qquad \tau_s^2 + \sum_{i=1}^{s} (-1)^i \delta_{s-i} \tau_{s+i} \tau_{s-i} = 0. \tag{24}$$

Take $k \ge 1$. Recall from Proposition 20 that

$$\mathrm{Rep}^{\mathbb{C}}_{\mathrm{poly}}(\mathrm{SO}(2k+1)) \simeq \mathbb{C}_{\mathrm{sym}}\left[(\bar h_1)^2, \ldots, (\bar h_k)^2\right].$$

Define a ring homomorphism (for any $1 \le k \le n$)

$$\bar\iota_k^n : \mathrm{Rep}^{\mathbb{C}}_{\mathrm{poly}}(\mathrm{SO}(2k+1)) \to \mathrm{Rep}^{\mathbb{C}}_{\mathrm{poly}}(L^B_{n-k})$$

by taking $f(\bar{\mathbf{h}}) \mapsto 1 \otimes f(\bar{\mathbf{t}})$, where $\bar{\mathbf{h}} := (\bar h_1, \ldots, \bar h_k)$, $\bar{\mathbf{t}} := (\bar t_{n-k+1}, \ldots, \bar t_n)$ and $f(\bar{\mathbf{t}})$ is the same polynomial written in the $\bar{\mathbf{t}}$-variables uner the transformation $\bar h_p \mapsto \bar t_{n-k+p}$. This gives rise to the map $\bar\xi_{n,k} := \xi^{P^B_{n-k}} \circ \bar\iota_k^n : \mathrm{Rep}^{\mathbb{C}}_{\mathrm{poly}}(\mathrm{SO}(2k+1)) \to H^*(\mathrm{OG}(n-k, 2n+1), \mathbb{C})$. Consider the following diagram, which is commutative because of Proposition 24.

$$\begin{array}{ccc}
 & & \uparrow \bar\pi^*_{n-1} \\
\mathrm{Rep}^{\mathbb{C}}_{\mathrm{poly}}(\mathrm{SO}(2k+1)) & \xrightarrow{\bar\xi_{n,k}} & H^*(\mathrm{OG}(n-k, 2n+1), \mathbb{C}) \\
 & \searrow^{\bar\xi_{n+1,k}} & \uparrow \bar\pi^*_{n} \\
 & & H^*(\mathrm{OG}(n-k+1, 2n+3), \mathbb{C}) \\
 & & \uparrow \bar\pi^*_{n+1}
\end{array}$$

The compatible ring homomorphisms $\bar\xi_{n,k} : \mathrm{Rep}^{\mathbb{C}}_{\mathrm{poly}}(\mathrm{SO}(2k+1)) \to H^*(\mathrm{OG}(n-k, 2n+1), \mathbb{C})$ combine to give a ring homomorphism

$$\bar\xi_k : \mathrm{Rep}^{\mathbb{C}}_{\mathrm{poly}}(\mathrm{SO}(2k+1)) \to \mathbb{H}^*(\mathrm{OG}_k, \mathbb{C}).$$

The following theorem is our second main result of the paper, which is analogous to Theorem 16.

Theorem 29. *Let $k \ge 1$ be an integer. The above ring homomorphism $\bar\xi_k : \mathrm{Rep}^{\mathbb{C}}_{\mathrm{poly}}(\mathrm{SO}(2k+1)) \to \mathbb{H}^*(\mathrm{OG}_k, \mathbb{C})$ takes the generators*

$$e_i\left((\bar h_1)^2, \ldots, (\bar h_k)^2\right) \mapsto c_i^2 + 2\sum_{j=1}^{i} (-1)^j c_{i+j} c_{i-j}, \quad \textit{for any } 1 \le i \le k, \tag{25}$$

where $c_i := c_i(Q_B)$.

In particular, $\bar\xi_k$ is injective.

Proof. The first part follows from Proposition 24.

We next prove the injectivity of $\bar{\xi}_k$:

By Proposition 20, $\mathrm{Rep}^{\mathbb{C}}_{\mathrm{poly}}(\mathrm{SO}(2k+1))$ is a polynomial ring over $\mathbb{C}$ generated by $\{e_1(\bar{\mathbf{h}}), \ldots, e_k(\bar{\mathbf{h}})\}$. Let $\mathrm{Rep}^{\mathbb{Z}}_{\mathrm{poly}}(\mathrm{SO}(2k+1))$ be the polynomial subring over $\mathbb{Z}$ generated by $\{e_1(\bar{\mathbf{h}}), \ldots, e_k(\bar{\mathbf{h}})\}$.

Let $\bar{\mathbb{H}}^*(\mathrm{OG}_k, \mathbb{Z}) \subset \mathbb{H}^*(\mathrm{OG}_k, \mathbb{Z})$ be the subring generated by $\{c_i\}_{i\geq 1}$. Then, by the identity (22),

$$\mathbb{C} \otimes_{\mathbb{Z}} \bar{\mathbb{H}}^*(\mathrm{OG}_k, \mathbb{Z}) = \mathbb{C} \otimes_{\mathbb{Z}} \mathbb{H}^*(\mathrm{OG}_k, \mathbb{Z}) = \mathbb{H}^*(\mathrm{OG}_k, \mathbb{C}). \tag{26}$$

Then, by the equation (25),

$$\bar{\xi}_k(\mathrm{Rep}^{\mathbb{Z}}_{\mathrm{poly}}(\mathrm{SO}(2k+1))) \subset \bar{\mathbb{H}}^*(\mathrm{OG}_k, \mathbb{Z}).$$

Thus, on restriction, we get the ring homomorphism

$$\bar{\xi}^{\mathbb{Z}}_k : \mathrm{Rep}^{\mathbb{Z}}_{\mathrm{poly}}(\mathrm{SO}(2k+1)) \to \bar{\mathbb{H}}^*(\mathrm{OG}_k, \mathbb{Z}).$$

Observe further that $\bar{\xi}^{\mathbb{Z}}_k$ is a homomorphism of graded rings if we assign degree $4i$ to each e_i (and the standard cohomological degree to $\bar{\mathbb{H}}^*(\mathrm{OG}_k, \mathbb{Z})$). Let $\bar{K}$ be the Kernel of $\bar{\xi}^{\mathbb{Z}}_k$. Since $\mathbb{H}^*(\mathrm{OG}_k, \mathbb{Z})$ is a free $\mathbb{Z}$-module of finite rank in each degree and hence so is $\bar{\mathbb{H}}^*(\mathrm{OG}_k, \mathbb{Z})$, the induced homomorphism

$$\mathbb{Z}/(2) \otimes_{\mathbb{Z}} \bar{K} \to \mathbb{Z}/(2) \otimes_{\mathbb{Z}} \mathrm{Rep}^{\mathbb{Z}}_{\mathrm{poly}}(\mathrm{SO}(2k+1)) \text{ is injective.} \tag{27}$$

We next observe that the induced homomorphism

$$\mathbb{Z}/(2) \otimes_{\mathbb{Z}} \mathrm{Rep}^{\mathbb{Z}}_{\mathrm{poly}}(\mathrm{SO}(2k+1)) \to \mathbb{Z}/(2) \otimes_{\mathbb{Z}} \bar{\mathbb{H}}^*(\mathrm{OG}_k, \mathbb{Z}) \text{ is injective.} \tag{28}$$

To prove this, observe that

$$\mathbb{Z}/(2) \otimes_{\mathbb{Z}} \mathrm{Rep}^{\mathbb{Z}}_{\mathrm{poly}}(\mathrm{SO}(2k+1)) \simeq \mathbb{Z}/(2)[e_1, \ldots, e_k]. \tag{29}$$

Moreover, by the defining relations $(\bar{S}^s)\,(s > k)$ of $\mathbb{H}^*(\mathrm{OG}_k, \mathbb{Z})$ as in equation (24) together with the identity (22), we can rewrite the equation (24) as:

$$(\hat{S}^s)\ (s > k): \qquad c_s^2 + 2\sum_{i=1}^{s}(-1)^i c_{s+i}c_{s-i} = 0.$$

Thus,

$$\mathbb{Z}/(2) \otimes_{\mathbb{Z}} \bar{\mathbb{H}}^*(\mathrm{OG}_k, \mathbb{Z}) \simeq \mathbb{Z}/(2)[c_1, \ldots, c_k] \otimes \frac{\mathbb{Z}/(2)[c_{k+1}, c_{k+2} \ldots]}{\langle c^2_{k+1}, c^2_{k+2} \ldots \rangle}. \tag{30}$$

Moreover, under the above identifications (29) and (30), by the first part of the theorem, the ring homomorphism $\bar{\xi}^{\mathbb{Z}}_k$ modulo 2 is given by

$$e_i \mapsto c_i^2, \quad \text{for any } 1 \leq i \leq k.$$

In particular, it is injective. From this we obtain that

$$\mathbb{Z}/(2) \otimes_{\mathbb{Z}} \bar{K} = 0.$$

But, since $\bar{K}$ is a finitely generated torsionfree $\mathbb{Z}$-module in each graded degree (thus free) we get that

$$\bar{K} = 0.$$

Since $\mathbb{C}$ is a torsionfree $\mathbb{Z}$-module, by the equation (26), this clearly gives the injectivity of $\bar{\xi}_k$ proving the theorem. □

Remark 30. The ring homomorphism $\bar{\xi}_k : \mathrm{Rep}^{\mathbb{C}}_{\mathrm{poly}}(\mathrm{SO}(2k+1)) \to \mathbb{H}^*(\mathrm{OG}_k, \mathbb{C})$ of the above Theorem 29 is *not* surjective, as can be easily seen since the domain is a finitely generated $\mathbb{C}$-algebra (by Proposition 20) whereas the range is not (for otherwise for each n, $H^*(\mathrm{OG}(n-k, 2n+1), \mathbb{C})$ would be generated by a fixed finite number of generators independent of n).

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