ON THE FACES OF THE TENSOR CONE OF SYMMETRIZABLE KAC-MOODY LIE ALGEBRAS

SHRAWAN KUMAR AND NICOLAS RESSAYRE

Abstract. In this paper, we are interested in the decomposition of the tensor product of two representations of a symmetrizable Kac-Moody Lie algebra $\mathfrak{g}$, or more precisely in the tensor cone of $\mathfrak{g}$. As usual, we parametrize the integrable, highest weight (irreducible) representations of $\mathfrak{g}$ by their highest weights. Then, the triples of such representations such that the last one is contained in the tensor product of the first two is a semigroup. This semigroup generates a rational convex cone $\Gamma(\mathfrak{g})$ called tensor cone. If $\mathfrak{g}$ is finite-dimensional, $\Gamma(\mathfrak{g})$ is a polyhedral convex cone. In 2006, Belkale and the first author described this cone by an explicit finite list of inequalities. In 2010, this list of inequalities was proved to be irredundent by the second author: each such inequality corresponds to a codimension one face. In general, $\Gamma(\mathfrak{g})$ is neither polyhedral, nor closed. Brown and the first author obtained a list of inequalities that describe $\Gamma(\mathfrak{g})$ conjecturally. Here, we prove that each of these inequalities corresponds to a codimension one face of $\Gamma(\mathfrak{g})$.

A propos des faces du cône tensoriel d'une algèbre de Kac-Moody symétrisable

Dans cet article, nous nous intéressons à la décomposition du produit tensoriel de deux représentations d'une algèbre de Kac-Moody symétrisable $\mathfrak{g}$, et plus précisément au cône tensoriel de $\mathfrak{g}$. Comme d'habitude, nous paramétrons les représentations irréductibles intégrables et de plus haut poids par ledit plus haut poids. Alors, les triplets de telles représentations telles que la troisième s'injecte dans le produit tensoriel des deux premières est un semi-groupe. Ces triplets engendrent un cône convexe rationnel $\Gamma(\mathfrak{g})$ que nous appelons le *cone tensoriel*. Lorsque $\mathfrak{g}$ est de dimension finie, $\Gamma(\mathfrak{g})$ est un cône convexe polyédral. En 2006, Belkale et le premier auteur ont décrit ce cône par une liste finie explicite d'inégalités linéaires. En 2010, le second auteur a montré que cette liste d'inégalités n'est pas redondante : chaque inégalité correspond à une face de codimension un. En général, $\Gamma(\mathfrak{g})$ n'est ni fermé, ni polyédral. Brown et le premier auteur ont obtenu une liste d'inégalités qui décrit conjecturalement le cône $\Gamma(\mathfrak{g})$. Nous montrons ici que chacune de ces inégalités correspond à une face de codimension un de $\Gamma(\mathfrak{g})$.

1. Introduction

Let A be a symmetrizable irreducible GCM (generalized Cartan matrix) of size $l+1$. Let $\mathfrak{h} \supset \{\alpha_0^\vee, \dots, \alpha_l^\vee\}$ and $\mathfrak{h}^* \supset \{\alpha_0, \dots, \alpha_l\} =: \Delta$ be a realization of A over the complex numbers $\mathbb{C}$. We fix an integral form $\mathfrak{h}_\mathbb{Z} \subset \mathfrak{h}$ containing each $\alpha_i^\vee$, such that $\mathfrak{h}_\mathbb{Z}^* := \mathrm{Hom}(\mathfrak{h}_\mathbb{Z}, \mathbb{Z})$ contains Δ and such that $\mathfrak{h}_\mathbb{Z}/\bigoplus_i \mathbb{Z}\alpha_i^\vee$ is torsion-free. Set $\mathfrak{h}_\mathbb{Q}^* = \mathfrak{h}_\mathbb{Z}^* \otimes \mathbb{Q} \subset \mathfrak{h}^*$, $P_{+,\mathbb{Q}} := \{\lambda \in \mathfrak{h}_\mathbb{Q}^* : \langle \alpha_i^\vee, \lambda\rangle \geq 0 \quad \forall i\}$, and $P_+ = \mathfrak{h}_\mathbb{Z}^* \cap P_{+,\mathbb{Q}}$.

Let $\mathfrak{g} = \mathfrak{g}(A)$ be the associated Kac-Moody Lie algebra over $\mathbb{C}$ with Cartan subalgebra $\mathfrak{h}$. For $\lambda \in P_+$, $L(\lambda)$ denotes the (irreducible) integrable, highest weight representation of $\mathfrak{g}$ with highest weight λ. Define the (rational) *tensor cone* as

$$\Gamma(\mathfrak{g}) := \{(\lambda_1, \lambda_2, \mu) \in P_{+,\mathbb{Q}}^3 : \exists N \geq 1 \,\text{such that}\, L(N\mu) \subset L(N\lambda_1) \otimes L(N\lambda_2)\}.$$

The aim of this paper is to describe facets (codimension one faces) of this cone. Before describing our result, we recall from [BK14] a conjectural description of $\Gamma(\mathfrak{g})$, due to Brown and the first author. We need some more notation.

Fix $\{x_0, \dots, x_l\} \in \mathfrak{h}$ to be dual of the roots: $\langle \alpha_j, x_i\rangle = \delta_i^j$. Let $Q = \bigoplus_{i=0}^l \mathbb{Z}\alpha_i$ denote the root lattice. Let $X = G/B$ be the standard full KM-flag variety associated to $\mathfrak{g}$, where G is the 'minimal' Kac-Moody group with Lie algebra $\mathfrak{g}$ and B is the standard Borel subgroup of G. For w in the Weyl group W of G, let $X_w = \overline{BwB/B} \subset X$ be the corresponding Schubert variety. Let $\{\varepsilon^w\}_{w \in W} \subset \mathrm{H}^*(X, \mathbb{Z})$ be the (Schubert) basis dual (with respect to the standard pairing) to the basis of the singular homology of X given by the fundamental classes of X_w.

Let $P \supset B$ be a (standard) parabolic subgroup and let $X_P := G/P$ be the corresponding partial flag variety. Let W_P be the Weyl group of P (which is, by definition, the Weyl group of the Levi L of P) and let W^P be the set of minimal length representatives of cosets in W/W_P. The projection map $X \to X_P$ induces an injective homomorphism $\mathrm{H}^*(X_P, \mathbb{Z}) \to \mathrm{H}^*(X, \mathbb{Z})$ and $\mathrm{H}^*(X_P, \mathbb{Z})$ has the Schubert basis $\{\varepsilon_P^w\}_{w \in W^P}$ such that ε_P^w goes to ε^w for any $w \in W^P$. As defined by Belkale and the first author [BK06, §6] in the finite-dimensional case and extended by the first author in [Kum08] for any symmetrizable Kac-Moody case, there is a new deformed product $\odot_0$ in $\mathrm{H}^*(X_P, \mathbb{Z})$, which is commutative and associative. Now, we are ready to state Brown-Kumar's conjecture [BK14].

Conjecture 1.1. *Let $\mathfrak{g}$ be any indecomposable symmetrizable Kac-Moody Lie algebra and let $(\lambda_1, \lambda_2, \mu) \in P_+^3$. Assume further that none* *of* λ_j *is* W*-invariant and* $\mu - \sum_{j=1}^{s} \lambda_j \in Q$*. Then, the following are equivalent:*

(a) $(\lambda_1, \lambda_2, \mu) \in \Gamma(\mathfrak{g})$*.*

(b) For every standard maximal parabolic subgroup P *in* G *and every choice of triples* $(w_1, w_2, v) \in (W^P)^3$ *such that* ε_P^v *occurs with coefficient* 1 *in the deformed product*

$$\varepsilon_P^{w_1} \odot_0 \varepsilon_P^{w_2} \in \big(\mathrm{H}^*(X_P, \mathbb{Z}), \odot_0\big),$$

the following inequality holds:

$$(I^P_{(w_1,w_2,v)}) \qquad \lambda_1(w_1 x_P) + \lambda_2(w_2 x_P) - \mu(v x_P) \geq 0,$$

where α_{i_P} *is the (unique) simple root not in the Levi of* P *and* $x_P := x_{i_P}$*.*

Note that if λ_1 is W-invariant, $L(\lambda_1)$ is one-dimensional and hence $L(\lambda_1) \otimes L(\lambda_2)$ is irreducible.

In the case that $\mathfrak{g}$ is a semisimple Lie algebra, Conjecture 1.1 was proved by Belkale and the first author in [BK06]. The following result is due to the second author.

Theorem 1.2. [Res21] *In the case that* $\mathfrak{g}$ *is affine untwisted, Conjecture 1.1 is true.*

The conjecture in the general symmetrizable case is still open. But it is conceivable that the inductive proof in the case of affine $\mathfrak{g}$ obtained by the second author might be amenable to handle the general symmetrizable case.

Let us come back to the case that $\mathfrak{g}$ is semisimple. Then, $\Gamma(\mathfrak{g})$ is a closed convex polyhedral cone, and Conjecture 1.1 (Belkale-Kumar's theorem) describes $\Gamma(\mathfrak{g})$ in $(\mathfrak{h}^*_{\mathbb{Q}})^3$ by (finitely many) explicit inequalities. (Recall that a rational cone $\mathcal{C}$ is called *convex* if for $x, y \in \mathcal{C}$ and $0 < \alpha < 1, \alpha \in \mathbb{Q}$, $\alpha x + (1-\alpha) y \in \mathcal{C}$.) In the case of $\mathfrak{g} = \mathfrak{sl}_n$, a larger set of inequalities describing $\Gamma(\mathfrak{g})$ was conjectured by Horn [Hor62] and proved by Klyachko [Kly98] (combining the saturation result of Knutson-Tao [KT99]). A larger set of inequalities describing $\Gamma(\mathfrak{g})$ for any semisimple $\mathfrak{g}$ was known earlier (see [BS00]). The irredundancy of the above set of inequalities $I^P_{(w_1,w_2,v)}$ was proved by Knutson-Tao-Woodward in type A [KTW04] and by the second author in general [Res10]. (See [Kum14, §1] for more details on the history.) The irredundancy assertion is the statement that each inequality $I^P_{(w_1,w_2,v)}$ in Conjecture 1.1 corresponds to a face of $\Gamma(\mathfrak{g})$ of codimension one. The aim of this paper is to extend this result to any symmetrizable Kac-Moody Lie algebra. We, in fact, prove the following (stronger) result for any (not necessarily maximal) standard parabolic subgroup P.

Theorem 1.3. *Let $\mathfrak{g}$ be any indecomposable symmetrizable Kac-Moody Lie algebra. Let P be a standard parabolic subgroup in G and let $(w_1, w_2, v) \in (W^P)^3$ be a triple such that ε_P^v occurs with coefficient* 1 *in the deformed product*

$$\varepsilon_P^{w_1} \odot_0 \varepsilon_P^{w_2} \in \big(\mathrm{H}^*(X_P, \mathbb{Z}), \odot_0\big).$$

Then, the set of $(\lambda_1, \lambda_2, \mu) \in \Gamma(\mathfrak{g})$ such that for all $\alpha_j \notin \Delta(P)$,

$$(I^j_{(w_1,w_2,v)}) \qquad \lambda_1(w_1 x_j) + \lambda_2(w_2 x_j) - \mu(v x_j) = 0$$

has codimension $\sharp(\Delta \backslash \Delta(P))$ in $\Gamma(\mathfrak{g})$, where $\Delta(P) \subset \Delta$ is the set of simple roots of the Levi subgroup L of P.

Let $\mathcal{C}$ denote the cone determined by the inequalities in Conjecture 1.1. For P maximal, Theorem 1.3 implies that if one removes any of the inequalities $I^P_{(w_1,w_2,v)}$, the cone thus obtained is strictly larger than $\mathcal{C}$.

Theorem 1.3 implies that $\mathcal{C}$ is locally polyhedral. This property of $\mathcal{C}$ plays an important role in the inductive proof of Theorem 1.2 from [Res21]. (Note that in [Res21], the local polyhedrality is proved in a totally different way.) As a consequence, one can hopefully think about Theorem 1.3 as a first step towards a proof of Conjecture 1.1.

Combining Theorems 1.2 and 1.3, we get the following.

Corollary 1.4. *For any untwisted affine Kac-Moody Lie algebra $\mathfrak{g}$, the inequalities $I^P_{(w_1,w_2,v)}$ in Conjecture 1.1 give an irredundent and complete set of inequalities determining the cone $\Gamma(\mathfrak{g})$.*

To prove Theorem 1.3 we will use (geometric) Theorem 1.5 below. Let us introduce some more notation.

Fix a standard parabolic subgroup P of G. For $w \in W^P$, we set

$$\Delta^-(w) = \{\alpha \in \Delta \,:\, \ell(s_\alpha w) = \ell(w) - 1\},$$

and

$$\Delta^+(w) = \{\alpha \in \Delta \,:\, \ell(s_\alpha w) = \ell(w) + 1 \text{ and } s_\alpha w \in W^P\},$$

where s_α is the (simple) reflection corresponding to the (simple) root α. It is easy to see that for any $\alpha \in \Delta^-(w), s_\alpha w \in W^P$.

Let B^- denote the Borel subgroup of G opposite to B. Consider the flag ind-variety $\mathcal{X} := (\mathrm{G}/B^-)^2 \times \mathrm{G}/B$ and $\mathrm{Pic}^{\mathrm{G}}(\mathcal{X})$ the group of G-linearized line bundles on $\mathcal{X}$. For $\lambda \in \mathfrak{h}^*_{\mathbb{Z}}$, denote the line bundle $\mathcal{L}^-(\lambda) := G \times^{B^-} \mathbb{C}_\lambda$ over G/B^- (resp. $\mathcal{L}(\lambda) := G \times^B \mathbb{C}_{-\lambda}$ over G/B) associated to the principal B^--bundle $G \to G/B^-$ (resp. the B-bundle $G \to G/B$) via the one-dimensional representation $\mathbb{C}_\lambda$ of B^- given by the character e^{λ} uniquely extended to a character of B^- (resp. the representation $\mathbb{C}_{-\lambda}$ of B given by the character $e^{-\lambda}$).

Fix $(\lambda_1, \lambda_2, \mu) \in P_+^3$. By an analogue of the Borel-Weil theorem for any Kac-Moody group G (cf. [Kum02, Corollary 8.3.12]), the G-linearized line bundle $\mathcal{L} := \mathcal{L}^-(\lambda_1) \boxtimes \mathcal{L}^-(\lambda_2) \boxtimes \mathcal{L}(\mu)$ on $\mathcal{X}$ is such that the dimension of the space $\mathrm{H}^0(\mathcal{X}, \mathcal{L})^G$ of G-invariant sections is the multiplicity of $L(\mu)$ in $L(\lambda_1) \otimes L(\lambda_2)$ (cf. [BK14, Proof of Theorem 3.2]). From this we see that $\Gamma(\mathfrak{g})$ is a convex subset of $P^3_{+,\mathbb{Q}}$.

Fix $(w_1, w_2, v) \in (W^P)^3$ as in Theorem 1.3 and let $L \supset T$ denote the standard Levi subgroup of P, where T is the standard maximal torus of G with Lie algebra $\mathfrak{h}$. The base point B/B in G/B is denoted by $\underline{o}$. Similarly, $\underline{o}^- = B^-/B^-$. Set

$$x_0 = (w_1^{-1}\underline{o}^-, w_2^{-1}\underline{o}^-, v^{-1}\underline{o}) \in \mathcal{X}.$$

For $\alpha \in \Delta^+(w_1)$, we set

$$x_{\alpha,1} = (w_1^{-1}s_\alpha\underline{o}^-, w_2^{-1}\underline{o}^-, v^{-1}\underline{o}) \in \mathcal{X}.$$

Similarly, we define $x_{\alpha,2}$ associated to $\alpha \in \Delta^+(w_2)$. For $\alpha \in \Delta^-(v)$, we set

$$x_{\alpha,3} = (w_1^{-1}\underline{o}^-, w_2^{-1}\underline{o}^-, v^{-1}s_\alpha\underline{o}) \in \mathcal{X}.$$

For any (α, i) as above, we denote by $\ell_{\alpha,i}$ the unique T-stable curve in $\mathcal{X}$ containing x_0 and $x_{\alpha,i}$; then $\ell_{\alpha,i} \simeq \mathbb{P}^1$ and x_0 and $x_{\alpha,i}$ are the two T-fixed points in $\ell_{\alpha,i}$. Explicitly,

$$\ell_{\alpha,1} = \left(w_1^{-1}P_\alpha^-\underline{o}^-, w_2^{-1}\underline{o}^-, v^{-1}\underline{o}\right) \subset \mathcal{X},$$

where P_α^- is the minimal (opposite) parabolic subgroup containing B^- and s_α. Similarly, $\ell_{\alpha,2}$ and $\ell_{\alpha,3}$ can be described explicitly.

Consider now

$$C = Lw_1^{-1}\underline{o}^- \times Lw_2^{-1}\underline{o}^- \times Lv^{-1}\underline{o},$$

acted on by L diagonally.

Theorem 1.5. *Let P and $(w_1, w_2, v) \in (W^P)^3$ be as in Theorem 1.3. Fix $(\lambda_1, \lambda_2, \mu) \in (\mathfrak{h}^*_{\mathbb{Z}})^3$ such that*

$$\forall \alpha_j \notin \Delta(P), \qquad \lambda_1(w_1 x_j) + \lambda_2(w_2 x_j) - \mu(v x_j) = 0.$$

Let $\mathcal{L} := \mathcal{L}^-(\lambda_1) \boxtimes \mathcal{L}^-(\lambda_2) \boxtimes \mathcal{L}(\mu)$ denote the associated line bundle on $\mathcal{X}$. We assume that, for any $i = 1, 2$ and $\alpha \in \Delta^+(w_i)$, the restriction of $\mathcal{L}$ to $\ell_{\alpha,i}$ is nonnegative. Similarly, we assume that for any $\alpha \in \Delta^-(v)$ the restriction of $\mathcal{L}$ to $\ell_{\alpha,3}$ is nonnegative.

Then, the restriction map induces an isomorphism:

$$\mathrm{H}^0(\mathcal{X}, \mathcal{L})^G \simeq \mathrm{H}^0(C, \mathcal{L})^L.$$

To prove Theorem 1.3, we have to produce line bundles $\mathcal{L}$ on $\mathcal{X}$ having nonzero G-invariant sections and satisfying the equalities $(I^i_{(w_1,w_2,v)})$. To do this we start with a line bundle $\mathcal{M}$ on $\mathcal{X}$ whose restriction $\mathcal{M}_{|C}$ admits an L-invariant section σ. Now, we want to extend σ to a regular G-invariant section on $\mathcal{X}$. The first step is to extend σ to a rational G-invariant section. Even though this rational section can have poles, we are able to kill them by adding an explicit line bundle $\mathcal{L}'$ to $\mathcal{M}$. An informed reader will notice that the strategy is similar to the one used by the second author in [Res10]. Nevertheless, there are numerous difficulties because of infinite-dimensional phenomena. For example, we have no abstract construction of line bundles arising from divisors; the order of a pole along a divisor is not so easy to define (and even if it is defined, such an order could be infinite) etc. In this paper, we overcome these difficulties by making various constructions more explicit which extend to our infinite-dimensional situation.

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2. Zariski's main theorem

We recall a consequence of the Zariski's main theorem for our later use.

Proposition 2.1. *Let $f : Y \longrightarrow Z$ be a proper birational morphism between two quasiprojective irreducible varieties. We assume that we have an open subset $\tilde{Y}$ of Y such that $f(Y \backslash \tilde{Y})$ has codimension at least two in Z and that Z is normal. Let $\mathcal{L}$ be a line bundle over Z.*

Then, $f^* : \mathrm{H}^0(Z, \mathcal{L}) \longrightarrow \mathrm{H}^0(Y, f^*(\mathcal{L}))$ and the restriction map $r : \mathrm{H}^0(Y, f^*(\mathcal{L})) \longrightarrow \mathrm{H}^0(\tilde{Y}, f^*(\mathcal{L}))$ are both isomorphisms.

Proof. To prove that f^* is an isomorphism, use the proof of Zariski's main theorem as in [Har77, Chap. III, Corollary 11.4].

To prove that r is an isomorphism, consider the following commutative diagram:

$$\begin{tikzcd}
\mathrm{H}^0(Z,\mathcal{L}) \arrow[rr, "\beta", "\sim"'] \arrow[d, "f^*"', "\wr"] & & \mathrm{H}^0(Z\backslash f(Y\backslash\tilde{Y}),\mathcal{L}) \arrow[d, "f^*", "\wr"'] \\
\mathrm{H}^0(Y,f^*\mathcal{L}) \arrow[rr, "\sim", "r_1"'] \arrow[dr, "r"'] & & \mathrm{H}^0(Y\backslash f^{-1}(f(Y\backslash\tilde{Y})),f^*\mathcal{L}) \\
& \mathrm{H}^0(\tilde{Y},f^*\mathcal{L}). \arrow[ur, "r_2"'] &
\end{tikzcd}$$

In the above diagram, β is an isomorphism since $f(Y\backslash\tilde{Y})$ is of codimension ≥ 2 and Z is normal. Thus, r_1 is an isomorphism. Further, since r_1 is an isomorphism and r and r_2 are injective, r is an isomorphism as well. $\square$

3. The span of the cone

Before being interested in the faces of $\Gamma(\mathfrak{g})$, we describe the span of it.

Proposition 3.1. *The tensor cone* $\Gamma(\mathfrak{g})$ *(which is, by definition, a rational cone) has nonempty interior in the following rational vector space*

$$E = E_{\mathfrak{g}} := \{(\lambda_1,\lambda_2,\mu) \in (\mathfrak{h}^*_{\mathbb{Q}})^3\ :\ \lambda_1+\lambda_2-\mu \in \mathrm{Span}_{\mathbb{Q}}(\Delta)\}.$$

In particular, E *has dimension* $2\dim\mathfrak{h} + \sharp\Delta$.

Proof. If $(\lambda_1,\lambda_2,\mu) \in \Gamma(\mathfrak{g})$ then some integral multiple $N(\lambda_1+\lambda_2-\mu)$ belongs to the root lattice. Hence,

$$\Gamma(\mathfrak{g}) \subset E. \tag{1}$$

Note that, for λ,μ in P^+, the point

$$(\lambda,\mu,\lambda+\mu) \in \Gamma(\mathfrak{g}). \tag{2}$$

We claim that for any simple root $\alpha_i \in \Delta$,

$$(\rho,\rho,2\rho-\alpha_i) \in \Gamma(\mathfrak{g}), \tag{3}$$

where $\rho \in \mathfrak{h}^*_{\mathbb{Q}}$ is any element satisfying $\rho(\alpha_i^\vee) = 1$ for all the simple coroots $\alpha_i^\vee$. Indeed, fix a highest weight vector v_+ in $L(\rho)$ and a nonzero e_j (resp. f_j) in $\mathfrak{g}_{\alpha_j}$ (resp. $\mathfrak{g}_{-\alpha_j}$) for any simple root α_j, where $\mathfrak{g}_\alpha$ denotes the corresponding root space. Consider the element in $L(\rho)\otimes L(\rho)$:

$$v = f_iv_+\otimes v_+ - v_+\otimes f_iv_+.$$

Clearly, $e_j v_+ = 0$ for any $j \neq i$. Also,

$$\begin{aligned} e_i v &= (e_i f_i v_+) \otimes v_+ - v_+ \otimes (e_i f_i v_+) \\ &= \alpha_i^\vee v_+ \otimes v_+ - v_+ \otimes \alpha_i^\vee v_+ \\ &= 0. \end{aligned}$$

It follows that v is a highest weight vector. But its weight is $2\rho - \alpha_i$, proving (3). Combined with (2), we get

$$(0, 0, \alpha_i) \in \langle \Gamma(\mathfrak{g}) \rangle, \qquad \forall \alpha_i \in \Delta, \tag{4}$$

where $\langle \Gamma(\mathfrak{g}) \rangle$ is the $\mathbb{Q}$-span of $\Gamma(\mathfrak{g})$ in $(\mathfrak{h}_{\mathbb{Q}}^*)^3$. Now, by (2) and (4), $\Gamma(\mathfrak{g})$ spans E. ☐

4. Construction of line bundles

Consider a subvariety $Z \subset \mathcal{X}$. If G and so $\mathcal{X}$ is finite-dimensional, Z can be realized as the zero set of a section of some line bundle on $\mathcal{X}$ if and only if Z has codimension one. If G is not finite-dimensional, then $\mathcal{X}$ is only an ind-variety and the codimension is not so easy to define. Moreover, even if there exists a filtration $\mathcal{X} = \bigcup_n \mathcal{X}_n$ by finite-dimensional closed subvarieties such that $Z \cap \mathcal{X}_n$ has codimension one in $\mathcal{X}_n$, Z is not necessarily the zero locus of a section of some line bundle on $\mathcal{X}$.

Nevertheless, if $Z = F_{\alpha,i}$ or $Z = E_{w_1,w_2,v}$ as defined by formula (5) (resp. (11)) below, we prove in this section that Z is the zero locus of a section of some line bundle.

4.1. **First divisors.** Fix once and for all fundamental weights $\varpi_{\alpha_0}, \dots, \varpi_{\alpha_l}$ in $\mathfrak{h}_{\mathbb{Z}}^*$ such that $\langle \varpi_{\alpha_i}, \alpha_j^\vee \rangle = \delta_i^j$.

Let M be a $\mathfrak{g}$-module such that, under the action of $\mathfrak{h}$, M decomposes as $\bigoplus_{\mu \in \mathfrak{h}^*} M_\mu$ with finite-dimensional weight spaces M_μ. Set $M^\vee = \bigoplus_\mu M_\mu^*$: it is a $\mathfrak{g}$-submodule of the full dual space M^*.

Recall that $\mathcal{X} = (G/B^-)^2 \times G/B$ and $\underline{o}^\pm = B^\pm/B^\pm$. Consider, for $\alpha \in \Delta$ and $i = 1, 2$,

$$F_{\alpha,i} = \{(x_1, x_2, g\underline{o}) \in \mathcal{X} \, : \, g^{-1} x_i \in \overline{B s_\alpha \underline{o}^-}\} \tag{5}$$

with the reduced ind-scheme structure. It is easy to see that $F_{\alpha,i}$ is ind-irreducible (i.e., union of finite-dimensional irreducible closed subsets). Let p_1, p_2 and p_3 denote the projections from $\mathcal{X}$ to the corresponding factor. Set, for $i = 1, 2$ and $\alpha \in \Delta$,

$$\mathcal{M}_{\alpha,i} = p_i^*(\mathcal{L}_{\varpi_\alpha}^-) \otimes p_3^*(\mathcal{L}_{\varpi_\alpha}).$$

Lemma 4.1. *The space* $\mathrm{H}^0(\mathcal{X}, \mathcal{M}_{\alpha,i})$ *contains a unique (up to scalar multiples) nonzero* G*-invariant section* $\sigma = \sigma_{\alpha,i}$*. Moreover, scheme-theoretically,*

$$F_{\alpha,i} = \{x \in \mathcal{X} \, : \, \sigma(x) = 0\}.$$

Proof. Our construction of $\mathcal{M}_{\alpha,i}$ and $\sigma_{\alpha,i}$ is completely explicit.

By the analogue of the Borel-Weil theorem for Kac-Moody groups (cf. [Kum02, Corollary 8.3.12]), we have (cf. [BK14, Proof of Theorem 3.2]):

$$\mathrm{H}^0(\mathcal{X}, \mathcal{M}_{\alpha,i}) \simeq \mathrm{Hom}_{\mathbb{C}}(L(\varpi_\alpha)^\vee \otimes L(\varpi_\alpha), \mathbb{C}). \tag{6}$$

Observe that

$$\mathrm{Hom}_{\mathbb{C}}(L(\varpi_\alpha)^\vee \otimes L(\varpi_\alpha), \mathbb{C}) \simeq \mathrm{Hom}_{\mathbb{C}}(L(\varpi_\alpha)^\vee, L(\varpi_\alpha)^*), \tag{7}$$

since $\mathrm{Hom}_{\mathbb{C}}(V \otimes W, \mathbb{C}) \simeq \mathrm{Hom}_{\mathbb{C}}(V, W^*)$ for any $\mathbb{C}$-vector spaces V and W. From the equations (6) and (7) it is easy to see that $\mathrm{H}^0(\mathcal{X}, \mathcal{M}_{\alpha,i})^G$ is one-dimensional spanned by the inclusion of $L(\varpi_\alpha)^\vee$ in $L(\varpi_\alpha)^*$ under the identifications (6) and (7). We now identify the zero locus of nonzero $\sigma \in \mathrm{H}^0(\mathcal{X}, \mathcal{M}_{\alpha,i})^G$:

Consider the isomorphism

$$\psi : G \times^{B^-} G/B \simeq G/B^- \times G/B, \;\; [g, h\underline{o}] \mapsto (g\underline{o}^-, gh\underline{o}), \text{ for } g, h \in G,$$

where $[g, h\underline{o}]$ denotes the B^--orbit of $(g, h\underline{o})$. Consider the B^--equivariant line bundle $\mathbb{C}_{\varpi_\alpha} \otimes \mathcal{L}_{\varpi_\alpha}$ over G/B, where $\mathbb{C}_{\varpi_\alpha}$ denotes the trivial line bundle over G/B with the B^--action given by the character e^{ϖ_α}. It is easy to see that

$$\psi^*(\mathcal{L}^-_{\varpi_\alpha} \boxtimes \mathcal{L}_{\varpi_\alpha}) = G \times^{B^-} (\mathbb{C}_{\varpi_\alpha} \otimes \mathcal{L}_{\varpi_\alpha}). \tag{8}$$

Let v_- be a fixed nonzero vector of $\mathbb{C}_{-\varpi_\alpha}$. Consider the section σ_o of $\mathcal{L}_{\varpi_\alpha}$ over G/B given by

$$\sigma_o(g\underline{o}) = [g, v_+^*(gv_+)v_-], \text{ for } g \in G, \tag{9}$$

where v_+ is a nonzero highest weight vector of $L(\varpi_\alpha)$ and $v_+^* \in L(\varpi_\alpha)^*$ is given by

$$v_+^*(v_+) = 1 \, \text{and} \, v_+^*(v) = 0, \text{ for any weight vector } v \text{ of } L(\varpi_\alpha) \text{ of weight } \neq \varpi_\alpha.$$

By the definition of σ_o, it is a character of B^- of weight $-\varpi_\alpha$ and hence $1 \otimes \sigma_o$ thought of as a section of $\mathbb{C}_{\varpi_\alpha} \otimes \mathcal{L}_{\varpi_\alpha}$ is B^--invariant. Thus, it canonically gives rise to a G-invariant section $\hat{\sigma}_o$ of $G \times^{B^-} (\mathbb{C}_{\varpi_\alpha} \otimes \mathcal{L}_{\varpi_\alpha})$.

We next claim that the zero set $Z(\sigma_o)$ of σ_o is given by

$$Z(\sigma_o) = \overline{B^- s_\alpha \underline{o}} \subset G/B. \tag{10}$$

By the definition of σ_o, $Z(\sigma_o)$ is left B^--stable (since $v_+^* \in L(\varpi_\alpha)^*$ is an eigenvector for the action of B^-). Take $w \in W$. Then,

$$\begin{aligned} w\underline{o} \in Z(\sigma_o) &\Leftrightarrow v_+^*(wv_+) = 0 \\ &\Leftrightarrow w\varpi_\alpha \neq \varpi_\alpha \\ &\Leftrightarrow w \notin \langle s_\beta \rangle_{\beta \in \Delta \setminus \{\alpha\}}, \text{ by [Kum02, Proposition 1.4.2 (a)]} \\ &\Leftrightarrow w \geq s_\alpha, \end{aligned}$$

where $\langle s_\beta \rangle \subset W$ denotes the subgroup generated by the elements s_β. This proves the equation (10) by the Birkhoff decomposition [Kum02, Theorem 6.2.8]. Thus, the zero set $Z(\hat{\sigma}_o)$ of $\hat{\sigma}_o$ is given by:

$$Z(\hat{\sigma}_o) = G \times^{B^-} \left(\overline{B^- s_\alpha \underline{o}}\right).$$

Moreover,

$$\psi\left(G \times^{B^-} \left(\overline{B^- s_\alpha \underline{o}}\right)\right) = \{(x, g\underline{o}) \in G/B^- \times G/B : g^{-1}x \in \overline{B s_\alpha \underline{o}^-}\}.$$

From this we obtain that $Z(\sigma) = F_{\alpha,i}$ set theoretically.

To prove that $Z(\sigma) = F_{\alpha,i}$ scheme-theoretically, it suffices to show that $Z(\sigma_o)$ (which is set theoretically $X^{s_\alpha} = \overline{B^- s_\alpha \underline{o}} \subset G/B$) is reduced. For any $v \in W$, consider $Z(\sigma_o) \cap X_v = X^{s_\alpha} \cap X_v$, which is an irreducible subset of codimension one in X_v. The Chern class of the line bundle $\mathcal{L}_{\varpi_\alpha | X_v}$ is the Schubert class $\varepsilon^{s_\alpha} \in \mathrm{H}^2(X_v, \mathbb{Z})$. If $Z(\sigma_o) \cap X_v$ were not reduced, say

$$Z(\sigma_o) \cap X_v = d(X^{s_\alpha} \cap X_v) \text{ (scheme-theoretically) for some } d > 1,$$

then $\frac{1}{d}\varepsilon^{s_\alpha} \in \mathrm{H}^2(X_v, \mathbb{Z})$, which is a contradiction. Hence $d = 1$, proving that $Z(\sigma_o) \cap X_v$ is reduced for any $v \in W$. Thus, $Z(\sigma_o)$ is reduced, proving the lemma. □

4.2. **Subvarieties of $\mathcal{X}$ from Schubert varieties.** Fix a standard parabolic subgroup P of G with Levi subgroup $L \supset T$, where T is the (standard) maximal torus of G with Lie algebra $\mathfrak{h}$. For $w \in W^P$, let

$$X_P^w := \overline{B^- wP/P} \subset X_P \text{ and } X_w^P := \overline{BwP/P} \subset X_P$$

be respectively the *opposite Schubert variety* and the *Schubert variety* associated to w.

For any triple $(w_1, w_2, v) \in (W^P)^3$, set

$$\bar{C}^+_{w_1,w_2,v} = \overline{Pw_1^{-1}\underline{o}^-} \times \overline{Pw_2^{-1}\underline{o}^-} \times \overline{Pv^{-1}\underline{o}} \subset \mathcal{X},$$

and

$$E_{w_1,w_2,v} = G.\bar{C}^+_{w_1,w_2,v} \subset \mathcal{X} \text{ under the diagonal action of } G. \tag{11}$$

Lemma 4.2. *For any triple* $(w_1, w_2, v) \in (W^P)^3$, *the set* $E_{w_1,w_2,v}$ *is closed and ind-irreducible in* $\mathcal{X}$.

Proof. Since G and $\bar{C}^+_{w_1,w_2,v}$ are ind-irreducible (see [Res21, before Lemma 3]), so is $E_{w_1,w_2,v}$. Note that

(12) $E_{w_1,w_2,v} = \{(g_1\underline{o}^-, g_2\underline{o}^-, g_3\underline{o}) \in \mathcal{X} \,:\, g_1X_P^{w_1} \cap g_2X_P^{w_2} \cap g_3X_v^P \neq \emptyset\}.$

By the following isomorphism

$$G \times_B (G/B^-)^2 \longrightarrow \mathcal{X},\ [g,x] = (gx, gB/B),$$

it is sufficient to prove that

$$\tilde{E} = \{(g_1\underline{o}^-, g_2\underline{o}^-) \,:\, g_1X_P^{w_1} \cap g_2X_P^{w_2} \cap X_v^P \neq \emptyset\}$$

is closed in $\mathcal{X}_\flat := (G/B^-)^2 \simeq (G/B^-)^2 \times \underline{o}$. Consider

$$\pi_\flat \,:\, \mathfrak{X}_\flat \longrightarrow \mathcal{X}_\flat,$$

where

$$\mathfrak{X}_\flat := \{(y, g_1\underline{o}^-, g_2\underline{o}^-, \underline{o}) \in G/P \times \mathcal{X} : y \in g_1X_P^{w_1} \cap g_2X_P^{w_2} \cap X_v^P\}$$

and $\pi_\flat$ is the projection to the last three factros. Note that $\tilde{E}$ is the image of $\mathfrak{X}_\flat$. Consider a filtration $\mathcal{X}_\flat = \bigcup_n \mathcal{X}_\flat^n$ by closed finite-dimensional subvarieties. Then, $\pi_\flat^{-1}(\mathcal{X}_\flat^n)$ is closed in $X_v^P \times \mathcal{X}_s^n$. Since X_v^P is projective, it follows that $\pi_\flat(\pi_\flat^{-1}(\mathcal{X}_\flat^n))$ is closed in $\mathcal{X}_\flat^n$. This concludes the proof since $\pi_\flat(\pi_\flat^{-1}(\mathcal{X}_\flat^n)) = \tilde{E} \cap \mathcal{X}_\flat^n$. $\square$

For $w \in W^P$, we set $\mathring{X}_P^w = B^-wP/P$ and $\mathring{X}_w^P = BwP/P$. Consider, for any triple $(w_1, w_2, v) \in (W^P)^3$,

$$\mathfrak{X} := \{(gP/P, x) \in G/P \times \mathcal{X} \,:\, g^{-1}x \in \bar{C}^+\}$$

(13)

$$= \{(y, g_1\underline{o}^-, g_2\underline{o}^-, g_3\underline{o}) \in G/P \times \mathcal{X} \,:\, y \in g_1X_P^{w_1} \cap g_2X_P^{w_2} \cap g_3X_v^P\}$$

and

$$\mathring{\mathfrak{X}} := \{(y, g_1\underline{o}^-, g_2\underline{o}^-, g_3\underline{o}) \in G/P \times \mathcal{X} \,:\, y \in g_1\mathring{X}_P^{w_1} \cap g_2\mathring{X}_P^{w_2} \cap g_3\mathring{X}_v^P\},$$

where $\bar{C}^+ = \bar{C}^+_{w_1,w_2,v}$. Observe that $\mathfrak{X}$ is closed in $G/P \times \mathcal{X}$ and it is irreducible (in its Zariski topology) since $\mathfrak{X} = G \cdot (P/P, \bar{C}^+)$.

Consider also the set $\mathring{\mathfrak{X}}^+$ of points $(y, g_1\underline{o}^-, g_2\underline{o}^-, g_3\underline{o}) \in \mathring{\mathfrak{X}}$ such that the linear map

$$\mathcal{T}_y(g_3\mathring{X}_v^P) \longrightarrow \frac{\mathcal{T}_y(G/P)}{\mathcal{T}_y(g_1\mathring{X}_P^{w_1})} \oplus \frac{\mathcal{T}_y(G/P)}{\mathcal{T}_y(g_2\mathring{X}_P^{w_2})}$$

is injective, i.e.,

$$\mathcal{T}_y(g_1\mathring{X}_P^{w_1}) \cap \mathcal{T}_y(g_2\mathring{X}_P^{w_2}) \cap \mathcal{T}_y(g_3\mathring{X}_v^P) = (0),$$

where $\mathcal{T}$ denotes the Zariski tangent space.

For $v \in W^P$, we denote $v' \to v$ if $v' \in W^P$, $\ell(v') = \ell(v) - 1$ and $v' \leq v$.

Lemma 4.3. *The subsets $\mathring{\mathfrak{X}}$ and $\mathring{\mathfrak{X}}^+$ are open in $\mathfrak{X}$ for any triple $(w_1, w_2, v) \in (W^P)^3$.*

In the definition of $\mathring{\mathfrak{X}}$ and $\mathring{\mathfrak{X}}^+$ if we replace $\mathring{X}_P^{w_i}$ (for any $i = 1, 2$) by any B^--stable open subset of $\mathring{X}_P^{w_i} \cup (\bigcup_{w_i \to w'_i \in W^P} \mathring{X}_P^{w'_i})$ and $\mathring{X}_v^P$ by any B-stable open subset of $\mathring{X}_v^P \cup (\bigcup_{v' \to v, v' \in W^P} \mathring{X}_{v'}^P)$, then the lemma still remains true.

Proof. Consider the projection

$$\pi : G^{\times 4} \to G/P \times \mathcal{X}, \ \ (g, g_1, g_2, g_3) \mapsto (gP/P, g_1\underline{o}^-, g_2\underline{o}^-, g_3\underline{o}),$$

and define $\tilde{\mathfrak{X}} := \pi^{-1}(\mathfrak{X})$ and $\tilde{\mathring{\mathfrak{X}}} := \pi^{-1}(\mathring{\mathfrak{X}})$. Then,

$$\tilde{\mathfrak{X}} = \{(g, g_1, g_2, g_3) \in G^{\times 4} : gP/P \in g_1 X_P^{w_1} \cap g_2 X_P^{w_2} \cap g_3 X_v^P\}, \tag{14}$$

and

$$\tilde{\mathring{\mathfrak{X}}} = \{(g, g_1, g_2, g_3) \in G^{\times 4} : gP/P \in g_1 \mathring{X}_P^{w_1} \cap g_2 \mathring{X}_P^{w_2} \cap g_3 \mathring{X}_v^P\}. \tag{15}$$

Define the morphism

$$\beta : \tilde{\mathfrak{X}} \to X_P^{w_1} \times X_P^{w_2} \times X_v^P, \ \ (g, g_1, g_2, g_3) \mapsto (g_1^{-1}gP/P, g_2^{-1}gP/P, g_3^{-1}gP/P).$$

Then,

$$\tilde{\mathring{\mathfrak{X}}} = \beta^{-1}\left(\mathring{X}_P^{w_1} \times \mathring{X}_P^{w_2} \times \mathring{X}_v^P\right)$$

and hence $\tilde{\mathring{\mathfrak{X}}}$ is open in $\tilde{\mathfrak{X}}$. Thus, π being an open map, $\mathring{\mathfrak{X}}$ is open in $\mathfrak{X}$.

We now prove that

$$\mathring{\mathfrak{X}}^+ \text{ is open in } \mathring{\mathfrak{X}} \text{ (and hence in } \mathfrak{X}).$$

By the equation (15)

$$\begin{aligned}\pi^{-1}(\mathring{\mathfrak{X}}) = \tilde{\mathring{\mathfrak{X}}} = \{&(g, g_1, g_2, g_3) \in G^{\times 4} : \\ &g^{-1}g_1 \in Pw_1^{-1}U^-,\ g^{-1}g_2 \in Pw_2^{-1}U^-,\ g^{-1}g_3 \in Pv^{-1}U\},\end{aligned} \tag{16}$$

and

$$\begin{aligned}\pi^{-1}(\mathring{\mathfrak{X}}^+) = \{&(g, g_1, g_2, g_3) \in \pi^{-1}(\mathring{\mathfrak{X}}) : \\ &\mathcal{T}_{\dot{e}}(g^{-1}g_1\mathring{X}_P^{w_1}) \cap \mathcal{T}_{\dot{e}}(g^{-1}g_2\mathring{X}_P^{w_2}) \cap \mathcal{T}_{\dot{e}}(g^{-1}g_3\mathring{X}_v^P) = (0)\},\end{aligned} \tag{17}$$

where $\dot{e} := P/P \in G/P$. Consider the morphism

$$\tilde{\mathring{\beta}} : \tilde{\mathring{\mathfrak{X}}} \to \tilde{\mathring{X}}_{w_1, w_2, v} := \tilde{\mathring{X}}_P^{w_1} \times \tilde{\mathring{X}}_P^{w_2} \times \tilde{\mathring{X}}_v^P, \ \ (g, g_1, g_2, g_3) \mapsto (g_1^{-1}g, g_2^{-1}g, g_3^{-1}g),$$

where $\tilde{\mathring{X}}_P^{w_i} := B^- w_i P \subset G$ and similarly $\tilde{\mathring{X}}_v^P := BvP \subset G$. Define the finite rank vector bundle $\mathcal{E}_i$ over $\tilde{\mathring{X}}_P^{w_i}$ $(i = 1, 2)$ by

$$\bigcup_{h_i \in \tilde{\mathring{X}}_P^{w_i}} \mathcal{T}_{\dot{e}}(G/P)/\mathcal{T}_{\dot{e}}(h_i^{-1} \mathring{X}_P^{w_i}) \to \tilde{\mathring{X}}_P^{w_i},$$

and similarly the finite rank vector bundle $\mathcal{E}_3$ over $\tilde{\mathring{X}}_v^P$ by

$$\bigcup_{h \in \tilde{\mathring{X}}_v^P} \mathcal{T}_{\dot{e}}(h^{-1} \mathring{X}_v^P) \to \tilde{\mathring{X}}_P^v,$$

and a morphism over $\tilde{\mathring{X}}_{w_1,w_2,v}$:

$$\varphi : \pi_3^*(\mathcal{E}_3) \to \pi_1^*(\mathcal{E}_1) \oplus \pi_2^*(\mathcal{E}_2)$$

induced by the canonical inclusion of $\mathcal{T}_{\dot{e}}(h^{-1} \mathring{X}_v^P) \hookrightarrow \mathcal{T}_{\dot{e}}(G/P)$, where π_i is the projection from $\tilde{\mathring{X}}_{w_1,w_2,v}$ to the i-th factor.. The set of points $Z \subset \tilde{\mathring{X}}_{w_1,w_2,v}$ where φ is injective is clearly open. But, it is easy to see that $(\tilde{\mathring{\beta}})^{-1}(Z) = \pi^{-1}(\mathring{\mathfrak{X}}^+)$, and hence $\pi^{-1}(\mathring{\mathfrak{X}}^+)$ is open in $\tilde{\mathring{\mathfrak{X}}}$ and thus $\mathring{\mathfrak{X}}^+$ is open in $\mathring{\mathfrak{X}}$. This proves the first part of the lemma.

The proof for the second (stronger) part of the lemma is identical.

□

4.3. **Divisors from Schubert varieties.** Fix $(w_1, w_2, v) \in (W^P)^3$ such that

(i) $w_1 \leq v$ and $w_2 \leq v$;
(ii) $\ell(v) = \ell(w_1) + \ell(w_2) - 1$;
(iii) there exist l_1, l_2 and l_3 in L such that the linear map

$$l_3 \mathcal{T}_v \longrightarrow \frac{\mathcal{T}}{l_1 \mathcal{T}^{w_1}} \oplus \frac{\mathcal{T}}{l_2 \mathcal{T}^{w_2}}$$

is injective, where the Zariski tangent spaces

$$\mathcal{T} = T_{\dot{e}}(G/P), \;\; \mathcal{T}^{w_i} = T_{\dot{e}}({w_i}^{-1} X_P^{w_i}), \;\; \text{and } \mathcal{T}_v = T_{\dot{e}}(v^{-1} X_v^P).$$

Proposition 4.4. *There exists a G-linearized line bundle $\mathcal{L}_{w_1,w_2,v}$ over $\mathcal{X}$ of the form $\mathcal{L}_{w_1,w_2,v} = \mathcal{L}^-(\lambda_1) \boxtimes \mathcal{L}^-(\lambda_2) \boxtimes \mathcal{L}(\mu)$ for some $(\lambda_1, \lambda_2, \mu) \in P_+^3$ and a nonzero G-invariant section $\sigma_{w_1,w_2,v}$ of $\mathcal{L}_{w_1,w_2,v}$ such that*

$$E_{w_1,w_2,v} = \{x \in \mathcal{X} \,:\, \sigma_{w_1,w_2,v}(x) = 0\}.$$

Before we come to the proof of the proposition, we need to prove some preparatory results.

Let U be the commutator subgroup $[B, B]$ of B and $U\underline{o}^-$ be the open cell in G/B^-. Set

$$\Omega = \{(x_1, x_2, g_3\underline{o}) \in \mathcal{X} \,:\, g_3^{-1}x_i \in U\underline{o}^- \text{ for } i = 1, 2\}.$$

It is easy to see that Ω is open in $\mathcal{X}$.

The construction of $\mathcal{L}_{w_1,w_2,v}$ and $\sigma_{w_1,w_2,v}$ is made in two steps:

(1) construct their restrictions to Ω by using a slice technique to reduce to the case of finite-dimensional varieties (see Lemma 4.6 below). Now, $E_{w_1,w_2,v}$ corresponds to the subvariety $\hat{E}$ (see (19) below) of an affine space. Lemma 4.5 proves that $\hat{E}$ is a closed divisor using Lemma 4.3.

(2) Twist the restriction $(\mathcal{L}_{w_1,w_2,v})_{|\Omega}$ to avoid components of the zero locus of $\sigma_{w_1,w_2,v}$ in the boundary $\mathcal{X} - \Omega$. This step uses Lemmas 4.6 and 4.7 below.

Observe that, by the Birkhoff decomposition [Kum02, Theorem 6.2.8],

$$\mathcal{X} = \Omega \sqcup \left(\bigcup_{\alpha\in\Delta,\, i=1,2} F_{\alpha,i} \right). \tag{18}$$

Consider the group homomorphism $\theta \,:\, U \longrightarrow \mathrm{Aut}(X_v^P)$ given by the action and let U_v be its image. Note that U_v is a finite-dimensional unipotent group. Set

$$\hat{E} := \{u \in U_v \,:\, (uX_v^{w_1}(P)) \cap X_v^{w_2}(P) \neq \emptyset\}, \text{ where } X_v^w(P) := X_v^P \cap X_P^w. \tag{19}$$

Lemma 4.5. *The subset $\hat{E}$ of U_v is a closed irreducible divisor of U_v.*

Proof. Consider the closed subset of $U_v \times X_v^{w_2}(P)$:

$$\hat{\mathfrak{X}} := \{(u, x) \in U_v \times X_v^{w_2}(P) \,:\, u^{-1}x \in X_v^{w_1}(P)\},$$

with its two projections p_1 and p_2 on U_v and $X_v^{w_2}(P)$ respectively. Since $X_v^{w_2}(P)$ is projective, p_1 is proper. In particular, $\hat{E} = p_1(\hat{\mathfrak{X}})$ is closed in U_v.

Recall the definition of $\mathfrak{X}$ from the equation (13) and as defined earlier in the proof of Lemma 4.2,

$$\begin{aligned} \mathfrak{X}_{\mathfrak{s}} &:= \mathfrak{X} \cap \left(G/P \times G/B^- \times G/B^- \times \{\underline{o}\}\right) \\ &= \{(y, g_1\underline{o}^-, g_2\underline{o}^-, \underline{o}) \in G/P \times \mathcal{X} : y \in (g_1X_P^{w_1}) \cap (g_2X_P^{w_2}) \cap X_v^P\}, \end{aligned}$$

its open subset

$$\mathring{\mathfrak{X}}_1 := \mathfrak{X}_{\mathfrak{s}} \cap \left(G/P \times (U \cdot \underline{o}^-) \times (U \cdot \underline{o}^-) \times \{\underline{o}\}\right),$$

and

$$\hat{\mathfrak{X}}_{\mathfrak{s}} := \pi_1^{-1}(\mathfrak{X}_{\mathfrak{s}}), \text{ where } \pi_1 : G \times \mathcal{X} \to G/P \times \mathcal{X} \text{ is the projection.}$$

Then,

$$(\overline{BvP})\times(\overline{Pw_1^{-1}B^-/B^-})\times(\overline{Pw_2^{-1}B^-/B^-}) \simeq \hat{\mathfrak{X}}_\flat,\ (g,x_1,x_2)\mapsto(g,gx_1,gx_2,\underline{o}).$$

Hence, $\hat{\mathfrak{X}}_\flat$ is irreducible and thus so is its quotient $\mathfrak{X}_\flat$. By the condition (i) of §4.3, $\mathring{\mathfrak{X}}_1$ is nonempty. By the condition (iii) of §4.3 and Lemma 4.3, $\mathfrak{X}_\flat \cap \mathring{\mathfrak{X}}^+$ is a nonempty open subset of $\mathfrak{X}_\flat$. Since $\mathfrak{X}_\flat$ is irreducible and $\mathfrak{X}_\flat\cap\mathring{\mathfrak{X}}^+$ and $\mathring{\mathfrak{X}}_1$ are nonempty open subsets of irreducible $\mathfrak{X}_\flat$, their intersection

$$\mathring{\mathfrak{X}}_1^+ := \mathring{\mathfrak{X}}_1 \cap \mathring{\mathfrak{X}}^+ \text{ is nonempty.}$$

Consider the ind-variety $Y = G/P \times U \times U$ and the morphism

$$\alpha : Y \to G/P^{\times 3},\ \ (y,u_1,u_2)\mapsto(u_1^{-1}y,u_2^{-1}y,y).$$

Let $Y' = Y_{(w_1,w_2,v)} \subset Y$ be the closed ind-subvariety

$$Y' := \alpha^{-1}\left(X_P^{w_1}\times X_P^{w_2}\times X_v^P\right).$$

Then, there is an isomorphism

$$\beta : \mathring{\mathfrak{X}}_1 \simeq Y',\ \ (y,u_1\underline{o}^-,u_2\underline{o}^-,\underline{o})\mapsto(y,u_1,u_2).$$

In particular, Y' is also irreducible. Let

$$Y'_+ := \beta(\mathring{\mathfrak{X}}_1^+)\subset Y' \text{ be the nonempty open subset.}$$

Consider the morphism

$$q : Y' \to \hat{\mathfrak{X}},\ \ (y,u_1,u_2)\mapsto(\theta(u_2^{-1}u_1),u_2^{-1}y).$$

Clearly, q is surjective. In particular, we obtain that $\hat{\mathfrak{X}}$ is irreducible and hence so is $\hat{E}=p_1(\hat{\mathfrak{X}})$.

We now determine the image of p_2: Let $x\in X_v^{w_2}(\mathrm{P})$ and let $v'\le v$ be such that $v'\in W^P$ and $x\in \mathring{X}_{v'}^P$. Then, $x\in \mathrm{Im}(p_2)$ if and only if $Ux\cap X_P^{w_1}\neq\emptyset$ if and only if $w_1\le v'$ (cf. [Kum02, Lemma 7.1.22]). We deduce that

$$\mathrm{Im}(p_2) = X_P^{w_2}\cap\left(\bigcup_{w_1\le v'\le v; v'\in W^P}\mathring{X}_{v'}^P\right). \tag{20}$$

In particular, it is open in $X_v^{w_2}(P)$.

We now analyze the fibers of p_2: Let $x\in\mathrm{Im}(p_2)$ and v' be as above. Then, $p_2^{-1}(x)$ is the set of points $u\in U_v$ such that $u^{-1}x\in X_P^{w_1}$. It is the pullback of $\mathring{X}_{v'}^P\cap X_P^{w_1}$ by the orbit map $u\mapsto u^{-1}x$. Since $\mathring{X}_{v'}^P\cap X_P^{w_1}$ is irreducible (cf. [Kum17, Proposition 6.6]) and the stabilizer of x in U_v

is, of course, irreducible (being a closed subgroup of a finite-dimensional unipotent group), so is $p_2^{-1}(x)$. Moreover,

$$\dim(p_2^{-1}(x)) \begin{aligned}[t] &= \ell(v') + \dim(\mathrm{Stab}_{U_v}(v'P/P)) - \ell(w_1) \\ &= \ell(v) + \dim(\mathrm{Stab}_{U_v}(vP/P)) - \ell(w_1), \end{aligned} \tag{21}$$

where $\mathrm{Stab}_{U_v}(v'P/P)$ denotes the stabilizer of $v'P/P$ in U_v.

Further, by equations (20) and (21),

$$\begin{aligned} \dim \hat{\mathfrak{X}} &= \ell(v) + \dim(\mathrm{Stab}_{U_v}(vP/P)) - \ell(w_1) + \ell(v) - \ell(w_2) \\ &= \dim U_v - 1, \text{ by the assumption (ii) of §4.3.} \end{aligned} \tag{22}$$

We return to the surjective map $q : Y' \twoheadrightarrow \hat{\mathfrak{X}}$ defined above. By Chevalley's theorem (cf. [Har77, Chap. II, Exercise 3.19(b)]), $q(Y'_+)$ contains a nonempty open subset (denoted by $\hat{\mathfrak{X}}^+$) of $\hat{\mathfrak{X}}$. By the definition of $\mathring{\mathfrak{X}}_1^+$, we get the following:

(23)

$$\mathcal{T}_x(u\mathring{X}_v^{w_1}(P)) \cap \mathcal{T}_x(\mathring{X}_v^{w_2}(P)) = (0), \text{ for any } (u,x) \in \hat{\mathfrak{X}}^+ \subset U_v \times \mathring{X}_v^{w_2}(P),$$

where

$$\mathring{X}_v^w(P) := \mathring{X}_P^w \cap \mathring{X}_v^P.$$

Observe that $\mathring{X}_v^{w_i}(P)$ is smooth (which follows from [Kum02, Lemma 7.3.10]). Consider the projection map

$$p_1^+ : \hat{\mathfrak{X}}^+ \to U_v, \text{ where } p_1^+ := p_{1|\hat{\mathfrak{X}}^+}.$$

From the above equation (23), we conclude that

$$(p_1^+)^{-1}(p_1^+(u,x)) \subset \{u\} \times \Big((u\mathring{X}_v^{w_1}(P)) \cap \mathring{X}_v^{w_2}(P)\Big)$$

is a finite set for any $(u,x) \in \hat{\mathfrak{X}}^+$. In particular, $\hat{E}$ being irreducible,

$$\dim(\hat{E}) = \dim(\overline{\mathrm{Im}\, p_1^+}) = \dim(\hat{\mathfrak{X}}^+) = \dim(\hat{\mathfrak{X}}) = \dim(U_v) - 1,$$

where the last equality follows from the equation (22). This proves that $\hat{E}$ is a divisor, proving the lemma. $\square$

Lemma 4.6. *There exists a G-equivariant line bundle $\mathcal{M} \in \mathrm{Pic}(\Omega)$ and $\tau \in \mathrm{H}^0(\Omega, \mathcal{M})^G$ such that*

$$\Omega \cap E = \{x \in \Omega \,:\, \tau(x) = 0\},$$

where $E = E_{w_1,w_2,v}$. In fact, we can take $\mathcal{M} = (p_{3|\Omega})^*\mathcal{L}_\chi$ for a character χ of B.

In particular, $E \cap \Omega$ is closed in Ω.

Proof. By definition,

$$\begin{aligned} E &= \{(g_1\underline{o}^-, g_2\underline{o}^-, g_3\underline{o}) \in \mathcal{X} \,:\, g_1X_P^{w_1} \cap g_2X_P^{w_2} \cap g_3X_v^P \neq \emptyset\} \\ &= \{(g_1\underline{o}^-, g_2\underline{o}^-, g_3\underline{o}) \,:\, (g_3^{-1}g_1X_P^{w_1}) \cap (g_3^{-1}g_2X_P^{w_2}) \cap X_v^P \neq \emptyset\}. \end{aligned}$$

Consider the isomorphism $\iota \,:\, U\underline{o}^- \longrightarrow U, u\underline{o}^- \mapsto u$. Then,

$$\begin{aligned} E\cap\Omega &= \{(x_1, x_2, g_3\underline{o}) \in \Omega \,:\, \iota(g_3^{-1}x_1)X_P^{w_1} \cap \iota(g_3^{-1}x_2)X_P^{w_2} \cap X_v^P \neq \emptyset\} \\ &= \{(x_1, x_2, g_3\underline{o}) \in \Omega \,:\, \big(\iota(g_3^{-1}x_1)X_v^{w_1}(P)\big) \cap \big(\iota(g_3^{-1}x_2)X_v^{w_2}(P)\big) \neq \emptyset\}, \end{aligned}$$

since X_v^P is U-stable. Here (as earlier) $X_v^{w_1}(P) := X_P^{w_1} \cap X_v^P$. Thus,

(24)

$$E\cap\Omega = \{(x_1, x_2, g_3\underline{o}) \in \Omega \,:\, \big([\iota(g_3^{-1}x_2)^{-1}\iota(g_3^{-1}x_1)]X_v^{w_1}(P)\big)\cap X_v^{w_2}(P) \neq \emptyset\}.$$

As earlier, consider the group homomorphism $\theta \,:\, U \longrightarrow \mathrm{Aut}(X_v^P)$ given by the action, and denote by U_v its image (which is a finite-dimensional unipotent group). Recall that

$$\hat{E} := \{u \in U_v \,:\, (uX_v^{w_1}(P)) \cap X_v^{w_2}(P) \neq \emptyset\}.$$

Note that the torus T acts by conjugation on U_v and that $\hat{E}$ is T-stable. Being a finite-dimensional unipotent group, U_v is isomorphic as a variety to an affine space. In particular, there exists $\hat{f} \in \mathbb{C}[U_v]$, unique up to scalar multiplication, such that $\mathrm{div}(\hat{f}) = \hat{E}$ (since $\hat{E}$ is an irreducible divisor by Lemma 4.5). Moreover, since $\hat{E}$ is T-stable, $\hat{f}$ is an eigenvector of T; denote by χ the corresponding character. We extend χ uniquely to a character of B.

Set $\tilde{E} = \tilde{\pi}^{-1}(E)$ and $\tilde{\Omega} := \tilde{\pi}^{-1}(\Omega)$, where $\tilde{\pi} : \tilde{\mathcal{X}} := G/B^- \times G/B^- \times G \to \mathcal{X}$ is the projection. Then, $\tilde{\Omega}$ and $\tilde{E}$ are stable by the following action of $G \times B$:

$$(g, b).(x_1, x_2, g') := (gx_1, gx_2, gg'b^{-1}).$$

Consider $\tilde{f} \,:\, \tilde{\Omega} \longrightarrow \mathbb{C}$ defined by

$$\tilde{f}(x_1, x_2, g) = \hat{f} \circ \theta(\iota(g^{-1}x_2)^{-1}\iota(g^{-1}x_1)).$$

Then, by the equation (24), $\tilde{E} \cap \tilde{\Omega}$ is the zero locus $Z(\tilde{f})$ of $\tilde{f}$ and for $b = ut \in B$ (where $u \in U, t \in T$):

$$\begin{aligned} \tilde{f}(x_1, x_2, gb) &:= \hat{f} \circ \theta(t^{-1}[\iota(g^{-1}x_2)^{-1}\iota(g^{-1}x_1)]t) \\ &= \chi(t)\tilde{f}(x_1, x_2, g) = \chi(b)\tilde{f}(x_1, x_2, g). \end{aligned} \tag{25}$$

We claim that $\tilde{f}$ induces a section $\tau_{\tilde{f}}$ of $(p_{3|\Omega})^*(\mathcal{L}_\chi)$, where $p_3 : \mathcal{X} \to G/B$ is the projection onto the third factor.

By the equation (25), $\tilde{f}$ gives rise to a section $\tau_{\tilde{f}}$ of the line bundle $\mathcal{L}_\Omega(\chi)$ associated to the principal B-bundle $\tilde{\Omega} \to \Omega$ (induced from the right $\cdot$ action of B on $\tilde{\Omega}$) via the character χ^{-1} of B. Clearly,

$$\mathcal{L}_\Omega(\chi) = (p_{3|\Omega})^*(\mathcal{L}_\chi).$$

By construction, the zero set $Z(\tau_{\tilde{f}}) = E \cap \Omega$. By the definition of $\tau_{\tilde{f}}$, it is easy to see that it is a G-invariant section. Taking $\tau = \tau_{\tilde{f}}$, we get the lemma. ☐

We now have a line bundle and a section τ on Ω with the expected zero locus. To avoid extra zero locus in the boundary $\mathcal{X}\backslash\Omega$ we need to twist by some line bundles given by Lemma 4.1. The key point to do this is the following finiteness result:

Lemma 4.7. *The valuation $v_{F_{\alpha,i}}(\tau)$ is finite for any $\alpha \in \Delta$ and $i = 1, 2$, where τ is the section taken from Lemma 4.6. (In the proof below we see that $F_{\alpha,i}$ is irreducible.)*

Proof. We are going to prove that $v_{F_{\alpha,i}}(\tau)$ can be computed in some finite-dimensional variety after taking a quotient by a unipotent group.

Fix a simple root $\alpha \in \Delta$ and $i = 1$ and consider

$$F = F_{\alpha,1} = \{(x_1, x_2, g_3\underline{o}) \in \mathcal{X} \, : \, g_3^{-1}x_1 \in \overline{Bs_\alpha\underline{o}^-}\}.$$

Consider the isomorphism

$$\varphi : \tilde{\mathcal{X}} \to \tilde{\mathcal{X}}, \quad (x_1, x_2, g) \mapsto (gx_1, gx_2, g).$$

Endow $\tilde{\mathcal{X}}$ with the following two right actions of B:

$$(x_1, x_2, g) \odot b = (b^{-1}x_1, b^{-1}x_2, gb)$$

and

$$(x_1, x_2, g) \cdot b = (x_1, x_2, gb).$$

Then, the morphism φ is B-equivariant with respect to the action $\odot$ on the domain and the action $\cdot$ on the range.

Clearly, $\tilde{\pi} : \tilde{\mathcal{X}} \to \mathcal{X}$ is a principal B-bundle with respect to the action $\cdot$. Define

$$\tilde{\Omega}' := \varphi^{-1}(\tilde{\Omega}).$$

By the definition of Ω,

$$\tilde{\Omega}' = U\underline{o}^- \times U\underline{o}^- \times G. \tag{26}$$

Let $\hat{f}$ and $\tilde{f}$ be as in the proof of Lemma 4.6. Set $\tilde{f}' = \tilde{f} \circ \varphi : \tilde{\Omega}' \to \mathbb{C}$. Thus,

$$\tilde{f}'(u_1\underline{o}^-, u_2\underline{o}^-, g) = \hat{f} \circ \theta(u_2^{-1}u_1), \qquad \text{for } u_1, u_2 \in U \text{ and } g \in G. \tag{27}$$

Set $F' := (\tilde{\pi} \circ \varphi)^{-1}(F) = \overline{Us_\alpha\underline{o}^-} \times G/B^- \times G$. Consider $V^\alpha := U\underline{o}^- \cup Us_\alpha\underline{o}^-$. It is an open subset of G/B^- (containing $\overline{Us_\alpha\underline{o}^-}$). By [Kum17, Lemma 6.1], there exists a closed normal subgroup $\mathcal{U}$ of U such that $V^\alpha \longrightarrow \mathcal{U}\backslash V^\alpha =: Y^\alpha$ is a principal $\mathcal{U}$-bundle and Y^α is a smooth finite-dimensional variety. Moreover, by intersecting with $\operatorname{Ker}\theta$, one can assume that $\mathcal{U}$ acts trivially on X_v.

Let $h_1, h_2 \in \mathcal{U}$. We have, for any $u_1, u_2 \in U$ and $g \in G$,

$$\begin{aligned}
\tilde{f}'(h_1u_1\underline{o}^-, h_2u_2\underline{o}^-, g) &= \hat{f}\circ\theta(u_2^{-1}h_2^{-1}h_1u_1), \text{ by equation (27)} \\
&= \hat{f}\circ\theta(u_2^{-1}u_1), \text{ since } \theta \text{ is a group homomorphism} \\
&\quad \text{and } h_1, h_2 \in \mathcal{U} \subset \operatorname{Ker}\theta \\
&= \tilde{f}'(u_1\underline{o}^-, u_2\underline{o}^-, g). \qquad (28)
\end{aligned}$$

Since the line bundle $p_3^*(\mathcal{L}_\chi)$ over $\mathcal{X}$ pulled to the principal B-bundle $\pi : \tilde{\mathcal{X}} \to \mathcal{X}$ is trivialized, to prove the finiteness of $v_F(\tau)$, it suffices to show that the function $\tilde{f} : \tilde{\Omega} \to \mathbb{C}$ has a pole of finite order along $\pi^{-1}(F)$. Equivalently, considering the isomorphism $\varphi : \tilde{\mathcal{X}} \to \tilde{\mathcal{X}}$, it suffices to show that the function

$$\tilde{f}' : \tilde{\Omega}' = U\underline{o}^- \times U\underline{o}^- \times G \to \mathbb{C}$$

has a pole of finite order along $F' = \overline{Us_\alpha\underline{o}^-} \times G/B^- \times G$.

The diagonal action of G on $\tilde{\mathcal{X}}$ pulled back via φ to the action $\odot$ of G on $\tilde{\mathcal{X}}$ is given by:

$$g \odot (x_1, x_2, h) = (x_1, x_2, gh), \text{ for } x_1, x_2 \in G/B^- \text{ and } g, h \in G.$$

The function $\tilde{f}' : U\underline{o}^- \times U\underline{o}^- \times G \to \mathbb{C}$ descends to a function $\hat{f}'$ on $U\underline{o}^- \times U\underline{o}^-$ by equation (27). So, to prove that the function $\tilde{f}'$ has a pole of finite order along F', it suffices to show that the function $\hat{f}' : U\underline{o}^- \times U\underline{o}^- \to \mathbb{C}$ has a pole of finite order along $\left(\overline{Us_\alpha\underline{o}^-}\right) \times G/B^-$. Consider the open embedding

$$\left(\mathcal{U}\backslash U\underline{o}^-\right) \times \left(\mathcal{U}\backslash U\underline{o}^-\right) \hookrightarrow \left(\mathcal{U}\backslash V^\alpha\right) \times \left(\mathcal{U}\backslash U\underline{o}^-\right).$$

By the equation (28), the function $\hat{f}'$ descends to a function $\hat{\phi}'$ on $(\mathcal{U}\backslash U\underline{o}^-) \times (\mathcal{U}\backslash U\underline{o}^-)$. Since $(\mathcal{U}\backslash V^\alpha) \times (\mathcal{U}\backslash U\underline{o}^-)$ is a (smooth) scheme of finite type over $\mathbb{C}$, the function $\hat{\phi}'$ has a pole of finite order along the divisor $(\mathcal{U}\backslash(Us_\alpha\underline{o}^-)) \times (\mathcal{U}\backslash U\underline{o}^-)$ and hence $\hat{f}'$ has a pole of finite order along the divisor $(Us_\alpha\underline{o}^-) \times U\underline{o}^-$. Since $Us_\alpha\underline{o}^-$ is an open subset of $\overline{Us_\alpha\underline{o}^-}$, we get that $\hat{f}'$ has a pole of finite order along $(\overline{Us_\alpha\underline{o}^-}) \times G/B^-$. This proves the finiteness of $v_{F_{\alpha,1}}(\tau)$ for any $\alpha \in \Delta$. The proof of the finiteness of $v_{F_{\alpha,2}}(\tau)$ is identical. $\square$

Proof of Proposition 4.4. By Lemma 4.6, there exist a G-equivariant line bundle $\mathcal{M}$ over Ω and a nonzero G-invariant section τ over Ω with $\operatorname{div}\tau = E \cap \Omega$. Moreover, the line bundle $\mathcal{M}$ is the restriction of the line bundle $p_3^*(\mathcal{L}_\chi)$ over $\mathcal{X}$. Then, τ is a (rational) section of $\mathcal{M}' := p_3^*(\mathcal{L}_\chi)$ regular over Ω.

Lemma 4.7 allows to consider the G-linearized line bundle

$$\mathcal{L}_{w_1,w_2,v} := \mathcal{M} \otimes \left(\bigotimes_{\alpha\in\Delta,\, i=1,2} \mathcal{M}_{\alpha,i}^{v_{F_{\alpha,i}}(\tau)} \right) \text{ over } \mathcal{X},$$

where the line bundle $\mathcal{M} := p_3^*(\mathcal{L}_\chi)$ is as in Lemma 4.6 and the line bundles $\mathcal{M}_{\alpha,i}$ are as in Lemma 4.1. In particular, $\mathcal{L}_{w_1,w_2,v}$ is of the form $\mathcal{L}^-(\lambda_1) \boxtimes \mathcal{L}^-(\lambda_2) \boxtimes \mathcal{L}(\mu)$ for some $\lambda_1, \lambda_2, \mu \in \mathfrak{h}_\mathbb{Z}^*$.

By Lemmas 4.1 and 4.6 and the decomposition (18), it has a nonzero G-invariant section

$$\sigma_{w_1,w_2,v} = \tau \otimes \left(\bigotimes_{\alpha\in\Delta,\, i=1,2} \sigma_{\alpha,i}^{v_{F_{\alpha,i}}(\tau)} \right). \tag{29}$$

Thus, by [Kum02, Corollary 8.3.12], $(\lambda_1, \lambda_2, \mu) \in P_+^3$. This proves the proposition by using the following Lemma 4.8. □

Observe that $\overline{E \cap \Omega} \subset E$ (since E is closed by Lemma 4.2). Moreover, since E is irreducible and $E \cap \Omega \neq \emptyset$ (as $(\underline{o}^-, \underline{o}^-, \underline{o}) \in E \cap \Omega$),

$$\overline{E \cap \Omega} = E. \tag{30}$$

Lemma 4.8. *The zero set $Z(\sigma_{w_1,w_2,v}) := \{x \in \mathcal{X} : \sigma_{w_1,w_2,v}(x) = 0\}$ is equal to E.*

Proof. Consider the map

$$\psi : \tilde{\mathcal{X}} := (G/B^-)^2 \times G \to \mathcal{X} := (G/B^-)^2 \times G/B, \;\; (x_1, x_2, g) \mapsto (gx_1, gx_2, g\underline{o}).$$

For any subset $Y \subset \mathcal{X}$, we set $\hat{Y}' := \psi^{-1}(Y)$. Then,

$$\hat{F}'_{\alpha,1} = \overline{Bs_\alpha\underline{o}^-} \times G/B^- \times G.$$

Take an increasing cofinal sequence $w_n \in W$ (i.e., $w_1 < w_2 < w_3 < \cdots$ and for any $w \in W$ there exists a w_n such that $w \leq w_n$). Take a filtration $(G_n)_{n\geq 0}$ of G by finite-dimensional irreducible subvarieties compatible with its ind-variety structure (cf. [Res21, above Lemma 2.3]). Now, define the increasing filtration

$$\tilde{\mathcal{X}}_n := X_{w_n}^- \times X_{w_n}^- \times G_n \text{ of } \tilde{\mathcal{X}}, \text{ where } X_w^- := \overline{B^- w\underline{o}^-}.$$

Then,

$$\tilde{\mathcal{X}}_n \cap \hat{F}'_{\alpha,1} = (X_{w_n}^- \cap \overline{Bs_\alpha\underline{o}^-}) \times X_{w_n}^- \times G_n, \tag{31}$$

and a similar expression for $\tilde{\mathcal{X}}_n \cap \hat{F}'_{\alpha,2}$. Thus, $\tilde{\mathcal{X}}_n \cap \hat{F}'_{\alpha,i}$ is irreducible. Abbreviate $Z = Z(\sigma_{w_1,w_2,v})$. Then, by Lemmas 4.1 and 4.6 and the identity (18), $Z \cap \Omega = E \cap \Omega$ and hence $Z \supset E$ by the identity (30). Write

$$\hat{Z}' = \hat{E}' \cup \left(\bigcup_{(\alpha,i)\in\Delta\times\{1,2\}} (\hat{Z}' \cap \hat{F}'_{\alpha,i}) \right), \text{ by the identity (18).}$$

Thus, for any $n \geq 0$,

$$\hat{Z}' \cap \tilde{\mathcal{X}}_n = (\hat{E}' \cap \tilde{\mathcal{X}}_n) \cup \left(\bigcup_{(\alpha,i)\in\Delta\times\{1,2\}} (\hat{Z}' \cap \hat{F}'_{\alpha,i} \cap \tilde{\mathcal{X}}_n) \right). \tag{32}$$

But, being the zero set of a section of a line bundle, $\hat{Z}' \cap \tilde{\mathcal{X}}_n$ is a divisor in $\tilde{\mathcal{X}}_n$ and so is $\hat{F}'_{\alpha,i} \cap \tilde{\mathcal{X}}_n$ and the latter is irreducible (divisor of $\tilde{\mathcal{X}}_n$) by the equation (31). From the definition of σ given by the equation (29), we get (for any $(\alpha, i) \in \Delta \times \{1, 2\}$)

$$\hat{Z}' \cap \hat{F}'_{\alpha,i} \cap \tilde{\mathcal{X}}_n \subsetneq \hat{F}'_{\alpha,i} \cap \tilde{\mathcal{X}}_n, \text{ for large enough } n. \tag{33}$$

Thus, $\hat{Z}' \cap \hat{F}'_{\alpha,i} \cap \tilde{\mathcal{X}}_n$ is of codimension ≥ 2 in $\tilde{\mathcal{X}}_n$ for large enough n. But, since $\hat{Z}' \cap \tilde{\mathcal{X}}_n$ is a divisor in $\tilde{\mathcal{X}}_n$, we get from the equation (32) that

$$\hat{Z}' \cap \hat{F}'_{\alpha,i} \cap \tilde{\mathcal{X}}_n \subset \hat{E}' \cap \tilde{\mathcal{X}}_n, \text{ for large enough } n.$$

Thus,

$$\hat{Z}' \cap \tilde{\mathcal{X}}_n = \hat{E}' \cap \tilde{\mathcal{X}}_n, \text{ for large enough } n \text{ which gives } \hat{Z}' = \hat{E}'.$$

Hence, $Z = E$ proving the lemma. □

5. Proof of Theorem 1.5

In this section, we fix P, (w_1, w_2, v) and $\mathcal{L}$ as in the theorem.

Let $\mathcal{D}$ denote the set of pairs $(\alpha, i) \in \Delta \times \{1, 2, 3\}$ coming from $\Delta^+(w_1)$, $\Delta^+(w_2)$ and $\Delta^-(v)$, i.e.,

$$\mathcal{D} \cap (\Delta \times \{i\}) = \Delta^+(w_i) \text{ for } i = 1, 2 \text{ and } \mathcal{D} \cap (\Delta \times \{3\}) = \Delta^-(v),$$

where $\Delta^+(w_i)$ and $\Delta^-(v)$ are defined in the Introduction.

5.1. **Strategy.** Note that the assumptions on the triple (w_1, w_2, v) differ from that of Section 4.3. Nevertheless, we use the same notation. We set

$$C = Lw_1^{-1}\underline{o}^- \times Lw_2^{-1}\underline{o}^- \times Lv^{-1}\underline{o},$$
$$C^+ = Pw_1^{-1}\underline{o}^- \times Pw_2^{-1}\underline{o}^- \times Pv^{-1}\underline{o},$$

and (as earlier)

$$\bar{C}^+ = \bar{C}^+_{w_1,w_2,v} := \overline{Pw_1^{-1}\underline{o}^-} \times \overline{Pw_2^{-1}\underline{o}^-} \times \overline{Pv^{-1}\underline{o}}.$$

Recall from equation (13):

$$\begin{aligned}\mathfrak{X} &:= \{(gP/P, x) \in G/P \times \mathcal{X} \,:\, g^{-1}x \in \bar{C}^+\}\\ &= \{(y, g_1\underline{o}^-, g_2\underline{o}^-, g_3\underline{o}) \in G/P \times \mathcal{X} \,:\, y \in g_1X_P^{w_1} \cap g_2X_P^{w_2} \cap g_3X_v^P\}.\end{aligned}$$

As a closed subset of $G/P \times \mathcal{X}$, it is a G-ind-variety with the diagonal action of G. Consider the projection

$$\eta : \mathfrak{X} \to \mathcal{X},\ \ (y, x) \mapsto x.$$

For each $(\alpha, i) \in \mathcal{D}$, consider the associated P^3-orbit $\partial C^+_{\alpha,i}$ in $\mathcal{X}$, where $\partial C^+_{\alpha,1} := Pw_1^{-1}s_\alpha\underline{o}^- \times Pw_2^{-1}\underline{o}^- \times Pv^{-1}\underline{o}$ and $\partial C^+_{\alpha,i}$ $(i = 2, 3)$ are defined similarly. Then, $\partial C^+_{\alpha,i}$ is open in an irreducible component of $\bar{C}^+\backslash C^+$. Set

$$\tilde{C}^+ = C^+ \cup \left(\bigcup_{(\alpha,i)\in\mathcal{D}} \partial C^+_{\alpha,i} \right).$$

It is open in $\bar{C}^+$. Similarly, we define the open subset of $X_P^{w_i}$:

$$\tilde{X}_P^{w_i} := (B^-w_iP/P) \cup \left(\bigcup_{\alpha\in\Delta^+(w_i)} B^-s_\alpha w_iP/P \right) \ \ (\text{for } i = 1, 2)$$

and the open subset of X_v^P:

$$\tilde{X}_v^P := (BvP/P) \cup \left(\bigcup_{\alpha\in\Delta^-(v)} Bs_\alpha vP/P \right).$$

We also set

$$\begin{aligned}\tilde{\mathfrak{X}}' &:= \{(gP/P, x) \in G/P \times \mathcal{X} \,:\, g^{-1}x \in \tilde{C}^+\}\\ &= \{(y, g_1\underline{o}^-, g_2\underline{o}^-, g_3\underline{o}) \in G/P \times \mathcal{X} \,:\, y \in g_1\tilde{X}_P^{w_1} \cap g_2\tilde{X}_P^{w_2} \cap g_3\tilde{X}_v^P\},\end{aligned}$$

which is an open subset of $\mathfrak{X}$ and hence irreducible (since so is $\mathfrak{X}$ as observed earlier below the equation (13)). We make use of a slice by setting

$$\mathcal{X}_{\flat} := (G/B^-)^2 \times \{\underline{o}\} \subset \mathcal{X},$$

and its B-stable open subset

$$\mathring{\mathring{\mathcal{X}}}_{\flat} := (B\underline{o}^- \cup \bigcup_{\alpha\in\Delta} s_\alpha B\underline{o}^-)^2 \times \{\underline{o}\} = \left(\bigcup_{\ell(w)\le 1} Bw\underline{o}^- \right)^2 \times \{\underline{o}\}.$$

Then, we have a G-equivariant isomorphism:

(34) $$G \times^B \mathcal{X}_{\flat} \simeq \mathcal{X},\ \ [g, x] \mapsto gx.$$

As defined in the proof of Lemma 4.2,

$\mathfrak{X}_\flat := \{(y, g_1\underline{o}^-, g_2\underline{o}^-, \underline{o}) \in G/P \times \mathcal{X}_\flat \,:\, y \in g_1X_P^{w_1} \cap g_2X_P^{w_2} \cap X_v^P\} \subset \mathfrak{X}$.

We also set

$$\mathring{\mathfrak{X}}_\flat := \mathfrak{X}_\flat \cap (G/P \times \mathring{\mathcal{X}}_\flat)$$

and

$$\tilde{\mathfrak{X}}_\flat := \{(y, g_1\underline{o}^-, g_2\underline{o}^-, \underline{o}) \in G/P \times \mathcal{X}_\flat \,:\, y \in g_1\tilde{X}_P^{w_1} \cap g_2\tilde{X}_P^{w_2} \cap \tilde{X}_v^P\}.$$

Then,

$$G \times^B \mathfrak{X}_\flat \simeq \mathfrak{X}, \;\; [g, x] \mapsto gx. \tag{35}$$

In particular, $\mathfrak{X}_\flat$ is irreducible since so is $\mathfrak{X}$. Hence, $\tilde{\mathfrak{X}}_\flat$ and $\mathring{\mathfrak{X}}_\flat$ (being open subsets of $\mathfrak{X}_\flat$) are irreducible.

We now consider the following commutative diagram ($\diamond$) for any G-equivariant line bundle $\mathcal{L}$ over $\mathcal{X}$ as in Theorem 1.5:

$$\begin{tikzcd}
\mathrm{H}^0(\mathcal{X}, \mathcal{L})^G \arrow[r, "\eta^*"] \arrow[d, "i_1^*"'] & \mathrm{H}^0(\mathfrak{X}, \mathcal{L})^G \arrow[r, "\alpha^*"] \arrow[d, "i_2^*"'] & \mathrm{H}^0(G \times^P \tilde{C}^+, \mathcal{L})^G \arrow[r, "\beta^*"] \arrow[d, "\gamma^*"] & \mathrm{H}^0(C, \mathcal{L})^L \\
\mathrm{H}^0(\mathcal{X}_\flat, \mathcal{L})^B \arrow[r, "\eta_1^*"] \arrow[d, "i_4^*"'] & \mathrm{H}^0(\mathfrak{X}_\flat, \mathcal{L})^B \arrow[r, "i_3^*"] \arrow[d, "i_5^*"'] & \mathrm{H}^0(\tilde{\mathfrak{X}}_\flat, \mathcal{L})^B \arrow[d, "i_6^*"] & \\
\mathrm{H}^0(\mathring{\mathcal{X}}_\flat, \mathcal{L})^B \arrow[r, "\eta_2^*"] & \mathrm{H}^0(\mathring{\mathfrak{X}}_\flat, \mathcal{L})^B \arrow[r, "i_7^*"] & \mathrm{H}^0(\tilde{\mathfrak{X}}_\flat \cap \mathring{\mathfrak{X}}_\flat, \mathcal{L})^B\,, &
\end{tikzcd}$$

where

$$\alpha : G \times^P \tilde{C}^+ \to \mathfrak{X}, \;\; [g, (x_1, x_2, x_3)] \mapsto (gP, gx_1, gx_2, gx_3)$$

is a G-equivariant open embedding with image $\tilde{\mathfrak{X}}'$,

$$\beta : C \hookrightarrow G \times^P \tilde{C}^+ \text{ is the } L\text{-equivariant morphism } x \mapsto [1, x],$$

$\gamma : \tilde{\mathfrak{X}}_\flat \longrightarrow G \times^P \tilde{C}^+, \;\; (gP, g_1\underline{o}^-, g_2\underline{o}^-, \underline{o}) \mapsto [g, (g^{-1}g_1\underline{o}^-, g^{-1}g_2\underline{o}^-, g^{-1}\underline{o})]$,

is the morphism (which is $\alpha^{-1}_{|\tilde{\mathfrak{X}}_\flat}$), η_1, η_2 are restrictions of η to $\mathfrak{X}_\flat$ and $\mathring{\mathfrak{X}}_\flat$ respectively. All the maps i_j are appropriate inclusion maps. In the above diagram $\mathcal{L}$ also denotes the induced line bundle on each of the above ind-varieties by pullback. Note that the ind-varieties with $\flat$ as subscript are B-ind-varieties with the B-action induced from the G-action of the ambient G-ind-varieties; in particular, the line bundle $\mathcal{L}$ over them is endowed with a natural B-action.

We now prove that all the maps in the above commutative diagram are isomorphisms.

5.2. **Various isomorphisms.** We first prove the following lemma for its use in the proof of Lemma 5.2.

Lemma 5.1. *Let U_P be the unipotent radical of P. Then,*
(a) Any regular map $U_P \to \mathbb{C}^*$ is constant.
(b) $\mathrm{Pic}(U_P) = (0)$.

Proof. (a) Consider the parabolic subgroup P^- opposite to P and the homogeneous space G/P^-. Then U_P can be seen as an open subset of G/P^-. For any Schubert variety $X_w^- = X_w^-(P) := \overline{B^-wP^-/P^-} \subset G/P^-$ (with $w \in W^P$), $X_w^- \cap U_P$ is contractible in the analytic topology (cf. [Kum02, Proposition 7.4.17 and its proof]). Now, by [KNR94, Lemma 2.5], we get that any regular map $X_w^- \cap U_P \to \mathbb{C}^*$ is a constant. From this (a) follows.

(b) By induction on $\ell(w)$, we show that the group of k-cycles modulo rational equivalence $A_k(X_w^- \cap U_P)$ is a finitely generated group. By [Ful98, Proposition 1.8], we have an exact sequence:

$$A_k((\partial X_w^-) \cap U_P) \to A_k(X_w^- \cap U_P) \to A_k\left((B^-wP^-/P^-) \cap U_P\right) \to 0.$$

Writing ∂X_w^- as a union $\bigcup_{\ell(v)=\ell(w)-1} X_v^-$ and applying [Ful98, Example 1.3.1(c)] and the induction hypotheis, we get that $A_k(\partial X_w^- \cap U_P)$ is finitely generated. Also, applying [Ful98, Proposition 1.8] again to the open subset $(B^-wP^-/P^-) \cap U_P$ of the affine space B^-wP^-/P^-, we get that $A_k\left((B^-wP^-/P^-) \cap U_P\right)$ is finitely generated since so is $A_k(B^-wP^-/P^-)$ (cf. [Ful98, Proposition 1.9]). Thus, from the above exact sequence, we get that $A_k(X_w^- \cap U_P)$ is finitely generated, completing the induction.

Consider the cohomology exact sequence (since $X_w^- \cap U_P$ is contractible in the analytic topology)

$$\begin{aligned}\mathrm{H}^1(X_w^- \cap U_P, \mathbb{Z}_m) = 0 \to \mathrm{H}^1(X_w^- \cap U_P, \mathscr{O}^*) = \mathrm{Pic}(X_w^- \cap U_P) \to \\ \mathrm{H}^1(X_w^- \cap U_P, \mathscr{O}^*) = \mathrm{Pic}(X_w^- \cap U_P) \to \mathrm{H}^2(X_w^- \cap U_P, \mathbb{Z}_m) = 0,\end{aligned}$$

induced from the sheaf exact sequence:

$$\mathbb{Z}_m \to \mathscr{O}^* \to \mathscr{O}^* \to 0,$$

where the map $\mathscr{O}^* \to \mathscr{O}^*$ takes $f \mapsto f^m$. From the above cohomology exact sequence we see that $\mathrm{Pic}(X_w^- \cap U_P)$ is a divisible group. But, since it is also a finitely generated abelian group (by [Ful98, Example 2.1.1]), it must be trivial. From this, taking limit, we obtain (b). $\square$

Since $\mathfrak{X}$ is irreducible and $\operatorname{Im}\alpha = \tilde{\mathfrak{X}}'$ is open in $\mathfrak{X}$, the restriction map

$$\mathrm{H}^0(\mathfrak{X},\mathcal{L}) \longrightarrow \mathrm{H}^0(G\times^P \tilde{C}^+,\mathcal{L}) \text{ is injective and hence so is } \alpha^*.$$

Lemma 5.2. *(a) The pullback induces an isomorphism:*

$$\eta^* : \mathrm{H}^0(\mathcal{X},\mathcal{L})^G \simeq \mathrm{H}^0(\mathfrak{X},\mathcal{L})^G.$$

(b) The restriction map

$$\mathrm{H}^0(\tilde{C}^+,\mathcal{L})^P \longrightarrow \mathrm{H}^0(C^+,\mathcal{L})^P$$

is an isomorphism.

(c) The restriction map

$$\mathrm{H}^0(C^+,\mathcal{L})^P \to \mathrm{H}^0(C,\mathcal{L})^L$$

is an isomorphism.

Proof. (a) follows by [Res21, Lemma 6.3].

The proof of (b) is analogous to the proof of [Res21, Lemma 6.5]. We sketch the proof: The map $\mathrm{H}^0(\tilde{C}^+,\mathcal{L})^P \longrightarrow \mathrm{H}^0(C^+,\mathcal{L})^P$ is obviously injective. Hence, it remains to prove that any P-invariant section σ of $\mathcal{L}$ on C^+ extends to $\tilde{C}^+$.

For $x\in W^P$, $Px^{-1}\underline{o}^-$ is contained in $\overline{Pw_i^{-1}\underline{o}^-}$ if and only if $x\geq w$. Moreover $\{z\in W\ :\ z\underline{o}^-\in Px^{-1}\underline{o}^-\}$ is the set of $z\in W$ that can be written as $z=yx^{-1}$ for some $y\in W_P$. Since $xy^{-1}\geq x$, such a point $z\underline{o}^-$ belongs to $\overline{Bw_i^{-1}\underline{o}^-}$. Then, $\overline{Pw_i^{-1}\underline{o}^-}$ and $\overline{Bw_i^{-1}\underline{o}^-}$ are B-stable and contain the same T-fixed points. We deduce that

$$\overline{Pw_i^{-1}\underline{o}^-} = \overline{Bw_i^{-1}\underline{o}^-}. \tag{36}$$

On the other hand, $\overline{Pv^{-1}\underline{o}} = \bigcup_{v_n\in W_P} X_{v_nv^{-1}}$, where v_n is an increasing cofinal sequence in W_P. We now construct an increasing filtration of $\bar{C}^+$ by products of finite-dimensional Richardson varieties:

$$\bar{C}^+ = \bigcup_{n\in\mathbb{N}} \bar{C}^+_n.$$

Explicitly

$$\bar{C}^+_n := (X^{w_1}_- \cap X^-_{w_n}) \times (X^{w_2}_- \cap X^-_{w_n}) \times X_{v_nv^{-1}},$$

where $\{w_n\}$ is a cofinal increasing sequence in W and $\overline{Pw_i^{-1}\underline{o}^-} = X^{w_i}_-$ by the equation (36), where $X^w_- := \overline{Bw^{-1}\underline{o}^-}$ and $X^-_w := \overline{B^-w\underline{o}^-}$. In particular, $\bar{C}^+_n$ are irreducible and normal (cf. [Kum17, Proposition 6.6]). Of course, $\bar{C}^+_n\cap C^+$ is open in $\bar{C}^+_n$ and nonempty for large enough

n. It remains to prove that $\sigma_{|\bar{C}_n^+\cap C^+}$ extends to a regular section on $\bar{C}_n^+\cap\tilde{C}^+$, for any n.

Fix $(\alpha,i)\in\mathcal{D}$. The irreducibility of the Richardson varieties implies that the intersection $\bar{C}_n^+\cap\overline{\partial C_{\alpha,i}^+}$ is either empty or irreducible. Since $\bar{C}_n^+$ is normal, to prove that $\sigma_{|\bar{C}_n^+\cap C^+}$ extends to $\bar{C}_n^+\cap\tilde{C}^+$, it is sufficient to prove that $\sigma_{|\bar{C}_n^+\cap C^+}$ has no pole along $\bar{C}_n^+\cap\overline{\partial C_{\alpha,i}^+}$ if $\bar{C}_n^+\cap\overline{\partial C_{\alpha,i}^+}$ has codimension 1 in $\bar{C}_n^+$.

Assume that $D_n:=\bar{C}_n^+\cap\overline{\partial C_{\alpha,i}^+}$ has codimension 1 in $\bar{C}_n^+$. Then, D_n is equal to either

(α) $(X_-^{\bar{u}_1}\cap X_{w_n}^-)\times(X_-^{w_2}\cap X_{w_n}^-)\times X_{v_nv^{-1}}$, for some $\bar{u}_1\geq w_1\in W^P$ and $\ell(\bar{u}_1)=\ell(w_1)+1$; or

(α') $(X_-^{w_1}\cap X_{w_n}^-)\times(X_-^{\bar{u}_2}\cap X_{w_n}^-)\times X_{v_nv^{-1}}$, for some $\bar{u}_2\geq w_2\in W^P$ and $\ell(\bar{u}_2)=\ell(w_2)+1$; or

(β) $(X_-^{w_1}\cap X_{w_n}^-)\times(X_-^{w_2}\cap X_{w_n}^-)\times X_{v_nv^{-1}s_\alpha}$.

Now, we construct an explicit affine open subset Ω_n in $\bar{C}_n^+$ that intersects D_n.

In case (α), set

$$\Omega_n=(X_-^{w_1}\cap X_{w_n}^-\cap(\bar{u}_1B\underline{o}^-))\times(X_-^{w_2}\cap\mathring{X}_{w_n}^-)\times\mathring{X}_{v_nv^{-1}},$$

where $\mathring{X}_w^-:=B^-w\underline{o}^-$ and $\mathring{X}_w:=Bw\underline{o}$ and similarly for the case (α'). In case (β),

$$\Omega_n=(X_-^{w_1}\cap\mathring{X}_{w_n}^-)\times(X_-^{w_2}\cap\mathring{X}_{w_n}^-)\times(X_{v_nv^{-1}}\cap(v_nv^{-1}s_\alpha B^-\underline{o})).$$

Fix $\tau=z^{\sum_{\alpha_i\notin\Delta(P)}d_ix_i}:\mathbb{C}^*\longrightarrow T$, where $d_i>0$ is an integer such that d_ix_i is in the coroot lattice. We now apply [Res21, Lemma 11.5] to Ω_n endowed with the action of $\mathbb{C}^*$ induced by τ. The checking of the assumptions $(i)-(iv)$ of [Res21, Lemma 11.5] are done in the proof of [Res21, Lemma 6.5]. The only remaining point, with the notation of [Res21, Lemma 11.5], is to prove that $k\geq 0$. This is done as in [Res21, Proof of Lemma 6.5, specifically the part 'The line bundle on the affine subvarieties']. Here, the non-negativity of k is due to the fact that $\mathcal{L}$ is nonnegative restricted to the projective lines $\ell_{\alpha,i}$ for any $(\alpha,i)\in\mathcal{D}$, which is our assumption (cf. Theorem 1.5). This proves (b).

We now come to the proof of (c). Since $\mathrm{H}^0(C^+,\mathcal{L})^P$ is contained in $\mathrm{H}^0(C^+,\mathcal{L})^\tau$, [Res21, Lemma 6.6] implies that the map (c) of the lemma is injective. We now prove its surjectivity:

Consider the map $\theta:P\longrightarrow L$, $p\longmapsto\lim_{t\to 0}\tau(t)p\tau(t^{-1})$, which is a surjective group homomorphism. This provides an action of P^3 on

C through the homomorphism θ. Then, the regular map $\gamma : C^+ \longrightarrow C$, $x \longmapsto \lim_{t\to 0} \tau(t)x$ is P^3-equivariant.

Take a G^3-equivariant lift of $\mathcal{L}$ over $\mathcal{X}$ under the componentwise action of G^3 on $\mathcal{X}$. (This is possible since any character of the diagonal of $(G/[G,G])^3$ extends to a character of $(G/[G,G])^3$. Thus, we will think of $\mathcal{L}$ as a G^3-equivariant line bundle over $\mathcal{X}$. Denote

$$x = (w_1^{-1}\underline{o}^-, w_2^{-1}\underline{o}^-, v^{-1}\underline{o}) \in C.$$

Then, $C = L^3 \cdot x$ and $C^+ = P^3 \cdot x$. Thus,

$$\operatorname{Pic}^{P^3}(C^+) \simeq X(P^3_x) \text{ and } \operatorname{Pic}^{L^3}(C) \simeq X(L^3_x), \tag{37}$$

where $X(\ )$ denotes the character group and P^3_x (resp. L^3_x) denotes the isotropy subgroup of P^3 (resp. L^3) at x. Now, it is easy to see that

$$P^3_x = L^3_x \cdot (U_{w_1} \times U_{w_2} \times U'_v)\,, \tag{38}$$

where U_w (resp. U'_v) is the finite-dimensional (resp. finite-codimensional) subgroup of the unipotent radical U_P of P with Lie algebra $\bigoplus_{\beta\in\Phi^+\cap w^{-1}\Phi^-} \mathfrak{g}_\beta$ (resp. $\bigoplus_{\beta\in(\Phi^+\setminus\Phi^+_L)\cap v^{-1}\Phi^+} \mathfrak{g}_\beta$), where Φ^+ (resp. Φ^+_L) is the set of positive roots of G (resp. L). Moreover, since L^3 normalizes U^3_P, L^3_x normalizes $U_{w_1} \times U_{w_2} \times U'_v$. Now, for a finite-dimensional unipotent group, any character is trivial and similarly U'_v has no nontrivial characters by the same proof as that of Lemma 5.1(a). Thus,

$$X(P^3_x) = X(L^3_x).$$

Hence, by combining the equations (37) and (38), we get

$$\operatorname{Pic}^{P^3}(C^+) \simeq \operatorname{Pic}^{L^3}(C). \tag{39}$$

We define the P^3-action on $\mathcal{L}_{|C}$ compatible with the action of P^3 on C by demanding that U^3_P acts trivially on $\mathcal{L}_{|C}$. Thus, we get a P^3-equivariant line bundle $\gamma^*(\mathcal{L}_{|C})$ over C^+. We also have a P^3-equivariant line bundle $\mathcal{L}_{|C^+}$. By the equation (39), we readily see that

$$\mathcal{L}_{|C^+} \simeq \gamma^*(\mathcal{L}_{|C}), \text{ as } P^3\text{-equivariant line bundles;}$$

in particular, as diagonal P-equivariant line bundles.

Thus, for $\sigma \in \mathrm{H}^0(C,\mathcal{L})^L$, $\gamma^*(\sigma) \in \mathrm{H}^0(C^+,\mathcal{L})^P$ and $\gamma^*(\sigma)_{|C} = \sigma$. We deduce thus that the restriction map $\mathrm{H}^0(C^+,\mathcal{L})^P \to \mathrm{H}^0(C,\mathcal{L})^L$ is surjective. This proves (c). □

We thus conclude that the first horozontal line in the above diagram ($\diamond$) satisfies:

$$\begin{tikzcd}
\mathrm{H}^0(\mathcal{X},\mathcal{L})^G \arrow[r, "\sim", "\eta^*"'] & \mathrm{H}^0(\mathfrak{X},\mathcal{L})^G \arrow[r, hook, "\alpha^*"] & \mathrm{H}^0(G\times^P \tilde{C}^+,\mathcal{L})^G \arrow[r, "\sim"] \arrow[dr, "\sim"', "\beta^*"'] & \mathrm{H}^0(\tilde{C}^+,\mathcal{L})^P \arrow[d, "\wr"] \\
 & & & \mathrm{H}^0(C,\mathcal{L})^L,
\end{tikzcd}$$

where η^* is an isomorphism and the last vertical map is an isomorphism (which follows from Lemma 5.2).

5.3. **Isomorphisms induced from slice.** Since $G\times^B \mathcal{X}_{\mathscr{s}} \simeq \mathcal{X}$ (cf. equation (34)), we get that $i_1^* : \mathrm{H}^0(\mathcal{X},\mathcal{L})^G \longrightarrow \mathrm{H}^0(\mathcal{X}_{\mathscr{s}},\mathcal{L})^B$ is an isomorphism. Similarly, i_2^* is an isomorphism by using equation (35). Further, γ^* is an isomorphism since $\alpha : G\times^P \tilde{C}^+ \to \tilde{\mathfrak{X}}'$ is a G-equivariant isomorphism and so is

$$G\times^B \tilde{\mathfrak{X}}_{\mathscr{s}} \simeq \tilde{\mathfrak{X}}', \;\; [g,x]\mapsto gx. \tag{40}$$

5.4. **Isomorphisms obtained from restriction to some open subsets.**

Lemma 5.3. *The restriction map* $\mathrm{H}^0(\mathcal{X}_{\mathscr{s}},\mathcal{L}) \longrightarrow \mathrm{H}^0(\overset{\circ\circ}{\mathcal{X}}_{\mathscr{s}},\mathcal{L})$ *is an isomorphism and hence* i_4^* *is an isomorphism.*

Proof. For any $w\in W$, consider the Schubert variety

$$X_w^- := \overline{B^-wB^-/B^-} \subset G/B^-.$$

For any $w_1, w_2 \in W$, consider the open embedding

$$i_{w_1,w_2} : \overset{\circ\circ}{\mathcal{X}}_{\mathscr{s}} \cap \left(X_{w_1}^- \times X_{w_2}^- \times \{\underline{o}\}\right) \hookrightarrow X_{w_1}^- \times X_{w_2}^- \times \{\underline{o}\}.$$

The complement

$$Y_{w_1,w_2} := \left(X_{w_1}^- \times X_{w_2}^- \times \{\underline{o}\}\right) \backslash \mathrm{Im}(i_{w_1,w_2})$$

has its irreducible components of the form

$$(X_{w_1}^- \cap \overline{BuB^-/B^-}) \times X_{w_2}^- \times \{\underline{o}\} \;\text{ or }\; X_{w_1}^- \times (X_{w_2}^- \cap \overline{BuB^-/B^-}) \times \{\underline{o}\}$$

for some $\ell(u) = 2$. But, by [Kum02, Lemma 7.3.10], each of these irreducible components have codimension 2 in (the finite-dimensional) $X_{w_1}^- \times X_{w_2}^- \times \{\underline{o}\}$. Thus, by the normality of X_w^- (cf. [Kum02, Theorem 8.2.2(b)], we see that the restriction map

$$\mathrm{H}^0(X_{w_1}^- \times X_{w_2}^- \times \{\underline{o}\},\mathcal{L}) \to \mathrm{H}^0(\overset{\circ\circ}{\mathcal{X}}_{\mathscr{s}} \cap (X_{w_1}^- \times X_{w_2}^- \times \{\underline{o}\}),\mathcal{L})$$

is an isomorphism. Taking limits over w_1, w_2, we get the lemma. $\square$

As observed earlier, $\tilde{\mathfrak{X}}'$ is irreducible and hence so is $\tilde{\mathfrak{X}}_{\delta}$ by the isomorphism (40) and $\tilde{\mathfrak{X}}_{\delta} \cap \mathring{\mathring{\mathfrak{X}}}_{\delta}$ is open in $\tilde{\mathfrak{X}}_{\delta}$. It follows thus that the map

$$i_6^* : \mathrm{H}^0(\tilde{\mathfrak{X}}_{\delta}, \mathcal{L})^B \longrightarrow \mathrm{H}^0(\tilde{\mathfrak{X}}_{\delta} \cap \mathring{\mathring{\mathfrak{X}}}_{\delta}, \mathcal{L})^B$$

is injective.

We now prove that the maps η_2^* and i_7^* are isomorphisms.

Lemma 5.4. *The map* $\mathrm{H}^0(\mathring{\mathring{\mathcal{X}}}_{\delta}, \mathcal{L}) \to \mathrm{H}^0(\mathring{\mathring{\mathfrak{X}}}_{\delta}, \mathcal{L})$ *induced from* η_2 *is an isomorphism and hence so is* η_2^* *.*

Proof. It is easy to see that the map η_2 is proper. Moreover, it is birational by [Res21, Lemma 6.2]. In particular, it is surjective. If $\mathring{\mathring{\mathcal{X}}}_{\delta}$ and $\mathring{\mathring{\mathfrak{X}}}_{\delta}$ are finite-dimensional, the lemma follows from Zariski's main theorem (see, e.g., [Har77, Chap. III, Corollary 11.4]). The argument used to prove [Res21, Lemma 6.3] allows us to prove that the above map $\mathrm{H}^0(\mathring{\mathring{\mathcal{X}}}_{\delta}, \mathcal{L}) \to \mathrm{H}^0(\mathring{\mathring{\mathfrak{X}}}_{\delta}, \mathcal{L})$ is an isomorphism. Indeed, the only specific assumption is that $\mathring{\mathring{\mathfrak{X}}}_{\delta}$ can be written as a union of irreducible finite-dimensional closed subsets (called ind-irreducible in [Res21]). To prove this, since $\mathring{\mathring{\mathfrak{X}}}_{\delta}$ is an open subset of $\mathfrak{X}_{\delta}$, by the isomorphism (35), it suffices to show that $\mathfrak{X}$ is ind-irreducible. Further, since $\mathfrak{X} = \overline{G \cdot (P/P, \bar{C}^+)}$ (see above Lemma 4.3), it suffices to show that $\overline{Pw_i^{-1}\underline{o}^-}$ and $\overline{Pv^{-1}\underline{o}}$ are ind-irreducible. But, as observed earlier in the proof of Lemma 5.2 equality (36), $\overline{Pw_i^{-1}\underline{o}^-} = \overline{Bw_i^{-1}\underline{o}^-}$. So, it is ind-irreducible. Similarly, $\overline{Pv^{-1}\underline{o}} = \bigcup_{v_n \in W_P} X_{v_n v^{-1}}$, where v_n is an increasing cofinal sequence in W_P. This shows that $\overline{Pv^{-1}\underline{o}}$ is also ind-irreducible. Thus, $\mathfrak{X}$ is ind-irreducible. $\square$

Lemma 5.5. *The restriction map* $\mathrm{H}^0(\mathring{\mathring{\mathfrak{X}}}_{\delta}, \mathcal{L}) \to \mathrm{H}^0(\tilde{\mathfrak{X}}_{\delta} \cap \mathring{\mathring{\mathfrak{X}}}_{\delta}, \mathcal{L})$ *is an isomorphism and hence so is* i_7^**.*

Proof. As earlier, consider the action of U on X_v^P:

$$\theta \,:\, U \longrightarrow \mathrm{Aut}(X_v^P).$$

Then, $\mathrm{Im}\,\theta$ is a finite-dimensional unipotent group U_v. As a consequence, $\mathrm{Ker}\,\theta$ is a normal subgroup of U of finite-codimension.

Consider now the group

$$U_1 = \mathrm{Ker}\,\theta \cap \left(\bigcap_{\alpha \in \Delta} s_\alpha U s_\alpha \right).$$

Then, U_1 is again a normal subgroup of U of finite-codimension (i.e., U/U_1 is a finite-dimensional group). There exists a closed subgroup $\mathcal{U}$ of U_1 of finite-codimension such that $\mathcal{U}$ is normal in U, $\mathcal{U}^2 := \mathcal{U} \times \mathcal{U}$

acts freely and properly on $\overset{\circ\circ}{\mathcal{X}}_{\mathfrak{s}}$ (under the action $(u_1,u_2)\cdot(x_1,x_2,\underline{o}) = (u_1x_1,u_2x_2,\underline{o})$) and the quotient map $\pi_{\mathcal{X}} : \overset{\circ\circ}{\mathcal{X}}_{\mathfrak{s}} \longrightarrow \mathcal{U}^2\backslash\overset{\circ\circ}{\mathcal{X}}_{\mathfrak{s}}$ is a principal $\mathcal{U}^2$-bundle (cf. [Kum17, Lemma 6.1]). Moreover, since η_2 is proper (cf. Proof of Lemma 5.4), $\mathcal{U}^2$ acts freely and properly on $\overset{\circ\circ}{\mathfrak{X}}_{\mathfrak{s}}$.

Consider the action of $\mathcal{U}^2$ on $X_v^P \times \mathcal{X}_{\mathfrak{s}}$ given by

$$(u_1,u_2).(y,g_1\underline{o}^-,g_2\underline{o}^-,\underline{o}) = (y,u_1g_1\underline{o}^-,u_2g_2\underline{o}^-,\underline{o}). \tag{41}$$

Since $\mathcal{U}$ acts trivially on X_v^P and $y\in X_v^P$, the condition $y\in u_ig_iX_P^{w_i}$ is equivalent to $y\in g_iX_P^{w_i}$. In particular, $\mathfrak{X}_{\mathfrak{s}}$, $\overset{\circ\circ}{\mathfrak{X}}_{\mathfrak{s}}$ and $\tilde{\mathfrak{X}}_{\mathfrak{s}}$ are all stable by the action of $\mathcal{U}^2$. Moreover, $\eta_2 : \overset{\circ\circ}{\mathfrak{X}}_{\mathfrak{s}} \to \overset{\circ\circ}{\mathcal{X}}_{\mathfrak{s}}$ is $\mathcal{U}^2$-equivariant.

We consider the associated quotients:

$$\begin{tikzcd}
\overset{\circ\circ}{\mathcal{X}}_{\mathfrak{s}} \arrow[d, "\pi_{\mathcal{X}}"'] & \overset{\circ\circ}{\mathfrak{X}}_{\mathfrak{s}} \arrow[l, "\eta_2"'] \arrow[d, "\pi_{\mathfrak{X}}"'] & \tilde{\mathfrak{X}}_{\mathfrak{s}}\cap\overset{\circ\circ}{\mathfrak{X}}_{\mathfrak{s}} \arrow[l, hook'] \arrow[d] \\
\mathcal{U}^2\backslash\overset{\circ\circ}{\mathcal{X}}_{\mathfrak{s}} & \mathcal{U}^2\backslash\overset{\circ\circ}{\mathfrak{X}}_{\mathfrak{s}} \arrow[l, "\bar\eta_2"'] & \mathcal{U}^2\backslash\left(\tilde{\mathfrak{X}}_{\mathfrak{s}}\cap\overset{\circ\circ}{\mathfrak{X}}_{\mathfrak{s}}\right). \arrow[l, hook']
\end{tikzcd}$$

Let $\Omega_{\mathcal{X}}$ be an open subset of $\mathcal{U}^2\backslash\overset{\circ\circ}{\mathcal{X}}_{\mathfrak{s}}$ such that the quotient $\pi_{\mathcal{X}}$ is trivial over $\Omega_{\mathcal{X}}$. (It can be seen that $\pi_{\mathcal{X}}$ is locally trivial.) Set $\Omega_{\mathfrak{X}} = \bar\eta_2^{-1}(\Omega_{\mathcal{X}})$. Choosing a section of $\pi_{\mathcal{X}}$ over $\Omega_{\mathcal{X}}$ and taking the induced section of $\pi_{\mathfrak{X}}$ over $\Omega_{\mathfrak{X}}$, we get

$$\pi_{\mathcal{X}}^{-1}(\Omega_{\mathcal{X}}) \simeq \mathcal{U}^2\times\Omega_{\mathcal{X}} \text{ and } \pi_{\mathfrak{X}}^{-1}(\Omega_{\mathfrak{X}}) \simeq \mathcal{U}^2\times\Omega_{\mathfrak{X}} \tag{42}$$

such that $\eta_{2|\pi_{\mathfrak{X}}^{-1}(\Omega_{\mathfrak{X}})}$ under the above isomorphism is given by

$$\eta_2(\tilde u,x) = (\tilde u,\bar\eta_2(x)), \text{ for } \tilde u\in\mathcal{U}^2 \text{ and } x\in\Omega_{\mathfrak{X}}.$$

Since $\mathcal{L}$ is G^3-equivariant with G^3 acting on $\mathcal{X}$ componentwise, we get that $\mathcal{L}_{|\overset{\circ\circ}{\mathcal{X}}_{\mathfrak{s}}}$ and $\mathcal{L}_{|\overset{\circ\circ}{\mathfrak{X}}_{\mathfrak{s}}}$ are $\mathcal{U}^2$-equivariant. Since $\mathcal{U}^2$ acts freely on $\overset{\circ\circ}{\mathcal{X}}_{\mathfrak{s}}$ (resp. $\overset{\circ\circ}{\mathfrak{X}}_{\mathfrak{s}}$), $\mathcal{L}_{|\overset{\circ\circ}{\mathcal{X}}_{\mathfrak{s}}}$ (resp. $\mathcal{L}_{|\overset{\circ\circ}{\mathfrak{X}}_{\mathfrak{s}}}$) descends to a unique line bundle $\bar{\mathcal{L}}$ over $\mathcal{U}^2\backslash\overset{\circ\circ}{\mathcal{X}}_{\mathfrak{s}}$ (resp. $\mathcal{U}^2\backslash\overset{\circ\circ}{\mathfrak{X}}_{\mathfrak{s}}$). Hence, under the decompositions (42),

$$\mathcal{L}_{|\mathcal{U}^2\times\Omega_{\mathcal{X}}} = \mathscr{O}_{\mathcal{U}^2}\boxtimes\bar{\mathcal{L}}_{|\Omega_{\mathcal{X}}}, \text{ and } \mathcal{L}_{|\mathcal{U}^2\times\Omega_{\mathfrak{X}}} = \mathscr{O}_{\mathcal{U}^2}\boxtimes\bar{\mathcal{L}}_{|\Omega_{\mathfrak{X}}}. \tag{43}$$

Now, the map

$$\bar\eta_2 : \mathcal{U}^2\backslash\overset{\circ\circ}{\mathfrak{X}}_{\mathfrak{s}} \to \mathcal{U}^2\backslash\overset{\circ\circ}{\mathcal{X}}_{\mathfrak{s}}$$

is proper. To prove this, consider the projection

$$\pi_2 : \mathcal{U}^2\backslash(X_v^P\times\overset{\circ\circ}{\mathcal{X}}_{\mathfrak{s}}) = X_v^P\times(\mathcal{U}^2\backslash\overset{\circ\circ}{\mathcal{X}}_{\mathfrak{s}}) \to \mathcal{U}^2\backslash\overset{\circ\circ}{\mathcal{X}}_{\mathfrak{s}}$$

with $\mathcal{U}^2$ acting on $X_v^P \times \mathring{\mathring{\mathcal{X}}}_s$ as in (41). This is clearly a projective morphism. Now,

$$\bar{\eta}_2 = (\pi_2)_{|\mathcal{U}^2\backslash\mathring{\mathring{\mathfrak{X}}}_s}.$$

Moreover, $\mathcal{U}^2\backslash\mathring{\mathring{\mathfrak{X}}}_s$ is a closed subset of $\mathcal{U}^2\backslash(X_v^P \times \mathring{\mathring{\mathcal{X}}}_s)$ (as can easily be seen) and hence $\bar{\eta}_2$ is a projective morphism.

Further, $\bar{\eta}_2$ is a birational map since so is η_2 (cf. Proof of Lemma 5.4).

By the following lemma, $\bar{\eta}_2\left(\mathcal{U}^2\backslash(\mathring{\mathring{\mathfrak{X}}}_s\backslash\tilde{\mathfrak{X}}_s)\right)$ is of codimension ≥ 2 in $\mathcal{U}^2\backslash\mathring{\mathring{\mathcal{X}}}_s$. Moreover, $\mathcal{U}^2\backslash\mathring{\mathring{\mathcal{X}}}_s$ is normal (cf. [KS09, Proposition 3.2]). In fact, it is smooth (cf. [Kum17, §10]). Thus, by Proposition 2.1, the restriction map

$$\mathrm{H}^0(\Omega_{\mathfrak{X}}, \bar{\mathcal{L}}) \to \mathrm{H}^0(\Omega'_{\mathfrak{X}}, \bar{\mathcal{L}}) \text{ is an isomorphism,} \tag{44}$$

for any open subset $\Omega_{\mathcal{X}} \subset \mathcal{U}^2\backslash\mathring{\mathring{\mathcal{X}}}_s$ over which $\pi_{\mathcal{X}}$ admits a section and $\Omega_{\mathfrak{X}} := \bar{\eta}_2^{-1}(\Omega_{\mathcal{X}})$, where $\Omega'_{\mathfrak{X}} := \Omega_{\mathfrak{X}} \cap \left(\mathcal{U}^2\backslash(\mathring{\mathring{\mathfrak{X}}}_s \cap \tilde{\mathfrak{X}}_s)\right)$. But, by the decomposition (43)

$$\begin{aligned} \mathrm{H}^0(\pi_{\mathfrak{X}}^{-1}(\Omega_{\mathfrak{X}}), \mathcal{L}) &\simeq \mathrm{H}^0(\mathcal{U}^2 \times \Omega_{\mathfrak{X}}, \mathscr{O}_{\mathcal{U}^2} \boxtimes \bar{\mathcal{L}}) \\ &= \varprojlim_n \mathbb{C}[\mathcal{U}_n^2] \otimes \mathrm{H}^0(\Omega_{\mathfrak{X}}, \bar{\mathcal{L}}), \end{aligned} \tag{45}$$

where $\{\mathcal{U}_n\}_{n\geq 0}$ is a filtration of $\mathcal{U}$ giving the ind-variety structure. Similarly,

$$\mathrm{H}^0(\pi_{\mathfrak{X}}^{-1}(\Omega'_{\mathfrak{X}}), \mathcal{L}) = \varprojlim_n \mathbb{C}[\mathcal{U}_n^2] \otimes \mathrm{H}^0(\Omega'_{\mathfrak{X}}, \bar{\mathcal{L}}). \tag{46}$$

Combining the equations (44) - (46), we get that the restriction map

$$\mathrm{H}^0(\pi_{\mathfrak{X}}^{-1}(\Omega_{\mathfrak{X}}), \mathcal{L}) \to \mathrm{H}^0(\pi_{\mathfrak{X}}^{-1}(\Omega'_{\mathfrak{X}}), \mathcal{L})$$

is an isomorphism. Since $\{\pi_{\mathfrak{X}}^{-1}(\Omega_{\mathfrak{X}})\}$ provides an open cover of $\mathring{\mathring{\mathfrak{X}}}_s$, we get that the restriction map

$$\mathrm{H}^0(\mathring{\mathring{\mathfrak{X}}}_s, \mathcal{L}) \to \mathrm{H}^0(\tilde{\mathfrak{X}}_s \cap \mathring{\mathring{\mathfrak{X}}}_s, \mathcal{L})$$

is an isomorphism. This proves the lemma modulo Lemma 5.6 below. □

5.5. **Smallness of the boundary of $\tilde{\mathfrak{X}}_s$.** The goal of this subsection is to prove the following lemma. We refer to [BKR, §7] for some parallel arguments.

Lemma 5.6. *With the notation as in the proof of Lemma 5.5, the image* $\bar{\eta}_2\left(\mathcal{U}^2\backslash(\mathring{\mathring{\mathfrak{X}}}_s\backslash\tilde{\mathfrak{X}}_s)\right)$ *is of codimension* ≥ 2 *in* $\mathcal{U}^2\backslash\mathring{\mathring{\mathcal{X}}}_s$.

This lemma will be a consequence of the nontransversality Corollary 5.8, which in turn is a consequence of Proposition 5.7.

Set, for $i \in \mathbb{N} := \{0, 1, 2, \dots\}$,

(47)
$$(\mathfrak{g}/\mathfrak{p})_i := \{\xi \in \mathfrak{g}/\mathfrak{p} \,:\, \mathrm{ad}(x_P)\cdot \xi = -i\xi\}, \text{ where } x_P := \sum_{\alpha_j \in \Delta\setminus\Delta(P)} x_j$$

and

(48)
$$(\mathfrak{g}/\mathfrak{p})_{\leq i} := \bigoplus_{j \leq i} (\mathfrak{g}/\mathfrak{p})_j.$$

Note that the $(\mathfrak{g}/\mathfrak{p})_{\leq i}$'s form a P-stable filtration of $\mathfrak{g}/\mathfrak{p}$.

Let $Z \subset G/P$ be a locally closed finite-dimensional subvariety of G/P and let z be a point of Z. Write $z = gP/P$. Set, for $i \in \mathbb{N}$,

(49)
$$d_i(z, Z) := \dim \left(T_{\dot{e}}(g^{-1}Z) \cap (\mathfrak{g}/\mathfrak{p})_{\leq i}\right), \text{ where } \dot{g} := gP/P \in G/P.$$

This indeed does not depend on the choice of g such that $z = gP/P$. Observe that $d_0(z, Z) = 0$, $d_n(z, Z) = \dim T_z Z$ for n large enough, and that $i \mapsto d_i(z, Z)$ is non-decreasing. Define, for any $i \in \mathbb{N}$,

$$\bar{d}_i(z, Z) = d_i(z, Z) - d_{i-1}(z, Z),$$

where we declare $d_{-1}(z, Z) = 0$. Thus, $\bar{d}_m(z, Z) = 0$, for $m > n$.

Similarly, let $z \in Z \subset G/P$, where Z has finite-codimension. Write $z = gP/P$. Set, for $i \in \mathbb{N}$,

(50)
$$d^i(z, Z) := \dim \left(\frac{T_{\dot{e}}(g^{-1}Z) + (\mathfrak{g}/\mathfrak{p})_{\leq i}}{T_{\dot{e}}(g^{-1}Z)} \right).$$

Again this does not depend on the choice of g such that $z = gP/P$. Observe that $d^0(z, Z) = 0$, that $d^n(z, Z)$ is the codimension of $T_z Z$ for n large enough, and that $i \mapsto d^i(z, Z)$ is non-decreasing.

Proposition 5.7. *Let $v \in W^P$ and β be a positive real root such that $w = s_\beta v \in W^P$.*

(i) *If $\ell(w) = \ell(v) - 1$, then*
$$d_i(\dot{w}, X_v^P) \geq d_i(\dot{v}, X_v^P), \ \ \forall i \in \mathbb{N}.$$
Moreover, if β is **not** *a simple root,*
$$d_{i_o}(\dot{w}, X_v^P) > d_{i_o}(\dot{v}, X_v^P), \ \ \textit{for some } i_o \in \mathbb{N}.$$

(ii) *If $\ell(w) = \ell(v) + 1$, then*
$$d^i(\dot{w}, X^v_P) \leq d^i(\dot{v}, X^v_P), \ \ \forall i \in \mathbb{N}.$$

Moreover, if β is **not** *a simple root,*

$$d^{i_o}(\dot{w}, X_P^v) < d^{i_o}(\dot{v}, X_P^v), \text{ for some } i_o \in \mathbb{N}.$$

Proof. We first translate the first assertion in a combinatorial statement in terms of roots. Given a T-vector space E, we denote by $\Phi(E)$ the set of weights of T acting on E.

Let Φ^+ (resp. Φ^-) be the set of positive (resp. negative) roots. Since $T_{\dot{e}}(v^{-1}X_v^P)$ is multiplicity free as a T-module, and $\Phi(T_{\dot{e}}(v^{-1}X_v^P)) = \{\theta \in \Phi^- : v\theta \in \Phi^+\}$, we have

$$d_i(\dot{v}, X_v^P) = \sharp\{\theta \in \Phi^- : v\theta \in \Phi^+ \text{ and } -\theta(x_P) \leq i\}, \ \forall i \geq 1. \tag{51}$$

Consider the unique T-stable curve ℓ containing both $\dot{v}$ and $\dot{w}$. Observe that ℓ is isomorphic to $\mathbb{P}^1$, $\Phi(T_{\dot{v}}\ell) = \{\beta\}$, $\Phi(T_{\dot{w}}\ell) = \{-\beta\}$ and ℓ is contained in X_v^P. Moreover, X_w^P is contained in X_v^P and

$$T_{\dot{w}}X_v^P = T_{\dot{w}}X_w^P \oplus T_{\dot{w}}\ell. \tag{52}$$

After translating by w^{-1}, equality (52) implies that

$$\Phi(T_{\dot{e}}(w^{-1}X_v^P)) = \Phi(T_{\dot{e}}(w^{-1}X_w^P)) \cup \{-w^{-1}\beta\}.$$

It follows that

$$d_i(\dot{w}, X_v^P) = \sharp\{\theta \in \Phi^- : w\theta \in \Phi^+ \text{ and } -\theta(x_P) \leq i\} + \delta_i^{(w^{-1}\beta)(x_P)}, \ \forall i \geq 1, \tag{53}$$

where $\delta_i^m = 1$ if $m \leq i$ and 0 otherwise.

We deduce that the first assertion of the proposition is equivalent to $\forall i \geq 1$:

$$\begin{aligned} &\sharp\{\theta \in \Phi^- : w\theta \in \Phi^+ \text{ and } -\theta(x_P) \leq i\} + \delta_i^{(w^{-1}\beta)(x_P)} \geq \\ &\sharp\{\theta \in \Phi^- : v\theta \in \Phi^+ \text{ and } -\theta(x_P) \leq i\}, \end{aligned} \tag{54}$$

and the existence of i_o with a strict inequality (54) if β is not simple.

We now translate the second assertion of the proposition in a combinatorial statement. First observe that, since $v \in W^P$,

$$\Phi\left(\frac{T_{\dot{e}}(G/P)}{T_{\dot{e}}(v^{-1}X_P^v)}\right) = \{\theta \in \Phi^- : v\theta \in \Phi^+\}.$$

We deduce that

$$d^i(\dot{v}, X_P^v) = \sharp\{\theta \in \Phi^- : v\theta \in \Phi^+ \text{ and } -\theta(x_P) \leq i\}, \ \forall i \geq 1. \tag{55}$$

Now, since $\ell(w) = \ell(v) + 1$, $X_P^v \supset X_P^w$, ℓ is contained in X_P^v, $\Phi(T_{\dot{w}}\ell) = \{\beta\}$ and $\Phi(T_{\dot{v}}\ell) = \{-\beta\}$. Moreover, we have the following exact sequence

$$0 \longrightarrow T_{\dot{w}}\ell \longrightarrow \frac{T_{\dot{w}}(G/P)}{T_{\dot{w}}X_P^w} \longrightarrow \frac{T_{\dot{w}}(G/P)}{T_{\dot{w}}X_P^v} \longrightarrow 0.$$

After translation by w^{-1}, we obtain that

$$\Phi\left(\frac{T_{\dot e}(G/P)}{T_{\dot e}(w^{-1}X_P^w)}\right) = \Phi\left(\frac{T_{\dot e}(G/P)}{T_{\dot e}(w^{-1}X_P^v)}\right) \sqcup \{w^{-1}\beta\}.$$

This implies that

(56)

$$d^i(\dot w, X_P^v) = \sharp\{\theta \in \Phi^- : w\theta \in \Phi^+ \text{ and } -\theta(x_P) \le i\} - \delta_i^{-(w^{-1}\beta)(x_P)}, \ \forall i \ge 1.$$

With (55) and (56), the second assertion of the proposition is equivalent to $\forall i \ge 1$:

$$\begin{aligned} &\sharp\{\theta \in \Phi^- : v\theta \in \Phi^+ \text{ and } -\theta(x_P) \le i\} \ \ge \\ &\sharp\{\theta \in \Phi^- : w\theta \in \Phi^+ \text{ and } -\theta(x_P) \le i\} - \delta_i^{-(w^{-1}\beta)(x_P)}, \end{aligned} \tag{57}$$

with a strict inequality for some i_o, if β is not simple.

Now, observe that given (v, w) such that $w = s_\beta v$ and $\ell(w) = \ell(v)+1$, one gets (v', w') such that $w' = s_\beta v'$ and $\ell(w') = \ell(v') - 1$ by setting $w' = v$ and $v' = w$. By (54) and (57), the first assertion for (v', w') implies the second one for (v, w) (note that $w'^{-1}\beta = -w^{-1}\beta$). It is now sufficient to prove the first assertion.

From now on, we assume that $\ell(w) = \ell(v)-1$. Recall that we denote $v' \to v$ if $v' \in W^P$, $\ell(v') = \ell(v) - 1$ and $v' \le v$. Set

$$\hat X_v^P = \mathring{X}_v^P \cup \left(\bigcup_{v'\to v} \mathring{X}_{v'}^P\right), \ \text{ where } \mathring{X}_v^P := BvP/P.$$

Then, $\hat X_v^P$ is a smooth open subset of X_v^P. Set

$$\hat Y_v^P = \pi^{-1}(\hat X_v^P),$$

where $\pi : G \longrightarrow G/P$ is the natural projection. Define two vector bundles over $\hat Y_v^P$:

$$\mathcal{V} := \bigcup_{g \in \hat Y_v^P} (\{g\} \times T_{\dot e}(g^{-1}X_v^P)) \longrightarrow \hat Y_v^P,$$

and the trivial bundle

$$\varepsilon_i := \hat Y_v^P \times \frac{T_{\dot e}(G/P)}{(\mathfrak{g}/\mathfrak{p})_{\le i}},$$

for any fixed $i \in \mathbb{N}$. The inclusion $T_{\dot e}(g^{-1}X_v^P) \subset T_{\dot e}(G/P)$ induces a bundle map

$$\varphi_i : \mathcal{V} \longrightarrow \varepsilon_i.$$

On the open subset BvP, the rank of φ_i is constant since

$$T_{\dot e}((bvp)^{-1}X_v^P) = T_{\dot e}(p^{-1}v^{-1}b^{-1}X_v^P) = p^{-1}T_{\dot e}(v^{-1}X_v^P),$$

and $(\mathfrak{g}/\mathfrak{p})_{\leq i}$ is P-stable.

On the other hand, the subset of points in $\hat{Y}_v^P$, where the rank of φ_i is maximum is open. Hence, this rank is maximum at any point of $BvP \subset \hat{Y}_v^P$; in particular, at v. This shows the inequalities of the first assertion of the proposition.

Note that

$$\rho - v^{-1}\rho = -\sum_{\theta\in\Phi^-\cap w^{-1}\Phi^+} \theta.$$

Hence,

$$(\rho - v^{-1}\rho)(x_P) = \sum_{\theta\in\Phi^-\cap v^{-1}\Phi^+} -\theta(x_P).$$

But by the equation (51),

$$\sharp\{\theta \in \Phi^- : v\theta \in \Phi^+ \text{ and } -\theta(x_P) = i\} = d_i(\dot v, X_v^P) - d_{i-1}(\dot v, X_v^P), \ \ \forall i \geq 1.$$

Hence,

$$\begin{aligned}(\rho - v^{-1}\rho)(x_P) &= \textstyle\sum_{j\geq 1} j\bar d_j(\dot v, X_v^P)\\ &= \ell(v) + \textstyle\sum_{j\geq 2}(j-1)\bar d_j(\dot v, X_v^P),\end{aligned}$$

since, by the equation (51), $d_m = \ell(v)$ for large enough m. Similarly,

$$(\rho - w^{-1}\rho)(x_P) = \ell(w) + \sum_{j\geq 2}(j-1)\bar d_j(\dot w, X_w^P).$$

Since $\ell(w) = \ell(v) - 1$, we get

(58)

$$(\rho - w^{-1}\rho - (\rho - v^{-1}\rho))(x_P) = -1 + \sum_{j\geq 2}(j-1)\left(\bar d_j(\dot w, X_w^P) - \bar d_j(\dot v, X_v^P)\right).$$

On the other hand, since $w = s_\beta v$, we get

$$\begin{aligned}\rho - w^{-1}\rho - (\rho - v^{-1}\rho) &= -w^{-1}\rho + w^{-1}s_\beta\rho\\ &= w^{-1}(s_\beta\rho - \rho)\\ &= -\langle\rho, \beta^\vee\rangle w^{-1}\beta. \qquad (59)\end{aligned}$$

Combining the equations (58) and (59), we get

$$1 + \sum_{j\geq 2}(j-1)\left(\bar d_j(\dot v, X_v^P) - \bar d_j(\dot w, X_w^P)\right) = \langle\rho, \beta^\vee\rangle(w^{-1}\beta)(x_P).$$

But by the equation (51) (for v replaced by w) and the equation (53), we have

$$d_i(\dot{w}, X_v^P) = d_i(\dot{w}, X_w^P) + \delta_i^{(w^{-1}\beta)(x_P)}, \ \ \forall i \geq 1$$

and hence (for $k := (w^{-1}\beta)(x_P)$)

$$1 + \sum_{j\geq 2, j\neq k} (j-1)\left(\bar{d}_j(\dot{v}, X_v^P) - \bar{d}_j(\dot{w}, X_v^P)\right) + (k-1)$$
$$\left(\bar{d}_k(\dot{v}, X_v^P) - \bar{d}_k(\dot{w}, X_v^P) + 1\right) = \langle \rho, \beta^\vee\rangle (w^{-1}\beta)(x_P). \tag{60}$$

If possible, assume that

$$d_j(\dot{v}, X_v^P) = d_j(\dot{w}, X_v^P), \ \ \forall j \geq 1. \tag{61}$$

Equivalently,

$$\bar{d}_j(\dot{v}, X_v^P) = \bar{d}_j(\dot{w}, X_v^P), \ \ \forall j \geq 1.$$

Then, the equation (60) implies that

$$k = \langle \rho, \beta^\vee\rangle k.$$

But, $\langle \rho, \beta^\vee\rangle \geq 1$. Hence,

$$\langle \rho, \beta^\vee\rangle = 1.$$

We deduce that β is simple if (61) holds. This ends the proof of the proposition. ☐

Corollary 5.8. *Let $w_1, w_2, v \in W^P$ be as in Theorem 1.5. In particular, $\ell(v) = \ell(w_1) + \ell(w_2)$. Let $x \in G/P$ and g, g_1, g_2 in G be such that x belongs to $g\hat{X}_v^P \cap g_1\hat{X}_P^{w_1} \cap g_2\hat{X}_P^{w_2}$, where $\hat{X}_v^P$ is as defined in the proof of Proposition 5.7 and*

$$\hat{X}_P^w := \mathring{X}_P^w \cup \left(\bigcup_{w\to w'} \mathring{X}_P^{w'}\right), \ \ \textit{where } \mathring{X}_P^{w'} := B^- w'P/P.$$

We assume that there exists a non-simple real root β such that one of the following two conditions holds:

(i) *$\ell(s_\beta v) = \ell(v) - 1$, $s_\beta v \in W^P$ and $x \in g\mathring{X}_{s_\beta v}^P$.*

(ii) *$\ell(s_\beta w_1) = \ell(w_1) + 1$, $s_\beta v \in W^P$ and $x \in g_1\mathring{X}_P^{s_\beta w_1}$.*

Then, the intersection $g\hat{X}_v^P \cap g_1\hat{X}_P^{w_1} \cap g_2\hat{X}_P^{w_2}$ is not transverse at x.

Proof. It suffices to prove that the standard linear map

$$\theta \,:\, T_x(gX_v^P) \longrightarrow \frac{T_x(G/P)}{T_x(g_1X_P^{w_1})} \oplus \frac{T_x(G/P)}{T_x(g_2X_P^{w_2})}$$

is not an isomorphism. Write $x = hP/P$. Up to changing (g, g_1, g_2) by $(h^{-1}g, h^{-1}g_1, h^{-1}g_2)$, we may assume that $h = e$.

Observe that

$$\theta\left(T_{\dot e}(gX_v^P)\cap(\mathfrak{g}/\mathfrak{p})_{\leq i}\right)\subset\frac{(T_{\dot e}(g_1X_P^{w_1}))+(\mathfrak{g}/\mathfrak{p})_{\leq i}}{T_{\dot e}(g_1X_P^{w_1})}\oplus\frac{(T_{\dot e}(g_2X_P^{w_2}))+(\mathfrak{g}/\mathfrak{p})_{\leq i}}{T_{\dot e}(g_2X_P^{w_2})}$$

Moreover, since ε_P^v occurs with coefficient 1 (in particular, nonzero) in the deformed product $\varepsilon_P^{w_1}\odot_0\varepsilon_P^{w_2}$ by assumption, $\forall i\in\mathbb{N}$,

$$\begin{aligned}\dim\left(T_{\dot e}(v^{-1}X_v^P)\cap(\mathfrak{g}/\mathfrak{p})_{\leq i}\right)=&\dim\left(\frac{T_{\dot e}(w_1^{-1}X_P^{w_1})+(\mathfrak{g}/\mathfrak{p})_{\leq i}}{T_{\dot e}(w_1^{-1}X_P^{w_1})}\right)\\&+\dim\left(\frac{T_{\dot e}(w_2^{-1}X_P^{w_2})+(\mathfrak{g}/\mathfrak{p})_{\leq i}}{T_{\dot e}(w_2^{-1}X_P^{w_2})}\right)\end{aligned}$$

(cf. [Res21, §7]). But, Proposition 5.7 implies that, for some i_o, the dimension of the first space is greater than that of the direct sum. Hence, the restriction of θ to $T_{\dot e}(gX_v^P)\cap(\mathfrak{g}/\mathfrak{p})_{\leq i}$ can not be injective. Thus, θ can not be an isomorphism. ☐

Lemma 5.9. *Let $f : Y\longrightarrow X$ be a dominant morphism between two quasi-projective irreducible varieties of the same dimension. Let $D\subset Y$ be an irreducible proper closed subset.*

Then, if $\overline{f(D)}$ has codimension one in X, then, for $x\in D$ general, $f^{-1}(f(x))$ is finite.

Proof. Otherwise, the general fibers of the restriction of f to $f^{-1}(\overline{f(D)})$ would have positive dimension. Since $\overline{f(D)}$ has codimension one, this implies that $\dim(f^{-1}(\overline{f(D)}))=\dim(Y)$ and hence $f^{-1}(\overline{f(D)})=Y$. But, f is assumed to be dominant. A contradiction. ☐

Proof of Lemma 5.6. For $(w_1',w_2',v')\in(W^P)^3$, we set

$$\mathring{\mathring{\mathfrak{X}}}_\flat(w_1',w_2',v'):=\{(x,g_1\underline{o}^-,g_2\underline{o}_-,\underline{o})\in X_{v'}^P\times\mathring{\mathring{\mathcal{X}}}_\flat\,:\,x\in g_1X_P^{w_1'}\cap g_2X_P^{w_2'}\}$$

and

$$\mathfrak{X}_\flat(w_1',w_2',v'):=\{(x,g_1\underline{o}^-,g_2\underline{o}_-,\underline{o})\in X_{v'}^P\times\mathcal{X}_\flat\,:\,x\in g_1X_P^{w_1'}\cap g_2X_P^{w_2'}\}.$$

The set $\mathring{\mathring{\mathfrak{X}}}_\flat\backslash\tilde{\mathfrak{X}}_\flat$ is the union of finitely many subsets of one of the following types:

Type I. $\mathring{\mathring{\mathfrak{X}}}_\flat(w_1',w_2',v')$, where $(w_1',w_2',v')\in(W^P)^3$, $w_1'\geq w_1$, $w_2'\geq w_2$, $v'\leq v$ and $\ell(w_1')+\ell(w_2')-\ell(v)\geq 2$.

Type II. $\mathring{\mathring{\mathfrak{X}}}_\flat(w_1,w_2,v')$, where $v'\in W^P$, $v'\leq v$, $\ell(v')=\ell(v)-1$ and $v'v^{-1}$ is not a simple reflection.

Type III. $\mathring{\mathring{\mathfrak{X}}}_\flat(w_1',w_2,v)$, where $w_1'\in W^P$, $w_1'\geq w_1$, $\ell(w_1')=\ell(w_1)+1$ and $w_1'w_1^{-1}$ is not a simple reflection.

Type IV. Like type III after exchanging w_1 and w_2.

It is sufficient to prove that the image by $\bar{\eta}_2$ of each one of these subsets has codimension at least two in $\mathcal{U}^2\backslash\overset{\circ\circ}{\mathcal{X}}_{\mathfrak{s}}$.

Consider (w_1', w_2', v') as in type I. There exists (w_1'', w_2'', v'') such that $w_1' \geq w_1'' \geq w_1$, $w_2' \geq w_2'' \geq w_2$, $v' \leq v'' \leq v$ and $\ell(w_1'')+\ell(w_2'')-\ell(v'') = 1$. The point $(\dot{v}'', v''(w_1'')^{-1}\underline{o}^-, v''(w_2'')^{-1}\underline{o}^-)$ belongs to $\mathfrak{X}_{\mathfrak{s}}(w_1'', w_2'', v'')$ and does not belong to $\mathfrak{X}_{\mathfrak{s}}(w_1', w_2', v')$. Hence, $\mathfrak{X}_{\mathfrak{s}}(w_1'', w_2'', v'')\backslash\mathfrak{X}_{\mathfrak{s}}(w_1', w_2', v')$ is open and nonempty in $\mathfrak{X}_{\mathfrak{s}}(w_1'', w_2'', v'')$.

To prove the lemma in this type, we can assume that $\overset{\circ\circ}{\mathfrak{X}}_{\mathfrak{s}}(w_1', w_2', v')$ is nonempty, then so is $\overset{\circ\circ}{\mathfrak{X}}_{\mathfrak{s}}(w_1'', w_2'', v'')$. Since $\mathfrak{X}_{\mathfrak{s}}(w_1'', w_2'', v'')$ is irreducible (cf. §5.1), we deduce that $(\mathfrak{X}_{\mathfrak{s}}(w_1'', w_2'', v'')\backslash\mathfrak{X}_{\mathfrak{s}}(w_1', w_2', v')) \cap (G/P \times \overset{\circ\circ}{\mathcal{X}}_s)$ is nonempty. Thus, we have a strict inclusion $\overset{\circ\circ}{\mathfrak{X}}_{\mathfrak{s}}(w_1', w_2', v') \subset \overset{\circ\circ}{\mathfrak{X}}_{\mathfrak{s}}(w_1'', w_2'', v'')$. Similarly, we have the strict inclusion:

$$\overset{\circ\circ}{\mathfrak{X}}_{\mathfrak{s}}(w_1'', w_2'', v'') \subset \overset{\circ\circ}{\mathfrak{X}}_{\mathfrak{s}}(w_1, w_2, v).$$

Combining the above two, we get the strict inclusions:

$$\overset{\circ\circ}{\mathfrak{X}}_{\mathfrak{s}}(w_1', w_2', v') \subset \overset{\circ\circ}{\mathfrak{X}}_{\mathfrak{s}}(w_1'', w_2'', v'') \subset \overset{\circ\circ}{\mathfrak{X}}_{\mathfrak{s}}(w_1, w_2, v).$$

Since these varieties are irreducible and $\mathcal{U}^2$-stable, we deduce that $\mathcal{U}^2\backslash\overset{\circ\circ}{\mathfrak{X}}_{\mathfrak{s}}(w_1', w_2', v')$ is of codimension at least two in $\mathcal{U}^2\backslash\overset{\circ\circ}{\mathfrak{X}}_{\mathfrak{s}}$. The lemma follows in this case since $\dim(\mathcal{U}^2\backslash\overset{\circ\circ}{\mathfrak{X}}_{\mathfrak{s}}) = \dim(\mathcal{U}^2\backslash\overset{\circ\circ}{\mathcal{X}}_{\mathfrak{s}})$ since $\bar{\eta}_2$ is a birational map (cf. Proof of Lemma 5.5).

Let now (w_1, w_2, v') be as in type II. Assume, for contradiction, that $\bar{\eta}_2(\mathcal{U}^2\backslash\overset{\circ\circ}{\mathfrak{X}}_{\mathfrak{s}}(w_1, w_2, v'))$ is a divisor. By Lemma 5.9, there exists $(g_1, g_2) \in G^2$ such that $X_v^P \cap g_1X_P^{w_1} \cap g_2X_P^{w_2}$ is finite and there exists $x \in X_{v'}^P$ such that $(x, g_1\underline{o}^-, g_2\underline{o}^-, \underline{o}) \in \overset{\circ\circ}{\mathfrak{X}}_{\mathfrak{s}}(w_1, w_2, v')$. By Corollary 5.8, the intersection $X_v^P \cap g_1X_P^{w_1} \cap g_2X_P^{w_2}$ is not transverse at x. Hence, the multiplicity of x in $X_v^P \cap g_1X_P^{w_1} \cap g_2X_P^{w_2}$ is at least 2. Since this intersection is finite, this implies that the coefficient of ε_P^v in $\varepsilon_P^{w_1} \cdot \varepsilon_P^{w_2}$: $n_{w_1,w_2}^v \geq 2$. A contradiction!

The last types III and IV work similarly. □

5.6. **Conclusion of the proof of Theorem 1.5.** Observe that $\tilde{\mathfrak{X}}_{\mathfrak{s}} \cap \overset{\circ\circ}{\mathfrak{X}}_{\mathfrak{s}}$ being open in the irreducible $\tilde{\mathfrak{X}}_{\mathfrak{s}}$, i_6^* is injective. Combining the results from Subsections 5.2 - 5.4, we get that

$$i_6^* \circ \gamma^* \circ \alpha^* \circ \eta^* : \mathrm{H}^0(\mathcal{X}, \mathcal{L})^G \to \mathrm{H}^0(\tilde{\mathfrak{X}}_{\mathfrak{s}} \cap \overset{\circ\circ}{\mathfrak{X}}_{\mathfrak{s}}, \mathcal{L})^B \text{ is injective}$$

and

$$i_7^* \circ \eta_2^* \circ i_4^* \circ i_1^* : \mathrm{H}^0(\mathcal{X}, \mathcal{L})^G \to \mathrm{H}^0(\tilde{\mathfrak{X}}_{\mathfrak{s}} \cap \overset{\circ\circ}{\mathfrak{X}}_{\mathfrak{s}}, \mathcal{L})^B \text{ is an isomorphism.}$$

From the commutative diagram $(\diamond)$ of Subsection 5.1, these two composite maps are equal forcing α^* to be an isomorphism. Thus, we get (from the top horizontal line of the commutative diagram ($\diamond$)) that the restriction map

$$\mathrm{H}^0(\mathcal{X},\mathcal{L})^G \to \mathrm{H}^0(C,\mathcal{L})^L \text{ is an isomorphism.}$$

This ends the proof of the theorem. □

6. Proof of Theorem 1.3

6.1. **The boundary coming from the weak Bruhat order.** In this section, P is still a standard parabolic subgroup (and not necessarily maximal). We fix $(w_1, w_2, v) \in (W^P)^3$ such that ε_P^v occurs with coefficient 1 in the deformed product

$$\varepsilon_P^{w_1} \odot_0 \varepsilon_P^{w_2} \in \big(\mathrm{H}^*(X_P,\mathbb{Z}), \odot_0\big).$$

In particular, $w_1, w_2 \le v$. Recall the definition of $\Delta^{\pm}(w)$ from Section 1 and of $\mathcal{D}$ from Section 5. We associate to any $(\alpha, i) \in \mathcal{D}$, a subvariety $E_{\alpha,i}$ using formula (11):

$$E_{\alpha,1} = E_{s_\alpha w_1, w_2, v}, \quad E_{\alpha,2} = E_{w_1, s_\alpha w_2, v}, \quad E_{\alpha,3} = E_{w_1,w_2,s_\alpha v}.$$

6.2. **On the relative position of $E_{\alpha,i}$ and C.**

Proposition 6.1. *For any $(\alpha, i) \in \mathcal{D}$, the ind-variety*

$$C := Lw_1^{-1}\underline{o}^- \times Lw_2^{-1}\underline{o}^- \times Lv^{-1}\underline{o}$$

is not contained in $E_{\alpha,i}$.

To prove Proposition 6.1 we need the following lemma.

Lemma 6.2. *Let $x \in G/P$ and $(g, g_1, g_2) \in G^3$ be such that the intersection*

$$g_1\mathring{X}_P^{w_1} \cap g_2\mathring{X}_P^{w_2} \cap g\mathring{X}_v^P$$

contains x and is transverse at this point. Such a choice is possible by [Res21, Lemma 4.2]. *Then,*

$$g_1X_P^{w_1} \cap g_2X_P^{w_2} \cap gX_v^P = \{x\}.$$

In fact, the lemma remains true if we replace $\mathring{X}_P^{w_i}$ (for any $i = 1, 2$) by any B^--stable open subset of $\mathring{X}_P^{w_i} \cup \Big(\bigcup_{w_i \to w_i' \in W^P} \mathring{X}_P^{w_i'}\Big)$ and $\mathring{X}_v^P$ by any B-stable open subset of $\mathring{X}_v^P \cup \Big(\bigcup_{v' \to v, v' \in W^P} \mathring{X}_{v'}^P\Big)$.

Proof. The strategy of the proof is to reduce the problem to a finite-dimensional situation (by quotient), and then to apply Zariski's main theorem.

Up to a translation, we may assume that g is trivial. Since $G/B^- = \bigcup_{w\in W} wU\underline{o}^-$, there exists, for $i = 1, 2$, $u_i \in W$ such that $g_i\underline{o}^- \in u_iU\underline{o}^-$. Consider now

$$\mathring{\mathfrak{X}}_\flat = \{(y, h_1\underline{o}^-, h_2\underline{o}^-) \in X_v^P \times u_1U\underline{o}^- \times u_2U\underline{o}^- \,:\, y \in h_1X_P^{w_1} \cap h_2X_P^{w_2}\}$$

and its projection η to $u_1U\underline{o}^- \times u_2U\underline{o}^-$.

Consider $\theta : U \longrightarrow \mathrm{Aut}(X_v^P)$ obtained by the action as before. Fix $i \in \{1, 2\}$. Then, $\mathrm{Ker}\,\theta$ has finite-codimension in U and $U \cap u_iUu_i^{-1}$ has finite-codimension in $u_iUu_i^{-1}$. It follows that there exists a closed normal subgroup $\mathcal{U}_i$ of $u_iUu_i^{-1}$ of finite-codimension such that

$$\mathcal{U}_i \subset u_iUu_i^{-1} \cap \mathrm{Ker}\,\theta.$$

Such a $\mathcal{U}_i$ can be obtained as a closed subgroup of U with Lie algebra

$$\mathrm{Lie}\,\mathcal{U}_i = \bigoplus_{\beta\in\Phi^+, |\beta|>N} \mathfrak{g}_\beta,$$

for large enough N (depending upon v and u_i), where, for $\beta = \sum_j n_j\alpha_j$, $|\beta| := \sum n_j$.

The group $\mathcal{U}_1 \times \mathcal{U}_2$ acts freely and properly on $u_1U\underline{o}^- \times u_2U\underline{o}^-$ (and hence on $\mathring{\mathfrak{X}}_\flat$). Moreover, η is $(\mathcal{U}_1 \times \mathcal{U}_2)$-equivariant. After quotient, one gets

$$\bar{\eta} : (\mathcal{U}_1 \times \mathcal{U}_2)\backslash\mathring{\mathfrak{X}}_\flat \longrightarrow (\mathcal{U}_1 \times \mathcal{U}_2)\backslash(u_1U\underline{o}^- \times u_2U\underline{o}^-).$$

Observe that $\mathcal{U}_i$ being closed subgroups of finite-codimension in $u_iUu_i^{-1}$ and X_v^P being finite-dimensional, the domain and the range of $\bar{\eta}$ are finite-dimensional varieties and range of $\bar{\eta}$ is smooth and irreducible.

Since the coefficient of ε_v^P in $\varepsilon_{w_1}^P \cdot \varepsilon_{w_2}^P : n_{w_1,w_2}^v = 1$, the general fiber of η is one point (see [Res21, §4.2]). Further, as observed below the equation (35), $\mathfrak{X}_\flat$ is irreducible and hence so is $(\mathcal{U}_1 \times \mathcal{U}_2)\backslash\mathring{\mathfrak{X}}_\flat$. Since the base field is $\mathbb{C}$, this implies that $\bar{\eta}$ is birational. Since X_v^P is projective and $X_P^{w_1}$ and $X_P^{w_2}$ are closed in G/P, it is easy to see that the map $\bar{\eta}$ is proper. Now, we can apply Zariski's main theorem [Har77, Chap. III, Corollary 11.4] to conclude that the fibers of $\bar{\eta}$ are connected. But, by assumption, $(g_1\underline{o}^-, g_2\underline{o}^-, x)$ is isolated in the fiber $\bar{\eta}^{-1}(g_1\underline{o}^-, g_2\underline{o}^-)$. Then, $\bar{\eta}^{-1}(g_1\underline{o}^-, g_2\underline{o}^-) = \{(g_1\underline{o}^-, g_2\underline{o}^-, x)\}$, that is

$$g_1X_P^{w_1} \cap g_2X_P^{w_2} \cap X_v^P = \{x\}.$$

This proves the first part of the lemma.

The proof for the 'In fact' statement in the lemma is identical. □

Proof of Proposition 6.1. Since ε_P^v occurs with coefficient 1 in the deformed product $\varepsilon_P^{w_1} \odot_0 \varepsilon_P^{w_2}$, by the proof of [Res21, Lemma 7.5], there exist $l_1, l_2, l_3 \in L$ such that the intersection

$$(l_1 w_1^{-1} \mathring{X}_P^{w_1}) \cap (l_2 w_2^{-1} \mathring{X}_P^{w_2}) \cap (l_3 v^{-1} \mathring{X}_v^P) \tag{62}$$

is transverse at P/P. Then, Lemma 6.2 implies that the intersection $(l_1 w_1^{-1} X_P^{w_1}) \cap (l_2 w_2^{-1} X_P^{w_2}) \cap (l_3 v^{-1} X_v^P)$ is reduced to $\{P/P\}$. In particular, if $w_1 \leq s_\alpha w_1$ and $s_\alpha w_1 \in W^P$,

$$(l_1 w_1^{-1} X_P^{s_\alpha w_1}) \cap (l_2 w_2^{-1} X_P^{w_2}) \cap (l_3 v^{-1} X_v^P) = \emptyset. \tag{63}$$

Then,

(64)

$$(l_1 w_1^{-1} \underline{o}^-, l_2 w_2^{-1} \underline{o}^-, l_3 v^{-1} \underline{o}) \notin G \cdot \left(\overline{P w_1^{-1} s_\alpha \underline{o}^-} \times \overline{P w_2^{-1} \underline{o}^-} \times \overline{P v^{-1} \underline{o}} \right).$$

This proves that $(l_1 w_1^{-1} \underline{o}^-, l_2 w_2^{-1} \underline{o}^-, l_3 v^{-1} \underline{o})$ does not belong to $E_{\alpha,1}$. The proposition follows for $(\alpha, 1)$. The proof for $(\alpha, i) \in \mathcal{D}$ for $i = 2, 3$ is identical. □

6.3. **Line bundles and $E_{\alpha,i}$.** For $(\alpha, i) \in \mathcal{D}$, we now want to describe $E_{\alpha,i}$ as vanishing locus of sections of line bundles. We consider three cases:

(i) Set

$$\mathcal{D}_1 := \left(\bigcup_{i=1,2} \{(\alpha, i) \in \mathcal{D} : s_\alpha w_i \leq v\} \right) \cup \{(\alpha, 3) \in \mathcal{D} : w_1, w_2 \leq s_\alpha v\}.$$

(ii) Set

$$\mathcal{D}_3 := \{(\alpha, 3) \in \mathcal{D} : w_1 \not\leq s_\alpha v \text{ and } w_2 \not\leq s_\alpha v\},$$

(iii) and $\mathcal{D}_2 = \mathcal{D} - (\mathcal{D}_1 \cup \mathcal{D}_3)$.

6.3.1. *The case of* $\mathcal{D}_1$. By definition, for $(\alpha, i) \in \mathcal{D}_1$, the corresponding triple $(s_\alpha w_1, w_2, v)$, $(w_1, s_\alpha w_2, v)$ or $(w_1, w_2, s_\alpha v)$ depending on $i = 1, 2$ or 3 satisfies condition (i) at the beginning of Subsection 4.3. The following Lemma 6.3 allows us to obtain the two other conditions:

Lemma 6.3. (i) *For $(\alpha, 1) \in \mathcal{D}$ (resp. $(\alpha, 2) \in \mathcal{D}$), the triple $(s_\alpha w_1, w_2, v)$ (resp. $(w_1, s_\alpha w_2, v)$) satisfies the conditions (ii) and (iii) at the beginning of Subsection 4.3.*

(ii) *For any $(\alpha, 3) \in \mathcal{D}$, the triple $(w_1, w_2, s_\alpha v)$ satisfies the conditions (ii) and (iii) at the beginning of Subsection 4.3.*

Proof. We prove (i) for $\alpha \in \Delta^+(w_1)$: The condition (ii) of Subsection 4.3 is clearly satisfied. Further, by the proof of [Res21, Lemma 7.5], there exists $l_1, l_2, l_3 \in L$ such that

$$l_3 \mathcal{T}_v \cap l_1 \mathcal{T}^{w_1} \cap l_2 \mathcal{T}^{w_2} = (0),$$

where $\mathcal{T}_v := T_{\dot{e}}(v^{-1}X_v^P)$ and $\mathcal{T}^w := T_{\dot{e}}(w^{-1}X_P^w)$. Now, $\mathcal{T}^{w_1} \supset \mathcal{T}^{s_\alpha w_1}$, since

$$\mathcal{T}^{w_1} = \bigoplus_{\beta\in\Phi^+\cap w_1^{-1}\Phi^+} \mathfrak{g}_{-\beta} \text{ and } \mathcal{T}^{s_\alpha w_1} = \bigoplus_{\beta\in\Phi^+\cap w_1^{-1}s_\alpha\Phi^+} \mathfrak{g}_{-\beta},$$

where Φ^+ is the set of positive roots of the Kac-Moody Lie algebra $\mathfrak{g}$. Thus,

$$l_3\mathcal{T}_v \cap l_1\mathcal{T}^{s_\alpha w_1} \cap l_2\mathcal{T}^{w_2} = (0).$$

This proves the condition (iii) of Subsection 4.3.

The proof of (i) for $\alpha \in \Delta^+(w_2)$ and also the proof of (ii) are identical. $\square$

Definition 6.4. *For $(\alpha, i) \in \mathcal{D}_1$, Proposition 4.4 and Lemma 6.3 give a line bundle $\mathcal{N}_{\alpha,i}$ over $\mathcal{X}$ and a G-invariant section $\mu_{\alpha,i}$ of $\mathcal{N}_{\alpha,i}$ such that*

$$Z(\mu_{\alpha,i}) = E_{\alpha,i}. \tag{65}$$

In the notation of Proposition 4.4,

$$\mathcal{N}_{\alpha,1} := \mathcal{L}_{s_\alpha w_1,w_2,v} \text{ and } \mu_{\alpha,1} := \sigma_{s_\alpha w_1,w_2,v}.$$

6.3.2. *The case of $\mathcal{D}_2$ and $\mathcal{D}_3$.* In these cases, $E_{\alpha,i}$ can be described in terms of the divisors $F_{\alpha,j}$ (for $j = 1, 2$) defined in Section 4.1:

Lemma 6.5. *(a) Let $(\alpha, i) \in \mathcal{D}_2$ with $i = 1$ or 2. Then, $E_{\alpha,i} = F_{\alpha,i}$, where $F_{\alpha,i}$ is defined by the equation (5).*

(b) Let $(\alpha, 3) \in \mathcal{D}_2$ and denote by $j \in \{1, 2\}$ the only one with $w_j \nleq s_\alpha v$. Then, $E_{\alpha,3} = F_{\alpha,j}$.

(c) For $(\alpha, 3) \in \mathcal{D}_3$, we have $E_{\alpha,3} = F_{\alpha,1} \cap F_{\alpha,2}$.

Proof. (a) Assume that $i = 2$. Recall from the equation (13):

$$\mathfrak{X} := \{(y, g_1\underline{o}^-, g_2\underline{o}^-, g\underline{o}) \in G/P \times \mathcal{X} \,:\, y \in g_1X_P^{w_1} \cap g_2X_P^{w_2} \cap gX_v^P\},$$

for the triple (w_1, w_2, v). Consider its analogue for w_2 replaced by $s_\alpha w_2$:

$$\mathfrak{X}' := \mathfrak{X}'_{\alpha,2} := \{(y, g_1\underline{o}^-, g_2\underline{o}^-, g\underline{o}) \in G/P\times\mathcal{X} \,:\, y \in g_1X_P^{w_1}\cap g_2X_P^{s_\alpha w_2}\cap gX_v^P\},$$

and $\mathfrak{X}'_{\alpha,i}$ has a similar meaning, where we place s_α in the i-th factor. Let $\eta : G/P \times \mathcal{X} \longrightarrow \mathcal{X}$ be the projection. By Lemma 4.2, $\eta(\mathfrak{X}') = E_{\alpha,2}$ (cf. the identity (12)) is closed in $\mathcal{X}$ and ind-irreducible. Define the open subset of $\mathcal{X}$:

$$\mathring{\mathcal{X}} := \{(x_1, x_2, x) \in \mathcal{X} \,:\, (x_1, x) \in G.(\underline{o}^-, \underline{o})\}.$$

Since $(\underline{o}^-, s_\alpha\underline{o}^-, \underline{o}) \in \mathring{\mathcal{X}} \cap F_{\alpha,2}$ and $F_{\alpha,2}$ is ind-irreducible (cf. §4.1), we have

$$\overline{\mathring{\mathcal{X}} \cap F_{\alpha,2}} = F_{\alpha,2}. \tag{66}$$

Since $w_1 \leq v$, the Richardson variety $X_v^{w_1}(P) := X_v^P \cap X_P^{w_1}$ is nonempty. Take $x \in X_v^{w_1}(P)$. There exists $g \in G$ such that $g^{-1}x \in X_P^{s_\alpha w_2}$. Then, $(\underline{o}^-, g\underline{o}^-, \underline{o})$ belongs to $\mathring{\mathcal{X}} \cap \eta(\mathfrak{X}')$. Since $\eta(\mathfrak{X}')$ is ind-irreducible, we deduce that

$$\overline{\mathring{\mathcal{X}} \cap \eta(\mathfrak{X}')} = \eta(\mathfrak{X}'). \tag{67}$$

By (66) and (67), it is sufficient to prove that

$$\mathring{\mathcal{X}} \cap \eta(\mathfrak{X}') = \mathring{\mathcal{X}} \cap F_{\alpha,2}. \tag{68}$$

But $G \times_T G/B^- \longrightarrow \mathring{\mathcal{X}}, [g : x] \longmapsto (g\underline{o}^-, gx, g\underline{o})$ is an isomorphism. Consider the intersection of $\mathfrak{X}$ with $G/P \times \underline{o}^- \times G/B^- \times \underline{o}$:

$$\mathfrak{X}_{\mathfrak{ss}} = \{(x, g\underline{o}^-) \in X_v^{w_1}(P) \times G/B^- \,:\, x \in gX_P^{w_2}\}$$

and

$$\mathfrak{X}'_{\mathfrak{ss}} = \{(x, g\underline{o}^-) \in X_v^{w_1}(P) \times G/B^- \,:\, x \in gX_P^{s_\alpha w_2}\}.$$

Since $\mathfrak{X}$ is closed in $G/P \times \mathcal{X}$ (see above Lemma 4.3), $\mathfrak{X}_{\mathfrak{ss}}$ and $\mathfrak{X}'_{\mathfrak{ss}}$ are closed in $X_v^{w_1}(P) \times G/B^-$. Note that

$$\mathfrak{X} \cap (G/P \times \mathring{\mathcal{X}}) \simeq G \times_T \mathfrak{X}_{\mathfrak{ss}}, \;\; \mathfrak{X}' \cap (G/P \times \mathring{\mathcal{X}}) \simeq G \times_T \mathfrak{X}'_{\mathfrak{ss}} \tag{69}$$

under the maps

$$\delta : [g : (x, h\underline{o}^-)] \mapsto (gx, g\underline{o}^-, gh\underline{o}^-, g\underline{o})$$

and $\mathring{\mathcal{X}} \cap F_{\alpha,2} \simeq G \times_T \overline{Bs_\alpha\underline{o}^-}$. Thus, to prove (68), it is sufficient to prove that

$$\hat{\mathcal{X}}_{\mathfrak{ss}} = \overline{Bs_\alpha\underline{o}^-}, \tag{70}$$

where $\hat{\mathcal{X}}_{\mathfrak{ss}} := \{g\underline{o}^- \in G/B^- \,:\, X_v^{w_1}(P) \cap gX_P^{s_\alpha w_2} \neq \emptyset\}$. By Lemma 4.2, $\hat{\mathcal{X}}_{\mathfrak{ss}}$ is closed in G/B^-.

Since $s_\alpha w_2 P/P \notin X_v^P$, the Richardson variety $X_v^{s_\alpha w_2}(P)$ is empty. Then, for any $b \in B$, $X_v^P \cap bX_P^{s_\alpha w_2} = b(X_v^P \cap X_P^{s_\alpha w_2})$ is empty. Hence, $\hat{\mathcal{X}}_{\mathfrak{ss}} \cap B\underline{o}^- = \emptyset$, and, by the Birkhoff decomposition,

$$\hat{\mathcal{X}}_{\mathfrak{ss}} \subset \bigcup_{\beta \in \Delta} \overline{Bs_\beta\underline{o}^-}. \tag{71}$$

Since $s_\alpha w_2 \not\leq v$, by [Kum02, Corollary 1.3.19] , $s_\alpha w_2 \leq s_\alpha v$ and hence $vP/P \in s_\alpha X_P^{s_\alpha w_2}$ and thus $X_v^{w_1}(P) \cap s_\alpha X_P^{s_\alpha w_2}$ is nonempty. This gives $s_\alpha\underline{o}^- \in \hat{\mathcal{X}}_{\mathfrak{ss}}$. From the ind-irreducibility of $\mathfrak{X}$ and $\mathfrak{X}'$, it is easy to see that $\mathfrak{X}_{\mathfrak{d}\mathfrak{d}}$, $\mathfrak{X}'_{\mathfrak{d}\mathfrak{d}}$ and $\hat{\mathcal{X}}_{\mathfrak{d}\mathfrak{d}}$ are ind-irreducible. Thus, we deduce from (71) that

$$\hat{\mathcal{X}}_{\mathfrak{d}\mathfrak{d}} \subset \overline{Bs_\alpha \underline{o}^-}. \tag{72}$$

Consider $(G/B^-)^\circ := s_\alpha U \underline{o}^-$, which is a neighborhood of $s_\alpha \underline{o}^-$ in G/B^-. By the ind-irreduciblity of $\hat{\mathcal{X}}_{\mathfrak{d}\mathfrak{d}}$ and $\overline{Bs_\alpha \underline{o}^-}$, to prove the equality (70), it is sufficient to prove that

$$\hat{\mathcal{X}}_{\mathfrak{d}\mathfrak{d}} \cap (G/B^-)^\circ = \overline{Bs_\alpha \underline{o}^-} \cap (G/B^-)^\circ. \tag{73}$$

Let $\mathcal{U} = \mathcal{U}_\alpha$ be the kernel of the action of $s_\alpha U s_\alpha \cap U$ on X_v^P. Consider the following commutative diagram:

$$\begin{tikzcd}
(s_\alpha U s_\alpha \cap U) \times (s_\alpha U s_\alpha \cap U^-) \arrow[r, "\sim"] \arrow[d] & s_\alpha U s_\alpha \arrow[r, "\sim"] \arrow[d] & (G/B^-)^\circ \arrow[d] \\
(\mathcal{U}\backslash(s_\alpha U s_\alpha \cap U)) \times (s_\alpha U s_\alpha \cap U^-) \arrow[r, "\sim"] & \mathcal{U}\backslash(s_\alpha U s_\alpha) \arrow[r, "\sim"] & \mathcal{U}\backslash(G/B^-)^\circ.
\end{tikzcd}$$

Set $\mathring{\mathfrak{X}}_{\mathfrak{d}\mathfrak{d}} := \mathfrak{X}_{\mathfrak{d}\mathfrak{d}} \cap (G/P \times (G/B^-)^\circ)$, $\mathring{\mathfrak{X}}'_{\mathfrak{d}\mathfrak{d}} := \mathfrak{X}'_{\mathfrak{d}\mathfrak{d}} \cap (G/P \times (G/B^-)^\circ)$ and $\mathring{\hat{\mathcal{X}}}_{\mathfrak{d}\mathfrak{d}} := \hat{\mathcal{X}}_{\mathfrak{d}\mathfrak{d}} \cap (G/B^-)^\circ$. All these spaces are nonempty. Consider the commutative diagram of finite-dimensional irreducible varieties:

$$\begin{tikzcd}
\mathcal{U}\backslash\mathring{\mathfrak{X}}_{\mathfrak{d}\mathfrak{d}} \arrow[r, "\eta_{\mathcal{U}}"] & \mathcal{U}\backslash(G/B^-)^\circ \\
\mathcal{U}\backslash\mathring{\mathfrak{X}}'_{\mathfrak{d}\mathfrak{d}} \arrow[r, "\eta'_{\mathcal{U}}"] \arrow[u, hook] & \mathcal{U}\backslash \mathring{\hat{\mathcal{X}}}_{\mathfrak{d}\mathfrak{d}}, \arrow[u, hook]
\end{tikzcd}$$

where $\mathcal{U}$ acts on $\mathring{\mathfrak{X}}_{\mathfrak{d}\mathfrak{d}}$ and $\mathring{\mathfrak{X}}'_{\mathfrak{d}\mathfrak{d}}$ via its action on the $(G/B^-)^\circ$-factor only.

Since ε_P^v occurs in $\varepsilon_P^{w_1} \odot_0 \varepsilon_P^{w_2}$ with coefficient 1, there exist l_1, l_2 and l in L such that $(P/P, l_1 w_1^{-1}\underline{o}^-, l_2 w_2^{-1}\underline{o}^-, l v^{-1}\underline{o})$ belongs to $\mathring{\mathfrak{X}}^+$, where $\mathring{\mathfrak{X}}^+$ is defined above Lemma 4.3 and it corresponds to the triple (w_1, w_2, v) (cf. [Res21, Proof of Lemma 7.5]).

Let

$$\mathfrak{X}_{\mathfrak{d}} := \mathfrak{X} \cap (G/P \times \underline{o}^- \times G/B^- \times G/B).$$

Then, $\mathfrak{X}_{\mathfrak{d}}$ is irreducible, which follows from the irreducibility of the open subset $\mathfrak{X} \cap (G/P \times (U \cdot \underline{o}^-) \times G/B^- \times G/B)$ of $\mathfrak{X}$. By Lemma 4.3, $\mathring{\mathfrak{X}}^+ \cap \mathfrak{X}_{\mathfrak{d}}$ is a (nonempty) open subset of $\mathfrak{X}_{\mathfrak{d}}$. Moreover, $\mathfrak{X} \cap (G/P \times \underline{o}^- \times$

$G/B^- \times U^- \cdot \underline{o})$ is a nonempty (by the parabolic analogue of [BK14, Proposition 3.5]) open subset of $\mathfrak{X}_{\mathfrak{s}}$. Thus,

$$\mathring{\mathfrak{X}}^+ \cap (G/P \times \underline{o}^- \times G/B^- \times U^- \cdot \underline{o}) \neq \emptyset.$$

From this we see that $\mathring{\mathfrak{X}}^+ \cap (G/P \times \underline{o}^- \times G/B^- \times \underline{o})$ is a nonempty open subset of $\mathfrak{X}_{\mathfrak{s}\mathfrak{s}}$ (since $U^- \cdot \underline{o}^- = \underline{o}^-$). Further,

$$\mathring{\mathfrak{X}}_{\mathfrak{s}\mathfrak{s}} := (G/P \times \underline{o}^- \times (G/B^-)^\circ \times \underline{o}) \cap \mathfrak{X}_{\mathfrak{s}\mathfrak{s}}$$

is a nonempty (again by the parabolic analogue of [BK14, Proposition 3.5]) open subset of $\mathfrak{X}_{\mathfrak{s}\mathfrak{s}}$. Moreover, as observed above, $\mathfrak{X}_{\mathfrak{s}\mathfrak{s}}$ is irreducible. Hence, $\mathring{\mathfrak{X}}^+ \cap (G/P \times \underline{o}^- \times (G/B^-)^\circ \times \underline{o})$ is a nonempty open subset of $\mathring{\mathfrak{X}}_{\mathfrak{s}\mathfrak{s}}$. By the parabolic analogue of [BK14, Proposition 3.5], $\eta_{\mathcal{U}} : \mathcal{U}\backslash\mathring{\mathfrak{X}}_{\mathfrak{s}\mathfrak{s}} \to \mathcal{U}\backslash(G/B^-)^\circ$ is surjective and, by Lemma 6.2, it is birational. In particular,

$$\dim\big(\mathcal{U}\backslash\mathring{\mathfrak{X}}_{\mathfrak{s}\mathfrak{s}}\big) = \dim\big(\mathcal{U}\backslash(G/B^-)^\circ\big). \tag{74}$$

Consider the set

$$\tilde{\mathfrak{X}}_{\mathfrak{s}\mathfrak{s}} := \{(x,g) \in X_v^{w_1}(P) \times G : g^{-1}x \in X_P^{w_2}\}$$

and similarly $\tilde{\mathfrak{X}}'_{\mathfrak{s}\mathfrak{s}}$. Define the morphism

$$p : \tilde{\mathfrak{X}}_{\mathfrak{s}\mathfrak{s}} \to X_P^{w_2},\ \ (x,g) \mapsto g^{-1}x.$$

By definition,

$$p^{-1}(X_P^{s_\alpha w_2}) = \tilde{\mathfrak{X}}'_{\mathfrak{s}\mathfrak{s}}.$$

Clearly, p is surjective. Consider the open subset of $X_P^{w_2}$:

$$\hat{\mathring{}}\, X_P^{w_2} := \mathring{X}_P^{w_2} \sqcup \mathring{X}_P^{s_\alpha w_2}.$$

Then, it is smooth (in the sense that there is a subgroup $U_{w_2}^-$ of finite-codimension of U^- acting properly and discontinuously on $\hat{\mathring{}}\, X_P^{w_2}$ such that $U_{w_2}^- \backslash \hat{\mathring{}}\, X_P^{w_2}$ is a smooth variety of finite type over $\mathbb{C}$, cf. [Kum17, Lemma 6.1]) and $\mathring{X}_P^{s_\alpha w_2}$ is a closed smooth subset of codimension 1. In particular, $\mathring{X}_P^{s_\alpha w_2}$ is a Cartier divisor of $\hat{\mathring{}}\, X_P^{w_2}$. Thus, $p^{-1}(\mathring{X}_P^{s_\alpha w_2})$ is a Cartier divisor of $p^{-1}(\hat{\mathring{}}\, X_P^{w_2})$. Let $\mathring{p}$ be the restriction of the map p to the nonempty open subset of $\tilde{\mathfrak{X}}_{\mathfrak{s}\mathfrak{s}}$:

$$\tilde{\mathring{}}\,\mathfrak{X}_{\mathfrak{s}\mathfrak{s}} := \{(x,g) \in X_v^{w_1}(P) \times (s_\alpha U \cdot B^-) : g^{-1}x \in X_P^{w_2}\}.$$

Then, $\mathring{p}$ is a dominant morphism. Since $\mathfrak{X}_{\mathfrak{s}\mathfrak{s}}$ is irreducible and hence so is $\tilde{\mathfrak{X}}_{\mathfrak{s}\mathfrak{s}}$. Thus,

$$p^{-1}(\hat{\mathring{}}\, X_P^{w_2}) \cap \tilde{\mathring{}}\,\mathfrak{X}_{\mathfrak{s}\mathfrak{s}} \neq \emptyset.$$

Since $p^{-1}(\mathring{X}_P^{s_\alpha w_2})$ and $\tilde{\mathring{\mathfrak{X}}}_{\mathfrak{ss}} \cap \tilde{\mathfrak{X}}'_{\mathfrak{ss}}$ are both nonempty (since so is $\mathring{\mathfrak{X}}'_{\mathfrak{ss}}$) open subsets of irreducible $\tilde{\mathfrak{X}}'_{\mathfrak{ss}}$, we get that their intersection is nonempty. In particular,

$$p^{-1}(\mathring{X}_P^{s_\alpha w_2}) \cap \tilde{\mathring{\mathfrak{X}}}_{\mathfrak{ss}} \neq \emptyset.$$

The map $\mathring{p} : \tilde{\mathring{\mathfrak{X}}}_{\mathfrak{ss}} \to X_P^{w_2}$ is $\mathcal{U}$-invariant (with the trivial action of $\mathcal{U}$ on $X_P^{w_2}$). From this it is easy to see that

$$\dim\big(\mathcal{U}\backslash\mathring{\mathfrak{X}}'_{\mathfrak{ss}}\big) = \dim\big(\mathcal{U}\backslash\mathring{\mathfrak{X}}_{\mathfrak{ss}}\big) - 1. \tag{75}$$

Since $X_P^{w_2}$ is P_α-stable, for any l_1, l_2 and l in L such that

$$(P/P, l_1 w_1^{-1}\underline{o}^-, l_2 w_2^{-1}\underline{o}^-, l v^{-1}\underline{o}) \text{ belongs to } \mathring{\mathfrak{X}}^+, \text{ we get}$$

$$(P/P, l_1 w_1^{-1}\underline{o}^-, l_2 w_2^{-1} s_\alpha \underline{o}^-, l v^{-1}\underline{o}) \in \hat{\mathring{\mathfrak{X}}}^+ \cap \mathfrak{X}',$$

where

$$\hat{\mathring{\mathfrak{X}}}^+ := \{(y, g_1\underline{o}^-, g_2\underline{o}^-, g_3\underline{o}) \in G/P \times \mathcal{X} \,:\, y \in g_1\mathring{X}_P^{w_1} \cap g_2 \hat{\mathring{X}}_P^{w_2} \cap g_3\mathring{X}_v^P \text{ and}$$

(76)

$$\mathcal{T}_y(g_1\mathring{X}_P^{w_1}) \cap \mathcal{T}_y(g_2 \hat{\mathring{X}}_P^{w_2}) \cap \mathcal{T}_y(g_3\mathring{X}_v^P) = (0)\}.$$

Then, $\hat{\mathring{\mathfrak{X}}}^+$ is open in $\mathfrak{X}$ (cf. Lemma 4.3).

Consider the surjective morphism

$$\eta'_N : \mathcal{U}\backslash\mathring{\mathfrak{X}}'_{\mathfrak{ss}} \to \mathcal{U}\backslash \hat{\mathring{\mathcal{X}}}_{\mathfrak{ss}}.$$

We next prove that

$$\mathring{\mathfrak{X}}'_{\mathfrak{ss}} \cap \hat{\mathring{\mathfrak{X}}}^+ \neq \emptyset \text{ open subset of } \mathring{\mathfrak{X}}'_{\mathfrak{ss}}. \tag{77}$$

As observed above, $\hat{\mathring{\mathfrak{X}}}^+ \cap \mathfrak{X}'$ contains (in fact, is) a nonempty open subset of $\mathfrak{X}'$. Moreover, $\mathfrak{X}' \cap (G/P \times \mathring{\mathcal{X}})$ is a nonempty open subset of $\mathfrak{X}'$ by (69) since $\mathfrak{X}'_{\mathfrak{ss}}$ is nonempty. But, since $\mathfrak{X}'$ is irreducible, their intersection $\mathfrak{X}' \cap \left(\hat{\mathring{\mathfrak{X}}}^+ \cap (G/P \times \mathring{\mathcal{X}})\right) \neq \emptyset$. But, it is easy to see that under the isomorphism (as in (69)) $\delta : G \times_T \mathfrak{X}'_{\mathfrak{ss}} \simeq \mathfrak{X}' \cap (G/P \times \mathring{\mathcal{X}})$, $\mathfrak{X}' \cap \hat{\mathring{\mathfrak{X}}}^+ \cap (G/P \times \mathring{\mathcal{X}})$ corresponds to $G \times_T \left(\mathfrak{X}'_{\mathfrak{ss}} \cap \hat{\mathring{\mathfrak{X}}}^+\right)$. In particular, $\mathfrak{X}'_{\mathfrak{ss}} \cap \hat{\mathring{\mathfrak{X}}}^+$ is a nonempty open subset of $\mathfrak{X}'_{\mathfrak{ss}}$. Also, $\mathring{\mathfrak{X}}'_{\mathfrak{ss}}$ is a nonempty open subset of $\mathfrak{X}'_{\mathfrak{ss}}$. Thus, $\mathfrak{X}'_{\mathfrak{ss}}$ being irreducible, $\mathring{\mathfrak{X}}'_{\mathfrak{ss}} \cap \hat{\mathring{\mathfrak{X}}}^+$ is nonempty proving (77).

Moreover, by Lemma 6.2, η'_N is one to one restricted to $\mathring{\mathfrak{X}}'_{\delta\delta} \cap \hat{\mathring{\ }}\mathfrak{X}^+$. Thus, η'_N is birational. In particular,

$$\begin{aligned}\dim\big(\mathcal{U}\backslash\ \hat{\mathring{\ }}\mathcal{X}_{\delta\delta}\big) &= \dim\big(\mathcal{U}\backslash\mathring{\mathfrak{X}}'_{\delta\delta}\big)\\ &= \dim\big(\mathcal{U}\backslash\mathring{\mathfrak{X}}_{\delta\delta}\big)-1, \text{ by (75)}\\ &= \dim\big(\mathcal{U}\backslash(G/B^-)^\circ\big)-1, \text{ by (74)}\\ &= \dim\big(\mathcal{U}\backslash\big((\overline{Bs_\alpha\underline{o}^-})\cap(G/B^-)^\circ\big)\big).\end{aligned}$$

Thus, from the inclusion (72), and the irreducibility of $\mathcal{U}\backslash\big((\overline{Bs_\alpha\underline{o}^-})\cap(G/B^-)^\circ\big)$, we get (73).

This completes the proof of the lemma for $(\alpha,2)\in\mathcal{D}_2$.

The proof in the case $(\alpha,i)\in\mathcal{D}_2$ for $i=1$ is identical.

(b) Without loss of generality take $j=2$. By Lemma 4.2, $E_{\alpha,3}$ is closed and ind-irreducible. Define the open subset of $\mathcal{X}$:

$$\mathring{\mathcal{X}} := \{(x_1,x_2,x)\in\mathcal{X} \,:\, (x_1,x)\in G.(\underline{o}^-,\underline{o})\}.$$

Since $(\underline{o}^-,s_\alpha\underline{o}^-,\underline{o})\in\mathring{\mathcal{X}}\cap F_{\alpha,2}$ and $F_{\alpha,2}$ is ind-irreducible (cf. §4.1), we have

$$\overline{\mathring{\mathcal{X}}\cap F_{\alpha,2}} = F_{\alpha,2}. \tag{78}$$

Since $w_1\le s_\alpha v$, the Richardson variety $X^{w_1}_{s_\alpha v}(P) := X^P_{s_\alpha v}\cap X^{w_1}_P$ is nonempty. Take $x\in X^{w_1}_{s_\alpha v}(P)$. There exists $g\in G$ such that $g^{-1}x\in X^{w_2}_P$. Then, $(\underline{o}^-,g\underline{o}^-,\underline{o})$ belongs to $\mathring{\mathcal{X}}\cap\eta(\mathfrak{X}')$. Since $\eta(\mathfrak{X}')$ is ind-irreducible, we deduce that

$$\overline{\mathring{\mathcal{X}}\cap\eta(\mathfrak{X}')} = \eta(\mathfrak{X}'). \tag{79}$$

By (78) and (79), it is sufficient to prove that

$$\mathring{\mathcal{X}}\cap\eta(\mathfrak{X}') = \mathring{\mathcal{X}}\cap F_{\alpha,2}. \tag{80}$$

But $G\times_T G/B^-\longrightarrow\mathring{\mathcal{X}}, [g:x]\longmapsto(g\underline{o}^-,gx,g\underline{o})$ is an isomorphism. Consider the intersection of $\mathfrak{X}$ with $G/P\times\underline{o}^-\times G/B^-\times\underline{o}$:

$$\mathfrak{X}_{\delta\delta} = \{(x,g\underline{o}^-)\in X^{w_1}_v(P)\times G/B^- \,:\, x\in gX^{w_2}_P\}$$

and its closed subset

$$\mathfrak{X}'_{\delta\delta} = \{(x,g\underline{o}^-)\in X^{w_1}_{s_\alpha v}(P)\times G/B^- \,:\, x\in gX^{w_2}_P\}.$$

Since $\mathfrak{X}$ is closed in $G/P\times\mathcal{X}$ (see above Lemma 4.3), $\mathfrak{X}_{\delta\delta}$ and $\mathfrak{X}'_{\delta\delta}$ are closed in $X^{w_1}_v(P)\times G/B^-$. Note that

$$\mathfrak{X}\cap(G/P\times\mathring{\mathcal{X}})\simeq G\times_T\mathfrak{X}_{\delta\delta},\ \ \mathfrak{X}'\cap(G/P\times\mathring{\mathcal{X}})\simeq G\times_T\mathfrak{X}'_{\delta\delta} \tag{81}$$

under the maps

$$\delta : [g : (x, h\underline{o}^-)] \mapsto (gx, g\underline{o}^-, gh\underline{o}^-, g\underline{o})$$

and $\mathring{\mathcal{X}} \cap F_{\alpha,2} \simeq G \times_T \overline{Bs_\alpha \underline{o}^-}$. Thus, to prove (80), it is sufficient to prove that

$$\hat{\mathcal{X}}_{ss} = \overline{Bs_\alpha \underline{o}^-}, \tag{82}$$

where $\hat{\mathcal{X}}_{ss} := \{g\underline{o}^- \in G/B^- : X^{w_1}_{s_\alpha v}(P) \cap gX_P^{w_2} \neq \emptyset\}$. By Lemma 4.2, $\hat{\mathcal{X}}_{ss}$ is closed in G/B^-.

Since $w_2P/P \notin X^P_{s_\alpha v}$, the Richardson variety $X^{w_2}_{s_\alpha v}(P)$ is empty. Then, for any $b \in B$, $X^P_{s_\alpha v} \cap bX_P^{w_2} = b(X^P_{s_\alpha v} \cap X_P^{w_2})$ is empty. Hence, $\hat{\mathcal{X}}_{ss} \cap B\underline{o}^- = \emptyset$, and, by the Birkhoff decomposition,

$$\hat{\mathcal{X}}_{ss} \subset \bigcup_{\beta \in \Delta} \overline{Bs_\beta \underline{o}^-}. \tag{83}$$

Since $w_2 \leq v$ and $w_1 \leq s_\alpha v$, we have $s_\alpha vP/P \in s_\alpha X_P^{w_2}$ and thus $X^{w_1}_{s_\alpha v}(P) \cap s_\alpha X_P^{w_2}$ is nonempty. This gives $s_\alpha \underline{o}^- \in \hat{\mathcal{X}}_{ss}$. From the ind-irreducibility of $\mathfrak{X}$ and $\mathfrak{X}'$, it is easy to see that $\mathfrak{X}_{ss}$, $\mathfrak{X}'_{ss}$ and $\hat{\mathcal{X}}_{ss}$ are ind-irreducible. Thus, we deduce from (83) the inclusion (72).

Now, following the exact same argument as in the proof of the (a)-part till the identity (74).

Define the surjective projection

$$p : \mathfrak{X}_{ss} \to X_v^{w_1}(P), \;\; (x, g\underline{o}^-) \mapsto x.$$

By definition,

$$p^{-1}(X^{w_1}_{s_\alpha v}(P)) = \mathfrak{X}'_{ss}.$$

Consider the smooth open subset of $X_v^{w_1}(P)$:

$$\hat{\mathring{X}}_v^{w_1}(P) := \mathring{X}_P^{w_1} \cap \hat{\mathring{X}}_v^P, \;\text{ where }\; \hat{\mathring{X}}_v^P := \mathring{X}_v^P \cup \mathring{X}^P_{s_\alpha v}$$

and its closed smooth subset of codimension 1:

$$\mathring{X}^{w_1}_{s_\alpha v}(P) := \mathring{X}_P^{w_1} \cap \mathring{X}^P_{s_\alpha v}$$

In particular, $\mathring{X}^{w_1}_{s_\alpha v}(P)$ is a Cartier divisor of $\hat{\mathring{X}}_v^{w_1}(P)$. Thus, $p^{-1}(\mathring{X}^{w_1}_{s_\alpha v}(P))$ is a Cartier divisor of $p^{-1}(\hat{\mathring{X}}_v^{w_1}(P))$. Let $\mathring{p}$ be the restriction of the map p to the nonempty open subset $\mathring{\mathfrak{X}}_{ss}$: Then, $\mathring{p}$ is a dominant morphism. Since $\mathfrak{X}_{ss}$ is irreducible,

$$p^{-1}(\hat{\mathring{X}}_v^{w_1}(P)) \cap \mathring{\mathfrak{X}}_{ss} \neq \emptyset.$$

Since $p^{-1}(\mathring{X}^{w_1}_{s_\alpha v}(P))$ and $\mathring{\mathfrak{X}}'_{\delta\delta}$ are both nonempty open subsets of irreducible $\mathfrak{X}'_{\delta\delta}$, we get that their intersection is nonempty. In particular,

$$p^{-1}(\mathring{X}^{w_1}_{s_\alpha v}(P)) \cap \mathring{\mathfrak{X}}_{\delta\delta} \neq \emptyset.$$

The map $\mathring{p} : \mathring{\mathfrak{X}}_{\delta\delta} \to X^{w_1}_v(P)$ is $\mathcal{U}$-invariant (with the trivial action of $\mathcal{U}$ on $X^{w_1}_v(P)$). From this it is easy to see that

$$\dim\left(\mathcal{U}\backslash\mathring{\mathfrak{X}}'_{\delta\delta}\right) = \dim\left(\mathcal{U}\backslash\mathring{\mathfrak{X}}_{\delta\delta}\right) - 1. \tag{84}$$

Since X^P_v is P_α-stable, for any l_1, l_2 and l in L such that

$(P/P, l_1 w_1^{-1}\underline{o}^-, l_2 w_2^{-1}\underline{o}^-, l v^{-1}\underline{o})$ belongs to $\mathring{\mathfrak{X}}^+$, we get

$$(P/P, l_1 w_1^{-1}\underline{o}^-, l_2 w_2^{-1}\underline{o}^-, l v^{-1} s_\alpha \underline{o}) \in \mathfrak{X}' \cap \ \hat{\mathring{}}\mathfrak{X}^+,$$

where

$$\hat{\mathring{}}\mathfrak{X}^+ := \{(y, g_1\underline{o}^-, g_2\underline{o}^-, g_3\underline{o}) \in G/P \times \mathcal{X} \ : \ y \in g_1\mathring{X}^{w_1}_P \cap g_2\mathring{X}^{w_2}_P \cap g_3 \ \hat{\mathring{}}X^P_v \text{ and}$$
$$\mathcal{T}_y(g_1\mathring{X}^{w_1}_P) \cap \mathcal{T}_y(g_2\mathring{X}^{w_2}_P) \cap \mathcal{T}_y(g_3 \ \hat{\mathring{}}X^P_v) = (0)\}.$$

Then, $\hat{\mathring{}}\mathfrak{X}^+$ is open in $\mathfrak{X}$ (cf. Lemma 4.3).

Now, follow the exact same argument starting 'Consider the surjective morphism $\cdots$' till the end of the proof in the (a)-part.

This completes the proof of the (b)-part.

(c) By Lemma 4.2, $E_{\alpha,3}$ is closed and ind-irreducible. Define the subset of $\mathcal{X}$:

$$\mathring{\mathcal{X}} := \{(x_1, x_2, x) \in \mathcal{X} \ : \ (x_1, x) \in G.(s_\alpha\underline{o}^-, \underline{o})\}.$$

Let $F := F_{\alpha,1} \cap F_{\alpha,2}$. Since $F = G.(\overline{Bs_\alpha\underline{o}^-} \times \overline{Bs_\alpha\underline{o}^-} \times \{\underline{o}\})$, it is ind-irreducible. But, $(s_\alpha\underline{o}^-, s_\alpha\underline{o}^-, \underline{o}) \in \mathring{\mathcal{X}} \cap F$ and hence we have

$$\overline{\mathring{\mathcal{X}} \cap F} = F, \text{ since } \mathring{\mathcal{X}} \cap F \text{ contains an open subset of } F. \tag{85}$$

Since $s_\alpha w_1 \leq s_\alpha v$, the variety $Y(w_1; s_\alpha v) := X^P_{s_\alpha v} \cap s_\alpha X^{w_1}_P$ is nonempty. Take $x \in Y(w_1; s_\alpha v)$. There exists $g \in G$ such that $g^{-1}x \in X^{w_2}_P$. Then, $(s_\alpha\underline{o}^-, g\underline{o}^-, \underline{o})$ belongs to $\mathring{\mathcal{X}} \cap \eta(\mathfrak{X}')$. Since $\eta(\mathfrak{X}')$ is ind-irreducible, we deduce that

$$\overline{\mathring{\mathcal{X}} \cap \eta(\mathfrak{X}')} = \eta(\mathfrak{X}'), \tag{86}$$

since $\mathring{\mathcal{X}} \cap \eta(\mathfrak{X}')$ contains an open subset of $\eta(\mathfrak{X}')$ as can be seen since $(Bs_\alpha\underline{o}^- \times Bs_\alpha\underline{o}^- \times \underline{o}) \cap \eta(\mathfrak{X}')$ is nonempty. By (85) and (86), it is sufficient to prove that

$$\mathring{\mathcal{X}} \cap \eta(\mathfrak{X}') = \mathring{\mathcal{X}} \cap F. \tag{87}$$

But $G \times_{B_\alpha} G/B^- \longrightarrow \mathring{\mathcal{X}}, [g : x] \longmapsto (gs_\alpha \underline{o}^-, gx, g\underline{o})$ is an isomorphism, where B_α is the subgroup of B generated by T and the one dimensional root subgroup G_α. Consider the intersection of $\mathfrak{X}$ with $G/P \times s_\alpha \underline{o}^- \times G/B^- \times \underline{o}$:

$$\mathfrak{X}_{\mathscr{ss}} = \{(x, g\underline{o}^-) \in Y(w_1; v) \times G/B^- \, : \, x \in gX_P^{w_2}\},$$

where $Y(w_1; v) := X_v^P \cap s_\alpha X_P^{w_1} = s_\alpha(X_v^P \cap X_P^{w_1})$, and its closed subset

$$\mathfrak{X}'_{\mathscr{ss}} = \{(x, g\underline{o}^-) \in Y(w_1; s_\alpha v) \times G/B^- \, : \, x \in gX_P^{w_2}\}.$$

Since $\mathfrak{X}$ is closed in $G/P \times \mathcal{X}$ (see above Lemma 4.3), $\mathfrak{X}_{\mathscr{ss}}$ and $\mathfrak{X}'_{\mathscr{ss}}$ are closed in $Y(w_1; v) \times G/B^-$. Note that

$$\mathfrak{X} \cap (G/P \times \mathring{\mathcal{X}}) \simeq G \times_{B_\alpha} \mathfrak{X}_{\mathscr{ss}}, \;\; \mathfrak{X}' \cap (G/P \times \mathring{\mathcal{X}}) \simeq G \times_{B_\alpha} \mathfrak{X}'_{\mathscr{ss}} \tag{88}$$

under the maps

$$\delta : [g : (x, h\underline{o}^-)] \mapsto (gx, gs_\alpha \underline{o}^-, gh\underline{o}^-, g\underline{o})$$

and $\mathring{\mathcal{X}} \cap F \simeq G \times_{B_\alpha} \overline{Bs_\alpha \underline{o}^-}$. Thus, to prove (87), it is sufficient to prove that

$$\hat{\mathcal{X}}_{\mathscr{ss}} = \overline{Bs_\alpha \underline{o}^-}, \tag{89}$$

where $\hat{\mathcal{X}}_{\mathscr{ss}} := \{g\underline{o}^- \in G/B^- \, : \, Y(w_1; s_\alpha v) \cap gX_P^{w_2} \neq \emptyset\}$. By the proof of Lemma 4.2, $\hat{\mathcal{X}}_{\mathscr{ss}}$ is closed in G/B^-.

Since $w_2 P/P \notin X_{s_\alpha v}^P$, the Richardson variety $X_{s_\alpha v}^{w_2}(P)$ is empty. Then, for any $b \in B$, $X_{s_\alpha v}^P \cap bX_P^{w_2} = b(X_{s_\alpha v}^P \cap X_P^{w_2})$ is empty. Hence, $\hat{\mathcal{X}}_{\mathscr{ss}} \cap B\underline{o}^- = \emptyset$, and, by the Birkhoff decomposition,

$$\hat{\mathcal{X}}_{\mathscr{ss}} \subset \bigcup_{\beta \in \Delta} \overline{Bs_\beta \underline{o}^-}. \tag{90}$$

Since $w_1, w_2 \leq v$, we get $s_\alpha \underline{o}^- \in \hat{\mathcal{X}}_{\mathscr{ss}}$. From the ind-irreducibility of $\mathfrak{X}'$, it is easy to see that $\mathfrak{X}'_{\mathscr{ss}}$ and $\hat{\mathcal{X}}_{\mathscr{ss}}$ are ind-irreducible. Moreover, $\mathfrak{X}_{\mathscr{ss}}$ is ind-irreducible since $\mathfrak{X}_{\mathscr{ss}} \simeq \mathfrak{X} \cap (G/P \times \underline{o}^- \times G/B^- \times \underline{o})$ and the latter was proved to be ind-irreducible earlier. Thus, we deduce from (90) the equation (72).

Now, following the exact same argument as in the proof of the (a)-part till ‘ and it corresponds to the triple (w_1, w_2, v) (cf. [Res21, Proof of Lemma 7.5]).’

Let

$$\mathfrak{X}_{\mathscr{s}} := \mathfrak{X} \cap (G/P \times s_\alpha \underline{o}^- \times G/B^- \times G/B).$$

Then, $\mathfrak{X}_{\mathscr{s}}$ is irreducible, which follows from the irreducibility of the open subset $\mathfrak{X} \cap (G/P \times (U \cdot \underline{o}^-) \times G/B^- \times G/B)$ of $\mathfrak{X}$. By Lemma 4.3,

${}^{\hat\circ}\mathfrak{X}^+ \cap \mathfrak{X}_{\mathfrak{s}}$ is a (nonempty) open subset of $\mathfrak{X}_{\mathfrak{s}}$, where ${}^{\hat\circ}\mathfrak{X}^+$ is defined as follows:

$${}^{\hat\circ}\mathfrak{X}^+ := \{(y, g_1\underline{o}^-, g_2\underline{o}^-, g_3\underline{o}) \in G/P \times \mathcal{X} \,:\, y \in g_1\mathring{X}_P^{w_1} \cap g_2\mathring{X}_P^{w_2} \cap g_3 \; {}^{\hat\circ}X_v^P \text{ and}$$
$$\mathcal{T}_y(g_1\mathring{X}_P^{w_1}) \cap \mathcal{T}_y(g_2\mathring{X}_P^{w_2}) \cap \mathcal{T}_y(g_3 \; {}^{\hat\circ}X_v^P) = (0)\}.$$

Moreover, $\mathfrak{X} \cap (G/P \times s_\alpha\underline{o}^- \times G/B^- \times U^- \cdot \underline{o})$ is a nonempty (by the parabolic analogue of [BK14, Proposition 3.5]) open subset of $\mathfrak{X}_{\mathfrak{s}}$. Thus,

$${}^{\hat\circ}\mathfrak{X}^+ \cap (G/P \times s_\alpha\underline{o}^- \times G/B^- \times U^- \cdot \underline{o}) \neq \emptyset.$$

From this we see that ${}^{\hat\circ}\mathfrak{X}^+ \cap (G/P \times s_\alpha\underline{o}^- \times G/B^- \times \underline{o})$ is a nonempty open subset of $\mathfrak{X}_{\mathfrak{s}\mathfrak{s}}$ (since ${}^{\hat\circ}X_v^P$ is stable under the left multiplication by s_α). Further,

$$\mathring{\mathfrak{X}}_{\mathfrak{s}\mathfrak{s}} := (G/P \times s_\alpha\underline{o}^- \times (G/B^-)^\circ \times \underline{o}) \cap \mathfrak{X}_{\mathfrak{s}\mathfrak{s}}$$

is a nonempty (again by the parabolic analogue of [BK14, Proposition 3.5]) open subset of $\mathfrak{X}_{\mathfrak{s}\mathfrak{s}}$. Moreover, as observed above, $\mathfrak{X}_{\mathfrak{s}\mathfrak{s}}$ is irreducible. Hence, ${}^{\hat\circ}\mathfrak{X}^+ \cap (G/P \times s_\alpha\underline{o}^- \times (G/B^-)^\circ \times \underline{o})$ is a nonempty open subset of $\mathring{\mathfrak{X}}_{\mathfrak{s}\mathfrak{s}}$. By the parabolic analogue of [BK14, Proposition 3.5], $\eta_{\mathcal{U}} : \mathcal{U}\backslash\mathring{\mathfrak{X}}_{\mathfrak{s}\mathfrak{s}} \to \mathcal{U}\backslash(G/B^-)^\circ$ is surjective and, by Lemma 6.2, it is birational. In particular,

$$\dim\big(\mathcal{U}\backslash\mathring{\mathfrak{X}}_{\mathfrak{s}\mathfrak{s}}\big) = \dim\big(\mathcal{U}\backslash(G/B^-)^\circ\big). \tag{91}$$

Define the surjective projection

$$p : \mathfrak{X}_{\mathfrak{s}\mathfrak{s}} \to Y(w_1; v), \;\; (x, g\underline{o}^-) \mapsto x.$$

By definition,

$$p^{-1}(Y(w_1; s_\alpha v)) = \mathfrak{X}'_{\mathfrak{s}\mathfrak{s}}.$$

Consider the smooth open subset of $Y(w_1; v)$:

$${}^{\hat\circ}Y(w_1; v) := (s_\alpha\mathring{X}_P^{w_1}) \cap {}^{\hat\circ}X_v^P$$

and its closed subset of codimension 1:

$$\mathring{Y}(w_1; s_\alpha v) := (s_\alpha\mathring{X}_P^{w_1}) \cap \mathring{X}_{s_\alpha v}^P.$$

In particular, $\mathring{Y}(w_1; s_\alpha v)$ is a Cartier divisor of ${}^{\hat\circ}Y(w_1; v)$. Thus, $p^{-1}(\mathring{Y}(w_1; s_\alpha v))$ is a Cartier divisor of $p^{-1}({}^{\hat\circ}Y(w_1; v))$. Let $\mathring{p}$ be the restriction of the map p to the nonempty open subset $\mathring{\mathfrak{X}}_{\mathfrak{s}\mathfrak{s}}$: Then, $\mathring{p}$ is a dominant morphism. Since $\mathfrak{X}_{\mathfrak{s}\mathfrak{s}}$ is irreducible,

$$p^{-1}({}^{\hat\circ}Y(w_1; v)) \cap \mathring{\mathfrak{X}}_{\mathfrak{s}\mathfrak{s}} \neq \emptyset.$$

Since $p^{-1}(\mathring{Y}(w_1; s_\alpha v))$ and $\mathring{\mathfrak{X}}'_{\mathit{ss}}$ are both nonempty open subsets of irreducible $\mathfrak{X}'_{\mathit{ss}}$, we get that their intersection is nonempty. In particular,

$$p^{-1}(\mathring{Y}(w_1; s_\alpha v)) \cap \mathring{\mathfrak{X}}_{\mathit{ss}} \neq \emptyset.$$

The map $\mathring{p} : \mathring{\mathfrak{X}}_{\mathit{ss}} \to Y(w_1; v)$ is $\mathcal{U}$-invariant (with the trivial action of $\mathcal{U}$ on $Y(w_1; v)$). From this it is easy to see that

$$\dim\left(\mathcal{U}\backslash\mathring{\mathfrak{X}}'_{\mathit{ss}}\right) = \dim\left(\mathcal{U}\backslash\mathring{\mathfrak{X}}_{\mathit{ss}}\right) - 1. \tag{92}$$

Since X_v^P is P_α-stable, for any l_1, l_2 and l in L such that

$$(P/P, l_1 w_1^{-1}\underline{o}^-, l_2 w_2^{-1}\underline{o}^-, l v^{-1}\underline{o}) \text{ belongs to } \mathring{\mathfrak{X}}^+, \text{ we get}$$

$$(P/P, l_1 w_1^{-1}\underline{o}^-, l_2 w_2^{-1}\underline{o}^-, l v^{-1} s_\alpha \underline{o}) \in \mathfrak{X}' \cap \;\hat{\mathring{}}\, \mathfrak{X}^+.$$

Now, following the exact same argument as in the proof of the (a)-part starting from 'Consider the surjective morphism $\cdots$ ' till the end of the proof of the (a)-part, replacing $G\times_T$ by $G\times_{B_\alpha}$, we get the first part of the (c)-part.

The 'In particular' part of the (c)-part follows from the proof of Proposition 6.1 (specifically the equation (64)). ☐

Definition 6.6. *For $i = 1, 2$ and $(\alpha, i) \in \mathcal{D}_2$, take (cf. Lemma 4.1)*

$$\mathcal{N}_{\alpha,i} := \mathcal{M}_{\alpha,i} \text{ and } \mu_{\alpha,i} := \sigma_{\alpha,i}.$$

By Lemmas 4.1 and 6.5, we have

$$Z(\mu_{\alpha,i}) = E_{\alpha,i} = F_{\alpha,i}. \tag{93}$$

For $(\alpha, 3) \in \mathcal{D}_2$ and $j \in \{1, 2\}$ such that $w_j \nleq s_\alpha v$, take

$$\mathcal{N}_{\alpha,3} := \mathcal{M}_{\alpha,j} \text{ and } \mu_{\alpha,3} := \sigma_{\alpha,j}.$$

By Lemmas 4.1 and 6.5, we have

$$Z(\mu_{\alpha,3}) = E_{\alpha,3} = F_{\alpha,j}. \tag{94}$$

Let $(\alpha, 3) \in \mathcal{D}_3$. By Proposition 6.1 and Lemma 6.5, there exists j such that C is not contained in $F_{\alpha,j}$. With notation of Lemma 4.1, we set

$$\mathcal{N}_{\alpha,3} := \mathcal{M}_{\alpha,j} \text{ and } \mu_{\alpha,3} := \sigma_{\alpha,j}.$$

Then,

$$Z(\mu_{\alpha,3}) = F_{\alpha,j} \supset E_{\alpha,3}. \tag{95}$$

6.4. **The line bundles $\mathcal{N}_{\alpha,i}$.** The goal of this subsection is to prove that $\mathcal{N}_{\alpha,i}$ belongs to the face considered in Theorem 1.3:

Proposition 6.7. *For any $(\alpha, i) \in \mathcal{D}$, the center $Z(L)$ of L acts trivially on the restriction of $\mathcal{N}_{\alpha,i}$ to C.*

In fact, for any L-equivariant line bundle $\mathcal{L}$ over C with $\mathrm{H}^0(C, \mathcal{L})^L \neq 0$, $Z(L)$ acts trivially on $\mathcal{L}$. In particular, if we write $\mathcal{N}_{\alpha,i} = \mathcal{L}^-(\lambda_1) \otimes \mathcal{L}^-(\lambda_2) \otimes \mathcal{L}(\mu)$, then for all $\alpha_j \notin \Delta(P)$,

$$\lambda_1(w_1 x_j) + \lambda_2(w_2 x_j) - \mu(v x_j) = 0 \tag{$I^j_{(w_1,w_2,v)}$}$$

Proof. Consider the G-invariant section $\mu_{\alpha,i}$ of $\mathcal{N}_{\alpha,i}$. If $(\alpha, i) \in \mathcal{D}_1 \cup \mathcal{D}_2$, then by the equations (65), (94) and (93), $Z(\mu_{\alpha,i}) = E_{\alpha,i}$. Then Proposition 6.1 implies that $\mu_{\alpha,i}$ restricts to a nonzero L-invariant section on C. If $(\alpha, i) \in \mathcal{D}_3$ (and hence $i = 3$), then by the equations (95), $Z(\mu_{\alpha,i}) = F_{\alpha,j}$. But $F_{\alpha,j}$ was chosen not to contain C. Thus $\mu_{\alpha,i}$ restricts to a nonzero L-invariant section on C.

Since $Z(L)$ acts trivially on C, it acts by a character on any line bundle over C. The existence of the nonzero $Z(L)$-invariant section implies that this character is trivial for the restriction of $\mathcal{N}_{\alpha,i}$.

Write $\mathcal{N}_{\alpha,i} = \mathcal{L}^-(\lambda_1) \otimes \mathcal{L}^-(\lambda_2) \otimes \mathcal{L}(\mu)$ and fix $\alpha_j \notin \Delta(P)$. There exists $d > 0$, such that dx_j is the differential at 1 of a one parameter subgroup of $Z(L)$. This one parameter subgroup acts with weight $\lambda_1(w_1 x_j)$, $\lambda_2(w_2 x_j)$ and $-\mu(v x_j)$ on the fiber over $w_1^{-1}\underline{o}^-$, $w_2^{-1}\underline{o}^-$ and $v^{-1}\underline{o}$ in $\mathcal{L}^-(\lambda_1)$, $\mathcal{L}^-(\lambda_2)$ and $\mathcal{L}(\nu)$ respectively. Thus, the equality $I^j_{(w_1,w_2,v)}$ follows proving Proposition 6.7. $\square$

6.5. **The line bundles $\mathcal{N}_{\alpha,i}$ and the lines $\ell_{\beta,j}$.** Recall the definition of the line $\ell_{\beta,j}$ from §1. We now study the restriction of the line bundle $\mathcal{N}_{\alpha,i}$ to the lines $\ell_{\beta,j}$. This will be used to apply Theorem 1.5.

Lemma 6.8. *Let $(\alpha, i) \in \mathcal{D}$ and $(\beta, j) \in \mathcal{D}$ be two distinct elements. Then,*

(i) *the degree of the restriction of $\mathcal{N}_{\alpha,i}$ to $\ell_{\alpha,i}$ is positive.*
(ii) *the degree of the restriction of $\mathcal{N}_{\alpha,i}$ to $\ell_{\beta,j}$ is nonnegative.*

Proof. Take $(\alpha, 1) \in \mathcal{D}$. Then, as in Section 1,

$$\ell_{\alpha,1} = (w_1^{-1} P_\alpha^- \underline{o}^-, w_2^{-1}\underline{o}^-, v^{-1}\underline{o}).$$

Since the line bundle $\mathcal{N}_{\alpha,i}$ has the form $\mathcal{L}^-(\lambda_1) \boxtimes \mathcal{L}^-(\lambda_2) \boxtimes \mathcal{L}(\mu)$ for some $(\lambda_1, \lambda_2, \mu) \in P_+^3$ (cf. Proposition 4.4 and Lemma 4.1),

$$\mathcal{N}_{\alpha,1|\ell_{\alpha,1}} \simeq \mathcal{L}^-(\lambda_1)_{|w_1^{-1} P_\alpha^- \underline{o}^-},$$

which is of degree

$$(w_1^{-1}\lambda_1)(w_1^{-1}\alpha^\vee) = \lambda_1(\alpha^\vee) \geq 0.$$

Assume, if possible, that $\lambda_1(\alpha^\vee) = 0$. Then, the zero set $Z(\mu_{\alpha,1})$ would be of the form $\pi_\alpha^{-1}(S)$ for some $S \subset G/P_\alpha^- \times G/B^- \times G/B$, where

$$\pi_\alpha : G/B^- \times G/B^- \times G/B \to G/P_\alpha^- \times G/B^- \times G/B$$

is the projection.

Then, by equations (65), (94) and (93),

$$Z(\mu_{\alpha,1}) = E_{\alpha,1} = G \cdot \bar{C}^+_{s_\alpha w_1, w_2, v},$$

and hence we would have

$$Z(\mu_{\alpha,1}) \supset G \cdot \bar{C}^+_{w_1,w_2,v} = \mathcal{X},$$

where the last equality follows from [BK14, Proposition 3.5] since ε_P^v occurs with nonzero coefficient in $\varepsilon_P^{w_1} \cdot \varepsilon_P^{w_2}$. This contradicts the non-vanishing of $\mu_{\alpha,1}$. Thus, $\lambda_1(\alpha^\vee) > 0$, proving (i) for $(\alpha, 1) \in \mathcal{D}$. The same proof works for any $(\alpha, i) \in \mathcal{D}$ to prove (i). The sole difference is that if $(\alpha, i) \in \mathcal{D}_3$, equation (95) gives only the inclusion

$$Z(\mu_{\alpha,3}) \supset G \cdot \bar{C}^+_{w_1,w_2,s_\alpha v},$$

which is sufficient to get a contradiction.

To prove (ii), we still take $(\alpha, 1) \in \mathcal{D}$ and $(\beta, j) \in \mathcal{D}$ for $j = 1, 2$. Then,

$$\mathcal{N}_{\alpha,1_{|\ell_{\beta,j}}} \simeq \mathcal{L}^-(\lambda_j)_{|w_j^{-1} P_\beta^- \underline{o}^-},$$

which is of degree

$$(w_j^{-1}\lambda_j)(w_j^{-1}\beta^\vee) = \lambda_j(\beta^\vee) \geq 0.$$

For $(\beta, 3) \in \mathcal{D}$,

$$\mathcal{N}_{\alpha,1_{|\ell_{\beta,3}}} \simeq \mathcal{L}(\mu)_{|v^{-1} P_\beta \underline{o}},$$

which is of degree

$$(v^{-1}\mu)(v^{-1}\beta^\vee) = \mu(\beta^\vee) \geq 0.$$

This proves (ii) for $(\alpha, 1) \in \mathcal{D}$. The same proof gives (ii) for any $(\alpha, i) \in \mathcal{D}$. □

6.6. **Conclusion of the proof of Theorem 1.3.** Let $w_1, w_2, v \in W^P$ be as in Theorem 1.3, i.e., ε_P^v occurs with coefficient 1 in the deformed product $\varepsilon_P^{w_1} \odot_0 \varepsilon_P^{w_2}$.

Set $d = 2\dim \mathfrak{h} + \sharp\Delta(P)$. Let $\mathcal{F} = \mathcal{F}^P_{w_1,w_2,v}$ be the convex cone generated by the weights $(\lambda_1, \lambda_2, \mu)$ as in Theorem 1.3. Since the linear forms $\{I^j_{(w_1,w_2,v)}\}_{\alpha_j \in \Delta\setminus\Delta(P)}$ restricted to $E_{\mathfrak{g}}$ (cf. Proposition 3.1) defining $\mathcal{F}$ are linearly independent, the dimension of $\mathcal{F}$ is at most d.

We now have to produce 'enough' points in $\mathcal{F}$. To do this we consider the restriction map $\mathrm{Pic}^{G^3}(\mathcal{X}) \longrightarrow \mathrm{Pic}^{L^3}(C)$ and we apply Theorem 1.5 to sufficiently many line bundles $\mathcal{L}$ such that $\mathrm{H}^0(C, \mathcal{L}_{|C})^L \neq \{0\}$.

Observe that, for any $w \in W^P$, the map

$$L/B_L^- \to Lw^{-1}\underline{o}^- \subset G/B^-, \ \ lB_L^- \mapsto lw^{-1}\underline{o}^- \ \text{ is an isomorphism}$$

and also the map

$$L/B_L \to Lw^{-1}\underline{o} \subset G/B, \ \ lB_L \mapsto lw^{-1}\underline{o} \ \text{ is an isomorphism,}$$

where $B_L := B \cap L$ is the standard Borel subgroup of L and $B_L^- := B^- \cap L$ is the standard opposite Borel subgroup of L. (To prove the above two isomorphisms, use the fact that $w\Delta_P \subset \Phi^+$.) Thus, the restriction map $\mathrm{Pic}^{G^3}(\mathcal{X}) \simeq (\mathfrak{h}_{\mathbb{Z}}^*)^3 \longrightarrow \mathrm{Pic}^{L^3}(C) \simeq (\mathfrak{h}_{\mathbb{Z}}^*)^3$ is an isomorphism. Let $\mathfrak{l}$ denote the Lie algebra of L.

Lemma 6.9. *There exist $\mathcal{L}_1, \dots, \mathcal{L}_d \in \mathrm{Pic}^{G^3}(\mathcal{X})$ such that*

(i) *$\mathcal{L}_1, \dots, \mathcal{L}_d \in \mathrm{Pic}^{G^3}(\mathcal{X}) \otimes \mathbb{Q}$ are linearly independent;*
(ii) *The restriction of each $\mathcal{L}_i$ to C belongs to $\Gamma(\mathfrak{l})$.*

Proof. By Proposition 3.1, $\Gamma(\mathfrak{l})$ has dimension d. Hence, $\Gamma(\mathfrak{l}) \subset \mathrm{Pic}^{L^3}(C)$ contains d linearly independent elements. Then, the lemma follows from the isomorphism $\mathrm{Pic}^{G^3}(\mathcal{X}) \simeq \mathrm{Pic}^{L^3}(C)$. □

Proof of Theorem 1.3: Up to taking tensor powers, we may assume that the restriction of $\mathcal{L}_i$ to C admits a nonzero L-invariant section σ_i. By Lemma 6.8, there exists $(a_{\alpha,i})_{(\alpha,i)\in\mathcal{D}} \in \mathbb{N}^{\mathcal{D}}$ such that $\mathcal{N} := \sum_{(\alpha,i)\in\mathcal{D}} a_{\alpha,i}\mathcal{N}_{\alpha,i}$ satisfies:

$\mathcal{L}_k \otimes \mathcal{N}$ is nonnegative for all k when restricted to any $\ell_{\beta,j}$ for $(\beta, j) \in \mathcal{D}$.

Moreover, up to changing $\mathcal{N}$ by $2\mathcal{N}$ if necessary, we may assume that $\mathcal{L}_1 \otimes \mathcal{N}, \dots, \mathcal{L}_d \otimes \mathcal{N} \in \mathrm{Pic}^{G^3}(\mathcal{X}) \otimes \mathbb{Q}$ are linearly independent.

By Proposition 6.1, $\mathcal{N}$ has a G-invariant section $\sigma_{\mathcal{N}}$ that does not vanish identically on C. Then,

$$\tilde{\sigma}_i \in \mathrm{H}^0(C, \mathcal{L}_i \otimes \mathcal{N})^L \backslash \{0\}, \ \text{ where } \tilde{\sigma}_i := (\sigma_i \otimes \sigma_{\mathcal{N}})_{|C}.$$

Moreover, since $\tilde{\sigma}_i$ is not identically zero on C, by Proposition 6.7, each $\mathcal{L}_i \otimes \mathcal{N}$ satisfies the identity $I^j_{(w_1,w_2,v)}$ of Theorem 1.3 for all $\alpha_j \in \Delta \backslash \Delta(P)$.

By Theorem 1.5, each $\tilde{\sigma}_i$ can be extended to a G-invariant section $\tilde{\sigma}_i$ of $\mathcal{L}_i \otimes \mathcal{N}$. In particular, $\mathcal{L}_i \otimes \mathcal{N}$ belongs to $\Gamma(\mathfrak{g})$. Thus, the dimension of $\mathcal{F}$ is at least d and hence it is exactly d. This proves the theorem.

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Shrawan Kumar: Department of Mathematics, University of North Carolina, Chapel Hill, NC 27599-3250, USA

E-mail address: shrawan@email.unc.edu

N. Ressayre: Institut Camille Jordan (ICJ), UMR CNRS 5208, Université Claude Bernard Lyon I, 43 boulevard du 11 novembre 1918, F - 69622 Villeurbanne cedex

E-mail address: ressayre@math.univ-lyon1.fr