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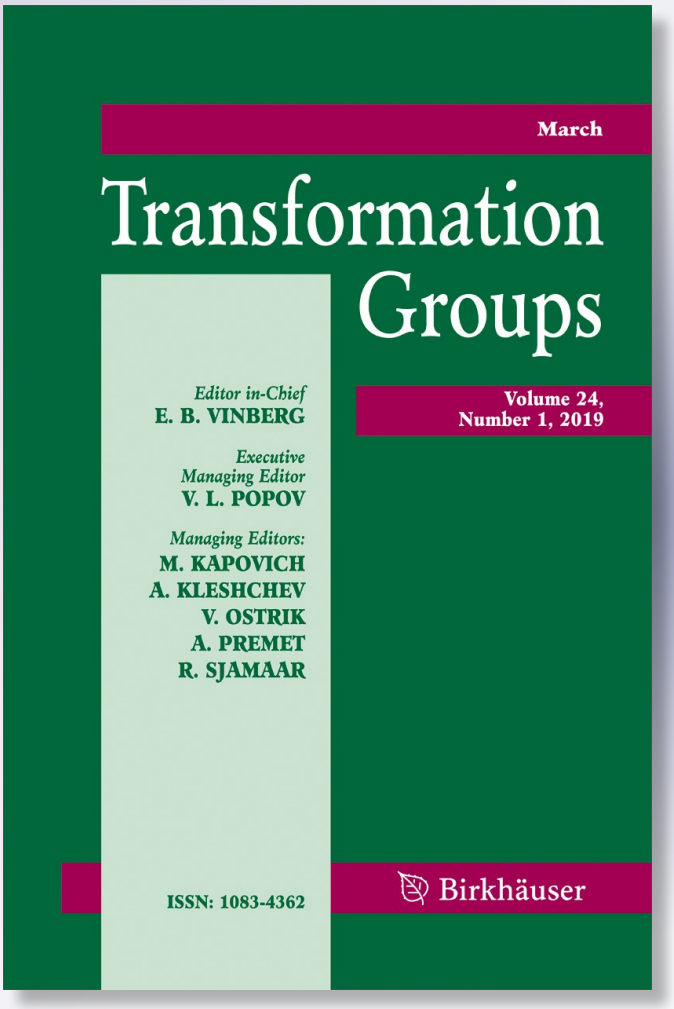

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A COMPLETE SET OF INTERTWINERS FOR ARBITRARY TENSOR PRODUCT REPRESENTATIONS VIA CURRENT ALGEBRAS

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Abstract. Let $\mathfrak{g}$ be a reductive Lie algebra and let $\vec{V}(\vec{\lambda})$ be a tensor product of d copies of finite-dimensional irreducible $\mathfrak{g}$-modules. Choosing d points in $\mathbb{C}$, $\vec{V}(\vec{\lambda})$ acquires a natural structure of the current algebra $\mathfrak{g} \otimes \mathbb{C}[t]$-module. Following a work of Rao [R], we produce an explicit and complete set of $\mathfrak{g}$-module intertwiners of $\vec{V}(\vec{\lambda})$ in terms of the action of the current algebra.

Introduction

Let $\mathfrak{g}$ be a finite-dimensional reductive Lie algebra over the complex numbers $\mathbb{C}$ and let A be any commutative $\mathbb{C}$-algebra with identity. Then, $\mathfrak{g} \otimes A$ acquires a natural structure of Lie algebra. Take any $\mathfrak{g}$-invariant $\theta \in \left[\mathfrak{g}^{\otimes k}\right]^{\mathfrak{g}}$, $\theta = \sum_i x_1^i \otimes x_2^i \otimes \cdots \otimes x_k^i$, and any $P_1, \ldots, P_k \in A$. Then, $\theta(P_1, \ldots, P_k) \in U(\mathfrak{g} \otimes A)$ defined by

$$\theta(P_1, \ldots, P_k) := \sum_i x_1^i(P_1) \ldots x_k^i(P_k),$$

commutes with $\mathfrak{g}$ (cf. Lemma 3), where $U(\mathfrak{g} \otimes A)$ is the enveloping algebra of $\mathfrak{g} \otimes A$ and $x_j^i(P_j)$ denotes $x_j^i \otimes P_j \in \mathfrak{g} \otimes A$.

In fact, we show that, via the above construction, we get all of $[U(\mathfrak{g} \otimes A)]^{\mathfrak{g}}$ as θ ranges over $\left[\mathfrak{g}^{\otimes k}\right]^{\mathfrak{g}}$ and P_i's range over the elements of A (see Proposition 8 for a more precise result).

Let us take $A = \mathbb{C}[t]$ and denote $\mathfrak{g} \otimes A$ by $\mathfrak{g}[t]$. For any $\vec{p} = (p_1, \ldots, p_d) \in \mathbb{C}^d$ and any irreducible $\mathfrak{g}$-modules $V(\lambda_1), \ldots, V(\lambda_d)$ with highest weights $\lambda_1, \ldots, \lambda_d$ respectively, consider the tensor product $\mathfrak{g}$-module

$$\vec{V}(\vec{\lambda}) := V(\lambda_1) \otimes \cdots \otimes V(\lambda_d), \text{ where } \vec{\lambda} = (\lambda_1, \ldots, \lambda_d).$$

Then, $\vec{V}(\vec{\lambda})$ acquires the structure of a $\mathfrak{g}[t]$-module (called an *evaluation module*):

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$$x(P)\cdot(v_1\otimes\cdots\otimes v_d) := \sum_{i=1}^{d} P(p_i)v_1\otimes\cdots\otimes x\cdot v_i\otimes\cdots\otimes v_d \text{ for } x\in\mathfrak{g}, P\in\mathbb{C}[t], v_i\in V(\lambda_i).$$

To emphasize the $\mathfrak{g}[t]$-module structure, we denote $\vec{V}(\vec{\lambda})$ by $\vec{V}_{\vec{p}}(\vec{\lambda})$.

Consider the Casimir element $\Omega\in\left[\mathfrak{g}^{\otimes 2}\right]^{\mathfrak{g}}$. Then, we calculate the action of $\Omega(P,Q)$ on $\vec{V}_{\vec{p}}(\vec{\lambda})$ for any $\vec{\lambda}=(\lambda_1,\lambda_2)$ (cf. Lemma 6).

We assume now that p_i are all distinct. In this case $\vec{V}_{\vec{p}}(\vec{\lambda})$ is an irreducible $\mathfrak{g}[t]$-module. Decompose $\vec{V}_{\vec{p}}(\vec{\lambda})$ into its isotypic components (as a $\mathfrak{g}$-module):

$$\vec{V}_{\vec{p}}(\vec{\lambda})=\bigoplus \vec{V}_{\vec{p}}(\vec{\lambda})[\mu],$$

where we denote by $\vec{V}_{\vec{p}}(\vec{\lambda})[\mu]$ the isotypic component corresponding to the highest weight μ.

Clearly, the action of $\mathfrak{g}$ commutes with the action of $[U(\mathfrak{g}[t])]^{\mathfrak{g}}$ on $\vec{V}_{\vec{p}}(\vec{\lambda})$. Thus, we get an action of $\mathfrak{g}\times[U(\mathfrak{g}[t])]^{\mathfrak{g}}$ on $\vec{V}_{\vec{p}}(\vec{\lambda})$ stabilizing each isotypic component $\vec{V}_{\vec{p}}(\vec{\lambda})[\mu]$.

The following theorem is one of our principal results of the paper (cf. Theorem 9).

Theorem 1. *Each isotypic component $\vec{V}_{\vec{p}}(\vec{\lambda})[\mu]$ is an irreducible module for $\mathfrak{g}\times[U(\mathfrak{g}[t])]^{\mathfrak{g}}$.*

In Sections 2, 4 and 5, we determine the spaces $\left[\mathfrak{g}^{\otimes k}\right]^{\mathfrak{g}}$ for $\mathfrak{g}=\mathfrak{gl}(n),\mathfrak{sp}(2n)$ and $\mathfrak{so}(n)$ respectively using the First Fundamental Theorem of Invariant Theory. (In the last case, for n even, we only determine $\left[\mathfrak{g}^{\otimes k}\right]^{\mathrm{O}(n)}$.)

Let $V=\mathbb{C}^n$ be the standard representation of $\mathfrak{g}=\mathfrak{gl}(n)$. For any positive integer k, the symmetric group Σ_k acts on the tensor product $V^{\otimes k}$ by permuting the factors. Clearly, this action of Σ_k commutes with the tensor product action of $\mathfrak{gl}(n)$ on $V^{\otimes k}$. Thus, we have an algebra homomorphism:

$$\Phi:\mathbb{C}[\Sigma_k]\to\mathrm{End}_{\mathfrak{g}}(V^{\otimes k}),$$

where $\mathrm{End}_{\mathfrak{g}}(V^{\otimes k})$ denotes the space of $\mathfrak{g}$-module endomorphisms of $V^{\otimes k}$. By the Schur–Weyl duality, the above map Φ is an (algebra) isomorphism. Choose $\vec{p}=(p_1,\ldots,p_k)\in\mathbb{C}^k$ such that p_i's are distinct. There is a surjective algebra homomorphism (for $\mathfrak{g}=\mathfrak{gl}(n)$)

$$\varphi^o:[U(\mathfrak{g}[t])]^{\mathfrak{g}}\twoheadrightarrow\mathrm{End}_{\mathfrak{g}}(V^{\otimes k})\simeq\mathbb{C}[\Sigma_k],$$

where the last identification is via Φ (cf. proof of Theorem 9). Thus, we get a surjective algebra homomorphism

$$\Xi:[U(\mathfrak{g}[t])]^{\mathfrak{g}}\twoheadrightarrow\mathbb{C}[\Sigma_k].$$

We give an explicit preimage of any reflection (i.e., transposition) $\tau=(r,s)\in\Sigma_k$ under Ξ (cf. Proposition 14).

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1. Intertwining operators — Main results

Let $\mathfrak{g}$ be a finite-dimensional reductive Lie algebra over the complex numbers $\mathbb{C}$ and let A be any commutative $\mathbb{C}$-algebra with identity. Then, $\mathfrak{g} \otimes A$ acquires a natural structure of Lie algebra:

$$[x(P), y(Q)] := [x, y](PQ), \text{ for } x, y \in \mathfrak{g},\ P, Q \in A,$$

where $x(P)$ denotes $x \otimes P$.

Definition 2. Take any $\mathfrak{g}$-invariant (under the adjoint action) $\theta \in \left[\mathfrak{g}^{\otimes k}\right]^{\mathfrak{g}}$, $\theta = \sum_i x_1^i \otimes x_2^i \otimes \cdots \otimes x_k^i$, and any $P_1, \ldots, P_k \in A$. Define $\theta(P_1, \ldots, P_k) \in U(\mathfrak{g} \otimes A)$ by

$$\theta(P_1, \ldots, P_k) := \sum_i x_1^i(P_1) \ldots x_k^i(P_k), \tag{1}$$

where $U(\mathfrak{g} \otimes A)$ denotes the enveloping algebra of $\mathfrak{g} \otimes A$.

Lemma 3. $[\mathfrak{g}, \theta(P_1, \ldots, P_k)] = 0$.

Proof. Let $\pi : T(\mathfrak{g} \otimes A) \twoheadrightarrow U(\mathfrak{g} \otimes A)$ be the canonical surjective homomorphism, where T is the tensor algebra. Consider the element $\hat{\theta}(P_1, \ldots, P_k) \in T(\mathfrak{g} \otimes A)$ defined by

$$\hat{\theta}(P_1, \ldots, P_k) = \sum_i x_1^i(P_1) \otimes \cdots \otimes x_k^i(P_k) \in T(\mathfrak{g} \otimes A).$$

For any $y \in \mathfrak{g}$,

$$\begin{aligned} [y, \hat{\theta}(P_1, \ldots, P_k)] &= \sum_i \sum_{j=1}^{k} x_1^i(P_1) \otimes \cdots \otimes [y, x_j^i](P_j) \otimes \cdots \otimes x_k^i(P_k) \\ &= 0, \quad \text{since} \quad [\mathfrak{g}, \theta] = 0, \end{aligned} \tag{2}$$

and $\mathfrak{g}(P_1) \otimes \cdots \otimes \mathfrak{g}(P_k) \simeq \mathfrak{g} \otimes \cdots \otimes \mathfrak{g}$ as $\mathfrak{g}$-modules under the adjoint action.

Now, since $\pi(\hat{\theta}(P_1, \ldots, P_k)) = \theta(P_1, \ldots, P_k)$, by the identity (2), the lemma follows. $\square$

Definition 4 (Evaluation modules)**.** From now on we take $A = \mathbb{C}[t]$ and denote $\mathfrak{g} \otimes A$ by $\mathfrak{g}[t]$. For any $\vec{p} = (p_1, \ldots, p_d) \in \mathbb{C}^d$ and any irreducible $\mathfrak{g}$-modules $V(\lambda_1), \ldots, V(\lambda_d)$ with highest weights $\lambda_1, \ldots, \lambda_d$ respectively, consider the tensor product $\mathfrak{g}$-module

$$\vec{V}(\vec{\lambda}) := V(\lambda_1) \otimes \cdots \otimes V(\lambda_d), \text{ where } \vec{\lambda} = (\lambda_1, \ldots, \lambda_d).$$

Then, $\vec{V}(\vec{\lambda})$ acquires the structure of a $\mathfrak{g}[t]$-module:

$$\begin{aligned} &x(P) \cdot (v_1 \otimes \cdots \otimes v_d) \\ &\qquad := \sum_{i=1}^{d} P(p_i) v_1 \otimes \cdots \otimes x \cdot v_i \otimes \cdots \otimes v_d \text{ for } x \in \mathfrak{g}, P \in \mathbb{C}[t], v_i \in V(\lambda_i). \end{aligned}$$

To emphasize the $\mathfrak{g}[t]$-module structure, we denote $\vec{V}(\vec{\lambda})$ by $\vec{V}_{\vec{p}}(\vec{\lambda})$ and it is called an *evaluation module*. It is well known (and easy to prove) that when $p_1, \ldots, p_d$ are distinct, then $\vec{V}_{\vec{p}}(\vec{\lambda})$ is an irreducible $\mathfrak{g}[t]$-module.

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Example 5. Take a basis $\{e_i\}$ and the dual basis $\{e^i\}$ of $\mathfrak{g}$ under an invariant non-degenerate symmetric form $\langle\ ,\ \rangle$ on $\mathfrak{g}$. Recall the Casimir element

$$\Omega := \sum_i e_i \otimes e^i \in \left[\mathfrak{g}^{\otimes 2}\right]^{\mathfrak{g}}.$$

Then, by Lemma 3, for any $P, Q \in A$, $\Omega(P,Q) \in [\mathfrak{g} \otimes A]^{\mathfrak{g}}$.

For any $\vec{p} = (p_1, p_2) \in \mathbb{C}^2$, consider the evaluation module $\vec{V}_{\vec{p}}(\vec{\lambda})$, where $\vec{\lambda} = (\lambda_1, \lambda_2)$ is a pair of dominant integral weights. Let

$$P(p_1) = w_1,\ \ P(p_2) = w_2$$
$$Q(p_1) = z_1,\ \ Q(p_2) = z_2.$$

Lemma 6. *For any v in the $\mathfrak{g}$-isotypic component of $V(\lambda_1) \otimes V(\lambda_2)$ of highest weight μ,*

$$\Omega(P,Q)(v) = \Big(w_1 z_1 C_{\lambda_1} + w_2 z_2 C_{\lambda_2} + \frac{w_1 z_2 + w_2 z_1}{2}(C_\mu - C_{\lambda_1} - C_{\lambda_2})\Big)v, \tag{3}$$

where C_μ denotes the scalar by which Ω acts on $V(\mu)$.

Proof. For $v_1 \in V(\lambda_1)$ and $v_2 \in V(\lambda_2)$,

$$\begin{aligned}\Big(\sum_i e_i(P)\cdot e^i(Q)\Big)\cdot(v_1\otimes v_2)&\\ = w_1 z_1 C_{\lambda_1}(v_1\otimes v_2) + w_2 z_2 C_{\lambda_2}(v_1\otimes v_2)&\\ + w_1 z_2 \sum_i e_i\cdot v_1 \otimes e^i\cdot v_2 + w_2 z_1 \sum_i e^i\cdot v_1\otimes e_i\cdot v_2.&\end{aligned} \tag{4}$$

Taking $w_1 = w_2 = z_1 = z_2 = 1$ in the above, we get

$$\Omega(v_1\otimes v_2) = C_{\lambda_1}(v_1\otimes v_2) + C_{\lambda_2}(v_1\otimes v_2) + \sum_i e_i\cdot v_1\otimes e^i\cdot v_2 + \sum_i e^i\cdot v_1\otimes e_i\cdot v_2. \tag{5}$$

Further, it is easy to see that

$$\sum_i e^i\cdot v_1\otimes e_i\cdot v_2 = \sum_i e_i\cdot v_1\otimes e^i\cdot v_2. \tag{6}$$

Thus, combining the equations (4) - (6), we get

$$\Big(\sum_i e_i(P)\cdot e^i(Q)\Big)\cdot v = \Big(w_1 z_1 C_{\lambda_1} + w_2 z_2 C_{\lambda_2} + \frac{w_1 z_2 + w_2 z_1}{2}(C_\mu - C_{\lambda_1} - C_{\lambda_2})\Big)v,$$

for v in the $\mathfrak{g}$-isotypic component of $V(\lambda_1)\otimes V(\lambda_2)$ of highest weight μ. This proves the lemma. $\square$

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Example 7. Take any simple Lie algebra $\mathfrak{g}$ and any

$$\theta \in \left[\wedge^k(\mathfrak{g})\right]^{\mathfrak{g}} \subset \left[\otimes^k \mathfrak{g}\right]^{\mathfrak{g}},$$

where we think of $\wedge^k(\mathfrak{g})$ as the subspace of $\otimes^k\mathfrak{g}$ consisting of alternating tensors. Recall that

$$[\wedge^\bullet(\mathfrak{g})]^{\mathfrak{g}} \simeq \wedge(\theta_1, \dots, \theta_\ell),$$

where $\ell := \operatorname{rank}\mathfrak{g}$ and $\theta_j \in \left[\wedge^{2b_j+1}(\mathfrak{g})\right]^{\mathfrak{g}}$ are primitive generators. Here $1 = b_1 < b_2 \le \dots \le b_\ell$ are the exponents of $\mathfrak{g}$. This gives rise to $\theta(P_1, \dots, P_k) \in [U(\mathfrak{g} \otimes A]^{\mathfrak{g}}$, for any $P_1, \dots P_k \in A$.

Similarly,

$$\left[S^k(\mathfrak{g})\right]^{\mathfrak{g}} \subset \left[\otimes^k \mathfrak{g}\right]^{\mathfrak{g}},$$

where we think of $S^k(\mathfrak{g})$ as the subspace of $\otimes^k\mathfrak{g}$ consisting of symmetric tensors. Recall that $[S^\bullet(\mathfrak{g})]^{\mathfrak{g}}$ is a polynomial algebra generated by certain homogeneous elements $\{\delta_j\}_{1\le j\le \ell}$, where $\delta_j \in \left[S^{b_j+1}(\mathfrak{g})\right]^{\mathfrak{g}}$.

Recall that, for any Lie algebra $\mathfrak{s}$, there is a canonical surjective algebra homomorphism from the tensor algebra to the enveloping algebra $\pi : T(\mathfrak{s}) \to U(\mathfrak{s})$ whose kernel is the two-sided ideal generated by $\{x \otimes y - y \otimes x - [x, y]; x, y \in \mathfrak{s}\}$.

In the following, let $\theta(n_1, \dots, n_k)$ denote $\theta(t^{n_1}, \dots, t^{n_k})$.

Proposition 8. *For any finite-dimensional reductive Lie algebra $\mathfrak{g}$, the subalgebra $[U(\mathfrak{g}[t])]^{\mathfrak{g}}$ is spanned (over $\mathbb{C}$) by $\{\theta(n_1, \dots, n_k)\}$, where θ runs over a homogeneous basis of $[T(\mathfrak{g})]^{\mathfrak{g}}$ and, for $k = \deg\theta$, $n_1 \le \dots \le n_k$ runs over non-negative integers.*

Further, the subalgebra $[U(\mathfrak{g}[t])]^{\mathfrak{g}}$ is generated (as an algebra) by $\{\theta(n_1, \dots, n_k)\}$, where θ runs over a set of homogeneous algebra generators of $[T(\mathfrak{g})]^{\mathfrak{g}}$ and, for $k = \deg\theta$, $n_1, \dots, n_k$ runs over non-negative integers.

Proof. As earlier, consider the surjective algebra homomorphism:

$$\pi : T(\mathfrak{g}[t]) \twoheadrightarrow U(\mathfrak{g}[t]). \tag{7}$$

Now, $\mathfrak{g}[t] = \bigoplus_{n=0}^\infty \mathfrak{g}(n)$, where $\mathfrak{g}(n) := \mathfrak{g} \otimes t^n$. Hence,

$$\begin{aligned} T(\mathfrak{g}[t]) &= \bigoplus_{k\ge 0} \bigoplus_{n_1,\dots,n_k\in Z_+} \mathfrak{g}(n_1) \otimes \dots \otimes \mathfrak{g}(n_k) \\ &\simeq \bigoplus_{k\ge 0} \bigoplus_{n_1,\dots,n_k\in Z_+} \mathfrak{g}^{\otimes k}[n_1, \dots, n_k] \quad \text{as } \mathfrak{g}\text{-modules}, \end{aligned} \tag{8}$$

where $\mathfrak{g}^{\otimes k}[n_1, \dots, n_k]$ simply means $\mathfrak{g}^{\otimes k}$ as a $\mathfrak{g}$-module, but as a subset of $T(\mathfrak{g}[t])$, it is defined to be $\mathfrak{g}(n_1) \otimes \dots \otimes \mathfrak{g}(n_k)$.

By (7), we get

$$[T(\mathfrak{g}[t])]^{\mathfrak{g}} \twoheadrightarrow [U(\mathfrak{g}[t])]^{\mathfrak{g}}.$$

From (8), we get

$$[T(\mathfrak{g}[t])]^{\mathfrak{g}} \simeq \bigoplus_{k\ge 0} \bigoplus_{n_1,\dots,n_k\in\mathbb{Z}_+} \left[\mathfrak{g}^{\otimes k}\right]^{\mathfrak{g}} [n_1, \dots, n_k].$$

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From this we see that $\{\hat{\theta}(n_1, \dots, n_k)\}$ spans (resp. generates) the algebra $[T(\mathfrak{g}[t])]^{\mathfrak{g}}$, where θ runs over a homogeneous basis (resp. homogeneous algebra generators) of $[T(\mathfrak{g})]^{\mathfrak{g}}$ and, for $k = \deg \theta$, $n_1, \dots, n_k$ runs over non-negative integers. (Here $\hat{\theta}(n_1, \dots, n_k)$ denotes the element in $[T(\mathfrak{g}[t])]^{\mathfrak{g}}$ as in the proof of Lemma 3.) From the surjectivity of the algebra homomorphism π, we get that $\{\theta(n_1, \dots, n_k)\}$ spans in the first case (resp. generates in the second case), where $n_1, \dots n_k$ runs over non-negative integers. Now, we can restrict to $0 \le n_1 \le \cdots \le n_k$ in the first case, which is easily seen from the commutation relation in $U(\mathfrak{g}[t])$. $\square$

Let $\vec{\lambda} = (\lambda_1, \dots, \lambda_d)$ be any tuple of dominant integral weights and let $\vec{p} = (p_1, \dots, p_d)$ be any tuple of *distinct* points in $\mathbb{C}$. Then, the evaluation module $\vec{V}_{\vec{p}}(\vec{\lambda})$ is an irreducible $\mathfrak{g}[t]$-module (cf. Definition 4). Decompose $\vec{V}_{\vec{p}}(\vec{\lambda})$ into its isotypic components (as a $\mathfrak{g}$-module):

$$\vec{V}_{\vec{p}}(\vec{\lambda}) = \bigoplus \vec{V}_{\vec{p}}(\vec{\lambda})[\mu],$$

where we denote by $\vec{V}_{\vec{p}}(\vec{\lambda})[\mu]$ the isotypic component corresponding to the highest weight μ.

Clearly, the action of $\mathfrak{g}$ commutes with the action of $[U(\mathfrak{g}[t])]^{\mathfrak{g}}$ on $\vec{V}_{\vec{p}}(\vec{\lambda})$. Thus, we get an action of $\mathfrak{g} \times [U(\mathfrak{g}[t])]^{\mathfrak{g}}$ on $\vec{V}_{\vec{p}}(\vec{\lambda})$ stabilizing each isotypic component $\vec{V}_{\vec{p}}(\vec{\lambda})[\mu]$.

The following theorem is one of our principal results of the paper. In view of Remark 12, the following theorem in the case of $\mathfrak{g} = \mathfrak{gl}(n)$ was stated as an open problem in [R, Rem. 4.9]. The following result is an instance of general 'double centralizer result', see, e.g., [D, Thm. 9.1.12].

Theorem 9. *Let $\mathfrak{g}$ be a reductive Lie algebra. With the notation as above, each isotypic component $\vec{V}_{\vec{p}}(\vec{\lambda})[\mu]$ is an irreducible module for $\mathfrak{g} \times [U(\mathfrak{g}[t])]^{\mathfrak{g}}$.*

Proof. Choose a Borel subalgebra $\mathfrak{b}$ of $\mathfrak{g}$ and let $W[\mu]^+$ be the $\mathfrak{b}$-eigen subspace of $W[\mu] := \vec{V}_{\vec{p}}(\vec{\lambda})[\mu]$. Then, $[U(\mathfrak{g}[t])]^{\mathfrak{g}}$ acts on $W[\mu]^+$. We have a ring homomorphism coming from the representation $W := \vec{V}_{\vec{p}}(\vec{\lambda})$:

$$\varphi : U(\mathfrak{g}[t]) \to \mathrm{End}_{\mathbb{C}}(W).$$

Since W is an irreducible $\mathfrak{g}[t]$-module, by Burnside's theorem (cf. [L, Cor. 1, Chap. XVII, §3]) φ is surjective. Taking the $\mathfrak{g}$-invariants, we get a surjective homomorphism

$$\varphi^o : [U(\mathfrak{g}[t])]^{\mathfrak{g}} \twoheadrightarrow \mathrm{End}_{\mathfrak{g}}(W) \simeq \prod_{\mu} \mathrm{End}_{\mathfrak{g}}(W[\mu]), \tag{9}$$

where $\mathrm{End}_{\mathfrak{g}}(W)$ is the space of $\mathfrak{g}$-module endomorphisms of W. On projection to $\mathrm{End}_{\mathfrak{g}}(W[\mu])$, we get a surjective morphism

$$\varphi^o_{\mu} : [U(\mathfrak{g}[t])]^{\mathfrak{g}} \twoheadrightarrow \mathrm{End}_{\mathfrak{g}}(W[\mu]) \simeq \mathrm{End}_{\mathbb{C}}(W[\mu]^+).$$

In particular, $W[\mu]^+$ is an irreducible module under the action of $[U(\mathfrak{g}[t])]^{\mathfrak{g}}$. From this the theorem follows. $\square$

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Remark 10. Since $[U(\mathfrak{g})]^{\mathfrak{g}} \subset [U(\mathfrak{g}[t])]^{\mathfrak{g}}$, the irreducible $[U(\mathfrak{g}[t])]^{\mathfrak{g}}$-modules $W[\mu]^+$ are mutually inequivalent as μ ranges over the highest weights of the components of $\vec{V}_{\vec{p}}(\vec{\lambda})$ by Harish-Chandra's theorem (cf. [H, Thm. 23.3]).

2. Determination of $[U(\mathfrak{g}[t])]^{\mathfrak{g}}$ for $\mathfrak{g} = \mathfrak{gl}(n)$

By Proposition 8, it suffices to determine $[T(\mathfrak{g})]^{\mathfrak{g}}$.

Let $V = \mathbb{C}^n$ with the standard basis $\{e_1, \dots, e_n\}$ and the standard representation of $\mathfrak{gl}(n)$. Let $\{e_1^*, \dots, e_n^*\}$ be the dual basis of V^*. Recall the isomorphism:

$$\beta : V^* \otimes V \simeq \operatorname{End} V = \mathfrak{gl}(n),$$

where $(\beta(f \otimes v))(w) = f(w)v$, for $v, w \in V, f \in V^*$. Moreover, the tensor product action of $\mathfrak{gl}(n)$ on $V^* \otimes V$ corresponds (under the identification β) to the adjoint action on $\mathfrak{gl}(n)$. Thus,

$$\left[\mathfrak{g}^{\otimes k}\right]^{\mathfrak{g}} \simeq \left[(V^* \otimes V)^{\otimes k}\right]^{\mathfrak{g}}.$$

By the tensor version of the *First Fundamental Theorem* (FFT) (cf. [GW, Thm. 5.3.1]), we get that $\{\theta_\sigma\}_{\sigma \in \Sigma_k}$ spans $\left[\mathfrak{g}^{\otimes k}\right]^{\mathfrak{g}}$, where

$$\begin{aligned}\theta_\sigma &:= \sum_{1 \le i_1, \dots, i_k \le n} (e^*_{i_{\sigma(1)}} \otimes e_{i_1}) \otimes \cdots \otimes (e^*_{i_{\sigma(k)}} \otimes e_{i_k}) \\ &= \sum_{1 \le i_1, \dots, i_k \le n} E_{i_1, i_{\sigma(1)}} \otimes \cdots \otimes E_{i_k, i_{\sigma(k)}} \quad \text{under the isomorphism } \beta,\end{aligned}$$

where $E_{i,j}$ is the $n \times n$-matrix with (i,j)th entry 1 and all other entries zero.

Thus, as a corollary of Proposition 8, we get:

Corollary 11. *Let $\mathfrak{g} = \mathfrak{gl}(n)$. Then, $[U(\mathfrak{g}[t])]^{\mathfrak{g}}$ is spanned by*

$$\bigcup_{k \ge 0} \bigcup_{0 \le n_1 \le \cdots \le n_k} \Big\{ \sum_{1 \le i_j \le n} E_{i_1, i_{\sigma(1)}}(n_1) \dots E_{i_k, i_{\sigma(k)}}(n_k) \Big\}_{\sigma \in \Sigma_k}.$$

Remark 12. From the explicit description of the standard generators of the center of $U(\mathfrak{gl}(n))$, the cycle $\sigma_k := (1, 2, \dots, k)$ plays a special role (also see [R, §4.2]). In this case

$$\theta_{\sigma_k} = \sum_{1 \le i_j \le n} E_{i_1, i_2} \otimes E_{i_2, i_3} \otimes \cdots \otimes E_{i_k, i_1}.$$

Considering the cycle decomposition of permutations in Σ_k, it is easy to see (by the above corollary) that the elements

$$\{\theta_{\sigma_k}(n_1, \dots, n_k) : 1 \le k \text{ and } n_1, \dots, n_k \ge 0\}$$

generate the algebra $[U(\mathfrak{gl}(n)[t])]^{\mathfrak{gl}(n)}$. This provides an affirmative answer to a question of Rao [R, Probl. 4.5 (1)]. Observe that in the above set of generators, we take $n_1, \dots, n_k$ to vary over $\mathbb{Z}_{\ge 0}$.

If we take the subset $\{\theta_{\sigma_k}(n_1, \dots, n_k) : 1 \le k \text{ and } 0 \le n_1 \le \cdots \le n_k\}$, then it does *not* generate the algebra $[U(\mathfrak{gl}(n)[t])]^{\mathfrak{gl}(n)}$.

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3. Realizing Schur–Weyl duality intertwiners in terms of current algebras

As in the last section, let $V = \mathbb{C}^n$ be the standard representation of $\mathfrak{g} = \mathfrak{gl}(n)$. For any positive integer k, the symmetric group Σ_k acts on the tensor product $V^{\otimes k}$ by permuting the factors. Clearly, this action of Σ_k commutes with the tensor product action of $\mathfrak{gl}(n)$ on $V^{\otimes k}$. Thus, we have an algebra homomorphism:

$$\Phi : \mathbb{C}[\Sigma_k] \to \mathrm{End}_{\mathfrak{g}}(V^{\otimes k}),$$

where $\mathrm{End}_{\mathfrak{g}}(V^{\otimes k})$ denotes the space of $\mathfrak{g}$-module endomorphisms of $V^{\otimes k}$.

The content of the Schur–Weyl duality is the following result (cf. [GW, §4.2.4]).

Theorem 13. *The above map Φ is an (algebra) isomorphism.*

Choose $\vec{p} = (p_1, \dots, p_k) \in \mathbb{C}^k$ such that p_i's are distinct. By the proof of Theorem 9 (see (9)), there is a surjective algebra homomorphism (for $\mathfrak{g} = \mathfrak{gl}(n)$)

$$\varphi^o : [U(\mathfrak{g}[t])]^{\mathfrak{g}} \twoheadrightarrow \mathrm{End}_{\mathfrak{g}}(V^{\otimes k}) \simeq \mathbb{C}[\Sigma_k],$$

where the last identification is via Φ. Thus, we get a surjective algebra homomorphism

$$\Xi : [U(\mathfrak{g}[t])]^{\mathfrak{g}} \twoheadrightarrow \mathbb{C}[\Sigma_k].$$

The following result is easy to prove using the definition of the map Ξ.

Proposition 14. *For any transposition $\tau = (r, s) \in \Sigma_k$, $r < s$, define the polynomials:*

$$P_\tau = (t - p_r + 1) \cdot \prod_{d \neq r} \frac{(t - p_d)}{(p_r - p_d)}, \quad Q_\tau = (t - p_s + 1) \cdot \prod_{d \neq s} \frac{(t - p_d)}{(p_s - p_d)}.$$

Then,

$$\Xi\Big(\sum_{1 \le i,j \le n} E_{i,j}(P_\tau) \cdot E_{j,i}(Q_\tau) \Big) = \tau.$$

4. Determination of $[U(\mathfrak{g}[t])]^{\mathfrak{g}}$ for the symplectic Lie algebra $\mathfrak{g} = \mathfrak{sp}(2n)$

Again we need to determine $\left[\mathfrak{g}^{\otimes k}\right]^{\mathfrak{g}}$ for $\mathfrak{g} = \mathfrak{sp}(2n)$.

Let $V = \mathbb{C}^{2n}$ be equipped with the nondegenerate symplectic form $\langle\ ,\ \rangle$ so that its matrix $\big(\langle e_i, e_j \rangle\big)_{1 \le i,j \le 2n}$ in the standard basis $\{e_1, \dots, e_{2n}\}$ is given by

$$\hat{J} = \begin{pmatrix} 0 & J \\ -J & 0 \end{pmatrix},$$

where J is the anti-diagonal matrix $(1, \dots, 1)$ of size n. Let

$$\mathrm{Sp}(2n) := \{g \in \mathrm{SL}(2n) : g \text{ leaves the form } \langle\ ,\ \rangle \text{ invariant}\}$$

be the associated symplectic group and $\mathfrak{sp}(2n)$ its Lie algebra. Thus, $\mathrm{Sp}(2n)$ has defining representation in V. The form $\langle\ ,\ \rangle$ allows us to identify $V \simeq V^*$ ($v \mapsto f_v(w) = \langle v, w \rangle$ for $v, w \in V$). Then, there is an identification

$$S^2(V) \simeq \mathfrak{sp}(2n),$$

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such that the standard action of $\mathfrak{sp}(2n)$ on the left corresponds to the adjoint action on the right. Hence, for $\mathfrak{g} = \mathfrak{sp}(2n)$,

$$\left[\mathfrak{g}^{\otimes k}\right]^{\mathfrak{g}} \simeq \left[\otimes^k (S^2(V))\right]^{\mathfrak{sp}(2n)}.$$

By the tensor version of FFT for $\mathrm{Sp}(2n)$ (cf. [GW, Thm. 5.3.3]), we get that $\left[\otimes^k (V^{\otimes 2})\right]^{\mathfrak{sp}(2n)}$ is spanned by

$$\Big\{\Theta_\sigma := \sum_{1\le i_1, i_3, \dots, i_{2k-1} \le 2n} (e_{i_{\sigma(1)}} \otimes e_{i_{\sigma(2)}}) \otimes \cdots \otimes (e_{i_{\sigma(2k-1)}} \otimes e_{i_{\sigma(2k)}})\Big\}_{\sigma \in \Sigma_{2k}},$$

where we require $e_{i_{2j}} = s(i_{2j-1}) e_{2n+1-i_{2j-1}}$ for $1 \le j \le k$ and $s : \{1, 2, \dots, 2n\} \to \{\pm 1\}$ is the function:

$$s(i) = \begin{cases} 1 & \text{if } 1 \le i \le n, \\ -1 & \text{if } i > n. \end{cases}$$

The standard symmetrization $V^{\otimes 2} \to S^2(V)$ gives rise to a surjective map

$$\gamma : \left[\otimes^k (V^{\otimes 2})\right]^{\mathfrak{sp}(2n)} \twoheadrightarrow \left[\otimes^k (S^2(V))\right]^{\mathfrak{sp}(2n)} \simeq \left[\mathfrak{g}^{\otimes k}\right]^{\mathfrak{g}}.$$

Under this identification, we get the element

$$\begin{aligned}\gamma(\Theta_\sigma) = \tfrac{1}{2} \sum_{1\le i_1, i_3, \dots, i_{2k-1} \le 2n} &\big(s(i_{\sigma(1)}) E_{i_{\sigma(2)}, 2n+1-i_{\sigma(1)}} + s(i_{\sigma(2)}) E_{i_{\sigma(1)}, 2n+1-i_{\sigma(2)}}\big) \otimes \\ &\cdots \otimes \big(s(i_{\sigma(2k-1)}) E_{i_{\sigma(2k)}, 2n+1-i_{\sigma(2k-1)}} + s(i_{\sigma(2k)}) E_{i_{\sigma(2k-1)}, 2n+1-i_{\sigma(2k)}}\big),\end{aligned}$$

where the square matrix $E_{i,j}$ is as defined in Section 2.

Thus, as a corollary of Proposition 8, we get:

Corollary 15. *Let $\mathfrak{g} = \mathfrak{sp}(2n)$. Then, $[U(\mathfrak{g}[t])]^{\mathfrak{g}}$ is spanned by*

$$\begin{aligned}\bigcup_{k\ge 0} \bigcup_{0 \le n_1 \le \cdots \le n_k} \Big\{ \sum_{1\le i_1, i_3, \dots, i_{2k-1} \le 2n} &\big(s(i_{\sigma(1)}) E_{i_{\sigma(2)}, 2n+1-i_{\sigma(1)}} \\ &+ s(i_{\sigma(2)}) E_{i_{\sigma(1)}, 2n+1-i_{\sigma(2)}}\big)(n_1) \otimes \\ \cdots \otimes \big(s(i_{\sigma(2k-1)}) &E_{i_{\sigma(2k)}, 2n+1-i_{\sigma(2k-1)}} \\ &+ s(i_{\sigma(2k)}) E_{i_{\sigma(2k-1)}, 2n+1-i_{\sigma(2k)}}\big)(n_k)\Big\}_{\sigma \in \Sigma_{2k}},\end{aligned}$$

where we require $e_{i_{2j}} = s(i_{2j-1}) e_{2n+1-i_{2j-1}}$ for $1 \le j \le k$, which gives the corresponding constraint on $E_{p,q} = e_q^* \otimes e_p$.

5. Determination of $[U(\mathfrak{so}(n)[t])]^{\mathrm{O}(n)}$

Let $V = \mathbb{C}^n$ be equipped with the nondegenerate symmetric form $\langle\ ,\ \rangle$ so that the standard basis $\{e_1, \dots, e_n\}$ is orthonormal. Let

$$\mathrm{O}(n) := \{g \in \mathrm{GL}(n) : g \text{ leaves the form } \langle\ ,\ \rangle \text{ invariant}\}$$

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be the associated orthogonal group and $\mathfrak{so}(n)$ its Lie algebra. Thus, $\mathrm{O}(n)$ has defining representation in V. The form $\langle\ ,\ \rangle$ allows us to identify $V \simeq V^*$ ($v \mapsto f_v(w) = \langle v, w\rangle$ for $v, w \in V$). Then, there is an identification

$$\wedge^2(V) \simeq \mathfrak{so}(n),$$

such that the standard action of $\mathrm{O}(n)$ on the left corresponds to the adjoint action on the right. Hence, for $\mathfrak{g} = \mathfrak{so}(n)$,

$$\left[\mathfrak{so}(n)^{\otimes k}\right]^{\mathrm{O}(n)} \simeq \left[\bigotimes^k(\wedge^2(V))\right]^{\mathrm{O}(n)}.$$

By the tensor version of FFT for $\mathrm{O}(n)$ (cf. [GW, Thm. 5.3.3]), we get that the space $\left[\otimes^k(V^{\otimes 2})\right]^{\mathrm{O}(n)}$ is spanned by

$$\left\{\Psi_\sigma := \sum_{1\le i_1, i_3, \dots, i_{2k-1}\le n} (e_{i_{\sigma(1)}} \otimes e_{i_{\sigma(2)}}) \otimes \cdots \otimes (e_{i_{\sigma(2k-1)}} \otimes e_{i_{\sigma(2k)}})\right\}_{\sigma\in\Sigma_{2k}},$$

where we require $i_{2j} = i_{2j-1}$ for $1 \le j \le k$.

The standard anti-symmetrization $V^{\otimes 2} \to \wedge^2(V)$ gives rise to a surjective map

$$\delta : \left[\otimes^k(V^{\otimes 2})\right]^{\mathrm{O}(n)} \twoheadrightarrow \left[\otimes^k(\wedge^2(V))\right]^{\mathrm{O}(n)} \simeq \left[\mathfrak{so}(n)^{\otimes k}\right]^{\mathrm{O}(n)}.$$

Under this identification, we get the element

$$\delta(\Psi_\sigma) = \tfrac{1}{2} \sum_{1\le i_1, i_3, \dots, i_{2k-1}\le n} \left(E_{i_{\sigma(1)}, i_{\sigma(2)}} - E_{i_{\sigma(2)}, i_{\sigma(1)}}\right) \otimes \cdots \otimes \left(E_{i_{\sigma(2k-1)}, i_{\sigma(2k)}} - E_{i_{\sigma(2k)}, i_{\sigma(2k-1)}}\right),$$

where the square matrix $E_{i,j}$ is as defined in Section 2.

Thus, as a corollary of Proposition 8, we get:

Corollary 16. *Let $\mathfrak{g} = \mathfrak{so}(n)$. Then, $[U(\mathfrak{g}[t])]^{\mathrm{O}(n)}$ is spanned by*

$$\bigcup_{k\ge 0} \bigcup_{0\le n_1\le\cdots\le n_k} \left\{ \sum_{1\le i_1, i_3, \dots, i_{2k-1}\le n} \left(E_{i_{\sigma(1)}, i_{\sigma(2)}} - E_{i_{\sigma(2)}, i_{\sigma(1)}}\right)(n_1) \otimes \cdots \otimes \left(E_{i_{\sigma(2k-1)}, i_{\sigma(2k)}} - E_{i_{\sigma(2k)}, i_{\sigma(2k-1)}}\right)(n_k)\right\}_{\sigma\in\Sigma_{2k}},$$

where we require $i_{2j} = i_{2j-1}$ for $1 \le j \le k$.

Remark 17. For n odd, observe that

$$\left[\mathfrak{so}(n)^{\otimes k}\right]^{\mathrm{O}(n)} \simeq \left[\mathfrak{so}(n)^{\otimes k}\right]^{so(n)}.$$

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