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COMPONENTS OF $V(\rho) \otimes V(\rho)$

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Abstract. Let ρ be half the sum of the positive roots of a root system. We prove that if λ is a dominant weight, $\lambda \leq 2\rho$ with respect to the dominance order, and d is a saturation factor for the complex Lie algebra associated to the root system, then the irreducible representation $V(d\lambda)$ appears in the tensor product $V(d\rho) \otimes V(d\rho)$.

Introduction

Let $\mathfrak{g}$ be any simple Lie algebra over $\mathbb{C}$. We fix a Borel subalgebra $\mathfrak{b}$ and a Cartan subalgebra $\mathfrak{t} \subset \mathfrak{b}$ and let ρ be the half sum of positive roots, where the roots of $\mathfrak{b}$ are called the positive roots. For any dominant integral weight $\lambda \in \mathfrak{t}^*$, let $V(\lambda)$ be the corresponding irreducible representation of $\mathfrak{g}$ with highest weight λ. B. Kostant initiated (and popularized) the study of the irreducible components of the tensor product $V(\rho) \otimes V(\rho)$. In fact, he conjectured the following.

Conjecture 1 (Kostant)**.** *Let λ be a dominant integral weight. Then, $V(\lambda)$ is a component of $V(\rho) \otimes V(\rho)$ if and only if $\lambda \leq 2\rho$ under the usual Bruhat–Chevalley order on the set of weights.*

It is, of course, clear that if $V(\lambda)$ is a component of $V(\rho) \otimes V(\rho)$, then $\lambda \leq 2\rho$.

One of the main motivations behind Kostant's conjecture was his result that the exterior algebra $\wedge\mathfrak{g}$, as a $\mathfrak{g}$-module under the adjoint action, is isomorphic with 2^r copies of $V(\rho) \otimes V(\rho)$, where r is the rank of $\mathfrak{g}$ (cf. [Ko]). Recall that $\wedge\mathfrak{g}$ is the

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underlying space of the standard chain complex computing the homology of the Lie algebra $\mathfrak{g}$, which is, of course, an object of immense interest.

Definition 2. An integer $d \geq 1$ is called a *saturation factor* for $\mathfrak{g}$, if for any $(\lambda, \mu, \nu) \in D^3$ such that $\lambda+\mu+\nu$ is in the root lattice and the space of $\mathfrak{g}$-invariants:

$$[V(N\lambda) \otimes V(N\mu) \otimes V(N\nu)]^{\mathfrak{g}} \neq 0$$

for some integer $N > 0$, then

$$[V(d\lambda) \otimes V(d\mu) \otimes V(d\nu)]^{\mathfrak{g}} \neq 0,$$

where $D \subset \mathfrak{t}^*$ is the set of dominant integral weights of $\mathfrak{g}$. Such a d always exists (cf. [Ku, Cor. 44]).

Recall that 1 is a saturation factor for $\mathfrak{g} = sl_n$, as proved by Knutson–Tao [KT]. By results of Belkale–Kumar [BK_2], Sam [S], and Hong–Shen [HS], d can be taken to be 2 for $\mathfrak{g}$ of types $\mathsf{B}_r, \mathsf{C}_r$, and d can be taken to be 4 for $\mathfrak{g}$ of type D_r by a result of Sam [S]. As proved by Kapovich–Millson [KM_1], [KM_2], the saturation factors d of $\mathfrak{g}$ of types $\mathsf{G}_2, \mathsf{F}_4, \mathsf{E}_6, \mathsf{E}_7, \mathsf{E}_8$ can be taken to be 2 (in fact, any $d \geq 2$), $144, 36, 144, 3600$ respectively. (For a discussion of saturation factors d, see [Ku, §10].)

Now, the following result (weaker than Conjecture 1) is our main theorem.

Theorem 3. *Let λ be a dominant integral weight such that $\lambda \leq 2\rho$. Then, $V(d\lambda) \subset V(d\rho) \otimes V(d\rho)$, where $d \geq 1$ is any saturation factor for $\mathfrak{g}$. In particular, for $\mathfrak{g} = sl_n$, $V(\lambda) \subset V(\rho) \otimes V(\rho)$.*

The proof uses a description of the eigencone of $\mathfrak{g}$ in terms of certain inequalities due to Berenstein–Sjamaar coming from the cohomology of the flag varieties associated to $\mathfrak{g}$, a 'non-negativity' result due to Belkale–Kumar and Proposition 9.

An interesting aspect of our work is that we make an essential use of a solution of the eigenvalue problem and saturation results for any $\mathfrak{g}$.

Remark 4. As informed by Papi, Berenstein–Zelevinsky had proved Conjecture 1 (by a different method) for $\mathfrak{g} = sl_n$ (cf. [BZ, Thm. 6]). They also determine in this case when $V(\lambda)$ appears in $V(\rho) \otimes V(\rho)$ with multiplicity one. To our knowledge, Conjecture 1 appears for the first time in this paper.

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1. Proof of Theorem 3

We now prove Theorem 3.

Proof. Let $\Gamma_3(\mathfrak{g})$ be the *saturated tensor semigroup* defined by

$$\Gamma_3(\mathfrak{g}) = \big\{(\lambda, \mu, \nu) \in D^3 : [V(N\lambda) \otimes V(N\mu) \otimes V(N\nu)]^{\mathfrak{g}} \neq 0 \text{ for some } N > 0\big\}.$$

To prove the theorem, it suffices to prove that $(\rho,\rho,\lambda^*)\in\Gamma_3(G)$, where λ^* is the dual weight $-w_0\lambda$, w_0 being the longest element of the Weyl group of $\mathfrak{g}$.

Let G be the connected, simply-connected complex algebraic group with Lie algebra $\mathfrak{g}$. Let B (resp. T) be the Borel subgroup (resp. maximal torus) of G with Lie algebra $\mathfrak{b}$ (resp. $\mathfrak{t}$). Let W be the Weyl group of G. For any standard parabolic subgroup $P\supset B$ with Levi subgroup L containing T, let W^P be the set of smallest length coset representatives in W/W_L, W_L being the Weyl group of L. Then, we have the Bruhat decomposition:

$$G/P=\bigsqcup_{w\in W^P}\Lambda_w^P,\quad \text{where } \Lambda_w^P:=BwP/P.$$

Let $\bar\Lambda_w^P$ denote the closure of Λ_w^P in G/P. We denote by $[\bar\Lambda_w^P]$ the Poincaré dual of its fundamental class. Thus, $[\bar\Lambda_w^P]$ belongs to the singular cohomology:

$$[\bar\Lambda_w^P]\in H^{2(\dim G/P-\ell(w))}(G/P,\mathbb{Z}),$$

where $\ell(w)$ is the length of w.

Let $\{x_j\}_{1\le j\le r}\subset\mathfrak{t}$ be the dual to the simple roots $\{\alpha_i\}_{1\le i\le r}$, i.e.,

$$\alpha_i(x_j)=\delta_{i,j}.$$

In view of [BS] (or [Ku, Thm. 10]), it suffices to prove that for any standard maximal parabolic subgroup P of G and triple $(u,v,w)\in(W^P)^3$ such that the cup product of the corresponding Schubert classes in G/P :

$$[\bar\Lambda_u^P]\cdot[\bar\Lambda_v^P]\cdot[\bar\Lambda_w^P]=k[\bar\Lambda_e^P]\in H^*(G/P,\mathbb{Z}),\ \text{ for some } k\neq 0, \tag{1}$$

the following inequality is satisfied:

$$\rho(ux_P)+\rho(vx_P)+\lambda^*(wx_P)\le 0. \tag{2}$$

Here, $x_P:=x_{i_P}$, where α_{i_P} is the unique simple root not in the Levi of P.

Now, by [BK$_1$, Prop. 17(a)] (or [Ku, Cor. 22 and identity (9)]), for any $u,v,w\in(W^P)^3$ such that the equation (1) is satisfied,

$$(\chi_{w_0ww_0^P}-\chi_u-\chi_v)(x_P)\ge 0, \tag{3}$$

where w_o^P is the longest element in the Weyl group of L and

$$\chi_w:=\rho-2\rho^L+w^{-1}\rho$$

(ρ^L being the half sum of positive roots in the Levi of P).

Now,

$$\begin{aligned}
&(\chi_{w_0ww_0^P}-\chi_u-\chi_v)(x_P)\\
&\qquad=(\rho-w_o^Pw^{-1}\rho-\rho-u^{-1}\rho-\rho-v^{-1}\rho)(x_P),\quad \text{since }\ \rho^L(x_P)=0\\
&\qquad=(-\rho-u^{-1}\rho-v^{-1}\rho-w^{-1}\rho)(x_P),\quad \text{since }\ w_o^P(x_P)=x_P.
\end{aligned} \tag{4}$$

Combining (3) and (4), we get

$$(\rho + u^{-1}\rho + v^{-1}\rho + w^{-1}\rho)(x_P) \leq 0 \quad \text{if (1) is satisfied.} \tag{5}$$

We next claim that for any dominant integral weight $\lambda \leq 2\rho$ and any $u, v, w \in (W^P)^3$,

$$\rho(ux_P) + \rho(vx_P) + \lambda^*(wx_P) \leq (\rho + u^{-1}\rho + v^{-1}\rho + w^{-1}\rho)(x_P), \tag{6}$$

which is equivalent to

$$\lambda^*(wx_P) \leq (\rho + w^{-1}\rho)(x_P). \tag{7}$$

Of course (5) and (6) together give (2). So, to prove the theorem, it suffices to prove (7). Since the assumption on λ in the theorem is invariant under the transformation $\lambda \mapsto \lambda^*$, we can replace λ^* by λ in (7). By Proposition 9, $\lambda = \rho + \beta$, where β is a weight of $V(\rho)$ (i.e., the weight space of $V(\rho)$ corresponding to the weight β is nonzero). Thus,

$$\lambda(wx_P) = \rho(wx_P) + \beta(x_P) \quad \text{for some weight } \beta \text{ of } V(\rho).$$

Hence

$$\lambda(wx_P) = \rho(wx_P) + \beta(x_P) \leq (w^{-1}\rho + \rho)(x_P) \quad \text{since } \beta \leq \rho.$$

This establishes (7) and hence the theorem is proved. □

We recall the following conjecture due to Kapovich–Millson [KM_1] (or [Ku, Conj. 47]).

Conjecture 5. *Let $\mathfrak{g}$ be a simple, simply-laced Lie algebra over $\mathbb{C}$. Then $d = 1$ is a saturation factor for $\mathfrak{g}$.*

The following theorem follows immediately by combining Theorem 3 and Conjecture 5.

Theorem 6. *For any simple, simply-laced Lie algebra $\mathfrak{g}$ over $\mathbb{C}$, assuming the validity of Conjecture 5, Conjecture 1 is valid for $\mathfrak{g}$, i.e., for any dominant integral weight $\lambda \leq 2\rho$, $V(\lambda)$ is a component of $V(\rho) \otimes V(\rho)$.*

Remark 7. By an explicit calculation using the program LIE, it is easy to see that Conjecture 1 has an affirmative answer for simple $\mathfrak{g}$ of types G_2 and F_4. Further, Paolo Papi has informed us that he has verified the validity of Conjecture 1 (by an explicit computer calculation using LIE again) for any simple $\mathfrak{g}$ of type $\mathsf{E}_6, \mathsf{E}_7$, and E_8 as well.

2. Determination of dominant weights $\leq 2\rho$

We follow the notation and assumptions from the previous sections. In particular, $\mathfrak{g}$ is a simple Lie algebra over $\mathbb{C}$ where we have fixed a Cartan subalgebra $\mathfrak{t}$ and a Borel subalgebra $\mathfrak{b} \supset \mathfrak{t}$. Let $\{\varpi_i\}_{i \in I}$ be the fundamental weights, $\{\alpha_i\}_{i \in I}$ the simple roots, and $\{s_i\}_{i \in I}$ the simple reflections, where $I := \{1 \leq i \leq r\}$. For any $J\subset I$, let W_J be the parabolic subgroup of the Weyl group W generated by s_j with $j\in J$, w_0^J be the longest element in W_J, Φ_J be the root system generated by the simple roots α_j with $j\in J$, and $\Phi_J^+\subset\Phi_J$ the subset of positive roots.

Let $A\subset\mathfrak{t}^*$ be the dominant cone, $B\subset\mathfrak{t}^*$ the cone generated by $\{\alpha_i : i\in I\}$ and $C:=2\rho-B$. We want to describe the vertices of the polytope $A\cap C$. For $J\subset I$, define

$$A_J:=\mathbb{R}_{\geq 0}[\varpi_j\ :\ j\in J],\quad B_J:=\mathbb{R}_{\geq 0}[\alpha_j\ :\ j\in J],\quad \text{and}\quad C_J:=2\rho-B_J.$$

The sets A_J and B_J are the faces of A and B. The vertices of the polytope $A\cap C$ are given by the zero-dimensional nonempty intersections of the form $A_J\cap C_H$. To describe these intersections, we introduce some notation. For any $J\subset I$, let

$$\rho_J:=\sum_{j\in J}\varpi_j,\quad b_J:=\sum_{\alpha\in\Phi_J^+}\alpha,\quad \text{and}\quad c_J:=2\rho-b_J;$$

in particular, $c_I=0$ and $c_\varnothing=2\rho$.

Lemma 8. *For each $J\subset I$, we have*

$$A_{I\smallsetminus J}\cap C_J=\{c_J\}.$$

Moreover, none of the other intersections $A_H\cap C_K$ give a single point. In particular, the intersection $A\cap C$ is the convex hull of the points $\{c_J\ :\ J\subset I\}$.

Proof. The lattice generated by $A_{I\smallsetminus J}$ and the one generated by B_J are orthogonal to each other so the intersection $A_{I\smallsetminus J}\cap C_J$ contains at most one point. Observe now that

$$b_J=2\rho_J+\sum_{\ell\notin J}a_\ell\varpi_\ell,\ \text{ where } a_\ell\leq 0.$$

Hence, $c_J\in A_{I\smallsetminus J}\cap C_J$.

Consider an intersection of the form $A_{I\smallsetminus H}\cap C_K$. Assume it is not empty and that $y=2\rho-x\in A_{I\smallsetminus H}\cap C_K$. Since $y\in A_{I\smallsetminus H}$, we have $x=2\rho_H+\sum_{\ell\notin H}a'_\ell\varpi_\ell$. Since $x\in B_K$, if $h\notin K$, the coefficient of ϖ_h in x cannot be positive. So, we must have $H\subset K$. If $H\subset K$ and $H\neq K$, then

$$A_{I\smallsetminus H}\cap C_K\supset\big(A_{I\smallsetminus H}\cap C_H\big)\cup\big(A_{I\smallsetminus K}\cap C_K\big)\supset\{c_H,c_K\}.$$

Hence, it is not a single point. $\square$

We apply this Lemma to obtain the following result about the weights below 2ρ.

Proposition 9. *Let $\lambda\leq 2\rho$ be a dominant integral weight. Then*

$$\lambda=\rho+\beta,$$

for some weight β of $V(\rho)$.

Proof. Let $Q \subset \mathfrak{t}^*$ be the root lattice and let H_ρ be the convex hull of the weights $\{w(\rho) : w \in W\}$. Recall that the weights of the module $V(\rho)$ are precisely the elements of the intersection

$$(\rho + Q) \cap H_\rho.$$

If λ is as in the Proposition, then it is clear that $\lambda - \rho \in \rho + Q$. So, we need to prove that it belongs to H_ρ. To check this, it is enough to check that $(A \cap C) - \rho \subset H_\rho$ or equivalently, by the previous Lemma, that

$$c_J - \rho \in H_\rho \quad \text{for all} \quad J \subset I.$$

Notice that $w_0^J(\Phi_J^+) = -\Phi_J^+$ and that, if α is a positive root, $w_o^J(\alpha)$ is a negative root if and only if $\alpha \in \Phi_J^+$. Hence,

$$w_0^J(\rho) = \tfrac{1}{2} \sum_{\alpha \in \Phi_I^+ \smallsetminus \Phi_J^+} \alpha - \tfrac{1}{2} \sum_{\alpha \in \Phi_J^+} \alpha = \rho - b_J,$$

and $c_J - \rho = \rho - b_J \in H_\rho$. □

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