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Richardson Varieties have Kawamata Log Terminal Singularities

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Let X_w^v be a Richardson variety in the full flag variety X associated to a symmetrizable Kac–Moody group G. Recall that X_w^v is the intersection of the finite-dimensional Schubert variety X_w with the finite-codimensional opposite Schubert variety X^v. We give an explicit $\mathbb{Q}$-divisor Δ on X_w^v and prove that the pair (X_w^v, Δ) has Kawamata log terminal singularities. In fact, $-K_{X_w^v} - \Delta$ is ample, which additionally proves that (X_w^v, Δ) is log Fano. We first give a proof of our result in the finite case (i.e., in the case when G is a finite-dimensional semisimple group) by a careful analysis of an explicit resolution of singularities of X_w^v (similar to the Bott–Samelson–Demazure–Hansen resolution of the Schubert varieties). In the general Kac–Moody case, in the absence of an explicit resolution of X_w^v as above, we give a proof that relies on the Frobenius splitting methods. In particular, we use Mathieu's result asserting that the Richardson varieties are Frobenius split, and combine it with a result of Hara and Watanabe [14] relating Frobenius splittings with log canonical singularities.

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1 Introduction

Let G be any symmetrizable Kac–Moody group over $\mathbb{C}$ (or any algebraically closed field of characteristic zero) with the standard Borel subgroup B, the standard negative Borel subgroup B^-, the maximal torus $T = B \cap B^-$ and the Weyl group W. Let $X = G/B$ be the full flag variety. For any $w \in W$, we have the Schubert variety

$$X_w := \overline{BwB/B} \subset G/B$$

and the opposite Schubert variety

$$X^w := \overline{B^- wB/B} \subset G/B.$$

For any $v \leq w$, consider the *Richardson variety* X_w^v which is defined to be the intersection of a Schubert variety and an opposite Schubert variety:

$$X_w^v := X^v \cap X_w$$

with the reduced subscheme structure. In this paper, we prove the following theorem.

Main Theorem (Theorem 3.1). With the notation as above, for any $v \leq w \in W$, there exists an effective $\mathbb{Q}$-divisor Δ on X_w^v such that (X_w^v, Δ) has Kawamata log terminal (for short KLT) singularities.

Furthermore, $-K_{X_w^v} - \Delta$ is ample which proves that (X_w^v, Δ) is in fact log Fano. □

This divisor Δ, described in Section 3, is built out of the boundary ∂X_w^v. As an immediate corollary of this result and Kawamata–Viehweg vanishing, we obtain the following cohomology vanishing (due to Brion–Lakshmibai in the finite case).

Main Corollary (Corollary 3.2). For a dominant integral weight λ, and any $v \leq w$,

$$H^i(X_w^v, \mathcal{L}(\lambda)_{|X_w^v}) = 0, \quad \text{for all } i > 0,$$

where the line bundle $\mathcal{L}(\lambda)$ on X is defined in Section 3. □

Note that KLT singularities are a refinement of rational singularities. In particular, every KLT singularity is also a rational singularity, but not conversely except in the Gorenstein case. We note that, in the finite case, Richardson varieties were known to have rational singularities [6, Theorem 4.2.1], even in positive characteristics [19, Appendix]. Indeed the singularities of generalizations of Richardson varieties have been a topic of interest lately [4, 19, 20].

On the other hand, KLT are the widest class of singularities for which the foundational theorems of the minimal model program over $\mathbb{C}$ are known to hold [21]. It is well known that toric varieties are KLT [9, Section 11.4] and more generally Alexeev and Brion [1] proved that spherical varieties are KLT. Recently, Anderson and Stapledon [2] proved that the Schubert varieties X_w are log Fano and thus also KLT, also see [17].

The proof of our main result in the finite case is much simpler than the general Kac–Moody case and is given in Section 4.3. In this case, we are able to directly prove that (X_w^v, Δ) is KLT through an explicit resolution of singularities of X_w^v due to Brion (similar to the Bott–Samelson–Demazure–Hansen (for short BSDH) desingularization of the Schubert varieties).

In the general symmetrizable Kac–Moody case, we are not aware of an explicit resolution of singularities of X_w^v to proceed as above. In the general case, we prove our main result by reduction to characteristic $p > 0$. In this case, we use an unpublished result of Mathieu asserting that the Richardson varieties X_w^v are Frobenius split compatibly splitting their boundary (cf. Proposition 5.3). This splitting together with results of Hara and Watanabe relating Frobenius splittings and log canonical singularities (cf. Theorem 5.7) allow us to conclude that the pair (X_w^v, Δ) as above is KLT.

2 Preliminaries and Definitions

We follow the notation from [21, Notation 0.4]. We fix X to be a normal scheme over an algebraically closed field.

Suppose that $\pi : \tilde{X} \to X$ is a proper birational map with $\tilde{X}$ normal. For any $\mathbb{Q}$-divisor $\Delta = \sum_i d_i D_i$ on X, we let $\Delta' = \pi_*^{-1}\Delta = \sum_i d_i D_i'$ denote the *strict transform* of Δ defined as the $\mathbb{Q}$-divisor on $\tilde{X}$, where D_i' is the prime divisor on $\tilde{X}$ which is birational to D_i under π. We let $Exc(\pi)$ of π be the exceptional set of π; the closed subset of $\tilde{X}$ consisting of those $x \in \tilde{X}$ where π is not biregular at x. We endow $Exc(\pi)$ with the reduced (closed) subscheme structure. An (integral) divisor $D = \sum n_i F_i$ is called a *canonical divisor* K_X of X if the restriction D^o of D, to the smooth locus X^o of X, represents the canonical line bundle ω_{X^o} of X^o.

Assume now that $K_X + \Delta$ is $\mathbb{Q}$-Cartier, that is, some multiple $n(K_X + \Delta)$ (for $n \in \mathbb{N} := \{1, 2, 3, \ldots\}$) is a Cartier divisor. We may choose $K_{\tilde{X}}$ that agrees with K_X wherever

π is an isomorphism and thus it follows that there exists a (unique) $\mathbb{Q}$-divisor $E_\pi(\Delta)$ on $\tilde{X}$ supported in $Exc(\pi)$ such that

$$n(K_{\tilde{X}} + \Delta') = \pi^*(n(K_X + \Delta)) + nE_\pi(\Delta). \tag{1}$$

A $\mathbb{Q}$-divisor $D = \sum d_i D_i$ on a smooth variety $\tilde{X}$ is called a *simple normal crossing divisor* if each D_i is smooth and they intersect transversally at each intersection point (in particular, this means that locally analytically the D_i can be thought of as coordinate hyperplanes).

Let X be an irreducible variety and D a $\mathbb{Q}$-divisor on X. A *log resolution* of (X, D) is a proper birational morphism $\pi : \tilde{X} \to X$ such that $\tilde{X}$ is smooth, $Exc(\pi)$ is a divisor and $Exc(\pi) \cup \pi^{-1}(SuppD)$ is a simple normal crossing divisor, where $SuppD$ is the support of D. Log resolutions exist for any (X, D) in characteristic zero by Hironaka [16].

Let X be a proper scheme. Then a $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor D is called *nef* (resp., *big*) if $D \cdot C \geq 0$, for every irreducible curve $C \subset X$ (resp., ND is the sum of an ample and an effective divisor, for some $N \in \mathbb{N}$; cf. [21, Sections 0.4 and 2.5]). Recall that an ample Cartier divisor is nef and big.

Definition 2.1. Let X be a normal irreducible variety over a field of characteristic zero and let $\Delta = \sum_i d_i D_i$ be a $\mathbb{Q}$-divisor with $d_i \in [0, 1)$. The pair (X, Δ) is called KLT, if the following two conditions are satisfied:

(a) $K_X + \Delta$ is a $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor, and
(b) there exists a log resolution $\pi : \tilde{X} \to X$ of (X, Δ) such that the $\mathbb{Q}$-divisor $E = E_\pi(\Delta) = \sum_i e_i E_i$, defined by (1), satisfies

$$-1 < e_i \quad \text{for all } i. \tag{2}$$

By Debarre [10, Remarks 7.25], (X, Δ) satisfying (a) is KLT if and only if for every proper birational map $\pi' : Y \to X$ with normal Y, the divisor $E_{\pi'}(\Delta) = \sum_j f_j F_j$ satisfies (2), that is, $-1 < f_j$, for all j. In fact, one may use this condition as a definition of KLT singularities in characteristic $p > 0$ (where it is an open question whether or not log resolutions exist). □

For a normal irreducible variety X of characteristic zero with a $\mathbb{Q}$-divisor $\Delta = \sum_i d_i D_i$ with $d_i \in [0, 1]$, the pair (X, Δ) is called *log canonical* if it satisfies the above conditions (a) and (b) with (2) replaced by

$$-1 \leq e_i \quad \text{for all } i. \tag{3}$$

Remark 2.2. Let X be a variety of characteristic zero. It is worth remarking that if (X, Δ) is KLT, then X has rational singularities [12], [21, Theorem 5.22]. Conversely, if K_X is Cartier and X has rational singularities, then $(X, 0)$ is KLT [21, Corollary 5.24]. □

We conclude this section with one final definition.

Definition 2.3. If X is projective, we say that a pair (X, Δ) is *log Fano* if (X, Δ) is KLT and $-K_X - \Delta$ is ample. □

3 Statement of the Main Result and Its Consequences

3.1 Notation

Let G be any symmetrizable Kac–Moody group over a field of characteristic zero with the standard Borel subgroup B, the standard negative Borel subgroup B^-, the maximal torus $T = B \cap B^-$ and the Weyl group W (cf. [22, Sections 6.1 and 6.2]). Let $X = G/B$ be the full flag variety, which is a projective ind-variety. For any $w \in W$, we have the Schubert variety

$$X_w := \overline{BwB/B} \subset G/B$$

and the opposite Schubert variety

$$X^w := \overline{B^- wB/B} \subset G/B.$$

Then, X_w is a (finite-dimensional) irreducible projective subvariety of G/B and X^w is a finite-codimensional irreducible projective ind-subvariety of G/B (cf. [22, Section 7.1]). For any integral weight λ (i.e., any character e^λ of T), we have a G-equivariant line bundle $\mathcal{L}(\lambda)$ on X associated to the character $e^{-\lambda}$ of T (cf. [22, Section 7.2] for a precise definition of $\mathcal{L}(\lambda)$ in the general Kac–Moody case). In the finite case, recall that $\mathcal{L}(\lambda)$ is the line bundle associated to the principal B-bundle $G \to G/B$ via the character $e^{-\lambda}$ of B

(any character of T uniquely extends to a character of B), that is,

$$\mathcal{L}(\lambda) = G \times^B \mathbb{C}_{-\lambda} \to G/B, \quad [g, v] \mapsto gB,$$

where $\mathbb{C}_{-\lambda}$ is the one-dimensional representation of B corresponding to the character $e^{-\lambda}$ of B and $[g, v]$ denotes the equivalence class of $(g, v) \in G \times \mathbb{C}_{-\lambda}$ under the B-action: $b \cdot (g, v) = (gb^{-1}, b \cdot v)$.

Let $\{\alpha_1, \ldots, \alpha_\ell\} \subset \mathfrak{t}^*$ be the set of simple roots and $\{\alpha_1^\vee, \ldots, \alpha_\ell^\vee\} \subset \mathfrak{t}$ the set of simple coroots, where $\mathfrak{t} = \mathit{Lie}\, T$. Let $\rho \in \mathfrak{t}^*$ be any integral weight satisfying

$$\rho(\alpha_i^\vee) = 1, \quad \text{for all } 1 \le i \le \ell.$$

When G is a finite-dimensional semisimple group, ρ is unique, but for a general Kac–Moody group G, it may not be unique.

For any $v \le w \in W$, consider the *Richardson variety*

$$X_w^v := X^v \cap X_w,$$

and its boundary

$$\partial X_w^v := ((\partial X^v) \cap X_w) \cup (X^v \cap \partial X_w)$$

both endowed with (reduced) subvariety structure, where $\partial X_w := X_w \backslash (BwB/B)$ and $\partial X^v := X^v \backslash (B^- vB/B)$.

Writing $\partial X_w^v = \cup_i X_i$ as the union of its irreducible components, the line bundle $\mathcal{L}(2\rho)_{|X_w^v}$ can be written as a (Cartier) divisor (for justification, see Section 4.3—the proof of Theorem 3.1):

$$\mathcal{L}(2\rho)_{|X_w^v} = \mathcal{O}_{X_w^v}\left(\sum_i b_i X_i\right), \quad b_i \in \mathbb{N}. \tag{4}$$

Now, take a positive integer N such that $N > b_i$ for all i, and consider the $\mathbb{Q}$-divisor on X_w^v:

$$\Delta = \sum_i \left(1 - \frac{b_i}{N}\right) X_i. \tag{5}$$

The following theorem is the main result of the paper.

Theorem 3.1. For any $v \le w \in W$, the pair (X_w^v, Δ) defined above is KLT. □

In fact, we will show in Lemma 4.2 that $\mathcal{O}_{X_w^v}(-N(K_{X_w^v} + \Delta)) \cong \mathcal{L}(2\rho)_{|X_w^v}$ is ample, which proves that (X_w^v, Δ) is log Fano.

We postpone the proof of this theorem until the next two sections. But we derive the following consequence proved earlier in the finite case (i.e., in the case when G is a finite-dimensional semisimple group) by Brion–Lakshmibai (see [8, Proposition 1]).

Corollary 3.2. For a dominant integral weight λ, and any $v \le w$,

$$H^i(X_w^v, \mathcal{L}(\lambda)_{|X_w^v}) = 0, \quad \text{for all } i > 0.$$ □

Proof. By (the subsequent) Lemma 4.2, $N(K_{X_w^v} + \Delta)$ is a Cartier divisor corresponding to the line bundle $\mathcal{L}(-2\rho)_{|X_w^v}$. Since λ is a dominant weight, the $\mathbb{Q}$-Cartier divisor D is nef and big, where ND is a Cartier divisor corresponding to the ample line bundle $\mathcal{L}(N\lambda + 2\rho)_{|X_w^v}$. Thus, the divisor $K_{X_w^v} + \Delta + D$ is Cartier and corresponds to the line bundle $\mathcal{L}(\lambda)_{|X_w^v}$. Hence, the corollary follows from the Logarithmic Kawamata–Viehweg vanishing theorem which we state below (cf. [10, Theorem 7.26] or [21, Theorem 2.70]). ■

Theorem 3.3. Let (X, Δ) be a KLT pair for a proper variety X and let D be a nef and big $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor on X such that $K_X + \Delta + D$ is a Cartier divisor. Then, we have

$$H^i(X, K_X + \Delta + D) = 0, \quad \text{for all } i > 0.$$ □

4 Proof of Theorem 3.1: Finite Case

In this section, except where otherwise noted, we assume that G is a finite-dimensional semisimple simply connected group. We refer to this as the *finite case*.

We first give a proof of Theorem 3.1 in the finite case. In this case, the proof is much simpler than the general (symmetrizable) Kac–Moody case proved in the next section. Unlike the general case, the proof in the finite case given below does not require any use of characteristic $p > 0$ methods.

Before we come to the proof of the theorem, we need some preliminaries on BSDH desingularization of Schubert varieties.

4.1 BSDH desingularization

For any $w \in W$, pick a reduced decomposition as a product of simple reflections:

$$w = s_{i_1} \dots s_{i_n}$$

and let $m_{\mathfrak{w}} : Z_{\mathfrak{w}} \to X_w$ be the BSDH desingularization (cf. [7, Section 2.2.1]), where $\mathfrak{w}$ is the word $(s_{i_1}, \dots, s_{i_n})$. This is a B-equivariant resolution, which is an isomorphism over the cell $C_w := BwB/B \subset X_w$.

Similarly, there is a B^--equivariant resolution

$$m^{\mathfrak{v}} : Z^{\mathfrak{v}} \to X^v,$$

obtained by taking a reduced word $\hat{\mathfrak{v}} = (s_{j_1}, \dots, s_{j_m})$ for $w_0 v$, that is, $w_0 v = s_{j_1} \dots s_{j_m}$ is a reduced decomposition, where $w_0 \in W$ is the longest element. Now, set

$$Z^{\mathfrak{v}} = Z_{\hat{\mathfrak{v}}},$$

which is canonically a B-variety. We define the action of B^- on $Z^{\mathfrak{v}}$ by twisting the B-action as follows:

$$b^- \odot z = (\dot{w}_0 b^- \dot{w}_0^{-1}) \cdot z, \quad \text{for } b^- \in B^- \text{ and } z \in Z^{\mathfrak{v}},$$

where $\dot{w}_0$ is a lift of w_0 in the normalizer $N(T)$ of the torus T. (Observe that this action does depend on the choice of the lift $\dot{w}_0$ of w_0.) Moreover, define the map

$$m^{\mathfrak{v}} : Z^{\mathfrak{v}} \to X^v = \dot{w}_0^{-1} X_{w_0 v} \quad \text{by } m^{\mathfrak{v}}(z) = \dot{w}_0^{-1}(m_{\hat{\mathfrak{v}}}(z)), \quad \text{for } z \in Z^{\mathfrak{v}}.$$

Clearly, $m^{\mathfrak{v}}$ is a B^--equivariant desingularization.

4.2 Desingularization of Richardson Varieties

We recall the construction of a desingularization of Richardson varieties communicated to us by Brion (also see [3, Section 1]). It is worked out in detail in any characteristic in [19, Appendix]. We briefly sketch the construction in characteristic zero. Consider the fiber product morphism

$$m^{\mathfrak{v}}_{\mathfrak{w}} := m^{\mathfrak{v}} \times_X m_{\mathfrak{w}} : Z^{\mathfrak{v}} \times_X Z_{\mathfrak{w}} \to X^v \times_X X_w = X^v_w,$$

which is a (smooth) desingularization (where $X = G/B$). It is an isomorphism over the intersection $C_w^v := C^v \cap C_w$ of the Bruhat cells, where $C^v := B^- vB/B \subset G/B$ and (as earlier) $C_w := BwB/B$. Moreover, the complement of C_w^v inside $Z_{\mathfrak{w}}^{\mathfrak{v}}$, considered as a reduced divisor, is a simple normal crossing divisor. (To prove these assertions, observe that by Kleiman's transversality theorem [15, Theorem 10.8, Chap. III], the fiber product $Z^{\mathfrak{v}} \times_X gZ_{\mathfrak{w}}$ is smooth for a general $g \in G$ and hence for some $g \in B^- B$. But since $Z_{\mathfrak{w}}$ is a B-variety and $Z^{\mathfrak{v}}$ is a B^--variety,

$$Z^{\mathfrak{v}} \times_X gZ_{\mathfrak{w}} \simeq Z^{\mathfrak{v}} \times_X Z_{\mathfrak{w}}.$$

Moreover, $Z^{\mathfrak{v}} \times_X Z_{\mathfrak{w}}$ is irreducible since each of its irreducible components is of the same dimension equal to $\ell(w) - \ell(v)$ and the complement of C_w^v in $Z^{\mathfrak{v}} \times_X Z_{\mathfrak{w}}$ is of dimension $< \ell(w) - \ell(v)$.)

Lemma 4.1. With the notation as above (still in characteristic zero), for any $v \leq w$, the Richardson variety X_w^v is irreducible, normal, and Cohen–Macaulay. □

This is proved in the finite case in [5, Lemma 2], [8, Lemma 1]. The same result (with a similar proof as in [5]) also holds in the Kac–Moody case (cf. [23, Proposition 6.5]). Also see [19] for some discussion in characteristic $p > 0$.

Lemma 4.2. For any symmetrizable Kac–Moody group G, and any $v \leq w \in W$, the canonical divisor $K_{X_w^v}$ of X_w^v is given by

$$K_{X_w^v} = \mathcal{O}_{X_w^v}[-\partial X_w^v],$$

where ∂X_w^v is considered as a reduced divisor. □

Proof. The finite case can be found in [6, Theorem 4.2.1(i)]. The detailed proof in the case of a general symmetrizable Kac–Moody group can be found in [23, Lemma 8.5]. We give a brief idea here.

Since X_w^v is Cohen–Macaulay by Kumar [23, Proposition 6.5] (in particular, so is X_w) and the codimension of X_w^v in X_w is $\ell(v)$, the dualizing sheaf

$$\omega_{X_w^v} \simeq \mathscr{E}xt^{\ell(v)}_{\mathcal{O}_{X_w}}(\mathcal{O}_{X_w^v}, \omega_{X_w}) \tag{6}$$

(cf. [11, Theorem 21.15]). Observing that $\operatorname{depth}(\omega_{X_w}) = \operatorname{depth}(\mathcal{O}_{X_w})$, as $\mathcal{O}_{X_w}$-modules (cf. [11, Theorem 21.8])

$$\mathscr{E}xt^{\ell(v)}_{\mathcal{O}_{X_w}}(\mathcal{O}_{X^v_w}, \omega_{X_w}) \simeq \mathscr{E}xt^{\ell(v)}_{\mathcal{O}_{\bar{X}}}(\mathcal{O}_{X^v}, \mathcal{O}_{\bar{X}}) \otimes_{\mathcal{O}_{\bar{X}}} \omega_{X_w}, \tag{7}$$

where $\bar{X}$ is the "thick" flag variety considered in [18] (cf. [23, Lemma 5.6]). By Graham and Kumar [13, Proposition 2.2], as T-equivariant sheaves,

$$\omega_{X_w} \simeq \mathbb{C}_{-\rho} \otimes \mathcal{L}(-\rho) \otimes \mathcal{O}_{X_w}(-\partial X_w). \tag{8}$$

Similarly, by Kumar [23, Theorem 10.4] (due to Kashiwara),

$$\omega_{X^v} \simeq \mathbb{C}_{\rho} \otimes \mathcal{L}(-\rho) \otimes \mathcal{O}_{X^v}(-\partial X^v). \tag{9}$$

Similar to the identity (6), we also have

$$\omega_{X^v} \simeq \mathscr{E}xt^{\ell(v)}_{\mathcal{O}_{\bar{X}}}(\mathcal{O}_{X^v}, \omega_{\bar{X}}). \tag{10}$$

Since $\omega_{\bar{X}} \simeq \mathcal{L}(-2\rho)$, combining the isomorphisms (6)–(10), we obtain

$$\omega_{X^v_w} \simeq \mathcal{O}_{X^v}(-\partial X^v) \otimes_{\mathcal{O}_{\bar{X}}} \mathcal{O}_{X_w}(-\partial X_w).$$

Now, the lemma follows since all the intersections $X^v \cap X_w$, $(\partial X^v) \cap X_w$, $X^v \cap \partial X_w$, and $\partial(X^v) \cap \partial X_w$ are proper. In fact, we need the corresponding local Tor vanishing result (cf. [23, Lemma 5.5]). ∎

We are now ready to prove Theorem 3.1 in the finite case. The basic strategy is similar to the proof that toric varieties have KLT singularities (and in fact are log Fano) [9, Section 11.4].

4.3 Proof of Theorem 3.1 in the finite case

Let us denote $Z^{\mathfrak{v}} \times_X Z_{\mathfrak{w}}$ by $Z^{\mathfrak{v}}_{\mathfrak{w}}$. Consider the desingularization

$$m^{\mathfrak{v}}_{\mathfrak{w}} : Z^{\mathfrak{v}}_{\mathfrak{w}} \to X^v_w,$$

as in Section 4.2.

By Brion and Kumar [7, Proposition 2.2.2], the canonical line bundle of $Z_{\mathfrak{w}}$ is isomorphic with the line bundle

$$\mathcal{L}_{\mathfrak{w}}(-\rho) \otimes \mathcal{O}_{Z_{\mathfrak{w}}}[-\partial Z_{\mathfrak{w}}],$$

where $\mathcal{L}_{\mathfrak{w}}(-\rho)$ is the pull-back of the line bundle $\mathcal{L}(-\rho)$ to $Z_{\mathfrak{w}}$ via the morphism $m_{\mathfrak{w}}$ and $\partial Z_{\mathfrak{w}}$ is the reduced divisor $Z_{\mathfrak{w}} \backslash C_w$.

Similarly, the canonical line bundle of $Z^{\mathfrak{v}}$ is isomorphic with

$$\mathcal{L}_{\mathfrak{v}}(-\rho) \otimes \mathcal{O}_{Z^{\mathfrak{v}}}[-\partial Z^{\mathfrak{v}}],$$

where $\mathcal{L}_{\mathfrak{v}}(-\rho)$ is the pull-back of the line bundle $\mathcal{L}(-\rho)$ to $Z^{\mathfrak{v}}$ via $m^{\mathfrak{v}}$ and $\partial Z^{\mathfrak{v}}$ is the reduced divisor $Z^{\mathfrak{v}} \backslash C^v$.

Thus, by adapting the proof of [5, Lemma 1],

$$\begin{aligned} \omega_{Z^{\mathfrak{v}}_{\mathfrak{w}}} &\cong \omega_{Z^{\mathfrak{v}}} \otimes_{\mathcal{O}_X} \omega_{Z_{\mathfrak{w}}} \otimes_{\mathcal{O}_X} (m^{\mathfrak{v}}_{\mathfrak{w}})^* \omega_X^{-1} \\ &= \mathcal{O}_{Z^{\mathfrak{v}}_{\mathfrak{w}}}[-\partial Z^{\mathfrak{v}}_{\mathfrak{w}}], \end{aligned} \tag{11}$$

where $\partial Z^{\mathfrak{v}}_{\mathfrak{w}}$ is the reduced divisor

$$(\partial Z^{\mathfrak{v}} \times_X Z_{\mathfrak{w}}) \cup (Z^{\mathfrak{v}} \times_X \partial Z_{\mathfrak{w}}).$$

Consider the desingularization

$$m^{\mathfrak{v}}_{\mathfrak{w}} : Z^{\mathfrak{v}}_{\mathfrak{w}} \to X^v_w.$$

Note that $m^{\mathfrak{v}}_{\mathfrak{w}}$ is an isomorphism outside of $\partial Z^{\mathfrak{v}}_{\mathfrak{w}}$.

By Lemma 4.2,

$$K_{X^v_w} = -\sum X_i,$$

where we have written $\partial X^v_w = \cup X_i$ as the union of prime divisors. Thus, by (5),

$$K_{X^v_w} + \Delta = -\frac{1}{N} \sum b_i X_i, \tag{12}$$

which is a $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor by (4).

We next calculate $Exc(m_{\mathfrak{w}}^{\mathfrak{v}})$ and the proper transform Δ' of Δ under the desingularization $m_{\mathfrak{w}}^{\mathfrak{v}} : Z_{\mathfrak{w}}^{\mathfrak{v}} \to X_w^v$.

Any irreducible component X_i of ∂X_w^v is of the form $X_w^{v'}$ or $X_{w'}^v$ for some $v \to v'$ and $w' \to w$, where the notation $w' \to w$ means that $\ell(w) = \ell(w') + 1$ and $w = s_\alpha w'$, for some reflection s_α through a positive (not necessarily simple) root α. Conversely, any $X_w^{v'}$ and $X_{w'}^v$ (for $v \to v'$ and $w' \to w$) is an irreducible component of ∂X_w^v. Thus, we have the prime decomposition

$$\partial X_w^v = \left(\bigcup_{v \to v'} X_w^{v'} \right) \cup \left(\bigcup_{w' \to w} X_{w'}^v \right).$$

We define Z_i as the prime divisor (of the resolution $Z_{\mathfrak{w}}^{\mathfrak{v}}$) $Z^{\mathfrak{v}'} \times_X Z_{\mathfrak{w}}$ if $X_i = X_w^{v'}$ or $Z^{\mathfrak{v}} \times_X Z_{\mathfrak{w}'}$ if $X_i = X_{w'}^v$. Thus, the strict transform of Δ can be written as

$$\Delta' = \sum_i \left(1 - \frac{b_i}{N} \right) Z_i. \tag{13}$$

We now calculate $E = E_{m_{\mathfrak{w}}^{\mathfrak{v}}}(\Delta)$. By definition (cf. (1)),

$$E = (K_{Z_w^v} + \Delta') - \frac{1}{N} (m_{\mathfrak{w}}^{\mathfrak{v}})^* (N(K_{X_w^v} + \Delta)). \tag{14}$$

Consider the prime decomposition of the reduced divisor $\partial Z_{\mathfrak{w}}^{\mathfrak{v}}$:

$$\partial Z_{\mathfrak{w}}^{\mathfrak{v}} = (\cup_i Z_i) \cup (\cup_j Z'_j), \tag{15}$$

where Z'_j are the irreducible components of $\partial Z_{\mathfrak{w}}^{\mathfrak{v}}$ which are not of the form Z_i. The line bundle $\mathcal{L}(2\rho)_{|X_w^v}$ has a section vanishing exactly on the set ∂X_w^v. To see this, consider the Borel–Weil isomorphism

$$\beta : V(\rho)^* \to H^0(G/B, \mathcal{L}(\rho)), \quad \beta(\chi)(gB) = [g, (g^{-1}\chi)_{|\mathbb{C}v_+}],$$

where $V(\rho)$ is the irreducible G-module with highest weight ρ and $v_+ \in V(\rho)$ is a highest weight vector. Let χ_v be the unique (up to a scalar multiple) vector of $V(\rho)^*$ with weight $-v\rho$. Now, take the section $\beta(\chi_v) \cdot \beta(\chi_w)$ of the line bundle $\mathcal{L}(2\rho)_{|X_w^v}$. Then, it has the zero set precisely equal to ∂X_w^v, since the zero set $Z(\beta(\chi_v)_{|X^v})$ of $\beta(\chi_v)_{|X^v}$ is given by

$$\begin{aligned} Z(\beta(\chi_v)_{|X^v}) &= \{gB \in X^v : \chi_v(gv_+) = 0\} \\ &= \bigcup_{v' > v} B^- v' B/B \\ &= \partial X^v. \end{aligned}$$

We fix $H = \sum_i b_i X_i$ to be the divisor corresponding to the section $\beta(\chi_v) \cdot \beta(\chi_w)$ of the line bundle $\mathcal{L}(2\rho)_{|X_w^v}$ as in (4). Observe that the coefficients of $m_{\mathfrak{w}}^{\mathfrak{v}\,*} H = \sum_i b_i Z_i + \sum_j d_j Z'_j$ are all strictly positive integers.

Thus, by combining the identities (4), (11)–(15), we obtain

$$\begin{aligned} E &= -\sum_i \frac{b_i}{N} Z_i - \sum_j Z'_j + \frac{1}{N} m_{\mathfrak{w}}^{\mathfrak{v}\,*}(H) \\ &= -\sum_i \frac{b_i}{N} Z_i - \sum_j Z'_j + \frac{1}{N} \sum_i b_i Z_i + \frac{1}{N} \sum d_j Z'_j, \\ &= \sum_j \left(\frac{d_j}{N} - 1 \right) Z'_j, \end{aligned}$$

for some $d_j \in \mathbb{N}$ (since the zero set of a certain section of $\mathcal{L}(2\rho)_{|X_w^v}$ is precisely equal to ∂X_w^v and all the Z'_j lie over ∂X_w^v). Thus, the coefficient e_j of Z'_j in E satisfies $-1 < e_j$.

Finally, observe that $Exc(m_{\mathfrak{w}}^{\mathfrak{v}}) + \Delta'$ is a $\mathbb{Q}$-divisor with simple normal crossings since $Supp(Exc(m_{\mathfrak{w}}^{\mathfrak{v}}) + \Delta') \subset \partial Z_{\mathfrak{w}}^{\mathfrak{v}}$ and the latter is a simple normal crossing divisor since

$$\partial Z_{\mathfrak{w}}^{\mathfrak{v}} = Z_{\mathfrak{w}}^{\mathfrak{v}} \setminus C_w^v$$

(cf. Section 4.2).

This completes the proof of Theorem 3.1 in the finite case. ■

Remark 4.3. The above proof crucially uses the explicit BSDH-type resolution of the Richardson varieties X_w^v given in Section 4.2. This resolution is available in the finite case, but we are not aware of such an explicit resolution in the Kac–Moody case. This is the main reason that we need to handle the general Kac–Moody case differently. □

5 Proof of Theorem 3.1 in the Kac–Moody Case

Our proof of Theorem 3.1 in the general Kac–Moody case is more involved. It requires the use of characteristic p methods; in particular, the Frobenius splitting.

For the construction of the flag variety $X = G/B$, Schubert subvarieties X_w, opposite Schubert subvarieties X^v (and thus the Richardson varieties X_w^v) associated to any Kac–Moody group G over an algebraically closed field k, we refer to [24, 25, 28–30].

Let k be an algebraically closed field of characteristic $p > 0$. Let $Y : Y_0 \subset Y_1 \subset \cdots$ be an ind-variety over k and let $\mathcal{O}_Y$ be its structure sheaf (cf. [22, Definition 4.1.1]). The *absolute Frobenius morphism*

$$F_Y : Y \longrightarrow Y$$

is the identity on the underlying space of Y, and the pth power map on the structure sheaf $\mathcal{O}_Y$. Consider the $\mathcal{O}_Y$-linear Frobenius map

$$F^{\#} : \mathcal{O}_Y \to F_* \mathcal{O}_Y,\ f \mapsto f^p,$$

where we have abbreviated F_Y by F.

Identical to the definition of Frobenius split varieties, we have the following definition for ind-varieties.

Definition 5.1. An ind-variety Y is called *Frobenius split* (or just *split*) if the $\mathcal{O}_Y$-linear map $F^{\#}$ splits, that is, there exists an $\mathcal{O}_Y$-linear map

$$\varphi : F_* \mathcal{O}_Y \longrightarrow \mathcal{O}_Y$$

such that the composition $\varphi \circ F^{\#}$ is the identity of $\mathcal{O}_Y$. Any such φ is called a *splitting*.

A closed ind-subvariety Z of Y is *compatibly split* under the splitting φ if

$$\varphi(F_* \mathcal{I}_Z) \subseteq \mathcal{I}_Z,$$

where $\mathcal{I}_Z \subset \mathcal{O}_Y$ is the ideal sheaf of Z.

Clearly, one way to obtain a splitting of Y is to give a sequence of splittings φ_n of Y_n such that φ_n compatibly splits Y_{n-1} inducing the splitting φ_{n-1} on Y_{n-1}. □

Let B be the standard Borel subgroup of any Kac–Moody group G over an algebraically closed field k of characteristic $p > 0$ and $T \subset B$ the standard maximal torus.

For any real root β, let U_β be the corresponding root subgroup. Then, there exists an algebraic group isomorphism $\varepsilon_\beta : \mathbb{G}_a \to U_\beta$ satisfying

$$t\varepsilon_\beta(z)t^{-1} = \varepsilon_\beta(\beta(t)z),$$

for $z \in \mathbb{G}_a$ and $t \in T$. For any B-locally finite algebraic representation V of B, $v \in V$ and $z \in \mathbb{G}_a$,

$$\varepsilon_\beta(z)v = \sum_{m \geq 0} z^m (e_\beta^{(m)} \cdot v),$$

where $e_\beta^{(m)}$ denotes the mth divided power of the root vector e_β, which is, by definition, the derivative of ε_β at 0.

Now, we come to the definition of B-canonical splittings for ind-varieties (cf. [7, Section 4.1] for more details in the finite case).

Definition 5.2. Let Y be a B-ind-variety, that is, B acts on the ind-variety Y via ind-variety isomorphisms. Let $\mathrm{End}_F(Y) := \mathrm{Hom}(F_*\mathcal{O}_Y, \mathcal{O}_Y)$ be the additive group of all the $\mathcal{O}_Y$-module maps $F_*\mathcal{O}_Y \to \mathcal{O}_Y$. Recall that $F_*\mathcal{O}_Y$ can canonically be identified with $\mathcal{O}_Y$ as a sheaf of abelian groups on Y; however, the $\mathcal{O}_Y$-module structure is given by $f \odot g := f^p g$, for $f, g \in \mathcal{O}_Y$. Since Y is a B-ind-variety, B acts on $\mathrm{End}_F(Y)$ by

$$(b * \psi)s = b(\psi(b^{-1}s)), \quad \text{for } b \in B,\ \psi \in \mathrm{End}_F(Y) \text{ and } s \in F_*\mathcal{O}_Y,$$

where the action of B on $F_*\mathcal{O}_Y$ is defined to be the standard action of B on $\mathcal{O}_Y$ under the identification $F_*\mathcal{O}_Y = \mathcal{O}_Y$ (as sheaves of abelian groups). We define the k-linear structure on $\mathrm{End}_F(Y)$ by

$$(z * \psi)s = \psi(zs) = z^{1/p}\psi(s),$$

for $z \in k$, $\psi \in \mathrm{End}_F(Y)$ and $s \in \mathcal{O}_Y$.

A splitting $\phi \in \mathrm{End}_F(Y)$ is called a B-*canonical splitting* if ϕ satisfies the following conditions:

(a) ϕ is T-invariant, that is,

$$t * \phi = \phi, \quad \text{for all } t \in T.$$

(b) For any simple root α_i, $1 \le i \le \ell$, there exist $\phi_{i,j} \in \mathrm{End}_F(Y)$, $0 \le j \le p-1$, such that

$$\varepsilon_{\alpha_i}(z) * \phi = \sum_{j=0}^{p-1} z^j * \phi_{i,j}, \quad \text{for all } z \in \mathbb{G}_a. \tag{16}$$

□

The definition of B^--canonical is of course parallel.

Before we come to the proof of Theorem 3.1 for the Kac–Moody case, we need the following results.

The following result in the symmetrizable Kac–Moody case is due to O. Mathieu (unpublished). (For a proof in the finite case, see [7, Theorem 2.3.2].) Since Mathieu's proof is unpublished, we briefly give an outline of his proof contained in [26].

Proposition 5.3. Consider the Richardson variety $X_w^v(k)$ (for any $v \le w$) over an algebraically closed field k of characteristic $p > 0$. Then, $X_w^v(k)$ is Frobenius split compatibly splitting its boundary ∂X_w^v. □

Proof. *Assertion I: The full flag variety $X = X(k)$ admits a B-canonical splitting.*

For any $w \in W$ and any reduced decomposition $\mathfrak{w}$ of w, consider the BSDH desingularization $Z_{\mathfrak{w}} = Z_{\mathfrak{w}}(k)$ of the Schubert variety X_w and the section $\sigma \in H^0(Z_{\mathfrak{w}}, \mathcal{O}_{Z_{\mathfrak{w}}}[\partial Z_{\mathfrak{w}}])$ with the associated divisor of zeroes $(\sigma)_0 = \partial Z_{\mathfrak{w}}$. Clearly, such a section is unique (up to a nonzero scalar multiple). Take the unique, up to a nonzero multiple, nonzero section $\theta \in H^0(Z_{\mathfrak{w}}, \mathcal{L}_{\mathfrak{w}}(\rho))$ of weight $-\rho$. (Such a section exists since $H^0(Z_{\mathfrak{w}}, \mathcal{L}_{\mathfrak{w}}(\rho)) \to H^0(\{1\}, \mathcal{L}_{\mathfrak{w}}(\rho)_{|\{1\}})$ is surjective by [7, Theorem 3.1.4], where $1 := [1, \dots, 1] \in Z_{\mathfrak{w}}$ denotes the $B^{\ell(w)}$-orbit of $(1, \dots, 1)$ as in [7, Definition 2.2.1]. Moreover, such a section is unique up to a scalar multiple since $H^0(Z_{\mathfrak{w}}, \mathcal{L}_{\mathfrak{w}}(\rho))^* \simeq H^0(X_w, \mathcal{L}(\rho)_{|X_w})^* \hookrightarrow V(\rho)$.) By the above, the section θ does not vanish at the base point 1. Thus, by [7, Propositions 1.3.11 and 2.2.2], $(\sigma\theta)^{p-1}$ provides a splitting $\hat{\sigma}_{\mathfrak{w}}$ of $Z_{\mathfrak{w}}$ compatibly splitting $\partial Z_{\mathfrak{w}}$. Since the Schubert variety X_w is normal, the splitting $\hat{\sigma}_{\mathfrak{w}}$ descends to give a splitting $\hat{\sigma}_w$ of X_w compatibly splitting all the Schubert subvarieties of X_w.

Now, the splitting $\hat{\sigma}_{\mathfrak{w}}$ is B-canonical and it is the unique B-canonical splitting of $Z_{\mathfrak{w}}$ (cf. [7, Exercise 4.1.E.2]; even though this exercise is for finite-dimensional G, the same proof works for the Kac–Moody case). We claim that the induced splittings $\hat{\sigma}_w$ of X_w are compatible to give a splitting of $X = \cup_w X_w$. Take $v, w \in W$ and choose $u \in W$ with $v \le u$ and $w \le u$. Choose a reduced word $\mathfrak{u}$ of u. Then there is a reduced subword $\mathfrak{v}$ (resp. $\mathfrak{w}$) of $\mathfrak{u}$ corresponding to v (resp. w). The B-canonical splitting $\hat{\sigma}_{\mathfrak{u}}$ of $Z_{\mathfrak{u}}$ (by the uniqueness of the B-canonical splittings of $Z_{\mathfrak{v}}$) restricts to the B-canonical splitting $\hat{\sigma}_{\mathfrak{v}}$ of $Z_{\mathfrak{v}}$ (and

$\hat{\sigma}_{\mathfrak{w}}$ of $Z_{\mathfrak{w}}$). In particular, the splitting $\hat{\sigma}_u$ of X_u restricts to the splitting $\hat{\sigma}_v$ of X_v (and $\hat{\sigma}_w$ of X_w). This proves the assertion that the splittings $\hat{\sigma}_u$ of X_u are compatible to give a B-canonical splitting $\hat{\sigma}$ of X. By the same proof as that of [7, Proposition 4.1.10], we obtain that the B-canonical splitting $\hat{\sigma}$ of X is automatically B^--canonical.

Assertion II: The splitting $\hat{\sigma}$ of X canonically splits the T-fixed points of X.

Take a T-fixed point $\dot{w}B \in X$ (for some $w \in W$) and consider the Schubert variety X_w. Then, $\dot{w}B \in X_w$ has an affine open neighborhood $U_w \simeq U_w \cdot \dot{w}B/B$, where U_w is the unipotent subgroup of G with Lie algebra $\mathfrak{g}_{\alpha_{\alpha\in R^+\cap wR^-}\bigoplus}$, where R^+ (resp., R^-) is the set of positive (resp., negative) roots. In particular, the ring $k[U_w\dot{w}B/B]$ of regular functions, as a T-module, has weights lying in the cone $\sum_{\alpha\in R^-\cap wR^+}\mathbb{Z}_+\alpha$ and the T-invariants $k[U_w\dot{w}B/B]^{\mathrm{T}}$ in $k[U_w\dot{w}B/B]$ are only the constant functions. Since any B-canonical splitting is T-equivariant by definition, it takes the λ-eigenspace $k[U_w\dot{w}B/B]_\lambda$ to $k[U_w\dot{w}B/B]_{\lambda/p}$ (cf. [7, Section 4.1.4]). This shows that the ideal of $\{\dot{w}B\}$ in $k[U_w\dot{w}B/B]$ is stable under $\hat{\sigma}$. Thus $\{\dot{w}B\}$ is compatibly split under $\hat{\sigma}$.

Assertion III: X^w is compatibly split under $\hat{\sigma}$.

Since $\hat{\sigma}$ is B^--canonical, by [7, Proposition 4.1.8], for any closed ind-subvariety Y of X which is compatibly split under $\hat{\sigma}$, the B^--orbit closure $\overline{B^-Y}\subset X$ is also compatibly split. In particular, the opposite Schubert variety $X^w := \overline{B^-\dot{w}B/B}$ is compatibly split.

Thus, we obtain that the Richardson varieties X_w^v (for $v \le w$) are compatibly split under the splitting $\hat{\sigma}$ of X. Since the boundary ∂X_w^v is a union of other Richardson varieties, the boundary also is compatibly split. This proves the proposition. ■

We need the following general result. First, we recall a definition.

Definition 5.4. Suppose that X is a normal variety over an algebraically closed field of characteristic $p>0$ and D is an effective $\mathbb{Q}$-divisor on X. The pair (X, D) is called *sharply F-pure* if, for every point $x\in X$, there exists an integer $e\ge 1$ such that e-iterated Frobenius map

$$\mathcal{O}_{X,x} \to F^e_*(\mathcal{O}_{X,x}(\lceil (p^e-1)D\rceil))$$

admits an $\mathcal{O}_{X,x}$-module splitting. In fact, if there exists a splitting for one $e>0$, by composing maps, we obtain a splitting for all sufficiently divisible $e>0$. □

Note that, by definition, if $\mathcal{O}_X \to F^e_*\mathcal{O}_X$ is split relative to an effective divisor D, then the pair $(X, \frac{1}{p^e-1}D)$ is sharply F-pure.

Note that being sharply F-pure is a purely local condition, unlike being split.

Proposition 5.5. Let X be an irreducible normal variety over a field of characteristic $p > 0$ and $D = \sum_i D_i$ a reduced divisor in X. Assume further that X is Frobenius split compatibly splitting *SuppD*. Then, the pair (X, D) is sharply F-pure. □

Proof. Note that we have a global splitting of $\mathcal{O}_X(-D) \to F_*(\mathcal{O}_X(-D))$. Twisting both sides by D and applying the projection formula give us a global splitting of $\mathcal{O}_X \to F_*(\mathcal{O}_X((p-1)D))$. We may localize this at any stalk and take $e = 1$. ■

By Lemma 4.1, the Richardson varieties X_w^v are normal in characteristic 0; in particular, they are normal in characteristics $p \gg 0$. Thus, combining Propositions 5.3 and 5.5, we obtain the following.

Corollary 5.6. With the notation as above, for any $v \leq w$, $(X_w^v, \partial X_w^v)$ is sharply F-pure in characteristics $p \gg 0$. □

We also recall the following from [14, Theorem 3.9]. It should be mentioned that even though in [14] the result is proved in the local situation, the same proof works for projective varieties. We sketch a proof for the convenience of the reader.

Theorem 5.7. Let X be an irreducible normal variety over a field of characteristic 0 and let D be an effective $\mathbb{Q}$-divisor on X such that $K_X + D$ is $\mathbb{Q}$-Cartier and such that $\lceil D \rceil$ is reduced (i.e., the coefficients of D are in $[0, 1]$). If the reduction (X_p, D_p) mod p of (X, D) is sharply F-pure for infinitely many primes p, then (X, D) is log canonical. □

Proof. (Sketch) Fix a log resolution $\pi : \tilde{X} \to X$ of (X, D) and write

$$E_\pi(D) - D' = K_{\tilde{X}} - \frac{1}{n}(\pi^*(nK_X + nD))$$

for a choice of $K_{\tilde{X}}$ agreeing with K_X wherever π is an isomorphism as in (1). We need to show that the coefficients of $E_\pi(D)$ are ≥ -1. We reduce the entire setup to some characteristic $p \gg 0$ where (X_p, D_p) is sharply F-pure (for a discussion of this process, see [14] or see [7, Section 1.6] in the special case when the varieties are defined over $\mathbb{Z}$).

Fix $x \in X_p$. We have a splitting ϕ:

$$\begin{tikzcd} \mathcal{O}_{X_p,x} \arrow[r, hook] & F^e_* \mathcal{O}_{X_p,x} \arrow[r, hook] \arrow[rr, bend right, "\phi"'] & F^e_*(\mathcal{O}_{X_p,x}(\lceil (p^e-1)D_{p,x} \rceil)) \arrow[r] & \mathcal{O}_{X_p,x} \end{tikzcd},$$

for some $e \geq 1$. This splitting $\phi \in \mathrm{Hom}(F^e_*\mathcal{O}_{X_p,x}, \mathcal{O}_{X_p,x})$ corresponds to a divisor $B_x \geq \lceil (p^e - 1)D_{p,x} \rceil$ on *Spec* $\mathcal{O}_{X_p,x}$, which is linearly equivalent to $(1-p^e)K_{X_p,x}$ (cf. [7, Section 1.3]). Set $\Delta = \frac{1}{p^e-1} B_x$. Observe that $(p^e-1)(K_{X_p,x} + \Delta)$ is linearly equivalent to 0; and thus $K_{X_p,x} + \Delta$ is $\mathbb{Q}$-Cartier. Since

$$\Delta = \frac{1}{p^e-1} B_x \geq \frac{1}{p^e-1} \lceil (p^e-1)D_{p,x} \rceil \geq D_{p,x},$$

we know

$$E_{\pi_p}(D_{p,x}) \geq E_{\pi_p}(\Delta).$$

Therefore, it is sufficient to prove that the coefficients of $E_{\pi_p}(\Delta)$ are ≥ -1. Note that it is possible that π_p is not a log resolution for Δ, but this will not matter for us.

We can factor the splitting ϕ as follows (we leave this verification to the reader):

$$\begin{tikzcd} F^e_*\mathcal{O}_{X_p,x} \arrow[r, hook] \arrow[rrr, bend right, "\phi"'] & F^e_*(\mathcal{O}_{X_p,x}(\lceil (p^e-1)D_{p,x} \rceil)) \arrow[r, hook] & F^e_*(\mathcal{O}_{X_p,x}((p^e-1)\Delta)) \arrow[r, "\psi"] & \mathcal{O}_{X_p,x} \end{tikzcd}.$$

Let C be any prime exceptional divisor of $\pi_p : \tilde{X}_p \to X_p$ with generic point η and let $\mathcal{O}_{\tilde{X}_p,\eta}$ be the associated valuation ring. Let $a \in \mathbb{Q}$ be the coefficient of C in $E_{\pi_p}(\Delta)$. There are two cases:

(i) $a > 0$.

(ii) $a \leq 0$.

Since we are trying to prove that $a \geq -1$, if we are in case (i), we are already done. Therefore, we may assume that $a \leq 0$. By tensoring ϕ with the fraction field $K(X_p) = K(\tilde{X}_p)$, we obtain a map $\phi_{K(X_p)} : F^e_* K(X_p) \to K(X_p)$.

Claim 5.8. By restricting $\phi_{K(X_p)}$ to the stalk $F^e_* \mathcal{O}_{\tilde{X}_p,\eta}$, we obtain a map ϕ_η which factors as:

$$\phi_\eta : F^e_* \mathcal{O}_{\tilde{X}_p,\eta} \hookrightarrow F^e_*(\mathcal{O}_{\tilde{X}_p,\eta}(-a(p^e-1)C)) \to \mathcal{O}_{\tilde{X}_p,\eta}. \qquad \square$$

Proof of the claim. Indeed, similar arguments are used to prove Grauert–Riemenschneider vanishing for Frobenius split varieties [27], [7, Theorem 1.3.14]. We briefly sketch the idea of the proof.

We identify ψ with a section $s \in \mathcal{O}_{X_p,x}((1-p^e)(K_{X_p,x}+\Delta)) \cong \mathrm{Hom}_{\mathcal{O}_{X_p,x}}(F^e_*(\mathcal{O}_{X_p,x}((p^e-1)\Delta)), \mathcal{O}_{X_p,x})$. Recall that $\pi_p^*((1-p^e)(K_{X_p,x}+\Delta)) = (1-p^e)K_{\tilde{X}_p,\eta} - (1-p^e)E_{\pi_p}(\Delta) + (1-p^e)\Delta'$, where Δ' is the strict transform of Δ. Thus, we can pull s back to a section

$$\begin{aligned}
t := \pi^* s &\in \mathcal{O}_{\tilde{X}_p,\eta}(\pi^*((1-p^e)(K_{X_p,x}+\Delta))) \\
&= \mathcal{O}_{\tilde{X}_p,\eta}((1-p^e)K_{\tilde{X}_p,\eta} - (1-p^e)E_{\pi_p}(\Delta) + (1-p^e)\Delta') \\
&= \mathcal{O}_{\tilde{X}_p,\eta}((1-p^e)K_{\tilde{X}_p,\eta} - (1-p^e)aC) \\
&\cong \mathrm{Hom}_{\mathcal{O}_{\tilde{X}_p,\eta}}(F^e_*(\mathcal{O}_{\tilde{X}_p,\eta}(-a(p^e-1)C)), \mathcal{O}_{\tilde{X}_p,\eta}).
\end{aligned}$$

It is not hard to see that the homomorphism $\psi_\eta : F^e_*(\mathcal{O}_{\tilde{X}_p,\eta}(-a(p^e-1)C)) \to \mathcal{O}_{\tilde{X}_p,\eta}$ corresponding to t can be chosen to agree with ψ on the fraction field $K(X) = K(\tilde{X})$, we leave this verification to the reader. It follows that ϕ_η is the composition $F^e_* \mathcal{O}_{\tilde{X}_p,\eta} \hookrightarrow F^e_*(\mathcal{O}_{\tilde{X}_p,\eta}(-a(p^e-1)C)) \xrightarrow{\psi_\eta} \mathcal{O}_{\tilde{X}_p,\eta}$. This concludes the proof of the claim. ∎

Now we complete the proof of Theorem 5.7. Note that ϕ_η is a splitting because ϕ was a splitting and both the maps agree on the field of fractions. Therefore, $0 \le -a(p^e-1) \le p^e - 1$ since the splitting along a divisor cannot vanish to order $> p^e - 1$. Dividing by $(1-p^e)$ proves that $a \ge -1$ as desired. ∎

We also recall the following.

Lemma 5.9. Let X be an irreducible normal projective variety over $\mathbb{C}$ and let $D = \sum a_i D_i$ be an effective $\mathbb{Q}$-divisor on X such that $X \backslash SuppD$ is smooth and (X, D) is log canonical. Now, consider a $\mathbb{Q}$-divisor $\Delta = \sum c_i D_i$ with $c_i \in [0, 1)$ and $c_i < a_i$ for all i, such that $K_X + \Delta$ is $\mathbb{Q}$-Cartier. Then, (X, Δ) is KLT. $\square$

Proof. We may choose a resolution of singularities $\pi : \tilde{X} \to X$ which is a log resolution for (X, D) (and hence also for (X, Δ)) by Hironaka [16]. Furthermore, we may assume that π is an isomorphism over $X \setminus \mathit{Supp}(D) \subseteq X \setminus \mathit{Supp}(\Delta)$. Therefore, we see that

$$E_\pi(\Delta) = K_{\tilde{X}} - \frac{1}{n}\pi^*(n(K_X + \Delta)) + \Delta' \geq K_{\tilde{X}} - \frac{1}{n}\pi^*(n(K_X + D)) + D' = E_\pi(D)$$

with strict inequality in every nonzero coefficient. Since every coefficient of $E_\pi(D)$ is ≥ -1, we are done. ∎

We now come to the proof of Theorem 3.1 in the Kac–Moody case.

Proof of Theorem 3.1. We begin by reducing our entire setup to characteristic $p \gg 0$.

By Corollary 5.6, $(X_w^v(k), \partial X_w^v(k))$ is sharply F-pure for any algebraically closed field k of characteristic $p \gg 0$. Moreover, $K_{X_w^v} + \partial X_w^v = 0$ by Lemma 4.2; in particular, it is Cartier.

Hence, returning now to characteristic zero, by Theorem 5.7, $(X_w^v, \partial X_w^v)$ is log canonical. Furthermore, the $\mathbb{Q}$-divisor $\Delta = \sum_i (1 - \frac{b_i}{N}) X_i$, where $\partial X_w^v = \sum X_i$, clearly satisfies all the assumptions of Lemma 5.9. Thus, (X_w^v, Δ) is KLT, proving Theorem 3.1 in the Kac–Moody case as well. ∎

Remark 5.10. One can define KLT singularities in positive characteristic too by considering all valuations on all normal birational models. In particular, a similar argument shows that any normal Richardson variety is KLT in characteristic $p > 0$ as well. (It is expected that, for any symmetrizable Kac–Moody group, all the Richardson varieties X_w^v are normal in any characteristic.) □

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