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An algebro-geometric realization of equivariant cohomology of some Springer fibers

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ABSTRACT

We give an explicit affine algebraic variety whose coordinate ring is isomorphic (as a W-algebra) with the equivariant cohomology of some Springer fibers.

1. Introduction

Let G be a connected simply-connected semisimple complex algebraic group with a Borel subgroup B and a maximal torus $T \subset B$. Let $P \supseteq B$ be a (standard) parabolic subgroup of G. Let $L \supset T$ be the Levi subgroup of P and let S be the connected center of L (i.e., S is the identity component of the center of L). Then, $S \subset T$. We denote the Lie algebras of G, T, B, P, L, S by the corresponding Gothic characters: $\mathfrak{g}$, $\mathfrak{t}$, $\mathfrak{b}$, $\mathfrak{p}$, $\mathfrak{l}$, $\mathfrak{s}$ respectively. Let W be the Weyl group of G and $W_L \subset W$ the Weyl group of L. Let $\sigma = \sigma_{\mathfrak{l}}$ be a principal nilpotent element of $\mathfrak{l}$. Let $X = G/B$ be the full flag variety of G and let $X_\sigma \subset X$ be the Springer fiber corresponding to the nilpotent element σ (i.e., X_σ is the subvariety of X fixed under the left multiplication by $\operatorname{Exp}\sigma$, endowed with the reduced subscheme structure). Observe that S keeps the variety X_σ stable under the left multiplication of S on X.

Definition 1.1. Let $Z_{\mathfrak{l}}$ be the reduced closed subvariety of $\mathfrak{t} \times \mathfrak{t}$ defined by

$$Z_{\mathfrak{l}} := \{(x, wx)\colon\ w \in W,\ x \in \mathfrak{s}\}.$$

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Since $Z_\mathfrak{l}$ is a cone inside $\mathfrak{t} \times \mathfrak{t}$, the affine coordinate ring $\mathbb{C}[Z_\mathfrak{l}]$ is a non-negatively graded algebra. Moreover, the projection $\pi_1 : Z_\mathfrak{l} \to \mathfrak{s}$ on the first factor gives rise to an $S(\mathfrak{s}^*)$-algebra structure on $\mathbb{C}[Z_\mathfrak{l}]$. Also, define an action of W on $Z_\mathfrak{l}$ by

$$v \cdot (x, wx) = (x, vwx), \quad \text{for } x \in \mathfrak{s},\ v, w \in W.$$

This action gives rise to a W-action on $\mathbb{C}[Z_\mathfrak{l}]$, commuting with the $S(\mathfrak{s}^*)$ action on $\mathbb{C}[Z_\mathfrak{l}]$.

In fact, even though we do not need it, W is precisely the automorphism group of $\mathbb{C}[Z_\mathfrak{l}]$ as $S(\mathfrak{s}^*)$-algebra.

For $\mathfrak{p} = \mathfrak{b}$, the Levi subalgebra $\mathfrak{l}$ coincides with $\mathfrak{t}$, $\sigma_\mathfrak{t} = 0$ and $X_\sigma = X$. In this case, $\mathfrak{s} = \mathfrak{t}$ and we abbreviate $Z_\mathfrak{t}$ by Z. Clearly, $Z_\mathfrak{l}$ (for any Levi subalgebra $\mathfrak{l}$) is a closed subvariety of Z.

The following theorem is our main result.

Theorem 1.2. *With the notation as above, assume that the canonical restriction map $H^*(X) \to H^*(X_\sigma)$ is surjective, where H^* denotes the singular cohomology with complex coefficients. Then, there is a graded $S(\mathfrak{s}^*)$-algebra isomorphism*

$$\phi_\mathfrak{l} : \mathbb{C}[Z_\mathfrak{l}] \to H^*_S(X_\sigma),$$

where H^*_S denotes the S-equivariant cohomology with complex coefficients.

Moreover, the following diagram is commutative:

$$\begin{array}{ccc} \mathbb{C}[Z] & \xrightarrow{\phi_\mathfrak{t}} & H^*_T(X) \\ \downarrow & & \downarrow \\ \mathbb{C}[Z_\mathfrak{l}] & \xrightarrow{\phi_\mathfrak{l}} & H^*_S(X_\sigma), \end{array} \tag{1}$$

where the vertical maps are the canonical restriction maps.

In particular, we get an isomorphism of graded algebras

$$\phi^o_\mathfrak{l} : \mathbb{C} \otimes_{S(\mathfrak{s}^*)} \mathbb{C}[Z_\mathfrak{l}] \to H^*(X_\sigma),$$

making the following diagram commutative:

$$\begin{array}{ccc} \mathbb{C} \otimes_{S(\mathfrak{t}^*)} \mathbb{C}[Z] & \xrightarrow{\phi^o_\mathfrak{t}} & H^*(X) \\ \downarrow & & \downarrow \\ \mathbb{C} \otimes_{S(\mathfrak{s}^*)} \mathbb{C}[Z_\mathfrak{l}] & \xrightarrow{\phi^o_\mathfrak{l}} & H^*(X_\sigma), \end{array} \tag{2}$$

where the vertical maps are the canonical restriction maps and $\mathbb{C}$ is considered as an $S(\mathfrak{s}^*)$-module under the evaluation at 0.

Moreover, the isomorphism $\phi^o_\mathfrak{l}$ is W-equivariant under the Springer's W-action on $H^*(X_\sigma)$ and the W-action on $\mathbb{C} \otimes_{S(\mathfrak{s}^*)} \mathbb{C}[Z_\mathfrak{l}]$ induced from the W-action on $\mathbb{C}[Z_\mathfrak{l}]$ defined above.

2. Proof of the theorem

Before we come to the proof of the theorem, we need the following lemma. (See, e.g., [C, Theorem 2].)

Lemma 2.1. *For any $w \in W$, there exists a unique $w' \in W_L$ such that*

$$w'wB \in X_\sigma^S \subset X.$$

Moreover, this induces a bijection

$$W_L \backslash W \leftrightarrow X_\sigma^S.$$

We also need the following simple (and well-known) result.

Lemma 2.2. *Let $S = S(V^*)$ be the symmetric algebra for a finite dimensional vector space V and let M, N, R be three S-modules. Assume that N and R are S-free of the same finite rank and M is an S-submodule of R. Then, any surjective S-module morphism $\phi : M \to N$ is an isomorphism.*

We now come to the proof of the theorem.

Proof of the theorem. Consider the equivariant Borel homomorphism

$$\beta : S(\mathfrak{t}^*) \to H_T(X)$$

obtained by $\lambda \mapsto c_1(\mathcal{L}_\lambda)$, where $\lambda \in \mathfrak{t}^*$ and $c_1(\mathcal{L}_\lambda)$ is the T-equivariant first Chern class of the line bundle $\mathcal{L}(\lambda)$ on X corresponding to the character e^λ, and extended as a graded algebra homomorphism. This gives rise to an algebra homomorphism

$$\chi : \mathbb{C}[\mathfrak{t} \oplus \mathfrak{t}] \simeq S(\mathfrak{t}^*) \otimes S(\mathfrak{t}^*) \to H_T(X), \quad p \otimes q \mapsto p \cdot \beta(q),$$

where $p\cdot$ denotes the multiplication in the T-equivariant cohomology by $p \in S(\mathfrak{t}^*) \simeq H_T(pt)$. It is well known that χ is surjective. Moreover, both the restriction maps

$$H_T(X) \twoheadrightarrow H_S(X) \twoheadrightarrow H_S(X_\sigma)$$

are surjective; this follows since both the spaces X and X_σ have cohomologies concentrated in even degrees (cf. [DLP]). (Use the degenerate Leray–Serre spectral sequence and the assumption that the restriction map $H^*(X) \to H^*(X_\sigma)$ is surjective.)

Consider the canonical surjective map $\theta : \mathbb{C}[\mathfrak{t} \oplus \mathfrak{t}] \twoheadrightarrow \mathbb{C}[Z_\mathfrak{l}]$. Then, of course,

$$\operatorname{Ker}\theta = \left\{ \sum_i p_i \otimes q_i\colon\ p_i, q_i \in S(\mathfrak{t}^*) \text{ and } \sum_i p_i(x) q_i(wx) = 0, \text{ for all } x \in \mathfrak{s} \text{ and } w \in W \right\}. \tag{3}$$

We claim that

$$\operatorname{Ker}\theta \subset \operatorname{Ker}\gamma, \tag{4}$$

where γ is the composite map

$$\mathbb{C}[\mathfrak{t} \oplus \mathfrak{t}] \xrightarrow{\chi} H_T(X) \to H_S(X_\sigma).$$

Since X_σ has cohomologies only in even degrees, by the degenerate Leray–Serre spectral sequence, $H_S(X_\sigma)$ is a free $S(\mathfrak{s}^*)$-module. In particular, by the Borel–Atiyah–Segal Localization Theorem (cf. [AP, Theorem 3.2.6]),

$$H_S(X_\sigma) \hookrightarrow H_S\big(X_\sigma^S\big).$$

Thus, to prove the claim (4), it suffices to prove that for any $\sum_i p_i \otimes q_i \in \operatorname{Ker}\theta$,

$$\gamma\left(\sum_i p_i \otimes q_i\right)_{|_{X_\sigma^S}} \equiv 0.$$

It is easy to see that the Borel homomorphism β restricted to the T-fixed points X^T satisfies:

$$\beta(q)(wB) = wq, \quad \text{for any } q \in S\big(\mathfrak{t}^*\big) \text{ and } w \in W.$$

Thus, for any $w \in W$,

$$\gamma\left(\sum_i p_i \otimes q_i\right)\big(w'wB\big) = \left(\sum_i (p_i)\big(w'wq_i\big)\right)_{|_{\mathfrak{s}}},$$

where w' is as in Lemma 2.1. From the description of $\operatorname{Ker}\theta$ given in (3), we thus get that the claim (4) is true. Hence, the map θ descends to a surjective $S(\mathfrak{s}^*)$-algebra homomorphism

$$\phi_{\mathfrak{l}} : \mathbb{C}[Z_{\mathfrak{l}}] \twoheadrightarrow H_S(X_\sigma).$$

Again using the Localization Theorem, the free $S(\mathfrak{s}^*)$-module $H_S(X_\sigma)$ is of rank $= \# W_L\backslash W$, since $\# X_\sigma^S = \# W_L\backslash W$ by Lemma 2.1. Also, the projection on the first factor $\pi_1 : Z_{\mathfrak{l}} \to \mathfrak{s}$ is a finite morphism with all its fibers of cardinality $\leqslant \# W_L\backslash W$. To see this, consider the surjective morphism $\alpha : \mathfrak{s} \times W/W_L \to Z_{\mathfrak{l}}$, $(x, wW_L) \mapsto (x, wx)$. Then, $\pi_1 \circ \alpha : \mathfrak{s} \times W/W_L \to \mathfrak{s}$ is again the projection on the first factor, which is clearly a finite morphism and hence so is π_1.

Now, taking $M = \mathbb{C}[Z_{\mathfrak{l}}]$, $N = H_S^*(X_\sigma)$, $R = \mathbb{C}[\mathfrak{s} \times W/W_L]$ and $V = \mathfrak{s}$ in Lemma 2.2, we get that $\phi_{\mathfrak{l}}$ is an isomorphism, where the inclusion $M \subset R$ is induced from the surjective morphism $\alpha : \mathfrak{s} \times W/W_L \to Z_{\mathfrak{l}}$.

The commutativity of the diagram (1) clearly follows from the above proof.

Since $H^*(X_\sigma)$ is concentrated in even degrees, by the degenerate Leray–Serre spectral sequence, we get that

$$H^*(X_\sigma) \simeq \mathbb{C} \otimes_{S(\mathfrak{s}^*)} H_S^*(X_\sigma).$$

From this the 'In particular' part of the theorem follows.

From the definition of the map $\phi_{\mathfrak{t}}$, it is clear that $\phi_{\mathfrak{t}}^o$ is W-equivariant with respect to the action of W on $\mathbb{C} \otimes_{S(\mathfrak{t}^*)} \mathbb{C}[Z]$ induced from the action of W on $\mathbb{C}[Z]$ as defined in Definition 1.1 and the standard action of W on $H^*(X)$. Moreover, the restriction map $H^*(X) \to H^*(X_\sigma)$ is W-equivariant with respect to the Springer's W action on $H^*(X_\sigma)$ (cf. [HS, Theorem 1.1]). Thus, the W-equivariance of $\phi_{\mathfrak{l}}^o$ follows from the commutativity of the diagram (2). This completes the proof of the theorem. □

Remark 2.3. (1) By the Jordan block decomposition, any nilpotent element $\sigma \in sl(N)$ (up to conjugacy) is a regular nilpotent element in a standard Levi subalgebra $\mathfrak{l}$ of $sl(N)$. Moreover, the canonical restriction map $H^*(X) \to H^*(X_\sigma)$ is surjective in this case. In fact, as proved by Spaltenstein [S], in this case there is a paving of X by affine spaces as cells such that X_σ is a closed union of cells (cf. also [DLP]). Thus, the above theorem, in particular, applies to any nilpotent element σ in any special linear Lie algebra $sl(N)$.

(2) A certain variant (though a less precise version) of our Theorem 1.2 for $\mathfrak{g} = sl(N)$ is obtained by Goresky and MacPherson [GM, Theorem 7.2].

(3) For a general semisimple Lie algebra $\mathfrak{g}$, it is not true that the restriction map $H^*(X) \to H^*(X_\sigma)$ is surjective for any regular nilpotent element in a Levi subalgebra $\mathfrak{l}$. Take, e.g., $\mathfrak{g}$ of type C_3 and σ corresponding to the Jordan blocks of size $(3, 3)$. In this case, the centralizer of σ in the symplectic group Sp(6) is connected and X_σ is two-dimensional. The cohomology of X_σ as a W-module is given as follows:

Of course, $H^0(X_\sigma)$ is the one-dimensional trivial W-module; $H^2(X_\sigma)$ is the sum of the three-dimensional reflection representation with a one-dimensional representation; and $H^4(X_\sigma)$ is a three-dimensional irreducible representation.

(4) It will be interesting to deduce the result by De Concini and Procesi on the identification of the cohomology of Springer fibers for $sl(N)$ as the coordinate ring of a certain scheme (cf. [DP, §4]) from our Theorem 1.2.

(5) Stroppel has identified the cohomology of Springer fibers for $sl(N)$ with the center of the proncipal block $\mathcal{O}_0^{\mathfrak{p}}$ of the corresponding parabolic category $\mathcal{O}$ (cf. [St]).

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