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A generalization of Fulton's conjecture for arbitrary groups

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Abstract We prove a generalization of Fulton's conjecture which relates intersection theory on an arbitrary flag variety to invariant theory.

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1 Introduction

1.1 The context of Fulton's original conjecture

Let L be a connected reductive complex algebraic group with a Borel subgroup B_L and maximal torus $H \subset B_L$. The isomorphism classes of finite dimensional irreducible representations of L are parametrized by the set $X(H)^+$ of L-dominant characters of H via the highest weight. For $\lambda \in X(H)^+$, let $V(\lambda) = V_L(\lambda)$ be the corresponding irreducible representation of L with highest weight λ. Define the *Littlewood-Richardson coefficients* $c_{\lambda,\mu}^{\nu}$ by:

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$$V(\lambda) \otimes V(\mu) = \sum_{\nu} c^{\nu}_{\lambda,\mu} V(\nu).$$

The following result was conjectured by Fulton and proved by Knutson–Tao–Woodward [13] (Subsequently, geometric proofs were given by Belkale [2] and Ressayre [17,18]).

Theorem 1.1 *Let* $L = \mathrm{GL}(r)$ *and let* $\lambda, \mu, \nu \in X(H)^+$. *Then, if* $c^{\nu}_{\lambda,\mu} = 1$, *we have* $c^{n\nu}_{n\lambda,n\mu} = 1$ *for every positive integer n.*

(Conversely, if $c^{n\nu}_{n\lambda,n\mu} = 1$ for some positive integer n, then $c^{\nu}_{\lambda,\mu} = 1$. This follows from the saturation theorem of Knutson–Tao.)

For $\lambda, \mu, \nu \in X(H)^+$, observe that the space of $\mathrm{SL}(r)$-invariants $[V(\lambda) \otimes V(\mu) \otimes V(\nu)]^{\mathrm{SL}(r)}$ is isomorphic with the space of $\mathrm{GL}(r)$-invariants $[V(\lambda) \otimes V(\mu) \otimes V(\nu + d\epsilon)]^{\mathrm{GL}(r)}$, for some $d \in \mathbb{Z}$ (where ϵ is the determinant character: $\epsilon(h) = \det(h)$, for $h \in H$). Moreover, if $[V(\lambda) \otimes V(\mu) \otimes V(\nu)]^{\mathrm{GL}(r)}$ is nonzero (for $\lambda, \mu, \nu \in X(H)^+$), then this space coincides with $[V(\lambda) \otimes V(\mu) \otimes V(\nu)]^{\mathrm{SL}(r)}$. Thus, replacing $V(\nu)$ by the dual $V(\nu)^*$, the above theorem is equivalent to the following:

Theorem 1.2 *Let* $L = \mathrm{GL}(r)$ *and let* $\lambda, \mu, \nu \in X(H)^+$. *Then, if the dimension* $\dim\big([V(\lambda) \otimes V(\mu) \otimes V(\nu)]^{\mathrm{SL}(r)}\big) = 1$, *we have* $\dim\big([V(n\lambda) \otimes V(n\mu) \otimes V(n\nu)]^{\mathrm{SL}(r)}\big) = 1$, *for every positive integer n.*

The direct generalization of the above theorem for an arbitrary reductive L is false (see Example 8.3(3)). It is also known that the saturation theorem fails for arbitrary reductive groups. It is a challenge to find an appropriate version of the above result for $\mathrm{GL}(r)$ which holds in the setting of general reductive groups.

The aim of this paper is to achieve one such generalization. This generalization is a relationship between the intersection theory of homogeneous spaces and the invariant theory. To obtain this generalization, we must first reinterpret the above result for $\mathrm{GL}(r)$ as follows.

Without loss of generality, we only consider the irreducible polynomial representations of $\mathrm{GL}(r)$. These are parametrized by the sequences $\lambda = (\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_r \geq 0)$, where we view any such λ as the dominant character, $\mathrm{diag}(t_1, \ldots, t_r) \mapsto t_1^{\lambda_1} \ldots t_r^{\lambda_r}$, of the standard maximal torus consisting of the diagonal matrices in $\mathrm{GL}(r)$. Let $\mathfrak{P}(r)$ be the set of such sequences (also called Young diagrams or partitions) $\lambda = (\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_r \geq 0)$ and let $\mathfrak{P}_k(r)$ be the subset of $\mathfrak{P}(r)$ consisting of those partitions λ such that $\lambda_1 \leq k$. Then, the Schubert cells in the Grassmannian $\mathrm{Gr}(r, r+k)$ of r-planes in $\mathbb{C}^{r+k}$ are parametrized by $\mathfrak{P}_k(r)$ (cf. [8, §9.4]). For $\lambda \in \mathfrak{P}_k(r)$, let σ_λ be the corresponding Schubert cell and $\bar{\sigma}_\lambda$ its closure. By a classical theorem (cf. loc. cit.), the structure constants for the intersection product in $H^*(\mathrm{Gr}(r, r+k), \mathbb{Z})$ in the basis $[\bar{\sigma}_\lambda]$ coincide with the corresponding Littlewood-Richardson coefficients for the representations of $\mathrm{GL}(r)$. Thus, the above theorem can be reinterpreted as follows:

Theorem 1.3 *Let* $L = \mathrm{GL}(r)$ *and let* $\lambda, \mu, \nu \in \mathfrak{P}_k(r)$ (*for some* $k \geq 1$) *be such that the intersection product*

$$[\bar{\sigma}_\lambda] \cdot [\bar{\sigma}_\mu] \cdot [\bar{\sigma}_\nu] = [\bar{\sigma}_{\lambda^o}] \text{ in } H^*(\mathrm{Gr}(r, r+k), \mathbb{Z}),$$

where $\lambda^o := (k \geq \cdots \geq k)$ *(r copies of k). Then,* $\dim\big([V(n\lambda) \otimes V(n\mu) \otimes V(n\nu)]^{\mathrm{SL}(r)}\big) = 1$, *for every positive integer n.*

1.2 Generalization for arbitrary groups

Our generalization of Fulton's conjecture to an arbitrary reductive group is by considering its equivalent formulation in Theorem 1.3. Moreover, the generalization replaces the intersection theory of the Grassmannians by the deformed product $\odot_0$ in the cohomology of G/P introduced in [3]. The role of the representation theory of $\mathrm{SL}(r)$ is replaced by the representation theory of the semisimple part L^{ss} of the Levi subgroup L of P.

To be more precise, let G be a connected reductive complex algebraic group with a Borel subgroup B and a maximal torus $H \subset B$. Let $P \supseteq B$ be a (standard) parabolic subgroup of G. Let $L \supset H$ be the Levi subgroup of P, $B_L := B \cap L$ the Borel subgroup of L and $L^{ss} = [L, L]$ the semisimple part of L. Let W be the Weyl group of G, W_P the Weyl group of P, and let W^P be the set of minimal length coset representatives in W/W_P. For any $w \in W^P$, let X_w be the corresponding Schubert variety and $[X_w] \in H^{2(\dim G/P - \ell(w))}(G/P, \mathbb{Z})$ the corresponding Poincaré dual class (cf. Sect. 2). Also, recall the definition of the deformed product $\odot_0$ in the singular cohomology $H^*(G/P, \mathbb{Z})$ from [3, Definition 18]. The following is our main theorem (cf. Theorem 8.2).

Theorem 1.4 *Let G be any connected reductive group and let P be any standard parabolic subgroup. Then, for any* $w_1, \ldots, w_s \in W^P$ *such that*

$$[X_{w_1}] \odot_0 \cdots \odot_0 [X_{w_s}] = [X_e] \in H^*(G/P), \tag{1}$$

we have, for every positive integer n,

$$\dim\Big(\big[V_L(n\chi_{w_1}) \otimes \cdots \otimes V_L(n\chi_{w_s})\big]^{L^{ss}}\Big) = 1, \tag{2}$$

where $V_L(\lambda)$ *is the irreducible representation of L with highest weight* λ *and* $\chi_w := \rho - 2\rho^L + w^{-1}\rho$ *(*ρ *and* ρ^L *being the half sum of positive roots of G and L respectively).*

Remark 1.5 Let $\mathcal{M}$ be the GIT quotient of $(L/B_L)^s$ by the diagonal action of L^{ss} linearized by $\mathcal{L}(\chi_{w_1}) \boxtimes \cdots \boxtimes \mathcal{L}(\chi_{w_s})$. Then, the conclusion of Theorem 1.4 is equivalent to the rigidity statement that $\mathcal{M} =$ point. Theorem 1.4 can therefore be interpreted as the statement "multiplicity one in intersection theory leads to rigidity in representation theory".

Our proof builds upon and further develops the connection between the deformed product $\odot_0$ and the representation theory of the Levi subgroup as established in [3]. In loc. cit., for any $w \in W^P$, the line bundle $\mathcal{L}_P(\chi_w)$ on P/B_L was constructed (see Sect. 6 for the definitions). Further, the following result was proved in there (cf. [3, Corollary 8 and Theorem 15]).

Proposition 1.6 *Let $w_1, \ldots, w_s \in W^P$ be such that*

$$[X_{w_1}] \odot_0 \cdots \odot_0 [X_{w_s}] = d[X_e] \in H^*(G/P), \textit{ for some } d \neq 0.$$

Then, $m := \dim\Big(H^0\big((L/B_L)^s, \big(\mathcal{L}_P(\chi_{w_1}) \boxtimes \cdots \boxtimes \mathcal{L}_P(\chi_{w_s})\big)_{|(L/B_L)^s}\big)^{L^{ss}}\Big) \neq 0.$

Note that, by the Borel–Weil theorem, for any $w \in W^P$, $H^0(L/B_L, \mathcal{L}_P(\chi_w)_{|(L/B_L)}) = V_L(\chi_w)^*$.

The condition (1) can be translated into the statement that a certain map of parameter spaces $X \to Y = (G/B)^s$ appearing in Kleiman's transversality theorem is birational ([10, Chap. III, §10], [12]). Here X is the "universal intersection space" of Schubert varieties. It is well known that, for any birational map $X \to Y$ between smooth projective varieties, no multiples of the ramification divisor R in X can move even infinitesimally (i.e., the corresponding Hilbert scheme is reduced, and of dimension 0 at nR for every positive integer n). We may therefore conclude that $h^0(X, \mathcal{O}(nR)) = 1$ for every positive integer n. In our situation, X is not smooth, and moreover $H^0(X, \mathcal{O}(nR))$ needs to be connected to invariant theory. We overcome these difficulties by taking a closer look at the codimension one Schubert cells inside the Schubert varieties.

The proof also brings into focus the largest (standard) parabolic subgroup Q_w acting on a Schubert variety $X_w \subseteq G/P$ (where $w \in W^P$), the open Q_w orbit $Y_w \subseteq X_w$ and the smooth locus $Z_w \subseteq X_w$. The difference $X_w \setminus Z_w$ is of codimension at least two in X_w (since X_w is normal) and can effectively be ignored.

The varieties Y_w give us the link to invariant theory (see Proposition 6.2). The difference $Z_w \backslash Y_w$ turns out to be quite subtle. A key result in the paper is that, in the setting of Proposition 6.2, the intersection $\cap_i g_i Z_{w_i}$ of translates is *non-transverse* 'essentially' at any point which lies in $\big(\cap_{i \neq j} g_i Z_{w_i}\big) \cap g_j(Z_{w_j} \setminus Y_{w_j})$ for some j (cf. Proposition 8.1 for a precise statement). This reveals the significance of Q_w in the intersection theory of G/P and, in particular, to the deformed product $\odot_0$. The "complexity" of the varieties $Z_w \backslash Y_w$ can therefore be expected to relate to the deformed product $\odot_0$. Note that by a result of Brion–Polo [5], if P is a cominuscule maximal parabolic subgroup (in particular, for the Grassmannians $\mathrm{Gr}(r, r+k)$), then $Y_w = Z_w$, and in this case the deformed cohomology product $\odot_0$ coincides with the standard intersection product as well (cf. [3, Lemma 19]).

In the case of $G = \mathrm{GL}(r+k)$ and $G/P = \mathrm{Gr}(r, r+k)$, the set W^P can be identified with $\mathfrak{P}_k(r)$. For any $\lambda \in W^P$, the corresponding irreducible representation of the Levi subgroup $L = \mathrm{GL}(r) \times \mathrm{GL}(k)$ with the highest weight χ_λ coincides with $V(\lambda)^* \boxtimes V(\tilde{\lambda})$ (cf. [1] for the identification of $\mathcal{L}(\chi_\lambda)$ as a determinant bundle), where $V(\lambda)$ is the irreducible representation of $\mathrm{GL}(r)$ as defined in Sect. 1.1 and $\tilde{\lambda}$ is the conjugate partition giving rise to the irreducible representation $V(\tilde{\lambda})$ of $\mathrm{GL}(k)$. Thus, if we specialize Theorem 1.4 to $G = \mathrm{GL}(r+k)$, we get Theorem 1.3.

Observe that in the case $G = \mathrm{GL}(r+k)$ and $G/P = \mathrm{Gr}(r, r+k)$, under the assumption of Proposition 1.6, from the above discussion and the discussion in Sect. 1.1, we get the stronger relation $m = d^2$. In general, however, there are no known numerical relations between m and d (cf. Example 8.3).

We remark that if we replace the condition (1) in Theorem 1.4 by the standard cohomology product, then the conclusion of the theorem is false in general (see Example 8.3(4)). Also, the converse to Theorem 1.4 is not true in general (cf. Example 8.3(1)).

2 Notation

Let G be a connected reductive complex algebraic group. We choose a Borel subgroup B and a maximal torus $H \subset B$ and let $W = W_G := N_G(H)/H$ be the associated Weyl group, where $N_G(H)$ is the normalizer of H in G. Let $P \supseteq B$ be a (standard) parabolic subgroup of G and let $U = U_P$ be its unipotent radical. Consider the Levi subgroup $L = L_P$ of P containing H, so that P is the semi-direct product of U and L. Then, $B_L := B \cap L$ is a Borel subgroup of L. Let $X(H)$ denote the character group of H, i.e., the group of all the algebraic group morphisms $H \to \mathbb{G}_m$. Then, B_L being the semidirect product of its commutator $[B_L, B_L]$ and H, any $\lambda \in X(H)$ extends uniquely to a character of B_L. We denote the Lie algebras of G, B, H, P, U, L, B_L by the corresponding Gothic characters: $\mathfrak{g}, \mathfrak{b}, \mathfrak{h}, \mathfrak{p}, \mathfrak{u}, \mathfrak{l}, \mathfrak{b}_L$ respectively. Let $R = R_{\mathfrak{g}}$ be the set of roots of $\mathfrak{g}$ with respect to the Cartan subalgebra $\mathfrak{h}$ and let R^+ be the set of positive roots (i.e., the set of roots of $\mathfrak{b}$). Similarly, let $R_{\mathfrak{l}}$ be the set of roots of $\mathfrak{l}$ with respect to $\mathfrak{h}$ and $R_{\mathfrak{l}}^+$ be the set of roots of $\mathfrak{b}_L$. Let $\Delta = \{\alpha_1, \ldots, \alpha_\ell\} \subset R^+$ be the set of simple roots, where ℓ is the semisimple rank of G (i.e., the dimension of $\mathfrak{h}' := \mathfrak{h} \cap [\mathfrak{g}, \mathfrak{g}]$). We denote by $\Delta(P)$ the set of simple roots contained in $R_{\mathfrak{l}}$. For any $1 \leq j \leq \ell$, define the element $x_j \in \mathfrak{h}'$ by

$$\alpha_i(x_j) = \delta_{i,j}, \ \forall\, 1 \leq i \leq \ell. \tag{3}$$

Recall that if W_P is the Weyl group of P (which is, by definition, the Weyl group W_L of L; thus $W_P := W_L$), then in each coset of W/W_P we have a unique member w of minimal length. This satisfies (cf. [14, Exercise 1.3.E]):

$$wB_Lw^{-1} \subseteq B. \tag{4}$$

Let W^P be the set of minimal length representatives in the cosets of W/W_P.

For any $w \in W^P$, define the Schubert cell:

$$C_w = C_w^P := BwP/P \subset G/P.$$

Then, it is a locally closed subvariety of G/P isomorphic to the affine space $\mathbb{A}^{\ell(w)}$, $\ell(w)$ being the length of w (cf. [11, Part II, Chapter 13]). Its closure is denoted by $X_w = X_w^P$, which is an irreducible (projective) subvariety of G/P of dimension $\ell(w)$. We denote the point $wP \in C_w$ by $\dot{w}$.

We also need the shifted Schubert cell:

$$\Lambda_w = \Lambda_w^P := w^{-1}BwP/P \subset G/P.$$

Let $\mu(X_w)$ denote the fundamental class of X_w considered as an element of the singular homology with integral coefficients $H_{2\ell(w)}(G/P, \mathbb{Z})$ of G/P [7, Chap. 19].

Then, from the Bruhat decomposition, the elements $\{\mu(X_w)\}_{w\in W^P}$ form a $\mathbb{Z}$-basis of $H_*(G/P,\mathbb{Z})$. Let $\{[X_w]\}_{w\in W^P}$ be the Poincaré dual basis of the singular cohomology with integral coefficients $H^*(G/P,\mathbb{Z})$. Thus, $[X_w]\in H^{2(\dim G/P-\ell(w))}(G/P,\mathbb{Z})$.

The tangent space $T^P = T_{\dot{e}}(G/P)$ of G/P at $\dot{e}\in G/P$ carries a canonical action of P induced from the left multiplication of P on G/P.

We recall the following definition from [3, Definition 4].

Definition 2.1 Fix a positive integer $s\geq 1$. Let $w_1,\ldots,w_s\in W^P$ be such that

$$\sum_{j=1}^{s}\operatorname{codim}\Lambda_{w_j}=\dim G/P. \tag{5}$$

This of course is equivalent to the condition:

$$\sum_{j=1}^{s}\ell(w_j)=(s-1)\dim G/P. \tag{6}$$

We then call the s-tuple $(w_1,\ldots,w_s)$ *Levi-movable* (for short *L-movable*) if, for generic $(l_1,\ldots,l_s)\in L^s$, the intersection $l_1\Lambda_{w_1}\cap\cdots\cap l_s\Lambda_{w_s}$ is transverse at $\dot{e}$.

All schemes considered are over the base field of complex numbers $\mathbb{C}$. By a variety, we mean a reduced (but not necessarily irreducible) scheme of finite type over $\mathbb{C}$.

3 A geometric result

Let $\pi : X\to Y$ be a birational morphism of smooth irreducible varieties (defined on the whole of X) with Y projective. Assume that we have a (not necessarily smooth) irreducible projective scheme $\bar{X}$ containing X as an open subscheme such that

(1) the codimension of each irreducible component of $\bar{X}\setminus X$ in $\bar{X}$ is at least two,
(2) π extends to a regular map $\bar{\pi} : \bar{X}\to Y$.

Let R be the *ramification divisor* of π in X (cf. [6, Section 1.10]). It is, by definition, the effective Cartier divisor obtained as the zero scheme of the section of the line bundle $\mathfrak{L}$ induced by the derivative map $D\pi_x : T_x(X)\to T_{\pi(x)}(Y)$, where the line bundle $\mathfrak{L}$ has base X and fiber $\mathfrak{L}_x$ at any $x\in X$ is given by:

$$\mathfrak{L}_x=\wedge^{\text{top}}(T_x(X)^*)\otimes\wedge^{\text{top}}(T_{\pi(x)}(Y)).$$

In the above set up, one has the following result.

Proposition 3.1 *For every $n\geq 1$, $h^0(X,\mathcal{O}(nR))=1$, where h^0 denotes the dimension of H^0.*

Proof Clearly $\pi_{|X\setminus R} : X\setminus R\to Y$ is an étale (and hence quasi-finite) birational morphism between smooth varieties. Hence, by the original form of Zariski's main theorem [16, Chap. III, §9], it is an open immersion, i.e., $\pi(X\backslash R)$ is open in Y and $\pi : X\backslash R \to \pi(X\backslash R)$ is an isomorphism. We will show that $V := Y \setminus \pi(X\backslash R)$ is of codimension at least two in Y. This will then imply that

$$H^0(X, \mathcal{O}(nR)) \subseteq H^0(X\backslash R, \mathcal{O}) = H^0(Y \setminus V, \mathcal{O}) = H^0(Y, \mathcal{O}) = \mathbb{C}.$$

Since $\bar{\pi}$ is surjective, a point $v \in V$ is either in $\bar{\pi}(\bar{X} \setminus X)$, or in $\pi(R)$, i.e., $V \subseteq \bar{\pi}(\bar{X} \setminus X) \cup \pi(R)$. We show that $\overline{\pi(R)}$ is of codimension at least two in Y and thus conclude the proof (by assumption (1)).

To do this let Z be the smallest closed subset of Y so that there exists a morphism $\sigma : Y\backslash Z \to \bar{X}$ representing the birational inverse to $\bar{\pi}$. It is known that the codimension of Z in Y is at least two (follow [10, Proof of Theorem 8.19 on page 181]). Clearly, $\bar{\pi} \circ \sigma = I$ on $Y\backslash Z$ and similarly $\sigma \circ \bar{\pi}$ is identity on $\bar{\pi}^{-1}(Y\backslash Z)$ (for the last, note that $\sigma \circ \bar{\pi}$ is well defined as a morphism $\bar{\pi}^{-1}(Y\backslash Z) \to \bar{X}$ which on an open subset is the identity). We therefore find that $\bar{\pi} : \bar{\pi}^{-1}(Y\backslash Z) \to Y\backslash Z$ is an isomorphism.

This tells us that $\bar{\pi}^{-1}(Y\backslash Z)$ is smooth and $\bar{\pi}^{-1}(Y\backslash Z) \cap R = \emptyset$. Hence, R is a subset of $\pi^{-1}(Z)$, or that $\pi(R) \subseteq Z$. ◻

4 Some remarks on ramification divisors

Consider a linear map $p : V \to W$ between vector spaces of the same dimension. Let

$$\mathrm{Det}(p) := (\wedge^{\mathrm{top}} V)^* \otimes \wedge^{\mathrm{top}} W = \mathrm{Hom}(\wedge^{\mathrm{top}} V, \wedge^{\mathrm{top}} W).$$

Denote by $\theta(p)$ the canonical element of $\mathrm{Det}(p)$ induced by p, i.e., $\theta(p)$ is the top exterior power of p. The following lemma is immediate.

Lemma 4.1 *Let $p : V \to W$ be as above and $\alpha : W' \to W$ a surjective map. Let $V' \subset V \oplus W'$ consist of (v, w') such that $p(v) = \alpha(w')$ (i.e., V' is the fiber product of p and α). Let $p' : V' \to W'$ be the projection. Then, the kernel of p' is identified with the kernel of p via the surjective projection $\pi : V' \to V$. Further, there is a canonical isomorphism of the vector spaces* $\mathrm{Det}(p)$ *and* $\mathrm{Det}(p')$ (*defined below*)*, which carries $\theta(p)$ to $\theta(p')$.* (*Observe that V' and W' have the same dimension.*)

Hence, for any fiber diagram of irreducible smooth varieties:

$$\begin{array}{ccc} X' & \xrightarrow{\hat{f}} & X \\ \Big\downarrow{\scriptstyle \pi'} & & \Big\downarrow{\scriptstyle \pi} \\ Y' & \xrightarrow{f} & Y, \end{array}$$

where f is a smooth morphism and X, Y are of the same dimension with π a dominant morphism, we have the following identity between the ramification divisors:

$$\hat{f}^*(R(\pi)) = R(\pi'). \tag{7}$$

The isomorphism $\xi : \mathrm{Det}(p) \to \mathrm{Det}(p')$ is given by:

$$\begin{aligned}&\xi(\theta)(e_1 \wedge \cdots \wedge e_d \wedge e_{d+1} \wedge \cdots \wedge e_n)\\ &\quad = p'(e_1) \wedge \cdots \wedge p'(e_d) \wedge \bar{\theta}(\pi(e_{d+1}) \wedge \cdots \wedge \pi(e_n)),\end{aligned}$$

for any $\theta \in \mathrm{Det}(p) = \mathrm{Hom}(\wedge^{\mathrm{top}} V, \wedge^{\mathrm{top}} W)$, where $\{e_1, \dots, e_n\}$ is any basis of V' such that $\{e_1, \dots, e_d\}$ is a basis of $\mathrm{Ker}(\pi)$ and $\bar{\theta} := \sigma \circ \theta$ (σ being any section of the map $\wedge^{n-d}(W') \to \wedge^{n-d}(W)$ induced from α). It is easy to see that ξ does not depend upon the choice of the basis and the section σ.

Let X be a variety endowed with an action of G. Let Y be a locally closed subvariety of X which is assumed to be stable by a closed subgroup H of G. Consider the fiber bundle $\mathfrak{Y} := G \times_H Y \to G/H$ with fiber Y, associated to the principal H-bundle $G \to G/H$. We can identify $\mathfrak{Y}$ (under the map $[g, y] \mapsto (gH, gy)$) with the set of pairs $(gH, x) \in G/H \times X$ such that $g^{-1}x \in Y$. In particular, $\mathfrak{Y}$ is a locally closed subset of $G/H \times X$.

We now assume that X is irreducible and smooth, and that $Y_1, \dots, Y_s$ are irreducible smooth locally closed subvarieties of X. Assume that X has a transitive action by a connected linear algebraic group G and let G_i be algebraic subgroups which keep Y_i stable. Assume further that $\sum_{i=1}^{s} \mathrm{codim}\, Y_i = \dim X$. Let $\mathfrak{Y}_i = G \times_{G_i} Y_i$ be the total space of the fiber bundle with fiber Y_i associated to the principal G_i-bundle $G \to G/G_i$. Consider the morphism $m_i : \mathfrak{Y}_i \to X$, $[g, y_i] \mapsto gy_i$. Since Y_i is smooth and G acts transitively on X, by the G-equivariance, m_i is a smooth morphism (cf. [10, Chap. III, Corollary 10.7]). Taking their Cartesian product, we get the smooth morphism $m : \mathfrak{Y}_1 \times \cdots \times \mathfrak{Y}_s \to X^s$. Let $\mathcal{Y}$ be the fiber product of m with the diagonal map $\delta : X \to X^s$. We get a smooth morphism $\hat{m} : \mathcal{Y} \to X$ by restricting m to $\mathcal{Y}$. In particular, $\mathcal{Y}$ is a smooth variety. We also have the morphism $\pi : \mathcal{Y} \to G/G_1 \times \cdots \times G/G_s$ obtained coordinatewise from the canonical projections $\pi_i : \mathfrak{Y}_i \to G/G_i$. For any $g_i \in G$ and $y_i \in Y_i$, the map $e_{y_i} : G \to X,\ g \mapsto gy_i$, induces the tangent map $\Psi_{(g_i, y_i)} : T_{g_i}(G) \to T_{g_i y_i}(X)$. Since Y_i is G_i-stable, this map induces the map $\bar{\Psi}_{(g_i, y_i)} : T_{\bar{g}_i}(G/G_i) \to T_{g_i y_i}(X)/T_{g_i y_i}(g_i Y_i)$, where $\bar{g}_i = g_i G_i$. Moreover, for any $h_i \in G_i$,

$$\bar{\Psi}_{(g_i, y_i)} = \bar{\Psi}_{(g_i h_i, h_i^{-1} y_i)}. \tag{8}$$

To see this, observe that the following diagram is commutative for any $g_i \in G$ and $h_i \in G_i$.

$$\begin{array}{ccc} T_{g_i}(G) & \xrightarrow{\quad DR_{h_i} \quad} & T_{g_i h_i}(G) \\ & \searrow \quad \swarrow & \\ & T_{\bar{g}_i}(G/G_i), & \end{array}$$

where $R_{h_i} : G \to G$ is the right multiplication by h_i. Thus, $\bar{\Psi}_{(g_i,y_i)}$ depends only upon the equivalence class $[g_i, y_i] \in G \times_{G_i} Y_i$ and we denote $\bar{\Psi}_{(g_i,y_i)}$ by $\Psi_{[g_i,y_i]}$. Since G acts transitively on X, $\Psi_{[g_i,y_i]}$ is surjective.

For any $\mathfrak{a} = ([g_1, y_1], \ldots, [g_s, y_s]) \in \mathcal{Y}$, we have the following diagram (for $x = \hat{m}(\mathfrak{a})$):

$$\begin{array}{ccc} T_{\mathfrak{a}}\mathcal{Y} & \xrightarrow{\pi_{\mathfrak{a}}} & T_{\bar{g}_1}(G/G_1) \oplus \cdots \oplus T_{\bar{g}_s}(G/G_s) \\ \downarrow D\hat{m} & & \downarrow \\ T_x X & \longrightarrow & \bigoplus_{i=1}^s \frac{T_x X}{T_x(g_i Y_i)}, \end{array} \tag{9}$$

where $\bar{g}_i := g_i G_i$, the bottom horizontal map is the canonical projection in each factor, $D\hat{m}$ is surjective since $\hat{m}$ is a smooth morphism and the right vertical map is the coordinatewise surjective map $\Psi_{[g_i,y_i]}$.

Lemma 4.2 *The above diagram is Cartesian.*

Proof By taking a path in $\mathcal{Y}$ representing a tangent vector and differentiating the relations, it is easy to see that the above diagram is commutative. Moreover, since $y_i = g_i^{-1}x$ for any $\mathfrak{a} = ([g_1, y_1], \ldots, [g_s, y_s]) \in \mathcal{Y}$ with $\hat{m}(\mathfrak{a}) = x$, $T_{\mathfrak{a}}(\mathcal{Y})$ is a subspace of the fiber product F of $T_x(X)$ and $T_{\bar{g}_1}(G/G_1) \oplus \cdots \oplus T_{\bar{g}_s}(G/G_s)$. Further,

$$\dim \mathcal{Y} = \dim X + \sum_{i=1}^{s} (\dim \mathfrak{Y}_i - \dim X) \tag{10}$$

$$= \dim X + \sum_{i=1}^{s} (\dim G/G_i + \dim Y_i - \dim X) \tag{11}$$

$$= \dim X + \sum_{i=1}^{s} (\dim G/G_i - \operatorname{codim} Y_i). \tag{12}$$

From this we see that $\dim F = \dim T_{\mathfrak{a}}(\mathcal{Y})$. This proves the lemma. □

5 Intersection of general translates of Schubert varieties

We follow the notation from Sect. 2. For $w \in W^P$, let Q_w be the stabilizer of the Schubert variety X_w inside G/P under the left multiplication of G on G/P. Then, clearly, Q_w is a standard parabolic subgroup of G. Let

$$Y_w := Q_w \dot{w} \subset X_w,$$

and let Z_w denote the smooth locus of X_w. Clearly

$$X_w \supset Z_w \supset Y_w \supset C_w,$$

and each of Z_w, Y_w, C_w is an open subset of X_w.

Remark 5.1 It is instructive to look at the example of $G/P = \mathrm{Gr}(r, n)$ with $G = SL(n)$. Let

$$F_\bullet : 0 \subsetneq F_1 \subsetneq F_2 \subsetneq \cdots \subsetneq F_n = \mathbb{C}^n$$

be the standard flag in $\mathbb{C}^n$, and let $I = \{i_1 < \cdots < i_r\}$ be a subset of $\{1, \ldots, n\}$ of cardinality r. Consider the (closed) Schubert variety $\Omega_I(F_\bullet) = \{V \in \mathrm{Gr}(r, n) \mid \dim V \cap F_{i_k} \geq k, k = 1, \ldots, r\}$. Let $J = \{i \in I : i + 1 \notin I\}$. It is easy to see that $\Omega_I(F_\bullet) = \{V \in \mathrm{Gr}(r, n) \mid \dim V \cap F_{i_k} \geq k, \forall i_k \in J\}$. So $I \setminus J$ is "redundant" for the definition of the closed Schubert variety $\Omega_I(F_\bullet)$.

It is easy to see that the stabilizer of the Schubert variety $\Omega_I(F_\bullet)$ is $Q_I := \{g \in \mathrm{SL}(n) : gF_j \subset F_j, \forall j \in J\}$. We may think of Q_I as the set of elements of $\mathrm{SL}(n)$ that preserve the parts of $F_\bullet$ "essential" for the definition of the closed Schubert variety $\Omega_I(F_\bullet)$.

It may be remarked that if P is a minuscule or cominuscule maximal parabolic, then $Z_w = Y_w$ (cf. [5]). Thus, in this case, Z_w is determined by (subsequent) Lemma 7.1.

We return to the general case. Fix a positive integer $s \geq 1$ and fix $w_1, \ldots, w_s \in W^P$, so that

$$[X_{w_1}] \cdot \ldots \cdot [X_{w_s}] = d[X_e] \in H^*(G/P), \text{ for some } d > 0. \tag{13}$$

There are four universal intersections that will be relevant here. Let $\delta : G/P \to (G/P)^s$ be the diagonal embedding. We denote its image by $\delta(G/P)$ and identify it with G/P. We consider a special case of the construction in Sect. 4 where we take $X = G/P$, $G_i = B$ and Y_i to be B-stable subvarieties of G/P. For a locally closed B-subvariety $A \subset G/P$, let $\mathfrak{A} := G \times_B A$ be the total space of the fiber bundle with fiber A associated to the principal B-bundle $G \to G/B$. Then, there is a G-equivariant morphism $m_A : \mathfrak{A} \to G/P$ defined by $[g, x] \mapsto gx$, which is a smooth morphism if A is smooth. Now, consider the product

$$\mathfrak{X} := \mathfrak{X}_{w_1} \times \cdots \times \mathfrak{X}_{w_s},$$

where $\mathfrak{X}_{w_i} = G \times_B X_{w_i}$, and similarly define $\mathfrak{Y}, \mathfrak{Z}, \mathfrak{C}$ by replacing X_{w_i} with $Y_{w_i}, Z_{w_i}, C_{w_i}$ respectively. Let $m_X : \mathfrak{X} \to (G/P)^s$ be the G-equivariant morphism $m_{X_{w_1}} \times \cdots \times m_{X_{w_s}}$ acting componentwise. We similarly define m_Y, m_Z, m_C.

Finally, we define the (universal intersection) G-scheme $\mathcal{X}$ as the fiber product of δ with m_X. We similarly define the G-schemes $\mathcal{Y}, \mathcal{Z}, \mathcal{C}$ by replacing m_X with m_Y, m_Z, m_C respectively. Since δ is a closed embedding, $\mathcal{X}$, $\mathcal{Y}$, $\mathcal{Z}, \mathcal{C}$ are the scheme theoretic inverse images of $\delta(G/P)$ under m_X, m_Y, m_Z, m_C respectively. Moreover, since m_Y, m_Z, m_C are smooth morphisms, $\mathcal{Y}$, $\mathcal{Z}, \mathcal{C}$ are reduced closed subschemes of $\mathfrak{Y}$, $\mathfrak{Z}$, $\mathfrak{C}$ respectively.

It is easy to see that (due to the assumption (13))

$$\dim \mathcal{X} = s \times \dim(G/B). \tag{14}$$

Observe that, set theoretically,

$$\mathcal{X} = \{(g_1B, \ldots, g_sB, x) \in (G/B)^s \times G/P : x \in \cap_{i=1}^s g_i X_{w_i}\}.$$

There is a similar description for $\mathfrak{Y}$, $\mathfrak{Z}$, $\mathfrak{C}$.

The open embeddings

$$C_{w_i} \subset Y_{w_i} \subset Z_{w_i} \subset X_{w_i}$$

give rise to G-equivariant open embeddings:

$$\mathcal{C} \subset \mathcal{Y} \subset \mathcal{Z} \subset \mathcal{X},$$

and $\mathcal{X}$ is projective.

Lemma 5.2 (1) *$\mathcal{X}$ is irreducible, and so are $\mathcal{Y}$, $\mathcal{Z}$ and $\mathcal{C}$.*
(2) *$\mathcal{Z}$ is a smooth variety (and hence so are $\mathcal{Y}$ and $\mathcal{C}$).*
(3) *The complement of $\mathcal{Z}$ in $\mathcal{X}$ is of codimension ≥ 2.*

Proof (1) It is easy to see that, for any $w \in W^P$, each fiber of $m_{X_w} : \mathfrak{X}_w \to G/P$ is irreducible. Thus, each fiber of $m_X : \mathfrak{X} \to (G/P)^s$ is also irreducible. Now, take an irreducible component $\mathcal{X}_1$ of $\mathcal{X}$ such that $\mathcal{X}_1$ contains the full fiber of m_X over the base point in $\delta(G/P)$. Since $\mathcal{X}_1$ is G-stable, $\mathcal{X}_1$ must contain the full fiber over any point in $\delta(G/P)$. Thus, $\mathcal{X}_1 = \mathcal{X}$, proving that $\mathcal{X}$ is irreducible. Since $\mathcal{Y}$, $\mathcal{Z}$ and $\mathcal{C}$ are open subsets of $\mathcal{X}$, they must be irreducible too.
(2) For the second part, observe that the canonical map $\mathcal{Z} \to \delta(G/P)$ is a smooth morphism. Since G/P is smooth, we get the smoothness of $\mathcal{Z}$.
(3) Since the Schubert varieties X_w are normal, the complement of Z_w in X_w is of codimension ≥ 2. Thus, the complement of $\mathfrak{Z}$ in $\mathfrak{X}$ is of codimension ≥ 2. From this it is easy to see that the complement of $\mathcal{Z}$ in $\mathcal{X}$ is of codimension ≥ 2. □

We have a natural G-equivariant projection $\pi : \mathcal{X} \to (G/B)^s$ obtained coordinatewise from the projections $\mathfrak{X}_{w_i} \to G/B$. As observed in the identity (14), the domain and the range of π have the same dimension. The following lemma follows from Lemma 4.2.

Lemma 5.3 *For any point $\mathfrak{a} = ([g_1, x_1], \ldots, [g_s, x_s]) \in \mathcal{Z}$, the derivative $(D\pi)_{\mathfrak{a}}$ of π at $\mathfrak{a}$ has*

$$\mathrm{Ker}(D\pi)_{\mathfrak{a}} \simeq \cap_{i=1}^s T_x(g_i Z_{w_i}),$$

where $x = g_1x_1 = \cdots = g_sx_s$.

In particular, π is regular at $\mathfrak{a}$ if and only if the intersection $\cap_{i=1}^s g_i Z_{w_i}$ in G/P is transverse at x.

Using Kleiman's transversality theorem [3, Proposition 3] and our assumption (13), the map $\pi_{|\mathcal{Z}} : \mathcal{Z} \to (G/B)^s$ is generically finite.

Lemma 5.4 *With the notation as above, $\pi_{|\mathcal{Z}}$ is birational if and only if $d = 1$, where d is given by Eq.* (13).

Proof By [3, Proposition 3], there exists a nonempty open subset $U \subset (G/B)^s$ such that for each $x = (g_1 B, \ldots, g_s B) \in U$, the intersection $\cap_{i=1}^s g_i Z_{w_i}$ is transverse at each point of the intersection and $\cap_{i=1}^s g_i Z_{w_i}$ is dense in $\cap_{i=1}^s g_i X_{w_i}$. Assume now that $d = 1$. Then, the intersection $\cap_{i=1}^s g_i Z_{w_i}$ consists of a single point for any $x \in U$. From this we see that $(\pi_{|\mathcal{Z}})^{-1}(x)$ consists of exactly one point for each $x \in U$ and, moreover, by Lemma 5.3, $(\pi_{|\mathcal{Z}})^{-1}(x) \subset \mathcal{Z} \setminus R$, where R is the ramification divisor of $\pi_{|\mathcal{Z}}$. Thus, $\pi_{|(\pi_{|\mathcal{Z}})^{-1}(U)} : (\pi_{|\mathcal{Z}})^{-1}(U) \to U$ is an isomorphism, proving that $\pi_{|\mathcal{Z}}$ is birational.

Conversely, assume that $\pi_{|\mathcal{Z}}$ is birational. Then, we can find a nonempty open subset $V \subset (G/B)^s$ such that $(\pi_{|\mathcal{Z}})^{-1}(V) \to V$ is an isomorphism and $(\pi_{|\mathcal{Z}})^{-1}(V) \subset \mathcal{Z} \backslash R$. By taking a point $x = (g_1 B, \ldots, g_s B) \in U \cap V$, we see that the intersection $\cap_{i=1}^s g_i Z_{w_i}$ is a single point and the intersection is transverse at that point. Moreover, $\cap_{i=1}^s g_i Z_{w_i}$ is dense in $\cap_{i=1}^s g_i X_{w_i}$. Thus, $d = 1$, proving the lemma. □

As above, let R be the ramification divisor for the map $\pi_{|\mathcal{Z}}$ (equipped with the scheme structure described in Sect. 3).

Corollary 5.5 *Assume that $d = 1$ in Eq.* (13). *Then, for every $n \geq 1$,*

$$h^0(\mathcal{Z}, \mathcal{O}(nR)) = 1. \tag{15}$$

Proof By Lemmas 5.2 and 5.4, all the hypotheses of Proposition 3.1 are satisfied. Thus, applying Proposition 3.1, we get the corollary. □

The aim now is to have Eq. (15) bear representation theoretic consequences. However, it is the space $H^0(\mathcal{Y}, \mathcal{O}(nR))$ which has clear relations to invariant theory.

6 Connecting $h^0(\mathcal{Y}, \mathcal{O}(nR))$ to invariant theory

We first prove that $\mathcal{Y}$ and $R \cap \mathcal{Y}$ are obtained from a base change with connected fibers. To do this, define

$$\mathfrak{Y}' := (G \times_{Q_{w_1}} Y_{w_1}) \times \cdots \times (G \times_{Q_{w_s}} Y_{w_s}).$$

Similar to the map m_Y, we define the map $m'_Y : \mathfrak{Y}' \to (G/P)^s$ obtained from the coordinatewise maps $G \times_{Q_{w_i}} Y_{w_i} \to G/P$, $[g, x] \mapsto gx$. Again, m'_Y is a smooth morphism. Now, let $\mathcal{Y}'$ be the fiber product of m'_Y with δ. Then, $\mathcal{Y}'$ is an irreducible smooth variety of the same dimension as that of $(G/Q_{w_1}) \times \cdots \times (G/Q_{w_s})$ (by virtue of the same proof given in the last section for the corresponding results for $\mathcal{Y}$). Similar to the map $\pi_{|\mathcal{Y}} : \mathcal{Y} \to (G/B)^s$, we have the map

$$\pi' : \mathcal{Y}' \to (G/Q_{w_1}) \times \cdots \times (G/Q_{w_s}).$$

It is easy to see that the following diagram is Cartesian:

$$\begin{array}{ccc} \mathcal{Y} & \xrightarrow{\theta} & \mathcal{Y}' \\ \Big\downarrow{\scriptstyle \pi} & & \Big\downarrow{\scriptstyle \pi'} \\ (G/B)^s & \longrightarrow & (G/Q_{w_1}) \times \cdots \times (G/Q_{w_s}), \end{array}$$

where the two horizontal maps are the canonical projections. (To prove this, observe that the above diagram is clearly Cartesian when $\mathcal{Y}$, $\mathcal{Y}'$ are replaced by $\mathfrak{Y}$, $\mathfrak{Y}'$ respectively.)

Since π is a dominant morphism, so is π'. Thus, by Lemma 4.1, the ramification divisor $S := R \cap \mathcal{Y}$ of $\pi|_{\mathcal{Y}}$ is the pull-back of the ramification divisor R' of π'. Therefore, the line bundle

$$\mathcal{O}(nR)|_{\mathcal{Y}} = \mathcal{O}(nS).$$

From the above Cartesian diagram, $\theta_*(\mathcal{O}_{\mathcal{Y}}) = \mathcal{O}_{\mathcal{Y}'}$, where θ is the top horizontal map. We therefore conclude that the G-equivariant pull-back map induces the following isomorphism.

Lemma 6.1 *For any $n \in \mathbb{Z}$, $H^0(\mathcal{Y}, \mathcal{O}(nR)|_{\mathcal{Y}}) \simeq H^0(\mathcal{Y}', \mathcal{O}(nR'))$, as G-modules.*

Define the P-variety (under the diagonal action of P):

$$\mathcal{P} = (P/(w_1^{-1} Q_{w_1} w_1 \cap P)) \times \cdots \times (P/(w_s^{-1} Q_{w_s} w_s \cap P)),$$

and define the G-equivariant morphism of G-varieties:

$$\phi : G \times_P \mathcal{P} \to \mathcal{Y}', \ \ [g, (\bar{p}_1, \ldots, \bar{p}_s)] \mapsto ([g p_1 w_1^{-1}, \dot{w}_1], \ldots, [g p_s w_s^{-1}, \dot{w}_s]),$$

where $\bar{p}_i = p_i(w_i^{-1} Q_{w_i} w_i \cap P)$.

It is easy to see that it is bijective. Since $\mathcal{Y}'$ is smooth and irreducible, ϕ is an isomorphism by a variant of Zariski's main theorem (cf., e.g., [14, Theorem A.11]).

For any $w \in W^P$, it is easy to see that the Borel B_L of the Levi subgroup L of P is contained in $w^{-1} Q_w w \cap L$ (in fact, it is contained in $w^{-1} B w$ by Eq. (4)).

For any $\lambda \in X(H)$, we have a P-equivariant line bundle $\mathcal{L}_P(\lambda)$ on P/B_L associated to the principal B_L-bundle $P \to P/B_L$ via the one dimensional B_L-module λ^{-1}. (As observed in Sect. 2, any $\lambda \in X(H)$ extends uniquely to a character of B_L.) The twist in the definition of $\mathcal{L}(\lambda)$ is introduced so that the dominant characters correspond to the dominant line bundles.

For $w \in W^P$, define the character $\chi_w \in \mathfrak{h}^*$ by

$$\chi_w = \sum_{\beta \in (R^+ \setminus R^+_{\mathfrak{l}}) \cap w^{-1} R^+} \beta.$$

Then, from [14, 1.3.22.3] and Eq. (4),

$$\chi_w = \rho - 2\rho^L + w^{-1}\rho, \tag{16}$$

where ρ (resp. ρ^L) is half the sum of roots in R^+ (resp. in $R^+_{\mathfrak{l}}$). It is easy to see that χ_w extends as a character of $w^{-1}Q_w w \cap P$.

Proposition 6.2 *Assume that the s-tuple $(w_1, \ldots, w_s)$ satisfying the condition (13) is Levi-movable. Then, for any $n \geq 1$,*

$$H^0(\mathcal{Y}, \mathcal{O}(nR)_{|\mathcal{Y}})^G \simeq \left[V_L(n(\chi_{w_1} - \chi_1))^* \otimes V_L(n\chi_{w_2})^* \otimes \cdots \otimes V_L(n\chi_{w_s})^*\right]^L,$$

where $V_L(\chi)$ is the irreducible L-module with highest weight χ. (Observe that for $w \in W^P$, χ_w is a L-dominant weight and so is $\chi_w - \chi_1$.)

Proof Applying Lemma 4.2 to the case when $X = G/P$, $Y_i = Y_{w_i}$, $G_i = Q_{w_i}$, and using the isomorphism $\phi : G \times_P \mathcal{P} \to \mathcal{Y}'$ as above, we get the following Cartesian diagram (for any $g \in G$ and $\mathbf{p} = (\bar{p}_1, \ldots, \bar{p}_s) \in \mathcal{P}$):

$$\begin{array}{ccc} T_{[g,\mathbf{p}]}(G \times_P \mathcal{P}) & \longrightarrow & \oplus_{i=1}^s T_{gp_i w_i^{-1} Q_{w_i}}(G/Q_{w_i}) \\ \downarrow & & \downarrow \pi' \\ T_{gP}(G/P) & \longrightarrow & \oplus_{i=1}^s \dfrac{T_{gP}(G/P)}{T_{gP}(gp_i w_i^{-1} Y_{w_i})}, \end{array}$$

where the top horizontal map is induced from the G-equivariant composite map $\pi' \circ \phi : G \times_P \mathcal{P} \to \prod_{i=1}^s (G/Q_{w_i})$ and the bottom horizontal map is the canonical projection in each factor. Thus, by Lemma 4.1, the ramification divisor $\phi^{-1}(R')$ is the same as the ramification divisor associated to the bundle map (between the vector bundles of the same rank over the base space $G \times_P \mathcal{P}$):

$$G \times_P (\mathcal{P} \times T^P) \to \bigoplus_{i=1}^s G \times_P \left(P \times_{(w_i^{-1} Q_{w_i} w_i \cap P)} (T^P/T^P_{w_i})\right),$$

which in turn is induced by a bundle map of P-equivariant vector bundles of equal rank on $\mathcal{P}$:

$$\mathcal{P} \times T^P \to \bigoplus_{i=1}^s P \times_{(w_i^{-1} Q_{w_i} w_i \cap P)} (T^P/T^P_{w_i}), \tag{17}$$

where T^P is the tangent space $T_{\dot{e}}(G/P)$, T^P_w is the tangent space $T_{\dot{e}}(\Lambda_w)$, P acts diagonally on $\mathcal{P} \times T^P$ and the map in the i-th factor is induced from the composite map

$$\mathcal{P} \times T^P \to (P/(w_i^{-1} Q_{w_i} w_i \cap P)) \times T^P$$
$$\simeq P \times_{(w_i^{-1} Q_{w_i} w_i \cap P)} T^P \to P \times_{(w_i^{-1} Q_{w_i} w_i \cap P)} (T^P / T^P_{w_i}).$$

Clearly, $\mathcal{L}_P(\chi_w)$, as a line bundle on P/B_L, equals the determinant of the vector bundle corresponding to the B_L-representation T^P/T^P_w. It is easy to see that the representation of B_L on T^P/T^P_w extends to a representation of $w^{-1}Q_w w \cap P \supseteq B_L$. In particular, the character χ_w extends to a character of $w^{-1}Q_w w \cap P$, and $\mathcal{L}_P(\chi_w)$ descends to a line bundle on $P/(w^{-1}Q_w w \cap P)$.

Taking the determinant of the bundle map (17) (for more details see the discussion in [3] following Lemma 6), and using Lemma 4.1, we see that the line bundle corresponding to the divisor $\phi^{-1}(R')$ is G-equivariantly isomorphic to the line bundle $G \times_P \mathcal{M}$ over the base space $G \times_P \mathcal{P}$, where

$$\mathcal{M} = \mathcal{L}_P(\chi_{w_1} - \chi_1) \boxtimes \mathcal{L}_P(\chi_{w_2}) \boxtimes \cdots \boxtimes \mathcal{L}_P(\chi_{w_s}).$$

Thus,

$$\begin{aligned} H^0(\mathcal{Y}, \mathcal{O}(nR)_{|\mathcal{Y}})^G &\simeq H^0(\mathcal{Y}', \mathcal{O}(nR'))^G, \text{ by Lemma 6.1} \\ &\simeq H^0(G \times_P \mathcal{P}, G \times_P \mathcal{M}^{\otimes n})^G \\ &\simeq H^0(\mathcal{P}, \mathcal{M}^{\otimes n})^P \\ &\simeq H^0(\mathfrak{L}, \mathcal{M}^{\otimes n}_{|\mathfrak{L}})^L, \end{aligned}$$

where

$$\mathfrak{L} := (L/(w_1^{-1} Q_{w_1} w_1 \cap L)) \times \cdots \times (L/(w_s^{-1} Q_{w_s} w_s \cap L))$$

and the last isomorphism follows from [3, Theorem 15 and Remark 31(a)].

Thus, the proposition follows from the Borel-Weil theorem. □

7 A study of codimension one cells in the Schubert varieties

We continue to follow the notation and assumptions from Sect. 2. The following lemma can be found in [5, §2.6]. However, we include its proof for completeness.

Lemma 7.1 *For any $w \in W^P$, the stabilizer Q_w of X_w satisfies*

$$\Delta(Q_w) = \Delta_w, \tag{18}$$

where $\Delta_w := \Delta \cap w(R^+_{\mathfrak{l}} \sqcup R^-)$ and R^- is the set of negative roots of $\mathfrak{g}$.

Thus,

$$\Delta(Q_w) = \Delta \cap (w w_o^P) R^-, \tag{19}$$

where w_o^P is the longest element of the Weyl group W_L of L. (Observe that $w w_o^P$ is the longest element $\hat{w}$ in the coset wW_L.)

Proof We first prove Eq. (18). Observe that

$$\begin{aligned} w\Big(R_{\mathfrak{l}}^+ \sqcup R^-\Big) &= w\Big(R_{\mathfrak{l}}^+ \sqcup R_{\mathfrak{l}}^- \sqcup (R^- \backslash R_{\mathfrak{l}}^-)\Big) \\ &= \hat{w}\Big(R_{\mathfrak{l}} \sqcup (R^- \backslash R_{\mathfrak{l}}^-)\Big). \end{aligned}$$

Thus,

$$\begin{aligned} \Delta_w &= \Delta \cap \hat{w}\Big(R_{\mathfrak{l}} \sqcup (R^- \backslash R_{\mathfrak{l}}^-)\Big) \\ &= \Delta \cap \hat{w}R^-, \qquad \text{since } \hat{w}(R_{\mathfrak{l}}^+) \subset R^-. \end{aligned} \tag{20}$$

Take $\alpha_i \in \Delta_w = \Delta \cap \hat{w}R^-$. Then,

$$\begin{aligned} s_i\, BwP/P &\subset (BwP/P) \cup (Bs_i wP/P) \\ &= (BwP/P) \cup (Bs_i\hat{w}P/P). \end{aligned}$$

But $s_i\hat{w} < \hat{w}$ since $(\hat{w})^{-1}\alpha_i \in R^-$. Hence,

$$s_i X_w \subset X_w.$$

This proves the inclusion $\Delta(Q_w) \supset \Delta_w = \Delta \cap \hat{w}R^-$.

Conversely, take $\alpha_i \in \Delta(Q_w)$, i.e., $s_i X_w \subset X_w$. Thus, $s_i\hat{w} < \hat{w}$ and hence $\hat{w}^{-1}\alpha_i \in R^-$. This proves the inclusion $\Delta(Q_w) \subset \Delta_w$ and hence Eq. (18) is proved. Now, Eq. (19) follows by combining Eqs. (18) and (20). ◻

Proposition 7.2 *Let $v \xrightarrow{\beta} w \in W^P$ (i.e., $v, w \in W^P$, $\beta \in R^+$ such that $w = s_\beta v$ and $\ell(w) = \ell(v) + 1$). Then, the (codimension one) cell C_v of X_w is contained in $Q_w wP/P$ if and only if $\beta \in \Delta_w$.*

In particular, β is a simple root in this case.

Proof We first prove the implication '⇐': If $\beta \in \Delta_w$, then $\beta \in \Delta(Q_w)$, by Lemma 7.1. Thus, $\dot{v} = s_\beta wP \in Q_w wP/P$.

Conversely, we prove the implication '⇒': Assume, if possible, that $\dot{v} \in Q_w wP/P$ but $\beta \notin \Delta_w$. We first show that X_v is stable under Q_w (assuming $\beta \notin \Delta_w$). By Lemma 7.1, it suffices to show that for any $\alpha_j \in \Delta_w = \Delta \cap \hat{w}R^-$, we have $\alpha_j \in \Delta_v$. Since $\hat{w}^{-1}\alpha_j \in R^-$, we get $s_j\hat{w} < \hat{w}$. Take a reduced decomposition $\hat{w} = s_j s_{i_1} \cdots s_{i_d}$. Since $v \xrightarrow{\beta} w$, then so is $\hat{v} \xrightarrow{\beta} \hat{w}$. Hence, there exists a (unique) $1 \le p \le d$ such that $\hat{v} = s_j s_{i_1} \cdots \hat{s}_{i_p} \cdots s_{i_d}$ and, of course, it is a reduced decomposition. (Here we have used the assumption that $\beta \notin \Delta_w$.)

Thus, $s_j\hat{v} < \hat{v}$, i.e., $\hat{v}^{-1}\alpha_j \in R^-$ and hence $\alpha_j \in \Delta_v$. This proves the assertion that X_v is stable under Q_w.

By assumption, $\dot{v} \in Q_w wP/P$, i.e., $\dot{v} = q\dot{w}$ for some $q \in Q_w$. Thus, $q^{-1}\dot{v} = \dot{w}$ and hence $\dot{w} \in Q_w X_v = X_v$, which is a contradiction. This contradiction shows that $\beta \in \Delta_w$ and hence completes the proof of the proposition. ◻

For $w \in W^P$, it is easy to see that the tangent space, as an H-module (induced from the left multiplication of H on X_w), is given by:

$$T_{\dot{w}}(X_w) \simeq \bigoplus_{\gamma \in R^+ \cap w R^-} \mathfrak{g}_\gamma, \tag{21}$$

where $\mathfrak{g}_\gamma$ is the root space of $\mathfrak{g}$ corresponding to the root γ. Hence,

$$T_{\dot{e}}\left(w^{-1} X_w\right) \simeq \bigoplus_{\gamma \in R^- \cap w^{-1} R^+} \mathfrak{g}_\gamma. \tag{22}$$

The following lemma determines the tangent space along codimension one cells.

Lemma 7.3 *For $v \xrightarrow{\beta} w \in W^P$, the tangent space, as an H-module, is given by:*

$$T_{\dot{v}}(X_w) \simeq \left(\bigoplus_{\gamma \in R^+ \cap v R^-} \mathfrak{g}_\gamma \right) \bigoplus \mathfrak{g}_{-\beta}.$$

Thus, as an H-module,

$$T_{\dot{e}}(v^{-1} X_w) \simeq \left(\bigoplus_{\gamma \in R^- \cap v^{-1} R^+} \mathfrak{g}_\gamma \right) \bigoplus \mathfrak{g}_{-v^{-1}\beta}.$$

(*Observe that $\dot{v}$ is a smooth point of X_w since X_w is normal; in particular, its singular locus is of codimension at least two.*)

Proof Since $\dot{v} \in X_v \subset X_w$, by (21),

$$\bigoplus_{\gamma \in R^+ \cap v R^-} \mathfrak{g}_\gamma \subset T_{\dot{v}}(X_w). \tag{23}$$

For any root $\alpha \in R$, let $U_\alpha := \mathrm{Exp}(\mathfrak{g}_\alpha) \subset G$ be the corresponding 1-dimensional unipotent group. Then,

$$U_\beta U_{-\beta} \dot{w} = U_\beta w U_{-w^{-1}\beta} \dot{e} = U_\beta \dot{w} \subset X_w \qquad (\text{since } w^{-1}\beta \in R^-).$$

Hence, $U_\beta H U_{-\beta} \dot{w} \subset X_w$. But, from the SL(2)-theory, $\overline{U_\beta H U_{-\beta}} \supset U_{-\beta} s_\beta H$. In particular,

$$U_{-\beta} s_\beta H \dot{w} \subset X_w, \text{ i.e., } U_{-\beta} \dot{v} \subset X_w.$$

This proves that

$$\mathfrak{g}_{-\beta} \subset T_{\dot{v}}(X_w). \tag{24}$$

Combining (23)–(24), we get

$$\left(\bigoplus_{\gamma\in R^+\cap vR^-} \mathfrak{g}_\gamma\right) \bigoplus \mathfrak{g}_{-\beta} \subset T_{\dot{v}}(X_w).$$

But, both the sides are of the same dimension $\ell(v)+1$, proving the lemma. □

As above, let P be any standard parabolic subgroup of G and let $x_P \in \mathfrak{h}' = \mathfrak{h}\cap[\mathfrak{g},\mathfrak{g}]$ be the element defined by

$$\begin{aligned} \alpha_i(x_P) &= 0, \qquad \text{for all the simple roots } \alpha_i \in \Delta(P)\\ &= 1, \qquad \text{for all the simple roots } \alpha_i \notin \Delta(P). \end{aligned}$$

Then, x_P is in the center of the Lie algebra $\mathfrak{l}$.

Set $m_o = \theta(x_P)$, where θ is the highest root of $\mathfrak{g}$. (Observe that $m_o \le 2$ for any maximal parabolic subgroup P of a classical group G.) Define a decomposition of $T_{\dot{e}}(G/P)$ as a direct sum of L-submodules as follows. First decompose $T_{\dot{e}}(G/P)$ as a direct sum of H-eigenspaces (induced from the canonical action of H on G/P):

$$T_{\dot{e}}(G/P) = \bigoplus_{\beta\in R^+\setminus R^+_{\mathfrak{l}}} T_{\dot{e}}(G/P)_{-\beta}.$$

For any $1 \le j \le m_o$, define

$$V_j = \bigoplus_{\substack{\beta\in R^+\setminus R^+_{\mathfrak{l}}:\\ \beta(x_P)=j}} T_{\dot{e}}(G/P)_{-\beta}.$$

Clearly, each V_j is a L-submodule of $T_{\dot{e}}(G/P)$ and we have the decomposition (as L-modules)

$$T_{\dot{e}}(G/P) = \bigoplus_{j=1}^{m_o} V_j.$$

Define an increasing filtration of $T_{\dot{e}}(G/P)$ by P-submodules given by

$$\mathcal{F}_1 \subset \mathcal{F}_2 \subset \mathcal{F}_3 \subset \cdots \subset \mathcal{F}_{m_o} = T_{\dot{e}}(G/P),$$

where

$$\mathcal{F}_d = \bigoplus_{j=1}^{d} V_j.$$

For any subvariety $Z \subset G/P$ such that $\dot{e}$ is a smooth point of Z, define

$$V_j(Z) := V_j \cap T_{\dot{e}}(Z).$$

Also, we get the increasing filtration $\mathcal{F}_j(Z)$ of $T_{\dot{e}}(Z)$ given by

$$\mathcal{F}_j(Z) := \mathcal{F}_j \cap T_{\dot{e}}(Z).$$

We set, for $1 \le j \le m_o$,

$$d_j(Z) = \text{dimension of } V_j(Z).$$

(Observe that $d_0(Z) = 0$ since $T_{\dot{e}}(Z) \subset T_{\dot{e}}(G/P) \simeq \bigoplus_{\alpha \in R^+ \setminus R_{\mathfrak{l}}^+} \mathfrak{g}_{-\alpha}$.)

Let $\mathfrak{z}(L)$ be the center of L. If Z, as above, is $\mathfrak{z}(L)$-stable, we get the decomposition (as $\mathfrak{z}(L)$-modules)

$$T_{\dot{e}}(Z) = \bigoplus_{j=1}^{m_o} V_j(Z).$$

Theorem 7.4 *For any $v \xrightarrow{\beta} w$ in W^P such that $\dot{v}$ is not in the Q_w-orbit of $\dot{w}$, there exists $1 \le j \le m_o$ such that*

$$d_j(w^{-1}X_w) \neq d_j(v^{-1}X_w). \tag{25}$$

Proof Let us set $\alpha := v^{-1}\beta \in R^+$. For any $1 \le j \le m_o$, we get (by Lemma 7.3 and Eq. (22) applied to v)

$$d_j(v^{-1}X_w) = d_j(v^{-1}X_v) + \delta_{j,\alpha(x_P)}. \tag{26}$$

By Eq. (22), the roots in $T_{\dot{e}}(w^{-1}X_w)$ are precisely $R^- \cap w^{-1}R^+$ $\left(\text{i.e., } T_{\dot{e}}(w^{-1}X_w) \simeq \bigoplus_{\gamma \in R^- \cap w^{-1}R^+} \mathfrak{g}_\gamma\right)$. Set

$$\Phi_{w^{-1}} := R^+ \cap w^{-1}R^-.$$

Then, as it is well known,

$$\sum_{\delta \in \Phi_{w^{-1}}} \delta = \rho - w^{-1}\rho,$$

where ρ is half the sum of all the positive roots.

Thus (abbreviating $d_j(w^{-1}X_w)$ by d_j and $d_j(v^{-1}X_v)$ by d_j'),

$$(\rho - w^{-1}\rho)(x_P) = d_1 + 2d_2 + \cdots + m_o d_{m_o}. \tag{27}$$

Similarly,

$$(\rho - v^{-1}\rho)(x_P) = d'_1 + 2d'_2 + \cdots + m_o d'_{m_o}. \tag{28}$$

Of course,

$$d_1 + d_2 + \cdots + d_{m_o} = \ell(w), \tag{29}$$

and

$$d'_1 + d'_2 + \cdots + d'_{m_o} = \ell(v) = \ell(w) - 1. \tag{30}$$

Now,

$$\begin{aligned}(\rho - w^{-1}\rho)(x_P) - (\rho - v^{-1}\rho)(x_P) &= (v^{-1}\rho - w^{-1}\rho)(x_P)\\ &= (v^{-1}\rho - s_\alpha v^{-1}\rho)(x_P), \text{ since } w = vs_\alpha\\ &= \langle v^{-1}\rho, \alpha^\vee\rangle\, \alpha(x_P)\\ &= \langle \rho, (v\alpha)^\vee\rangle\, \alpha(x_P)\\ &= \langle \rho, \beta^\vee\rangle\, \alpha(x_P).\end{aligned} \tag{31}$$

On the other hand, by (27)–(30),

$$\begin{aligned}&(\rho - w^{-1}\rho)(x_P) - (\rho - v^{-1}\rho)(x_P)\\ &\quad = (d_1 - d'_1) + 2(d_2 - d'_2) + \cdots + m_o(d_{m_o} - d'_{m_o})\\ &\quad = 1 + (d_2 - d'_2) + 2(d_3 - d'_3) + \cdots + (m_o - 1)(d_{m_o} - d'_{m_o}).\end{aligned} \tag{32}$$

Combining (31)–(32), we get

$$1 + (d_2 - d'_2) + 2(d_3 - d'_3) + \cdots + (m_o - 1)(d_{m_o} - d'_{m_o}) = \langle \rho, \beta^\vee\rangle\, \alpha(x_P). \tag{33}$$

If (25) were false, we would get

$$d_j = d_j(v^{-1}X_w), \qquad \text{for all } 1 \le j \le m_o,$$

i.e., by Eq. (26), we would get

$$d_j = d'_j \quad \text{for all } j \neq \alpha(x_P) \text{ and } \quad d_{\alpha(x_P)} = d'_{\alpha(x_P)} + 1.$$

Combining this with Eq. (33), we would get

$$\begin{aligned}1 + \alpha(x_P) - 1 = \langle \rho, \beta^\vee\rangle\, \alpha(x_P), \text{ i.e.,}\\ \alpha(x_P) = \langle \rho, \beta^\vee\rangle\, \alpha(x_P).\end{aligned} \tag{34}$$

But, by the definition of β, it is easy to see that if β were a simple root, then $\beta \in \Delta_w$. Since, by assumption, $\dot{v}$ is not in the Q_w-orbit of $\dot{w}$, this contradicts Proposition 7.2. Hence, β is not a simple root and this contradicts Eq. (34). (Observe that $\alpha(x_P) \neq 0$, since $v, w \in W^P$ and $w = vs_\alpha$.) This contradiction arose because we assumed that (25) was false. This proves (25) and hence the theorem is proved. □

8 The main theorem and its proof

We follow the notation and assumptions from Sect. 5. In particular, let $w_1, \ldots, w_s \in W^P$ be such that identity (13) is satisfied for some $d > 0$. We assume further that the s-tuple $(w_1, \ldots, w_s)$ is Levi-movable. This will be our assumption through this section.

Proposition 8.1 *Under the above assumption, there exists a closed subset A of $\mathcal{Z}$ such that*

$$\mathcal{Z} \setminus \mathcal{Y} \subseteq R \cup A, \ \operatorname{codim}(A, \mathcal{Z}) \geq 2. \tag{35}$$

Proof Let $\mathcal{Z}^o := \mathcal{Z} \setminus R$. It suffices to show that for $u_1, \ldots, u_s \in W^P$ such that $u_i = w_i$ for all $i \neq i_o$ and $u_{i_o} \to w_{i_o}$ for some $1 \leq i_o \leq s$ and $\dot{u}_{i_o} \notin Y_{w_{i_o}}$,

$$\mathcal{Z}^o \cap (\mathfrak{C}_{u_1} \times \cdots \times \mathfrak{C}_{u_s}) = \emptyset.$$

Since the s-tuple $(w_1, \ldots, w_s)$ is Levi-movable, there exist $l_1, \ldots, l_s \in L$ such that the standard quotient map

$$T_{\dot{e}}(G/P) \to \bigoplus_{i=1}^{s} T_{\dot{e}}(G/P)/T_{\dot{e}}(l_i \Lambda_{w_i})$$

is an isomorphism. Hence, the eigenspaces corresponding to any eigenvalue $1 \leq j \leq m_o$ under the action of x_P also are isomorphic, i.e.,

$$V_j(G/P) \simeq \bigoplus_{i=1}^{s} V_j(G/P)/V_j(l_i \Lambda_{w_i}),$$

where V_j is as in Sect. 7. (Here we have used the fact that $l_i \Lambda_{w_i}$ is $\mathfrak{z}(L)$-stable.) In particular, since the filtration $\mathcal{F}_j$ of $T_{\dot{e}}(G/P)$ is P-stable, for any $p_1, \ldots, p_s \in P$,

$$\dim \mathcal{F}_j = \sum_{i=1}^{s} (\dim \mathcal{F}_j - \dim(\mathcal{F}_j(l_i \Lambda_{w_i})) = \sum_{i=1}^{s} (\dim \mathcal{F}_j - \dim(\mathcal{F}_j(p_i \Lambda_{w_i})). \tag{36}$$

If nonempty, take $\mathfrak{a} = ([g_1, x_1], \ldots, [g_s, x_s]) \in \mathcal{Z}^o \cap (\mathfrak{C}_{u_1} \times \cdots \times \mathfrak{C}_{u_s})$, for $g_i \in G$ and $x_i \in C_{u_i}$. In particular, $g_1 x_1 = \cdots = g_s x_s$. Let us denote this common element by gP. From the G-equivariance, we can assume that $g = e$. By Lemma 5.3, the quotient map

$$T_{\dot{e}}(G/P) \to \bigoplus_{i=1}^{s} T_{\dot{e}}(G/P)/T_{\dot{e}}(p_i u_i^{-1} Z_{w_i})$$

is an isomorphism, where $p_i \in P$ is any element chosen such that $g_i \in p_i u_i^{-1} B$. In particular, for any j, the quotient map

$$\mathcal{F}_j \to \bigoplus_{i=1}^{s} \mathcal{F}_j/(T_{\dot{e}}(p_i u_i^{-1} Z_{w_i}) \cap \mathcal{F}_j)$$

is injective. Thus, for any j,

$$\dim \mathcal{F}_j \le \sum_{i=1}^{s} (\dim \mathcal{F}_j - \dim(\mathcal{F}_j(p_i u_i^{-1} Z_{w_i})) = \sum_{i=1}^{s} (\dim \mathcal{F}_j - \dim(\mathcal{F}_j(u_i^{-1} Z_{w_i})). \tag{37}$$

Take any $w \in W^P$ and $1 \le j \le m_o$. Let $\hat{Z}_w := \gamma^{-1}(Z_w)$, where $\gamma : G \to G/P$ is the standard projection. Consider the vector bundle

$$\mathcal{V}_w := \sqcup_{g \in \hat{Z}_w} T_{\dot{e}}(g^{-1} Z_w)$$

over $\hat{Z}_w$. The inclusion $T_{\dot{e}}(g^{-1} Z_w) \subset T_{\dot{e}}(G/P)$ gives rise to a bundle map $\phi : \mathcal{V}_w \to \epsilon_j(w)$, where $\epsilon_j(w)$ is the trivial bundle $\hat{Z}_w \times \left(T_{\dot{e}}(G/P)/\mathcal{F}_j\right) \to \hat{Z}_w$. Since the set of points of $\hat{Z}_w$, where the rank of ϕ is the highest, is an open subset and the rank of ϕ remains constant on the open subset $B\dot{w}P \subset \hat{Z}_w$ (because $\mathcal{F}_j$ is P-stable), for any $u, w \in W^P$ such that $\dot{u} \in Z_w$ and any j, we have

$$\dim \mathcal{F}_j(w^{-1} Z_w) \le \dim \mathcal{F}_j(u^{-1} Z_w). \tag{38}$$

Now, let j_o be an integer such that

$$\dim \mathcal{F}_{j_o}(w_i^{-1} Z_{w_i}) \ne \dim \mathcal{F}_{j_o}(u_i^{-1} Z_{w_i}) \text{ for } i = i_o. \tag{39}$$

It is possible to find such a j_o by virtue of Theorem 7.4. (Observe that $w_i, u_i \in Z_{w_i}$ and hence $\mathcal{F}_j(w_i^{-1} Z_{w_i}) = \mathcal{F}_j(w_i^{-1} X_{w_i})$ and $\mathcal{F}_j(u_i^{-1} Z_{w_i}) = \mathcal{F}_j(u_i^{-1} X_{w_i})$.) This contradicts the inequality (37) for $j = j_o$ (by using (36), (38)–(39)). Hence the proposition is proved. □

Recall the definition of the deformed product $\odot_0$ in the singular cohomology $H^*(G/P, \mathbb{Z})$ from [3, Definition 18]. We now come to our main theorem.

Theorem 8.2 *Let G be any connected reductive group and let P be any standard parabolic subgroup. Then, for any $w_1, \ldots, w_s \in W^P$ such that*

$$[X_{w_1}] \odot_0 \cdots \odot_0 [X_{w_s}] = [X_e] \in H^*(G/P),$$

we have (for any $n \geq 1$)

$$\dim \mathrm{Hom}_L(V_L(n\chi_1), V_L(n\chi_{w_1}) \otimes \cdots \otimes V_L(n\chi_{w_s})) = 1, \tag{40}$$

where χ_w is defined by Eq. (16).

Equivalently, we have (for the commutator subgroup $L^{ss} := [L, L]$):

$$\dim\big(\big[V_L(n\chi_{w_1}) \otimes \cdots \otimes V_L(n\chi_{w_s})\big]^{L^{ss}}\big) = 1, \ \forall n \geq 1. \tag{41}$$

Proof By [3, Theorem 15 and Proposition 17], the s-tuple $(w_1, \ldots, w_s)$ is L-movable. Hence, by Proposition 6.2, we have

$$H^0(\mathcal{Y}, \mathcal{O}(nR)_{|\mathcal{Y}})^G \simeq \big[V_L(n(\chi_{w_1} - \chi_1))^* \otimes V_L(n\chi_{w_2})^* \otimes \cdots \otimes V_L(n\chi_{w_s})^*\big]^L.$$

Moreover, by Proposition 8.1,

$$H^0(\mathcal{Y}, \mathcal{O}(nR)_{|\mathcal{Y}}) \hookrightarrow H^0(\mathcal{Z}, \mathcal{O}(m(n)R)), \text{ for some } m(n) > 0.$$

Finally, by Corollary 5.5, for any $m \geq 1$,

$$h^0(\mathcal{Z}, \mathcal{O}(mR)) = 1.$$

But, since the constants belong to $H^0(\mathcal{Y}, \mathcal{O}(nR)_{|\mathcal{Y}})$, we have

$$\dim(H^0(\mathcal{Y}, \mathcal{O}(nR)_{|\mathcal{Y}})^G) \geq 1.$$

Since $\chi_1 := 2(\rho - \rho^L)$ is a character of L, $V_L(n\chi_1)$ is a one dimensional representation and hence

$$V_L(n(\chi_{w_1} - \chi_1))^* \simeq V_L(n\chi_{w_1})^* \otimes V_L(n\chi_1).$$

Thus,

$$\begin{aligned}&\big[V_L(n(\chi_{w_1} - \chi_1))^* \otimes V_L(n\chi_{w_2})^* \otimes \cdots \otimes V_L(n\chi_{w_s})^*\big]^L \\ &\simeq \mathrm{Hom}_L(V_L(n\chi_1), V_L(n\chi_{w_1}) \otimes \cdots \otimes V_L(n\chi_{w_s})),\end{aligned}$$

which proves the identity (40). The equivalence of (40) with (41) follows from [3, Theorem 15] by observing that χ_1 is a trivial character when restricted to L^{ss}. □

Example 8.3 (1) The converse to the above theorem is false in general. For example, consider $G = \mathrm{Sp}(2\ell)$, G/P the Lagrangian Grassmannian $LG(\ell, 2\ell)$. It is cominuscule, so the structure constants for the singular cohomology and the deformed cohomology $\odot_0$ are the same. The cells in $LG(\ell, 2\ell)$ are parametrized by the strict partitions $\mathbf{a} : (a_1 > a_2 > \cdots > a_r > 0)$ and $a_1 \le \ell, r \le \ell$ (cf. [9, Page 29]).

The corresponding Levi subgroup is $GL(\ell)$, so the Fulton conjecture (Theorem 1.1) holds. Now, take $\ell = 3$ and consider the cells in $LG(3, 6)$ corresponding to the strict partitions (1), $(2 > 1)$, (2). The corresponding intersection number is 2. The corresponding representations of the Levi subgroup have Young diagrams $(2 \ge 0 \ge 0)$, $(3 \ge 3 \ge 0)$ and $(3 \ge 1 \ge 0)$ respectively. Hence, the dimension of the L^{ss}-invariant subspace for the corresponding tensor product is 1.

(2) In the above example, the intersection number is strictly larger than the dimension of the invariant subspace for the corresponding tensor product. We also have examples where the intersection number is strictly smaller than the dimension of the invariant subspace for the corresponding tensor product. Take, for G/P the Lagrangian Grassmannian $LG(5, 10)$ and consider the cells corresponding to the strict partitions $(3 > 1)$, $(3 > 2)$, $(4 > 2)$. The intersection number is 4. The corresponding representations of the Levi subgroup have Young diagrams $(4 \ge 3 \ge 1 \ge 0 \ge 0)$, $(4 \ge 4 \ge 2 \ge 0 \ge 0)$ and $(5 \ge 4 \ge 2 \ge 1 \ge 0)$ respectively. Hence, the dimension of the invariant subspace for the corresponding tensor product is 5.

(3) Following the convention in [4], for L of type G_2, $\dim([V(6\omega_1) \otimes V(6\omega_2) \otimes V(7\omega_2)]^L) = 1$, and $\dim([V(12\omega_1) \otimes V(12\omega_2) \otimes V(14\omega_2)]^L) = 2$. Similarly, $\dim([V(6\omega_1) \otimes V(6\omega_2) \otimes V)(10\omega_1 + \omega_2)]^L) = 1$ and $\dim([V(12\omega_1) \otimes V(12\omega_2) \otimes V(20\omega_1 + 2\omega_2)]^L) = 3$, where $\{\omega_1, \omega_2\}$ are the fundamental weights. Thus, the direct generalization of the Fulton conjecture is false for general semi-simple L.

(4) There are examples of $w_1, w_2, w_3 \in W^P$ such that

$$[X_{w_1}] \cdot [X_{w_2}] \cdot [X_{w_3}] = [X_e] \in H^*(G/P), \tag{42}$$

but (41) is false. Take, for example, $G = \mathrm{Sp}(6)$ and P to be the maximal parabolic with $\Delta \setminus \Delta(P) = \{\alpha_2\}$ (following the convention in [4]). Now, take $w_1 = w_2 = s_1 s_3 s_2 s_1 s_3 s_2$, $w_3 = s_3 s_2$, where $\{s_i\}_{1 \le i \le 3}$ are the simple reflections. Then, (42) is satisfied (cf. [15, Theorem 4.6]). In this case, restricted to the Cartan of L^{ss}, we have $\chi_{w_1} = \chi_{w_2} = \omega_1 + \omega_3$, $\chi_{w_3} = 3\omega_1 + \omega_3$. Thus, for any $n \ge 1$,

$$\dim\left(\left[V_L(n\chi_{w_1}) \otimes V_L(n\chi_{w_2}) \otimes V_L(n\chi_{w_3})\right]^{L^{ss}}\right) = 0.$$

Remark 8.4 If we specialize Theorem 8.2 to $G = \mathrm{GL}(m)$ and P any maximal parabolic subgroup, then (as explained in the introduction) we readily obtain a proof of Fulton's conjecture proved by Knutson–Tao–Woodward [13] (Belkale [2] and Ressayre [17, 18] gave other geometric proofs) asserting the following:

Let $V_L(\lambda_1), \ldots, V_L(\lambda_s)$ be finite dimensional irreducible representations of $L = \mathrm{GL}(r)$ with highest weights $\lambda_1, \ldots, \lambda_s$ respectively. Assume that $\left[V_L(\lambda_1) \otimes \cdots \otimes V_L(\lambda_s)\right]^{L^{ss}}$ is one dimensional. Then, for any $n \geq 1$, $\left[V_L(n\lambda_1) \otimes \cdots \otimes V_L(n\lambda_s)\right]^{L^{ss}}$ again is one dimensional.

In fact, since any maximal parabolic subgroup in $GL(m)$ is cominuscule, by a result of Brion–Polo [5], we have $\mathcal{Z} = \mathcal{Y}$. Hence, Proposition 8.1 and the results from Sect. 7 are *not* needed in this case.

References

1. Belkale, P.: Invariant theory of GL(n) and intersection theory of Grassmannians. IMRN **69**, 3709–3721 (2004)
2. Belkale, P.: Geometric proof of a conjecture of Fulton. Adv. Math. **216**, 346–357 (2007)
3. Belkale, P., Kumar, S.: Eigenvalue problem and a new product in cohomology of flag varieties. Invent. Math. **166**, 185–228 (2006)
4. Bourbaki, N.: Groupes et Algèbres de Lie, Chapters 4–6. Masson, Paris (1981)
5. Brion, M., Polo, P.: Generic singularities of certain Schubert varieties. Math. Z. **231**, 301–324 (1999)
6. Debarre, O.: Higher-Dimensional Algebraic Geometry. Springer, New York (2001)
7. Fulton, W.: Intersection Theory, 2nd edn. Springer, New York (1998)
8. Fulton, W.: Young Tableaux. London Mathematical Society, Cambridge University Press, Cambridge (1997)
9. Fulton, W., Pragacz, P.: Schubert varieties and degeneracy loci. Lecture Notes in Mathematics, vol. 1689. Springer, Berlin (1998)
10. Hartshorne, R.: Algebraic Geometry. Springer, New York (1977)
11. Jantzen, J.C.: Representations of Algebraic Groups, 2nd edn. Am. Math. Soc. (2003)
12. Kleiman, S.L.: The transversality of a general translate. Compos. Math. **28**, 287–297 (1974)
13. Knutson, A., Tao, T., Woodward, C.: The honeycomb model of $\mathrm{GL}_n(\mathbb{C})$ tensor products II: Puzzles determine facets of the Littlewood-Richardson cone. J. Am. Math. Soc. **17**, 19–48 (2004)
14. Kumar, S.: Kac-Moody Groups, their Flag Varieties and Representation Theory. Progress in Mathematics, vol. 204. Birkhäuser, Boston (2002)
15. Kumar, S., Leeb, B., Millson, J.: The generalized triangle inequalities for rank 3 symmetric spaces of noncompact type. Contemp. Math. **332**, 171–195 (2003)
16. Mumford, D.: The Red Book of Varieties and Schemes. Springer Lecture Notes in Mathematics, vol. 1358 (1988)
17. Ressayre, N.: Geometric invariant theory and the generalized eigenvalue problem. Invent. Math. **180**, 389–441 (2010)
18. Ressayre, N.: A short geometric proof of a conjecture of Fulton. L' Enseign. Math. **57**, 103–115 (2011)
19. Ressayre, N.: A cohomology free description of eigencones in type A, B and C. Preprint, (arXiv:09084557) (2009). (to appear in IMRN)