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Tensor Product Decomposition

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Abstract

Let G be a semisimple connected complex algebraic group. We study the tensor product decomposition of irreducible finite-dimensional representations of G. The techniques we employ range from representation theory to algebraic geometry and topology. This is mainly a survey of author's various results on the subject obtained individually or jointly with Belkale, Kapovich, Leeb, Millson and Stembridge.

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Dedicated to the memory of my beloved mother

1. Introduction

Let G be a semisimple connected complex algebraic group with Lie algebra $\mathfrak{g}$. The irreducible finite-dimensional representations of G are parametrized by the set Λ^+ of dominant characters of T, where T is a maximal torus of G. For $\lambda \in \Lambda^+$, let $V(\lambda)$ be the corresponding (finite-dimensional) irreducible representation of G. By the complete reducibility theorem, for any $\lambda, \mu \in \Lambda^+$, we can decompose

$$V(\lambda) \otimes V(\mu) = \bigoplus_{\nu \in \Lambda^+} m^{\nu}_{\lambda,\mu} V(\nu), \tag{1}$$

where $m^{\nu}_{\lambda,\mu}$ (called the *Littlewood-Richardson coefficients*) denotes the multiplicity of $V(\nu)$ in the tensor product $V(\lambda) \otimes V(\mu)$. We say that $V(\nu)$ *occurs in* $V(\lambda) \otimes V(\mu)$ (or $V(\nu)$ is a *component* of $V(\lambda) \otimes V(\mu)$) if $m^{\nu}_{\lambda,\mu} > 0$. The numbers $m^{\nu}_{\lambda,\mu}$ are also called the *tensor product multiplicities*.

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From the orthogonality relations, (1) is equivalent to the decomposition

$$\operatorname{ch} V(\lambda) \cdot \operatorname{ch} V(\mu) = \sum_{\nu \in \Lambda^+} m^{\nu}_{\lambda,\mu} \operatorname{ch} V(\nu). \tag{2}$$

One of the major goals of the 'tensor product problem' is to determine (all) the components of $V(\lambda) \otimes V(\mu)$. Of course, a more refined problem is to determine the components together with their multiplicities. In general, even the first problem is very hard.

We will also discuss a weaker 'saturated tensor product problem.' We say that $V(\nu)$ is a *saturated component* of $V(\lambda) \otimes V(\mu)$ if $V(N\nu)$ occurs in the tensor product $V(N\lambda) \otimes V(N\mu)$ for some integer $N \geq 1$.

The aim of this note is to give an overview of some of our results on the tensor product decomposition obtained individually or jointly with others over the last more than twenty years. We give enough details of many of the proofs to make this note more accessible.

We begin by setting the notation in Section 2 to be used through the paper. We recall some fairly well known basic facts (including some results of Kostant and Steinberg) about the tensor product decomposition in Section 3.

In Section 4, we recall the existence of 'root components' in the tensor product, conjectured by Wahl (and proved in [K_3]). Roughly, the result asserts that for any $\lambda, \mu \in \Lambda^+$ and any positive root β such that $\lambda + \mu - \beta \in \Lambda^+$, $V(\lambda + \mu - \beta)$ is a component of $V(\lambda) \otimes V(\mu)$ (cf. Theorem (4.1)). This result has a geometric counterpart in the surjectivity of the Wahl map for the flag varieties G/P (cf. Theorem (4.2)).

In Section 5, we study a solution of the Parthasarathy-Ranga Rao-Varadarajan-Kostant (for short PRVK) conjecture asserting that for $\lambda, \mu \in \Lambda^+$ and any $w \in W$, the irreducible G-module $V(\overline{\lambda + w\mu})$ occurs in the G-submodule $U(\mathfrak{g}) \cdot (v_\lambda \otimes v_{w\mu})$ of $V(\lambda) \otimes V(\mu)$ with multiplicity exactly 1, where W is the Weyl group of G, $\overline{\lambda + w\mu}$ denotes the unique element in Λ^+ in the W-orbit of $\lambda + w\mu$ and v_λ is a nonzero weight vector of $V(\lambda)$ of weight λ (cf. Theorem (5.13) and also its refinement Theorem (5.15)). We have outlined its more or less a complete proof except the proof of a crucial cohomology vanishing result for Bott-Samelson-Demazure-Hansen varieties (see Theorem (5.2)).

Section 6: This section is based on the work [BK_1] due to Belkale-Kumar. Since the existence of a component $V(\nu)$ in $V(\lambda) \otimes V(\mu)$ is equivalent to the nonvanishing of the G-invariant space $[V(\lambda) \otimes V(\mu) \otimes V(\nu^*)]^G$, the tensor product problem can be restated (replacing ν by ν^*) in a more symmetrical form of determining when $[V(\lambda) \otimes V(\mu) \otimes V(\nu)]^G \neq 0$. We generalize this problem from $s = 3$ to any $s \geq 1$ and define the *tensor product semigroup:*

$$\bar{\Gamma}_s(G) := \{(\lambda_1, \dots, \lambda_s) \in (\Lambda^+)^s : [V(\lambda_1) \otimes \cdots \otimes V(\lambda_s)]^G \neq 0\}.$$

Similarly, define the *saturated tensor product semigroup*:

$$\Gamma_s(G) := \{(\lambda_1, \dots, \lambda_s) \in (\Lambda^+)^s : [V(N\lambda_1) \otimes \cdots \otimes V(N\lambda_s)]^G \neq 0 \text{ for some } N > 0\}.$$

By virtue of the convexity result in symplectic geometry, there exists a (unique) convex polyhedral cone $\Gamma_s(G)_{\mathbb{R}} \subset (\Lambda^+_{\mathbb{R}})^s$ such that $\Gamma_s(G) = \Gamma_s(G)_{\mathbb{R}} \cap \Lambda^s$, where $\Lambda^+_{\mathbb{R}}$ is the dominant chamber in $\Lambda_{\mathbb{R}} := \Lambda \otimes_{\mathbb{Z}} \mathbb{R}$. The main result of this section (cf. Theorem (6.3)) determines a system of inequalities describing the cone $\Gamma_s(G)_{\mathbb{R}}$ explicitly in terms of a certain deformed product in the cohomology of the flag varieties G/P for maximal parabolic subgroups P. Moreover, as proved by Ressayre (cf. Theorem (6.4)), this system of inequalities is an irredundant system. We have outlined a more or less complete proof of Theorem (6.3), which makes essential use of Geometric Invariant Theory, specifically the Hilbert-Mumford criterion for semistability and Kempf's maximally destabilizing one parameter subgroups associated to unstable points. In addition, the notion of 'Levi-movability' plays a fundamental role in the proofs.

In Section 7, which is a joint work with Stembridge, we exploit isogenies between semisimple groups over algebraically closed fields of finite char. to get inequalities between the dimensions of invariants in tensor products of representations of complex semisimple groups (cf. Theorem (7.2)). As a corollary, we obtain that $\Gamma_s(\mathrm{Sp}(2\ell)) = \Gamma_s(\mathrm{SO}(2\ell+1))$ (cf. Corollary (7.5)).

Section 8 describes the 'saturation problem,' which provides a comparison between the semigroups $\Gamma_s(G)$ and $\bar{\Gamma}_s(G)$. We recall here the result due to Knutson-Tao on the saturation for the group $SL(n)$ and the results and conjectures of Kapovich-Millson and Belkale-Kumar.

Section 9 is devoted to recalling the classical Littlewood-Richardson theorem for the tensor product decomposition of irreducible polynomial representations of $\mathrm{GL}(n)$ and its generalization by Littelmann for any G via his *LS path model*. In addition, we recall the formula given by Berenstein-Zelevinsky, which determines the tensor product multiplicities as the number of lattice points in some convex polytope.

For the tensor product multiplicities, there is an approach by Lusztig [Lu] via his *canonical bases*. Similarly, there is an approach by Kashiwara [Ka] via his *crystal bases*.

There are some software programs to calculate the tensor product multiplicities (e.g., see [LCL], [St_1]). Also, for some explicit tensor product decompositions for $SL(n)$ see [BCH], [ST_2]; for E_8 see [MMP], [GP]; and for all the classical groups, see [Koi] and [L_1].

2. Notation

Let G be a semisimple connected complex algebraic group. We choose a Borel subgroup B and a maximal torus $T \subset B$ and let $W = W_G := N_G(T)/T$ be the associated Weyl group, where $N_G(T)$ is the normalizer of T in G. Let $P \supseteq B$ be a (standard) parabolic subgroup of G and let $U = U_P$ be its unipotent radical. Consider the Levi subgroup $L = L_P$ of P containing T, so that P is the semi-direct product of U and L. Then, $B_L := B \cap L$ is a Borel subgroup of L. Let $\Lambda = \Lambda(T)$ denote the character group of T, i.e., the group of all the algebraic group morphisms $T \to \mathbb{G}_m$. Clearly, W acts on Λ. We denote the Lie algebras of G, B, T, P, U, L, B_L by the corresponding Gothic characters: $\mathfrak{g}, \mathfrak{b}, \mathfrak{t}, \mathfrak{p}, \mathfrak{u}, \mathfrak{l}, \mathfrak{b}_L$ respectively. We will often identify an element λ of Λ (via its derivative $\dot{\lambda}$) by an element of $\mathfrak{t}^*$. Let $R = R_\mathfrak{g} \subset \mathfrak{t}^*$ be the set of roots of $\mathfrak{g}$ with respect to the Cartan subalgebra $\mathfrak{t}$ and let R^+ be the set of positive roots (i.e., the set of roots of $\mathfrak{b}$). Similarly, let $R_\mathfrak{l}$ be the set of roots of $\mathfrak{l}$ with respect to $\mathfrak{t}$ and $R_\mathfrak{l}^+$ be the set of roots of $\mathfrak{b}_L$. Let $\Delta = \{\alpha_1, \dots, \alpha_\ell\} \subset R^+$ be the set of simple roots, $\{\alpha_1^\vee, \dots, \alpha_\ell^\vee\} \subset \mathfrak{t}$ the corresponding simple coroots and $\{s_1, \dots, s_\ell\} \subset W$ the corresponding simple reflections, where ℓ is the rank of G. We denote the corresponding simple root vectors by $\{e_1, \dots, e_\ell\}$, i.e., $e_i \in \mathfrak{g}_{\alpha_i}$. We denote by $\Delta(P)$ the set of simple roots contained in $R_\mathfrak{l}$. For any $1 \le j \le \ell$, define the element $x_j \in \mathfrak{t}$ by

$$\alpha_i(x_j) = \delta_{i,j}, \ \forall\ 1 \le i \le \ell. \tag{3}$$

Recall that if W_P is the Weyl group of P (which is, by definition, the Weyl Group W_L of L), then in each coset of W/W_P we have a unique member w of minimal length. This satisfies (cf. [K$_4$, Exercise 1.3.E]):

$$wB_Lw^{-1} \subseteq B. \tag{4}$$

Let W^P be the set of the minimal length representatives in the cosets of W/W_P.

For any $w \in W^P$, define the Schubert cell:

$$C_w^P := BwP/P \subset G/P.$$

Then, it is a locally closed subvariety of G/P isomorphic with the affine space $\mathbb{A}^{\ell(w)}, \ell(w)$ being the length of w (cf. [J, Part II, Chapter 13]). Its closure is denoted by X_w^P, which is an irreducible (projective) subvariety of G/P of dimension $\ell(w)$. We denote the point $wP \in C_w^P$ by $\dot{w}$. We abbreviate X_w^B by X_w.

Let $\mu(X_w^P)$ denote the fundamental class of X_w^P considered as an element of the singular homology with integral coefficients $H_{2\ell(w)}(G/P, \mathbb{Z})$ of G/P. Then, from the Bruhat decomposition, the elements $\{\mu(X_w^P)\}_{w \in W^P}$ form a $\mathbb{Z}$-basis of $H_*(G/P, \mathbb{Z})$. Let $\{[X_w^P]\}_{w \in W^P}$ be the Poincaré dual basis of the singular cohomology with integral coefficients $H^*(G/P, \mathbb{Z})$. Thus, $[X_w^P] \in H^{2(\dim G/P - \ell(w))}(G/P, \mathbb{Z})$.

An element $\lambda \in \Lambda$ is called dominant (resp. dominant regular) if $\dot{\lambda}(\alpha_i^\vee) \ge 0$ (resp. $\dot{\lambda}(\alpha_i^\vee) > 0$) for all the simple coroots $\alpha_i^\vee$. Let Λ^+ (resp. Λ^{++}) denote the set of all the dominant (resp. dominant regular) characters. The set of isomorphism classes of irreducible (finite-dimensional) representations of G is parametrized by Λ^+ via the highest weight of an irreducible representation. For $\lambda \in \Lambda^+$, we denote by $V(\lambda)$ the corresponding irreducible representation (of highest weight λ). The dual representation $V(\lambda)^*$ is isomorphic with $V(\lambda^*)$, where λ^* is the weight $-w_o\lambda$; w_o being the longest element of W. The μ-weight space of $V(\lambda)$ is denoted by $V(\lambda)_\mu$. For $\lambda \in \Lambda^+$, let $P(\lambda)$ be the set of weights of $V(\lambda)$. The W-orbit of any $\lambda \in \Lambda$ contains a unique element in Λ^+, which we denote by $\bar{\lambda}$. We also have the shifted action of W on Λ via $w * \lambda = w(\lambda+\rho)-\rho$, where ρ is half the sum of positive roots. (Observe that, even though ρ may not belong to Λ, $w\rho - \rho$ does.)

For any $\lambda \in \Lambda$, we have a G-equivariant line bundle $\mathcal{L}(\lambda)$ on G/B associated to the principal B-bundle $G \to G/B$ via the one-dimensional B-module λ^{-1}. (Any $\lambda \in \Lambda$ extends uniquely to a character of B.) The one-dimensional B-module λ is also denoted by $\mathbb{C}_\lambda$.

All the schemes are considered over the base field of complex numbers $\mathbb{C}$. The varieties are reduced (but not necessarily irreducible) schemes.

3. Some Basic Results

We follow the notation from the last section; in particular, G is a semisimple connected complex algebraic group. The aim of this section is to recall some fairly well known basic results on the tensor product decomposition. We begin with the following.

Lemma (3.1). *For $\lambda, \mu \in \Lambda^+$, $V(\lambda+\mu)$ occurs in $V = V(\lambda) \otimes V(\mu)$ with multiplicity 1.*

The unique submodule $V(\lambda+\mu)$ is called the Cartan component *of V.*

Proof. Let $v_\lambda \in V(\lambda)$ (resp. $v_\mu \in V(\mu)$) be a nonzero highest weight vector. Then, the line $\mathbb{C}v_\lambda \otimes v_\mu \subset V$ is clearly stable under the Borel subalgebra. From this, we easily see that the G-submodule generated by $v_\lambda \otimes v_\mu$ is isomorphic with $V(\lambda+\mu)$.

The weight space of V corresponding to the weight $\lambda+\mu$ is clearly one-dimensional. Hence, the multiplicity of $V(\lambda+\mu)$ in V is at most one. □

The following result is due to Kostant [Ko].

Proposition (3.2). *For $\lambda, \mu \in \Lambda^+$, any component $V(\nu)$ of $V = V(\lambda) \otimes V(\mu)$ is of the form $\nu = \lambda + \mu_1$, for some $\mu_1 \in P(\mu)$. Moreover, its multiplicity $m^\nu_{\lambda,\mu} \leq \dim V(\mu)_{\mu_1}$.*

Proof. Clearly, the multiplicity $m^\nu_{\lambda,\mu}$ is equal to the dimension of

$$\begin{aligned}\mathrm{Hom}_{\mathfrak{g}}\big(V(\nu), V(\lambda) \otimes V(\mu)\big) &\simeq \mathrm{Hom}_{\mathfrak{b}}\big(\mathbb{C}_\nu, V(\lambda) \otimes V(\mu)\big)\\ &\simeq \mathrm{Hom}_{\mathfrak{b}}\big(\mathbb{C}_\nu \otimes V(\lambda)^*, V(\mu)\big).\end{aligned}$$

But, $V(\lambda)^*$ is generated, as a $\mathfrak{b}$-module, by its lowest weight vector $v_{-\lambda}$ of weight $-\lambda$. Hence, any homomorphism $\phi \in \mathrm{Hom}_{\mathfrak{b}}(\mathbb{C}_\nu \otimes V(\lambda)^*, V(\mu))$ is completely determined by $\phi(\mathbb{C}_\nu \otimes v_{-\lambda})$, which must be a weight vector of weight $-\lambda+\nu \in P(\mu)$. □

We have the following general result due to Steinberg [S].

Theorem (3.3). *For $\lambda, \mu, \nu \in \Lambda^+$, the multiplicity $m^\nu_{\lambda,\mu} = \sum_{w\in W} \varepsilon(w) n_{(w*\nu)-\lambda}(\mu)$, where $n_{\lambda'}(\mu)$ is the dimension of the λ'-weight space in $V(\mu)$.*

Proof. Define the $\mathbb{Z}$-linear operator $D : R(T) \to R(T)$ by $D(e^\gamma) = \frac{\sum_{w\in W} \varepsilon(w) e^{w*\gamma}}{\sum_{w\in W} \varepsilon(w) e^{w*0}}$, where $R(T)$ is the representation ring of the torus T. Then, D is linear over the invariant subring $R(T)^W$ (under the standard action of $W : v\cdot e^\gamma = e^{v\gamma}$). Moreover, $D(e^{v*\gamma}) = \varepsilon(v) D(e^\gamma)$, for any $v \in W$. In particular, $D(e^\gamma) = 0$ if $\gamma + \rho$ is not regular (equivalently, if γ has nontrivial isotropy under the shifted action of W). For any $\gamma \in \Lambda$ such that $\gamma + \rho$ is regular, let $w_\gamma \in W$ be the unique element such that $w_\gamma * \gamma \in \Lambda^+$.

By the Weyl character formula, for any $\lambda \in \Lambda^+$, $\operatorname{ch} V(\lambda) = D(e^\lambda)$, where $\operatorname{ch} V(\lambda)$ denotes the character of $V(\lambda)$. Thus,

$$\begin{aligned} \operatorname{ch}(V(\lambda)\otimes V(\mu)) &= \operatorname{ch} V(\lambda)\cdot \operatorname{ch} V(\mu) \\ &= D(e^\lambda \cdot \operatorname{ch} V(\mu)), \qquad \text{since } \operatorname{ch} V(\mu) \in R(T)^W \\ &= \sum_\gamma n_\gamma(\mu) D(e^\lambda \cdot e^\gamma) \\ &= \sum_{\substack{\gamma:\lambda+\gamma+\rho \\ \text{is regular}}} \varepsilon(w_{\lambda+\gamma})\, n_\gamma(\mu)\, D\Big(e^{w_{\lambda+\gamma} * (\lambda+\gamma)}\Big) \\ &= \sum_{\nu\in\Lambda^+} \left(\sum_{w\in W} \varepsilon(w)\, n_{(w*\nu)-\lambda}(\mu) \right) D(e^\nu), \text{ since } \varepsilon(w) = \varepsilon(w^{-1}). \end{aligned}$$

Thus, from the equivalence of (1) and (2) in Section 1, the theorem follows. □

Corollary (3.4). *For $\lambda, \mu \in \Lambda^+$, if $\lambda + \mu'$ is nearly dominant (i.e., $(\lambda + \mu')(\alpha_i^\vee) \geq -1$ for all the simple coroots $\alpha_i^\vee$) for all μ' in $P(\mu)$, then the multiplicity of $V(\nu)$ in $V(\lambda)\otimes V(\mu)$:*

$$m^\nu_{\lambda,\mu} = n_{\nu-\lambda}(\mu).$$

Of course, by Proposition (3.2), $V(\nu)$ occurs in $V(\lambda)\otimes V(\mu)$ only if $\nu = \lambda + \mu'$ for some $\mu' \in P(\mu)$.

Proof. By the above theorem,

$$m^\nu_{\lambda,\mu} = \sum_{w\in W} \varepsilon(w)\, n_{(w*\nu)-\lambda}(\mu).$$

For $w \neq 1$, we claim that $n_{(w*\nu)-\lambda}(\mu) = 0$. Equivalently, $(w*\nu) - \lambda \notin P(\mu)$. Since any weight in $\lambda + P(\mu)$ is nearly dominant (by assumption) and ν is dominant, we have $w * \nu \notin \lambda + P(\mu)$ for any $w \neq 1$. □

As a corollary of the above corollary, we get the following (cf. [Kas], [K_2, Proposition 1.5]).

Corollary (3.5). *For $\lambda, \mu \in \Lambda^+$ such that $V(\mu)$ is minuscule (i.e., $P(\mu)$ is a single W-orbit), we have the decomposition*

$$V(\lambda) \otimes V(\mu) \simeq \bigoplus_{\substack{\bar{w} \in W/W_\mu: \\ \lambda + w\mu \in \Lambda^+}} V(\lambda + w\mu), \qquad (*)$$

each occuring with multiplicity 1, where $W_\mu := \{w \in W : w\mu = \mu\}$ is the isotropy group of μ. Moreover, the number of irreducible components in $V(\lambda) \otimes V(\mu)$ is equal to the cardinality $\# W_\lambda \backslash W / W_\mu$.

Proof. By [Bo, Exercise 24, p. 226], $\lambda + \mu'$ is nearly dominant for any $\mu' \in P(\mu)$. Thus, by the above corollary, the decomposition $(*)$ follows. For the second part, define

$$f : (W/W_\mu)^+ \to W_\lambda \backslash W / W_\mu, \quad f(wW_\mu) = W_\lambda w W_\mu,$$

where $(W/W_\mu)^+ := \{\bar{w} \in W/W_\mu : \lambda + w\mu \in \Lambda^+\}$. It is easy to see that f is injective, and, for any w of minimal element in its double coset $W_\lambda w W_\mu$, $wW_\mu \in (W/W_\mu)^+$. □

As another corollary of Theorem (3.3), we get the following.

Corollary (3.6). *For $\lambda, \mu, \nu \in \Lambda^+$,*

$$m^\nu_{\lambda,\mu} = \sum_{v,w \in W} \varepsilon(v)\, \varepsilon(w)\, \mathcal{P}\big(v(\mu + \rho) - w(\nu + \rho) + \lambda\big),$$

where $\mathcal{P}$ is the Kostant's partition function.

Proof. Use Kostant's formula for any dominant character μ and any integral character λ':

$$n_{\lambda'}(\mu) = \sum_{v \in W} \varepsilon(v)\, \mathcal{P}((v * \mu) - \lambda').$$

□

The following result is due to Kostant [Ko, Lemma 4.1].

Theorem (3.7). *For any $\lambda, \mu, \nu \in \Lambda^+$, the multiplicity*

$$m^\nu_{\lambda,\mu} = \dim \Big\{ v \in V(\mu)_{\nu - \lambda} : e_i^{\lambda(\alpha_i^\vee) + 1} v = 0, \ \textit{for all simple roots } \alpha_i \Big\}.$$

Proof. Of course, by the proof of Proposition (3.2),

$$\begin{aligned} m^\nu_{\lambda,\mu} &= \dim \operatorname{Hom}_{\mathfrak{g}}(V(\nu), V(\lambda) \otimes V(\mu)) \\ &= \dim \operatorname{Hom}_{\mathfrak{b}}(\mathbb{C}_\nu \otimes V(\lambda)^*, V(\mu)). \end{aligned}$$

Let $v_{-\lambda} \in V(\lambda)^*$ be a nonzero lowest weight vector. Then, by a result due to Harish-Chandra,

$$\phi : U(\mathfrak{n}) \longrightarrow V(\lambda)^*, \quad X \mapsto X \cdot v_{-\lambda},$$

is surjective with kernel

$$\ker \phi = \sum_{1 \le i \le \ell} U(\mathfrak{n}) \cdot e_i^{\lambda(\alpha_i^\vee)+1},$$

where $\mathfrak{n}$ is the nil-radical of $\mathfrak{b}$. (This also follows immediately from the BGG resolution.) This proves the theorem. □

The following corollary follows immediately from the above theorem and SL(2)-representation theory.

Corollary (3.8). *For any $\lambda, \mu \in \Lambda^+$ and $w \in W$ such that $\lambda + w\mu$ is dominant, we have $m_{\lambda,\mu}^{\lambda+w\mu} = 1$.*

Lemma (3.9). *For any $\lambda, \mu, \nu, \lambda', \mu', \nu' \in \Lambda^+$ such that $m_{\lambda',\mu'}^{\nu'} \ge 1$, we have*

$$m_{\lambda+\lambda',\mu+\mu'}^{\nu+\nu'} \ge m_{\lambda,\mu}^{\nu}.$$

Proof. We have

$$\begin{aligned} \operatorname{Hom}_{\mathfrak{g}}\big(V(\nu), V(\lambda) \otimes V(\mu)\big) &\simeq \operatorname{Hom}_{\mathfrak{g}}\big(V(\lambda)^* \otimes V(\mu)^* \otimes V(\nu^*)^*, \mathbb{C}\big) \\ &\simeq \big[V(\lambda)^* \otimes V(\mu)^* \otimes V(\nu^*)^*\big]^{\mathfrak{g}} \\ &\simeq H^0\big((G/B)^3, \mathcal{L}(\lambda \boxtimes \mu \boxtimes \nu^*)\big)^G, \end{aligned}$$

where the last isomorphism follows from the Borel-Weil theorem: $H^0(G/B, \mathcal{L}(\lambda)) \simeq V(\lambda)^*$ (for any $\lambda \in \Lambda^+$), and $\mathcal{L}(\lambda \boxtimes \mu \boxtimes \nu^*)$ denotes the external tensor product line bundle $\mathcal{L}(\lambda) \boxtimes \mathcal{L}(\mu) \boxtimes \mathcal{L}(\nu^*)$ on $(G/B)^3$. Take a nonzero $\sigma_o \in H^0\big((G/B)^3, \mathcal{L}(\lambda' \boxtimes \mu' \boxtimes \nu'^*)\big)^G$. Then, the map

$$H^0\big((G/B)^3, \mathcal{L}(\lambda \boxtimes \mu \boxtimes \nu^*)\big)^G \longrightarrow H^0\big((G/B)^3, \mathcal{L}\big((\lambda+\lambda') \boxtimes (\mu+\mu') \boxtimes (\nu^*+\nu'^*)\big)\big)^G,$$

$\sigma \mapsto \sigma \cdot \sigma_o$, is clearly injective. □

4. Root Components in the Tensor Product

In this section, we assume that G is a semisimple simply-connected complex algebraic group and follow the notation from Section 2. The aim of this section is to state the existence of certain tensor product components coming from the positive roots, conjectured by Wahl [W]. Specifically, we have the following result due to Kumar [K_3], a proof of which can be found in loc. cit. The proof is purely representation theoretic (based on Theorem (3.7)) and unfortunately requires some case by case analysis. For any $\lambda \in \Lambda$, define $S_\lambda = \{1 \le i \le \ell : \lambda(\alpha_i^\vee) = 0\}$. Also, for any $\beta \in R^+$, define $F_\beta = \{1 \le i \le \ell : \beta - \alpha_i \notin R^+ \cup \{0\}\}$.

Theorem (4.1). *Take any $\lambda, \mu \in \Lambda^+$ and $\beta \in R^+$ satisfying:*

(P_1) $\lambda + \mu - \beta \in \Lambda^+$, *and*

(P_2) $S_\lambda \cup S_\mu \subset F_\beta$.

Then, $V(\lambda + \mu - \beta)$ *is a component of* $V(\lambda) \otimes V(\mu)$.

Observe that if G_2 does not occur as a component of $\mathfrak{g}$, then the conditions $(P_1) - (P_2)$ are automatically satisfied for any $\lambda, \mu \in \Lambda^{++}$.

Let X be a smooth projective variety with line bundles $\mathcal{L}_1$ and $\mathcal{L}_2$ on X. Consider the *Wahl map* defined by him (which he called the *Gaussian map*) $\Phi_{\mathcal{L}_1,\mathcal{L}_2} : H^0(X \times X, \mathcal{I}_D \otimes (\mathcal{L}_1 \boxtimes \mathcal{L}_2)) \to H^0(X, \Omega^1_X \otimes \mathcal{L}_1 \otimes \mathcal{L}_2)$, which is induced from the projection $\mathcal{I}_D \to \mathcal{I}_D/\mathcal{I}_D^2$ by identifying the $\mathcal{O}_{X\times X}/\mathcal{I}_D \simeq \mathcal{O}_D$-module $\mathcal{I}_D/\mathcal{I}_D^2$ (supported in D) with the sheaf of 1-forms Ω^1_X on $D \simeq X$ (cf. [W]), where $\mathcal{I}_D$ is the ideal sheaf of the diagonal D.

The following Theorem is a geometric counterpart of Theorem (4.1). It was conjectured by Wahl and proved by him for $X = \mathrm{SL}(n)/B$ and also for any minuscule G/P (cf. [W]). Kumar proved it for any G/P (cf. [K$_3$]) by using Theorem (4.1). In fact, he showed that Theorems (4.1) and (4.2) are 'essentially' equivalent. Theorem (4.2) is proved in an arbitrary char. for Grassmannians by Mehta-Parameswaran [MP]; for orthogonal and symplectic Grassmannians in odd char. by Lakshmibai-Raghavan-Sankaran [LRS]; and for minuscule G/P in any char. by Brown-Lakshmibai [BL].

Theorem (4.2). *The Wahl map* $\Phi_{\mathcal{L}_1,\mathcal{L}_2}$ *is surjective for any flag variety* $X = G/P$ *(where G is any semisimple simply-connected group and $P \subset G$ a parabolic subgroup) and any ample homogeneous line bundles $\mathcal{L}_1$ and $\mathcal{L}_2$ on X.*

Equivalently, $H^p(G/P \times G/P, \mathcal{I}_D^2 \otimes (\mathcal{L}_1 \boxtimes \mathcal{L}_2)) = 0,$ *for all* $p > 0$.

5. Proof of Parthasarathy-Ranga Rao-Varadarajan-Kostant Conjecture

In this section, we assume that G is a semisimple simply-connected complex algebraic group and follow the notation from Section 2. We begin with the following result due to Parthasarathy-Ranga Rao-Varadarajan [PRV, Corollary 1 to Theorem 2.1].

Theorem (5.1). *For any $\lambda, \mu \in \Lambda^+$, the irreducible module* $V(\overline{\lambda + w_o\mu})$ *occurs with multiplicity one in the tensor product $V(\lambda)\otimes V(\mu)$, where w_o is the longest element of W.*

Proof. Denote $\nu = \overline{\lambda + w_o\mu}$. We clearly have

$$\mathrm{Hom}_{\mathfrak{g}}(V(\lambda) \otimes V(\mu), V(\nu)) \simeq \mathrm{Hom}_{\mathfrak{b}}(\mathbb{C}_\lambda \otimes V(\mu), V(\nu)).$$

Moreover, as in the proof of Theorem (3.7), the map $\phi : U(\mathfrak{n}) \to V(\mu), X \mapsto X \cdot v_{w_o\mu}$, is surjective with kernel

$$\ker \phi = \sum_{1\le i\le \ell} U(\mathfrak{n}) e_i^{-(w_o\mu)(\alpha_i^\vee)+1}, \tag{5}$$

where $v_{w_o\mu}$ is a nonzero lowest weight vector of $V(\mu)$. Since the weight space $V(\nu)_{\lambda+w_o\mu}$ is one-dimensional, $\dim \operatorname{Hom}_{\mathfrak{b}}(\mathbb{C}_\lambda \otimes V(\mu), V(\nu)) \le 1$. Moreover, by (5), the map $v_\lambda \otimes v_{w_o\mu} \mapsto v_{\lambda+w_o\mu}$ extends to a $\mathfrak{b}$-module map $\mathbb{C}_\lambda \otimes V(\mu) \to V(\nu)$ iff $e_i^{-(w_o\mu)(\alpha_i^\vee)+1} v_{\lambda+w_o\mu} = 0$ for all $1 \le i \le \ell$. But the latter holds, as can be easily seen from the representation theory of SL_2. □

Now, we prove a vast generalization of the above theorem.

For any B-variety X, we denote by $\widetilde{X}$ the G-variety $G \underset{B}{\times} X$, i.e., it is the total space of the fiber bundle with fiber X, associated to the principal B-bundle $G \to G/B$. For any B-varieties X, Y and a B-morphism $\phi : X \to Y$, there is a canonical G-morphism $\widetilde{\phi} : \widetilde{X} \to \widetilde{Y}$.

For any sequence (not necessarily reduced) $\mathfrak{w} = (s_{i_1}, \dots, s_{i_n})$ of simple reflections, let $Z_{\mathfrak{w}}$ be the Bott-Samelson-Demazure-Hansen (for short BSDH) variety defined as the quotient $Z_{\mathfrak{w}} = P_{i_1} \times \cdots \times P_{i_n}/B^n$ under the right action of B^n on $P_{i_1} \times \cdots \times P_{i_n}$ via:

$$(p_1, \dots, p_n)(b_1, \dots, b_n) = (p_1 b_1, b_1^{-1} p_2 b_2, \dots, b_{n-1}^{-1} p_n b_n),$$

for $p_j \in P_{i_j}, b_j \in B$, where P_{i_j} is the standard minimal parabolic with $\Delta(P_{i_j}) = \{\alpha_{i_j}\}$. We denote the B^n-orbit of $(p_1, \dots, p_n)$ by $[p_1, \dots, p_n]$. Then, $Z_{\mathfrak{w}}$ is a smooth B-variety (in fact a P_{i_1}-variety) under the left multiplication on the first factor. For any $1 \le j \le n$, consider the subsequence $\mathfrak{w}(j) := (s_{i_1}, \dots, \hat{s}_{i_j}, \dots, s_{i_n})$. Then, we have a B-equivariant embeding $Z_{\mathfrak{w}(j)} \hookrightarrow Z_{\mathfrak{w}}, [p_1, \dots, \hat{p}_j, \dots, p_n] \mapsto [p_1, \dots, p_{j-1}, 1, p_{j+1}, \dots, p_n]$. Thus, we have the G-varieties $\widetilde{Z}_{\mathfrak{w}}$ and $\widetilde{Z}_{\mathfrak{w}(j)}$ and a canonical inclusion $\widetilde{Z}_{\mathfrak{w}(j)} \hookrightarrow \widetilde{Z}_{\mathfrak{w}}$. Of course, $\widetilde{Z}_{\mathfrak{w}}$ (and $\widetilde{Z}_{\mathfrak{w}(j)}$) is smooth. For any $w \in W$, we also have the G-variety $\widetilde{X}_w$, where X_w is the Schubert variety as in Section 2. Moreover, for any $v \le w$, we have a canonical inclusion $\widetilde{X}_v \hookrightarrow \widetilde{X}_w$, induced from the inclusion $X_v \hookrightarrow X_w$. Further, there are G-morphisms (G acting on $G/B \times G/B$ diagonally):

$$\widetilde{\theta}_{\mathfrak{w}} : \widetilde{Z}_{\mathfrak{w}} \to G/B \times G/B \text{ and } \widetilde{d}_w : \widetilde{X}_w \to G/B \times G/B,$$

defined by

$$\widetilde{\theta}_{\mathfrak{w}}[g, z] = (gB, g\theta_{\mathfrak{w}}(z)), \text{ for } g \in G, z \in Z_{\mathfrak{w}}, \text{ and}$$

$$\widetilde{d}_w[g, x] = (gB, gx), \text{ for } g \in G, x \in X_w,$$

where the map $\theta_{\mathfrak{w}} : Z_{\mathfrak{w}} \to G/B$ is defined by $[p_1, \dots, p_n] \mapsto p_1 \dots p_n B$. Clearly, the map $\widetilde{\theta}_{\mathfrak{w}}$ (resp. $\widetilde{d}_w$) is well defined, i.e., it descends to $\widetilde{Z}_{\mathfrak{w}}$ (resp. $\widetilde{X}_w$). It can be easily seen that the map $\widetilde{d}_w$ is a closed immersion and its image is the closure of the G-orbit of the point $(e, \dot{w})$ in $G/B \times G/B$, where $\dot{w}$ is the point $wB \in G/B$. The sequence $\mathfrak{w} = (s_{i_1}, \dots, s_{i_n})$ is said to be *reduced* if $m(\mathfrak{w}) := s_{i_1} \dots s_{i_n}$ is a reduced decomposition.

For any $\lambda, \mu \in \Lambda$, we denote by $\mathcal{L}(\lambda \boxtimes \mu)$ the line bundle on $G/B \times G/B$ which is the external tensor product of the line bundles $\mathcal{L}(\lambda)$ and $\mathcal{L}(\mu)$ respectively. We further denote by $\mathcal{L}_{\mathfrak{w}}(\lambda \boxtimes \mu)$ (resp. $\mathcal{L}_w(\lambda \boxtimes \mu)$) the pull-back of $\mathcal{L}(\lambda \boxtimes \mu)$ by the map $\widetilde{\theta}_{\mathfrak{w}}$ (resp. $\widetilde{d}_w$). The following cohomology vanishing result (rather its Corollary (5.4)) is crucial to the proof of the PRVK conjecture.

Theorem (5.2). *Let $\mathfrak{w} = (s_{i_1}, \dots, s_{i_n})$ be any sequence of simple reflections and let $1 \le j \le k \le n$ be such that the subsequence $(s_{i_j}, \dots, s_{i_k})$ is reduced. Then, for any $\lambda, \mu \in \Lambda^+$, we have:*

$$H^p\Big(\widetilde{Z}_{\mathfrak{w}}, \mathcal{L}_{\mathfrak{w}}(\lambda \boxtimes \mu) \otimes \mathcal{O}_{\widetilde{Z}_{\mathfrak{w}}}\big[-\cup_{q=j}^{k} \widetilde{Z}_{\mathfrak{w}(q)}\big]\Big) = 0, \text{ for all } p > 0.$$

The proof of this theorem is identical to the proof of the analogous result for $Z_{\mathfrak{w}}$ given in [K_4, Theorem 8.1.8] if we observe the following simple:

Lemma (5.3). *For any sequence $\mathfrak{w} = (s_{i_1}, \dots, s_{i_n})$ (not necessarily reduced), the canonical bundle $K_{\widetilde{Z}_{\mathfrak{w}}}$ of $\widetilde{Z}_{\mathfrak{w}}$ is isomorphic with*

$$\mathcal{L}_{\mathfrak{w}}((-\rho) \boxtimes (-\rho)) \otimes \mathcal{O}_{\widetilde{Z}_{\mathfrak{w}}}\big[-\partial \widetilde{Z}_{\mathfrak{w}}\big], \text{ where } \partial \widetilde{Z}_{\mathfrak{w}} := \cup_{q=1}^{n} \widetilde{Z}_{\mathfrak{w}(q)}.$$

Applying Theorem (5.2) to the cohomology exact sequence, corresponding to the sheaf sequence:

$$0 \to \mathcal{O}_{\widetilde{Z}_{\mathfrak{w}}}[-\widetilde{Z}_{\mathfrak{w}(j)}] \to \mathcal{O}_{\widetilde{Z}_{\mathfrak{w}}} \to \mathcal{O}_{\widetilde{Z}_{\mathfrak{w}(j)}} \to 0$$

tensored with the locally free sheaf $\mathcal{L}_{\mathfrak{w}}(\lambda \boxtimes \mu)$, we get the following:

Corollary (5.4). *Let $\mathfrak{w} = (s_{i_1}, \dots, s_{i_n})$ be any sequence. Then, for any $1 \le j \le n$ and any $\lambda, \mu \in \Lambda^+$, the canonical restriction map $H^0\big(\widetilde{Z}_{\mathfrak{w}}, \mathcal{L}_{\mathfrak{w}}(\lambda \boxtimes \mu)\big) \to H^0\big(\widetilde{Z}_{\mathfrak{w}(j)}, \mathcal{L}_{\mathfrak{w}(j)}(\lambda \boxtimes \mu)\big)$ is surjective.*

In the case when $\mathfrak{w}$ is a reduced sequence, the image of the map $\widetilde{\theta}_{\mathfrak{w}} : \widetilde{Z}_{\mathfrak{w}} \to G/B \times G/B$ is precisely equal to $\widetilde{d}_w(\widetilde{X}_w)$, where $w = m(\mathfrak{w})$. By slight abuse of notation, we denote the map $\widetilde{\theta}_{\mathfrak{w}}$, considered as a map $\widetilde{Z}_{\mathfrak{w}} \to \widetilde{X}_w$, again by $\widetilde{\theta}_{\mathfrak{w}}$. Then, $\widetilde{\theta}_{\mathfrak{w}}$ is a birational surjective morphism. As a consequence of the above corollary, we get the following:

Corollary (5.5). *For any $v \le w \in W$, and $\lambda, \mu \in \Lambda^+$, the canonical restriction map $H^0\big(\widetilde{X}_w, \mathcal{L}_w(\lambda \boxtimes \mu)\big) \to H^0\big(\widetilde{X}_v, \mathcal{L}_v(\lambda \boxtimes \mu)\big)$ is surjective.*

Proof. Take a reduced sequence $\mathfrak{w}$ such that $m(\mathfrak{w}) = w$. Then, we can find a reduced subsequence $\mathfrak{v}$ such that $m(\mathfrak{v}) = v$ (cf. [K_4, Lemma 1.3.16]). Since any Schubert variety X_w is normal (cf. [BrK, Theorem 3.2.2]), then so is $\widetilde{X}_w$. Hence, by the projection formula [H, Exercise 5.1, Chap. II] and the Zariski's main theorem [H, Corollary 11.4 and its proof, Chap. III] applied to the projective birational morphism $\widetilde{\theta}_{\mathfrak{w}}$, we get the isomorphism

$$\widetilde{\theta}_{\mathfrak{w}}^* : H^0\big(\widetilde{X}_w, \mathcal{L}_w(\lambda \boxtimes \mu)\big) \simeq H^0\big(\widetilde{Z}_{\mathfrak{w}}, \mathcal{L}_{\mathfrak{w}}(\lambda \boxtimes \mu)\big).$$

Now, the corollary follows by successively applying the last corollary. □

Remark (5.6). Even though we do not need, we also get (from Theorem (5.2)) that for any locally free sheaf $\mathcal{L}$ on $\widetilde{X}_w$, one has:

$$H^p(\widetilde{X}_w, \mathcal{L}) \simeq H^p(\widetilde{Z}_{\mathfrak{w}}, \widetilde{\theta}_{\mathfrak{w}}^*(\mathcal{L})), \text{ for all } p \geq 0,$$

and $H^p\big(\widetilde{X}_w, \mathcal{L}_w(\lambda \boxtimes \mu)\big) = 0$ for all $p > 0$ and any $\lambda, \mu \in \Lambda^+$. These cohomological results hold even in an arbitrary char. via Frobenius splitting methods (cf. [BrK, Theorems 3.1.2 and 3.3.4]).

The following result is a special case of a theorem of Bott [Bot, Theorem I], who proved the result for an arbitrary $H^p(G/B, \mathcal{M})$ in terms of the Lie algebra cohomology (cf. [K_4, Exercise 8.3.E.4] for the statement and the idea of a short proof).

Theorem (5.7). *For any finite-dimensional algebraic B-module M, there is a G-module isomorphism:*

$$H^0(G/B, \mathcal{M}) \simeq \bigoplus_{\theta \in \Lambda^+} V(\theta)^* \otimes [V(\theta) \otimes M]^{\mathfrak{b}},$$

where we put the trivial G-module structure on the $\mathfrak{b}$-invariants and $\mathcal{M}$ denotes the locally free sheaf on G/B associated to the B-module M.

Proof. By the Peter-Weyl theorem and Tannaka-Krein duality (cf. [BD, Chap. III]), the affine coordinate ring $\mathbb{C}[G]$ (as a $G \times G$-module) is given by:

$$\mathbb{C}[G] \simeq \bigoplus_{\theta \in \Lambda^+} V(\theta)^* \otimes V(\theta),$$

where $G \times G$ acts on $\mathbb{C}[G]$ via $\big((g,h).f\big)(x) = f(g^{-1}xh)$ and $G \times G$ acts on $V(\theta)^* \otimes V(\theta)$ factorwise. From this, the theorem follows easily. □

As a consequence of the above theorem, we derive the following:

Theorem (5.8). *For any $w \in W, \lambda \in \Lambda$ and $\mu \in \Lambda^+$, $H^0\big(\widetilde{X}_w, \mathcal{L}_w(\lambda \boxtimes \mu)\big)$ is canonically G-module isomorphic with*

$$\bigoplus_{\theta \in \Lambda^+} V(\theta)^* \otimes \operatorname{Hom}_{\mathfrak{b}}(\mathbb{C}_\lambda \otimes V_w(\mu), V(\theta)),$$

where we put the trivial G-module structure on $\operatorname{Hom}_{\mathfrak{b}}(\mathbb{C}_\lambda \otimes V_w(\mu), V(\theta))$ and $V_w(\mu) \subset V(\mu)$ is the Demazure submodule, which is, by definition, the $U(\mathfrak{b})$-span of the extremal weight vector $v_{w\mu}$ of weight $w\mu$ in $V(\mu)$.

Proof. By the definition of the direct image sheaf π_*, corresponding to the canonical fibration $\pi = \pi_w : \widetilde{X}_w \to G/B$, we get that $H^0\big(\widetilde{X}_w, \mathcal{L}_w(\lambda \boxtimes \mu)\big) \simeq H^0\big(G/B, \pi_*\mathcal{L}_w(\lambda \boxtimes \mu)\big)$. Since the line bundle $\mathcal{L}_w(\lambda \boxtimes \mu)$ on the G-space $\widetilde{X}_w$ is a G-equivariant line bundle and the map π is G-equivariant, the direct image sheaf $\pi_*\mathcal{L}_w(\lambda \boxtimes \mu)$ is a locally free sheaf on G/B associated to the B-module $M_w := \mathbb{C}_{-\lambda} \otimes H^0(X_w, \mathcal{L}_w(\mu))$, where $\mathcal{L}_w(\mu) := \mathcal{L}(\mu)|_{X_w}$. This gives the following:

$$H^0\big(\widetilde{X}_w, \mathcal{L}_w(\lambda \boxtimes \mu)\big) \simeq H^0(G/B, \mathcal{M}_w), \tag{6}$$

where $\mathcal{M}_w$ is the locally free sheaf on G/B associated to the B-module M_w. Now, by Theorem (5.7), we get by the isomorphism (6):

$$H^0\big(\widetilde{X}_w, \mathcal{L}_w(\lambda \boxtimes \mu)\big) \simeq \bigoplus_{\theta \in \Lambda^+} V(\theta)^* \otimes \big[V(\theta) \otimes M_w\big]^{\mathfrak{b}} \tag{7}$$

$$\simeq \bigoplus_{\theta \in \Lambda^+} V(\theta)^* \otimes \mathrm{Hom}_{\mathfrak{b}}(M_w^*, V(\theta)). \tag{8}$$

Now, the theorem follows from the isomorphism:

$$H^0(X_w, \mathcal{L}_w(\mu))^* \simeq V_w(\mu), \text{ for } \mu \in \Lambda^+,$$

which of course is a consequence of the Demazure character formula (cf., e.g., [K_4, Corollary 8.1.26]). □

We recall the following result due to Joseph [Jo, §3.5], which is a generalization of Harish-Chandra's theorem used in the proof of Theorem (3.7).

Theorem (5.9). *For any $w \in W$ and $\mu \in \Lambda^+$, the map $U(\mathfrak{n}) \to V_w(\mu)$, defined by $X \mapsto X.v_{w\mu}$, has kernel precisely equal to the left $U(\mathfrak{n})$-ideal $\sum_{\alpha \in R^+} U(\mathfrak{n}) X_\alpha^{k_\alpha+1}$, where X_α is any nonzero root vector in the root space $\mathfrak{g}_\alpha$ and k_α is defined as follows:*

$$k_\alpha = k_\alpha^\mu(w) = 0, \ \textit{if} \, (w\mu)(\alpha^\vee) \geq 0 \tag{9}$$

$$= -(w\mu)(\alpha^\vee), \ \textit{otherwise.} \tag{10}$$

Corollary (5.10). *For any $w \in W$ and $\lambda, \mu \in \Lambda^+$, $\mathrm{Hom}_{\mathfrak{b}}(\mathbb{C}_\lambda \otimes V_w(\mu), V(\overline{\lambda + w\mu}))$ is one-dimensional (over $\mathbb{C}$).*

Proof. Since $V_w(\mu)$ is a $U(\mathfrak{n})$-cyclic module generated by the element $v_{w\mu}$ of weight $w\mu$, $\mathbb{C}_\lambda$ is of weight λ, and the $\lambda + w\mu$ weight space in $V(\overline{\lambda + w\mu})$ is one-dimensional, we clearly have

$$\dim \mathrm{Hom}_{\mathfrak{b}}(\mathbb{C}_\lambda \otimes V_w(\mu), V(\overline{\lambda + w\mu})) \leq 1.$$

By the above theorem, the map $v_\lambda \otimes v_{w\mu} \mapsto v_{\lambda+w\mu}$ clearly extends uniquely to a $\mathfrak{b}$-module map, where $v_{\lambda+w\mu}$ is some fixed nonzero vector of weight $\lambda + w\mu$ in $V(\overline{\lambda + w\mu})$. □

For any $\lambda, \mu \in \Lambda^+$, by the Borel-Weil theorem, there is a G-module (in fact a $G \times G$-module) isomorphism $\xi : \big(V(\lambda) \otimes V(\mu)\big)^* \simeq H^0(G/B \times G/B, \mathcal{L}(\lambda \boxtimes \mu))$. On composition with the canonical restriction map $H^0(G/B \times G/B, \mathcal{L}(\lambda \boxtimes \mu)) \to H^0(\widetilde{X}_w, \mathcal{L}_w(\lambda \boxtimes \mu))$, we get a G-module map

$$\xi_w : \big(V(\lambda) \otimes V(\mu)\big)^* \to H^0(\widetilde{X}_w, \mathcal{L}_w(\lambda \boxtimes \mu)).$$

Since $\widetilde{C}_w = G \underset{B}{\times} C_w$ sits as a (Zariski) open subset of $\widetilde{X}_w$ (where $C_w = C_w^B$ is the Bruhat cell as in Section 2), the following lemma is trivial to prove.

Lemma (5.11). $\ker \xi_w = \{f \in \big(V(\lambda) \otimes V(\mu)\big)^* : f_{|U(\mathfrak{g})(v_\lambda \otimes v_{w\mu})} = 0\}$, *where* $U(\mathfrak{g})(v_\lambda \otimes v_{w\mu})$ *denotes the* $U(\mathfrak{g})$*-span of the vector* $v_\lambda \otimes v_{w\mu}$ *in* $V(\lambda) \otimes V(\mu)$.

By Corollary (5.5) (applied to $w = w_o$ and $v = w$), the map ξ_w is surjective and hence dualizing the above lemma, we get the following crucial:

Proposition (5.12). *For any* $w \in W$ *and* $\lambda, \mu \in \Lambda^+$,

$$H^0(\widetilde{X}_w, \mathcal{L}_w(\lambda \boxtimes \mu))^* \simeq U(\mathfrak{g})(v_\lambda \otimes v_{w\mu}) \hookrightarrow V(\lambda) \otimes V(\mu).$$

Now combining Theorem (5.8) with Corollary (5.10) and Proposition (5.12), we get the following most important result of this section. In the nineteen sixties, Parthasarathy-Ranga Rao-Varadarajan (for short PRV) conjectured (unpublished) the 'In particular' part of the following theorem (and proved it for $w = w_o$; cf. Theorem 5.1). Then, Kostant (in the mid eighties) came up with a more precise form of their conjecture (known as the PRVK conjecture), which is the first part of the following theorem. It was proved by Kumar [K_1] (using only char. 0 methods) and was extended by Mathieu [M_1] to an arbitrary char. The proof given here follows that of Kumar. Subsequently, other proofs of the original PRV conjecture appeared. Lusztig's results on the intersection homology of generalized Schubert varieties associated to affine Kac-Moody groups give a proof of the PRV conjecture; Rajeswari [Ra] gave a proof for classical G using Standard Monomial Theory; Littelmann [L_2] gave a proof using his LS path models.

Theorem (5.13). *For any finite-dimensional semisimple Lie algebra* $\mathfrak{g}$, *any* $\lambda, \mu \in \Lambda^+$, *and* $w \in W$, *the irreducible* $\mathfrak{g}$*-module* $V(\overline{\lambda + w\mu})$ *(with extremal weight* $\lambda + w\mu$*) occurs with multiplicity exactly one inside the* $\mathfrak{g}$*-submodule* $U(\mathfrak{g})(v_\lambda \otimes v_{w\mu})$ *of* $V(\lambda) \otimes V(\mu)$.

In particular, the $\mathfrak{g}$*-module* $V(\overline{\lambda + w\mu})$ *occurs with multiplicity at least one in* $V(\lambda) \otimes V(\mu)$.

Remark (5.14). (a) As proved in [K_1, Proposition 2.13], $V(\overline{\lambda + w\mu})$ occurs 'for the first time' in $U(\mathfrak{g})(v_\lambda \otimes v_{w\mu})$ if λ and μ are both regular. Precisely, for $\lambda, \mu \in \Lambda^{++}$, the $\mathfrak{g}$-module $V(\overline{\lambda + w\mu})$ does not occur in $U(\mathfrak{g})(v_\lambda \otimes v_{v\mu})$, for any $v < w$.

(b) Following recent works of Dimitrov-Roth and Ressayre (cf. [DR$_1$], [DR$_2$], [R$_3$]), one obtains the following: Let $\lambda, \mu, \nu \in \Lambda^+$ be such that there exists $w \in W$ with $\nu = \overline{\lambda + w\mu}$. Then, the following are equivalent:

(i) For all $k \geq 1$, $V(k\nu)$ appears in $V(k\lambda) \otimes V(k\mu)$ with multiplicity 1.

(ii) there exist $w_1, w_2, w_3 \in W$ such that $\ell(w_3) = \ell(w_1) + \ell(w_2)$, $w_3 * \nu = w_1 * \lambda + w_2 * \mu$ and the canonical product map

$$H^{\ell(w_1)}\big(G/B, \mathcal{L}(w_1 * \lambda)\big) \otimes H^{\ell(w_2)}\big(G/B, \mathcal{L}(w_2 * \mu)\big) \longrightarrow H^{\ell(w_3)}\big(G/B, \mathcal{L}(w_3 * \nu)\big)$$

is nonzero.

The following is a refinement of Theorem (5.13) proved by Kumar [K$_2$, Theorem 1.2] (which was conjectured by D.N. Verma).

Theorem (5.15). *Fix $\lambda, \mu \in \Lambda^+$ and consider the map $\eta : W_\lambda \backslash W / W_\mu \to \Lambda^+$, defined by $\eta(W_\lambda v W_\mu) = \overline{\lambda + v\mu}$, for any $v \in W$, where W_λ is the stabilizer of λ in W. Then, for any $w \in W$, the irreducible $\mathfrak{g}$-module $V(\overline{\lambda + w\mu})$ occurs in $V(\lambda) \otimes V(\mu)$ with multiplicity at least equal to $\#\eta^{-1}(\eta(W_\lambda w W_\mu))$, where $\#$ denotes the order.*

In particular, the number of irreducible components of $V(\lambda) \otimes V(\mu)$ (counted with multiplicities) is at least as much as the order of the double coset space $W_\lambda \backslash W / W_\mu$. (Of course, $W_\lambda = W_\mu = \{e\}$, if we assume λ and μ to be both regular.)

Proof. Fix a $w \in W$ and let $\{W_\lambda w_1 W_\mu, \dots, W_\lambda w_n W_\mu\}$ be the distinct double cosets such that $\eta(W_\lambda w_i W_\mu) = \overline{\lambda + w\mu}$, for all $1 \leq i \leq n$, and such that each w_i is of minimal length in its double coset. By [BrK, Remark 3.1.3], the restriction map

$$H^0\big(G/B \times G/B, \mathcal{L}(\lambda \boxtimes \mu)\big) \to H^0\big(\mathcal{Y}, \mathcal{L}(\lambda \boxtimes \mu)_{|\mathcal{Y}}\big)$$

is surjective, where $\mathcal{Y} := \bigcup_{i=1}^n \widetilde{X}_{w_i}$ is the closed subvariety (equipped with the reduced subscheme structure) of $G/B \times G/B$. For $1 \leq j \leq n$, define $\mathcal{Y}_j = \bigcup_{i=1}^j \widetilde{X}_{w_i}$. Now, the theorem follows from the following proposition due to Kumar [K$_2$, Proposition 2.5] together with [BrK, Exercise 3.3.E.3]. (This proposition is obtained by considering the ideal sheaf of $\mathcal{Y}_j$ in $\mathcal{Y}_{j+1}$ and induction on j.) ☐

Proposition (5.16). *For any $1 \leq j \leq n$, the irreducible $\mathfrak{g}$-module $V(\overline{\lambda + w\mu})$ occurs in $H^0\big(\mathcal{Y}_j, \mathcal{L}(\lambda \boxtimes \mu)_{|\mathcal{Y}_j}\big)^*$ with multiplicity exactly equal to j.*

6. Determination of the Saturated Tensor Cone

This section is based on the work [BK$_1$] due to Belkale-Kumar. We follow the notation and assumptions from Secton 2; in particular, G is a semisimple connected complex algebraic group.

For any $\lambda, \mu, \nu \in \Lambda^+$,

$$\mathrm{Hom}_G\big(V(\nu), V(\lambda) \otimes V(\mu)\big) \simeq [V(\lambda) \otimes V(\mu) \otimes V(\nu^*)]^G,$$

and hence the *tensor product problem* of determining the components $V(\nu)$ in the tensor product $V(\lambda)\otimes V(\mu)$ can be restated (replacing ν by ν^*) in a more symmetrical form of determining when $[V(\lambda)\otimes V(\mu)\otimes V(\nu)]^G \neq 0$. We generalize this problem from $s=3$ to any $s\geq 1$ and define the *tensor product semigroup:*

$$\bar{\Gamma}_s(G) := \{(\lambda_1,\dots,\lambda_s)\in(\Lambda^+)^s : [\lambda_1,\dots,\lambda_s]^G \neq 0\},$$

where $[\lambda_1,\dots,\lambda_s]^G$ denotes the dimension of the space of G-invariants $[V(\lambda_1)\otimes \cdots\otimes V(\lambda_s)]^G$. By Lemma (3.9), it is indeed a semigroup. Some general results on $\bar{\Gamma}_3(G)$ are obtained in the paper [KM$_1$] by Kapovich-Millson. The determination of $\bar{\Gamma}_s(G)$ in general is very hard, so we look at the weaker 'saturated tensor product problem' and define the *saturated tensor product semigroup*:

$$\Gamma_s(G) := \{(\lambda_1,\dots,\lambda_s)\in(\Lambda^+)^s : [N\lambda_1,\dots,N\lambda_s]^G \neq 0 \,\text{for some}\, N>0\}.$$

Let $\Lambda^+_{\mathbb{R}} := \{\lambda\in\Lambda\otimes_{\mathbb{Z}}\mathbb{R} : \lambda(\alpha_i^\vee)\geq 0 \,\text{for all the simple coroots}\, \alpha_i^\vee\}$. By virtue of the convexity result in symplectic geometry, there exists a (unique) convex polyhedral cone $\Gamma_s(G)_{\mathbb{R}}\subset(\Lambda^+_{\mathbb{R}})^s$ such that

$$\Gamma_s(G) = \Gamma_s(G)_{\mathbb{R}}\cap\Lambda^s.$$

The aim of this section is to find the inequalities describing the cone $\Gamma_s(G)_{\mathbb{R}}$ explicitly. Observe that the cone $\Gamma_s(G)_{\mathbb{R}}$ depends only upon the Lie algebra $\mathfrak{g}$ of G.

The following deformation of the cohomology product in $H^*(G/P)$ is due to Belkale-Kumar [BK$_1$, §6]. This deformed product is crucially used in the determination of $\Gamma_s(G)$.

Definition (6.1). Let P be any standard parabolic subgroup of G. Write the standard cup product in $H^*(G/P,\mathbb{Z})$ in the $\{[X_w^P]\}$ basis as follows:

$$\left[X_u^P\right]\cdot\left[X_v^P\right] = \sum_{w\in W^P} c_{u,v}^w\left[X_w^P\right]. \tag{11}$$

Introduce the indeterminates τ_i for each $\alpha_i\in\Delta\setminus\Delta(P)$ and define a deformed cup product $\odot$ as follows:

$$\left[X_u^P\right]\odot\left[X_v^P\right] = \sum_{w\in W^P}\left(\prod_{\alpha_i\in\Delta\setminus\Delta(P)}\tau_i^{(w^{-1}\rho-u^{-1}\rho-v^{-1}\rho-\rho)(x_i)}\right)c_{u,v}^w\left[X_w^P\right],$$

where ρ is the (usual) half sum of positive roots of $\mathfrak{g}$ and x_i is defined in Section 2.

By (subsequent) Corollary (6.17), whenever $c_{u,v}^w$ is nonzero, the exponent of τ_i in the above is a nonnegative integer. Moreover, it is easy to see that the product $\odot$ is associative and clearly commutative. This product should not be confused with the small quantum cohomology of G/P.

The cohomology algebra of G/P obtained by setting each $\tau_i = 0$ in $(H^*(G/P,\mathbb{Z})\otimes\mathbb{Z}[\tau_i],\odot)$ is denoted by $(H^*(G/P,\mathbb{Z}),\odot_0)$. Thus, as a $\mathbb{Z}$-module, it is the same as the singular cohomology $H^*(G/P,\mathbb{Z})$ and under the product $\odot_0$ it is associative (and commutative). Moreover, it continues to satisfy the Poincaré duality (cf. [BK$_1$, Lemma 16(d)]).

The cohomology algebra $H^*(G/P)$ under the product $\odot_0$ is intimately connected with the Lie algebra cohomology of the nil-radical $\mathfrak{u}$ of the parabolic subalgebra $\mathfrak{p}$ (cf. [BK$_1$, Theorem 43]).

We recall the following lemma from [BK$_1$, Lemma 19].

Lemma (6.2). *Let P be a cominuscule maximal standard parabolic subgroup of G (i.e., the simple root $\alpha_P \in \Delta\setminus\Delta(P)$ appears with coefficient 1 in the highest root of R^+). Then, the product $\odot$ coincides with the cup product in $H^*(G/P)$.*

Given a standard maximal parabolic subgroup P, let ω_P denote the corresponding fundamental weight, i.e., $\omega_P(\alpha_i^\vee) = 1$, if $\alpha_i \in \Delta\setminus\Delta(P)$ and 0 otherwise.

The following theorem due to Belkale-Kumar [BK$_1$, Theorem 22] determines the semigroup $\Gamma_s(G)$ 'most efficiently'. For $G = \mathrm{SL}(n)$, every maximal parabolic subgroup P is cominuscule and hence, by the above lemma, the deformed product $\odot_0$ in $H^*(G/P)$ coincides with the standard cup product. In this case, the following theorem was obtained by Klyachko [Kly] with a refinement by Belkale [B$_1$]. If we replace the product $\odot_0$ in (b) of the following theorem by the standard cup product, then the equivalence of (a) and (b) for general G was proved by Kapovich-Leeb-Millson [KLM] following an analogous slightly weaker result proved by Berenstein-Sjamaar [BS]. It may be mentioned that replacing the product $\odot_0$ in (b) by the standard cup product, we get, in general, 'far more' inequalities for simple groups other than SL_n. For example, for G of type B_3 (or C_3), the standard cup product gives rise to 135 inequalities, whereas the product $\odot_0$ gives only 102 inequalities (cf. [KuLM]).

Theorem (6.3). *Let $(\lambda_1,\dots,\lambda_s)\in(\Lambda^+)^s$. Then, the following are equivalent:*

(a) $(\lambda_1,\dots,\lambda_s)\in\Gamma_s(G)$.

(b) For every standard maximal parabolic subgroup P in G and every choice of s-tuples $(w_1,\dots,w_s)\in(W^P)^s$ such that

$$[X^P_{w_1}]\odot_0\cdots\odot_0[X^P_{w_s}] = [X^P_e]\in\big(H^*(G/P,\mathbb{Z}),\odot_0\big),$$

the following inequality holds:

$$\sum_{j=1}^s \lambda_j(w_j x_{i_P}) \le 0, \qquad (I^P_{(w_1,\dots,w_s)})$$

where α_{i_P} is the (unique) simple root in $\Delta\setminus\Delta(P)$.

The following result is due to Ressayre [R_1]. In the case $G = \mathrm{SL}(n)$, it was earlier proved by Knutson-Tao-Woodward [KTW].

Theorem (6.4). *The set of inequalities provided by the (b)-part of Theorem (6.3) is an irredundant system of inequalities describing the cone $\Gamma_s(G)_{\mathbb{R}}$ inside $(\Lambda^+_{\mathbb{R}})^s$, i.e., the hyperplanes given by the equality in $I^P_{(w_1,\dots,w_s)}$ are precisely those facets of the cone $\Gamma_s(G)_{\mathbb{R}}$ which intersect the interior of $(\Lambda^+_{\mathbb{R}})^s$.*

As a preparation towards the proof of Theorem (6.3), we first recall the following transversality theorem due to Kleiman (cf. [BK_1, Proposition 3]).

Theorem (6.5). *Let a connected algebraic group G act transitively on a smooth variety X and let $X_1, \dots, X_s$ be irreducible locally closed subvarieties of X. Then, there exists a nonempty open subset $U \subseteq G^s$ such that for $(g_1, \dots, g_s) \in U$, the intersection $\bigcap_{j=1}^s g_j X_j$ is proper (possibly empty) and dense in $\bigcap_{j=1}^s g_j \bar{X}_j$.*

Moreover, if X_j are smooth varieties, we can find such a U with the additional property that for $(g_1, \dots, g_s) \in U$, $\bigcap_{j=1}^s g_j X_j$ is transverse at each point of intersection.

We need the shifted Bruhat cell:

$$\Phi^P_w := w^{-1} B w P \subset G/P.$$

Let $T^P = T(G/P)_{\dot{e}}$ be the tangent space of G/P at $\dot{e} \in G/P$. It carries a canonical action of P. For $w \in W^P$, define T^P_w to be the tangent space of Φ^P_w at $\dot{e}$. We shall abbreviate T^P and T^P_w by T and T_w respectively when the reference to P is clear. By (4), B_L stabilizes Φ^P_w keeping $\dot{e}$ fixed. Thus,

$$B_L T_w \subset T_w. \tag{12}$$

The following result follows easily from the above transversality theorem and [F_1, Proposition 7.1 and §12.2] by observing that $g\Phi^P_w$ passes through $\dot{e} \Leftrightarrow g\Phi^P_w = p\Phi^P_w$ for some $p \in P$.

Proposition (6.6). *Take any $(w_1, \dots, w_s) \in (W^P)^s$ such that*

$$\sum_{j=1}^s \operatorname{codim} \Phi^P_{w_j} \le \dim G/P. \tag{13}$$

Then, the following three conditions are equivalent:

(a) $[X^P_{w_1}] \cdot \ldots \cdot [X^P_{w_s}] \neq 0 \in H^*(G/P)$.

(b) For generic $(p_1, \dots, p_s) \in P^s$, the intersection $p_1\Phi^P_{w_1} \cap \cdots \cap p_s\Phi^P_{w_s}$ is transverse at $\dot{e}$.

(c) *For generic* $(p_1, \ldots, p_s) \in P^s$,

$$\dim(p_1 T_{w_1} \cap \cdots \cap p_s T_{w_s}) = \dim G/P - \sum_{j=1}^{s} \operatorname{codim} \Phi^P_{w_j}.$$

The set of s-tuples in (b) as well as (c) is an open subset of P^s.

The definition of the deformed product $\odot_0$ was arrived at from the following crucial concept.

Definition (6.7). Let $w_1, \ldots, w_s \in W^P$ be such that

$$\sum_{j=1}^{s} \operatorname{codim} \Phi^P_{w_j} = \dim G/P. \tag{14}$$

We then call the s-tuple $(w_1, \ldots, w_s)$ *Levi-movable* for short *L-movable* if, for generic $(l_1, \ldots, l_s) \in L^s$, the intersection $l_1\Phi_{w_1} \cap \cdots \cap l_s\Phi_{w_s}$ is transverse at $\dot{e}$.

By Proposition (6.6), if $(w_1, \ldots, w_s)$ is L-movable, then $[X^P_{w_1}] \cdot \ldots \cdot [X^P_{w_s}] = d[X^P_e]$ in $H^*(G/P)$, for some nonzero d.

A Review of Geometric Invariant Theory. We need to consider the Geometric Invariant Theory (GIT) in a nontraditional setting, where a *nonreductive* group acts on a *nonprojective* variety. First we recall the following definition due to Mumford.

Definition (6.8). Let S be any (not necessarily reductive) algebraic group acting on a (not necessarily projective) variety $\mathbb{X}$ and let $\mathbb{L}$ be an S-equivariant line bundle on $\mathbb{X}$. Let $O(S)$ be the set of all one parameter subgroups (for short OPS) in S. Take any $x \in \mathbb{X}$ and $\delta \in O(S)$ such that the limit $\lim_{t\to 0} \delta(t)x$ exists in $\mathbb{X}$ (i.e., the morphism $\delta_x : \mathbb{G}_m \to X$ given by $t \mapsto \delta(t)x$ extends to a morphism $\widetilde{\delta}_x : \mathbb{A}^1 \to X$). Then, following Mumford, define a number $\mu^{\mathbb{L}}(x, \delta)$ as follows: Let $x_o \in X$ be the point $\widetilde{\delta}_x(0)$. Since x_o is $\mathbb{G}_m$-invariant via δ, the fiber of $\mathbb{L}$ over x_o is a $\mathbb{G}_m$-module; in particular, is given by a character of $\mathbb{G}_m$. This integer is defined as $\mu^{\mathbb{L}}(x, \delta)$.

We record the following standard properties of $\mu^{\mathbb{L}}(x, \delta)$ (cf. [MFK, Chap. 2, §1]):

Proposition (6.9). *For any* $x \in \mathbb{X}$ *and* $\delta \in O(S)$ *such that* $\lim_{t\to 0} \delta(t)x$ *exists in* $\mathbb{X}$, *we have the following (for any S-equivariant line bundles* $\mathbb{L}, \mathbb{L}_1, \mathbb{L}_2$*):*

(a) $\mu^{\mathbb{L}_1 \otimes \mathbb{L}_2}(x, \delta) = \mu^{\mathbb{L}_1}(x, \delta) + \mu^{\mathbb{L}_2}(x, \delta)$.

(b) *If there exists* $\sigma \in H^0(\mathbb{X}, \mathbb{L})^S$ *such that* $\sigma(x) \neq 0$, *then* $\mu^{\mathbb{L}}(x, \delta) \geq 0$.

(c) *If* $\mu^{\mathbb{L}}(x, \delta) = 0$, *then any element of* $H^0(\mathbb{X}, \mathbb{L})^S$ *which does not vanish at x does not vanish at* $\lim_{t\to 0} \delta(t)x$ *as well.*

(d) *For any S-variety $\mathbb{X}'$ together with an S-equivariant morphism $f : \mathbb{X}' \to \mathbb{X}$ and any $x' \in \mathbb{X}'$ such that $\lim_{t\to 0} \delta(t)x'$ exists in $\mathbb{X}'$, we have $\mu^{f^*\mathbb{L}}(x', \delta) = \mu^{\mathbb{L}}(f(x'), \delta)$.*

(e) *(Hilbert-Mumford criterion) Assume that $\mathbb{X}$ is projective, S is connected and reductive and $\mathbb{L}$ is ample. Then, $x \in \mathbb{X}$ is semistable (with respect to $\mathbb{L}$) if and only if $\mu^{\mathbb{L}}(x, \delta) \geq 0$, for all $\delta \in O(S)$.*

For an OPS $\delta \in O(S)$, let $\dot{\delta} \in \mathfrak{s}$ be its derivative at 1. Also, define the *associated parabolic subgroup* $P(\delta)$ of S by

$$P(\delta) := \left\{ g \in S : \lim_{t\to 0} \delta(t) g \delta(t)^{-1} \text{ exists in } S \right\}.$$

Definition (6.10). (Maximally destabilizing one parameter subgroups) We recall the definition of Kempf's OPS attached to an unstable point, which is in some sense 'most destabilizing' OPS. Let $\mathbb{X}$ be a projective variety with the action of a connected reductive group S and let $\mathbb{L}$ be a S-linearized ample line bundle on $\mathbb{X}$. Introduce the set $M(S)$ of fractional OPS in S. This is the set consisting of the ordered pairs (δ, a), where $\delta \in O(S)$ and $a \in \mathbb{Z}_{>0}$, modulo the equivalence relation $(\delta, a) \simeq (\gamma, b)$ if $\delta^b = \gamma^a$. The equivalence class of (δ, a) is denoted by $[\delta, a]$. An OPS δ of S can be thought of as the element $[\delta, 1] \in M(S)$. The group S acts on $M(S)$ via conjugation: $g \cdot [\delta, a] = [g\delta g^{-1}, a]$. Choose a S-invariant norm $q : M(S) \to \mathbb{R}_+$. We can extend the definition of $\mu^{\mathbb{L}}(x, \delta)$ to any element $\hat{\delta} = [\delta, a] \in M(S)$ and $x \in \mathbb{X}$ by setting $\mu^{\mathbb{L}}(x, \hat{\delta}) = \frac{\mu^{\mathbb{L}}(x,\delta)}{a}$. We note the following elementary property: If $\hat{\delta} \in M(S)$ and $p \in P(\delta)$ then

$$\mu^{\mathbb{L}}(x, \hat{\delta}) = \mu^{\mathbb{L}}(x, p\hat{\delta}p^{-1}). \tag{15}$$

For any unstable (i.e., nonsemistable) point $x \in \mathbb{X}$, define

$$q^*(x) = \inf_{\hat{\delta} \in M(S)} \{ q(\hat{\delta}) \mid \mu^{\mathbb{L}}(x, \hat{\delta}) \leq -1 \},$$

and the *optimal class*

$$\Lambda(x) = \{ \hat{\delta} \in M(S) \mid \mu^{\mathbb{L}}(x, \hat{\delta}) \leq -1, q(\hat{\delta}) = q^*(x) \}.$$

Any $\hat{\delta} \in \Lambda(x)$ is called *Kempf's OPS associated to x.*

By a theorem of Kempf (cf. [Ki, Lemma 12.13]), $\Lambda(x)$ is nonempty and the parabolic $P(\hat{\delta}) := P(\delta)$ (for $\hat{\delta} = [\delta, a]$) does not depend upon the choice of $\hat{\delta} \in \Lambda(x)$. The parabolic $P(\hat{\delta})$ for $\hat{\delta} \in \Lambda(x)$ will be denoted by $P(x)$ and called the *Kempf's parabolic associated to the unstable point x.*

We recall the following theorem due to Ramanan-Ramanathan [RR, Proposition 1.9].

Theorem (6.11). *For any unstable point $x \in \mathbb{X}$ and $\hat{\delta} = [\delta, a] \in \Lambda(x)$, let*

$$x_o = \lim_{t \to 0} \delta(t) \cdot x \in \mathbb{X}.$$

Then, x_o is unstable and $\hat{\delta} \in \Lambda(x_o)$.

Now, we return to the setting of Section 2. Let P be any standard parabolic subgroup of G acting on P/B_L via the left multiplication. We call $\delta \in O(P)$ P-*admissible* if, for all $x \in P/B_L$, $\lim_{t\to 0} \delta(t) \cdot x$ exists in P/B_L.

Observe that, B_L being the semidirect product of its commutator $[B_L, B_L]$ and T, any $\lambda \in \Lambda$ extends uniquely to a character of B_L. Thus, for any $\lambda \in \Lambda$, we have a P-equivariant line bundle $\mathcal{L}_P(\lambda)$ on P/B_L associated to the principal B_L-bundle $P \to P/B_L$ via the one-dimensional B_L-module λ^{-1}. The following lemma is easy to establish (cf. [BK$_1$, Lemma 14]). It is a generalization of the corresponding result in [BS, Section 4.2].

Lemma (6.12). *Let $\delta \in O(T)$ be such that $\dot{\delta} \in \mathfrak{t}_+ := \{x \in \mathfrak{t} : \alpha_i(x) \in \mathbb{R}_+ \forall$ the simple roots $\alpha_i\}$. Then, δ is P-admissible and, moreover, for any $\lambda \in \Lambda$ and $x = ulB_L \in P/B_L$ (for $u \in U, l \in L$), we have the following formula:*

$$\mu^{\mathcal{L}_P(\lambda)}(x, \delta) = -\lambda(w\dot{\delta}),$$

where $w \in W_P$ is the unique element such that $l^{-1} \in B_L w B_L$.

Definition (6.13). Let $w \in W^P$. Since T_w is a B_L-module (by (12)), we have the P-equivariant vector bundle $\mathcal{T}_w := P \underset{B_L}{\times} T_w$ on P/B_L. In particular, we have the P-equivariant vector bundle $\mathcal{T} := P \underset{B_L}{\times} T$ and $\mathcal{T}_w$ is canonically a P-equivariant subbundle of $\mathcal{T}$. Take the top exterior powers $\det(\mathcal{T}/\mathcal{T}_w)$ and $\det(\mathcal{T}_w)$, which are P-equivariant line bundles on P/B_L. Observe that, since T is a P-module, the P-equivariant vector bundle $\mathcal{T}$ is P-equivariantly isomorphic with the product bundle $P/B_L \times T$ under the map $\xi : P/B_L \times T \to \mathcal{T}$ taking $(pB_L, v) \mapsto [p, p^{-1}v]$, for $p \in P$ and $v \in T$; where P acts on $P/B_L \times T$ diagonally. We will often identify $\mathcal{T}$ with the product bundle $P/B_L \times T$ under ξ.

For $w \in W^P$, define the character $\chi_w \in \Lambda$ by

$$\chi_w = \sum_{\beta \in (R^+ \setminus R^+_{\mathfrak{l}}) \cap w^{-1} R^+} \beta \, .$$

Then, from [K$_4$, 1.3.22.3] and (4),

$$\chi_w = \rho - 2\rho^L + w^{-1}\rho, \tag{16}$$

where ρ (resp. ρ^L) is half the sum of roots in R^+ (resp. in $R^+_{\mathfrak{l}}$).

The following lemma is easy to establish.

Lemma (6.14). *For $w \in W^P$, as P-equivariant line bundles on P/B_L, we have:* $\det(\mathcal{T}/\mathcal{T}_w) = \mathcal{L}_P(\chi_w)$.

Let $\mathcal{T}_s$ be the P-equivariant product bundle $(P/B_L)^s \times T \to (P/B_L)^s$ under the diagonal action of P on $(P/B_L)^s \times T$. Then, $\mathcal{T}_s$ is canonically P-equivariantly isomorphic with the pull-back bundle $\pi_j^*(\mathcal{T})$, for any $1 \leq j \leq s$, where $\pi_j : (P/B_L)^s \to P/B_L$ is the projection onto the j-th factor. For any $w_1, \dots, w_s \in W^P$, we have a P-equivariant map of vector bundles on $(P/B_L)^s$:

$$\Theta = \Theta_{(w_1,\dots,w_s)} : \mathcal{T}_s \to \oplus_{j=1}^s \pi_j^*(\mathcal{T}/\mathcal{T}_{w_j}) \tag{17}$$

obtained as the direct sum of the projections $\mathcal{T}_s \to \pi_j^*(\mathcal{T}/\mathcal{T}_{w_j})$ under the identification $\mathcal{T}_s \simeq \pi_j^*(\mathcal{T})$. Now, assume that $w_1, \dots, w_s \in W^P$ satisfies the condition (14). In this case, we have the same rank bundles on the two sides of the map (17). Let θ be the bundle map obtained from Θ by taking the top exterior power:

$$\theta = \det(\Theta) : \det\big(\mathcal{T}_s\big) \to \det\big(\mathcal{T}/\mathcal{T}_{w_1}\big) \boxtimes \cdots \boxtimes \det\big(\mathcal{T}/\mathcal{T}_{w_s}\big). \tag{18}$$

Clearly, θ is P-equivariant and hence one can view θ as a P-invariant element in

$$H^0\left((P/B_L)^s, \det(\mathcal{T}_s)^* \otimes \Big(\det\big(\mathcal{T}/\mathcal{T}_{w_1}\big) \boxtimes \cdots \boxtimes \det\big(\mathcal{T}/\mathcal{T}_{w_s}\big)\Big)\right).$$

$$= H^0\big((P/B_L)^s, \mathcal{L}_P(\chi_{w_1} - \chi_1) \boxtimes \mathcal{L}_P(\chi_{w_2}) \boxtimes \cdots \boxtimes \mathcal{L}_P(\chi_{w_s})\big). \tag{19}$$

The following lemma follows easily from Proposition (6.6).

Lemma (6.15). *Let $(w_1, \dots, w_s)$ be an s-tuple of elements of W^P satisfying the condition (14). Then, we have the following:*

1. *The section θ is nonzero if and only if $[X_{w_1}^P] \cdot \ldots \cdot [X_{w_s}^P] \neq 0 \in H^*(G/P)$.*
2. *The s-tuple $(w_1, \dots, w_s)$ is L-movable if and only if the section θ restricted to $(L/B_L)^s$ is not identically 0.*

Proposition (6.16). *Assume that $(w_1, \dots, w_s) \in (W^P)^s$ satisfies condition (14). Then, the following are equivalent.*

(a) $(w_1, \dots, w_s)$ is L-movable.

(b) $[X_{w_1}^P] \cdot \ldots \cdot [X_{w_s}^P] = d[X_e^P]$ in $H^*(G/P)$, for some nonzero d, and for each $\alpha_i \in \Delta \setminus \Delta(P)$, we have

$$\left(\left(\sum_{j=1}^s \chi_{w_j}\right) - \chi_1\right)(x_i) = 0.$$

Proof. (a)⇒(b): Let $(w_1, \dots, w_s) \in (W^P)^s$ be L-movable. Consider the restriction $\hat{\theta}$ of the P-invariant section θ to $(L/B_L)^s$. Then, $\hat{\theta}$ is non-vanishing by the above lemma. But, for

$$H^0\left((L/B_L)^s, (\mathcal{L}_P(\chi_{w_1} - \chi_1) \boxtimes \mathcal{L}_P(\chi_{w_2}) \boxtimes \cdots \boxtimes \mathcal{L}_P(\chi_{w_s}))_{|(L/B_L)^s}\right)^L$$

to be nonzero, the center of L should act trivially (under the diagonal action) on $\mathcal{L}_P(\chi_{w_1} - \chi_1) \boxtimes \mathcal{L}_P(\chi_{w_2}) \boxtimes \cdots \boxtimes \mathcal{L}_P(\chi_{w_s})$ restricted to $(L/B_L)^s$. This gives $\sum_{j=1}^s \chi_{w_j}(h) = \chi_1(h)$, for all h in the Lie algebra $\mathfrak{z}_L$ of the center of L; in particular, for $h = x_i$ with $\alpha_i \in \Delta \setminus \Delta(P)$.

(b)⇒(a): By the above lemma, $\theta(\bar{p}_1, \dots, \bar{p}_s) \neq 0$, for some $\bar{p}_j \in P/B_L$. Consider the central OPS of L: $\delta(t) := \prod_{\alpha_i \in \Delta \setminus \Delta(P)} t^{x_i}$. For any $x = ulB_L \in P/B_L$, with $u \in U$ and $l \in L$,

$$\lim_{t \to 0} \delta(t)x = \lim_{t \to 0} \delta(t)u\delta(t)^{-1}(\delta(t)l)B_L.$$

But, since $\beta(\dot{\delta}) > 0$, for all $\beta \in R^+ \setminus R^+_{\mathfrak{l}}$, we get $\lim_{t\to 0} \delta(t)u\delta(t)^{-1} = 1$. Moreover, since $\delta(t)$ is central in $L, \delta(t)lB_L$ equals lB_L. Thus, $\lim_{t\to 0} \delta(t)x$ exists and lies in L/B_L.

Now, let $\mathbb{L}$ be the P-equivariant line bundle $\mathcal{L}_P(\chi_{w_1} - \chi_1) \boxtimes \mathcal{L}_P(\chi_{w_2}) \boxtimes \cdots \boxtimes \mathcal{L}_P(\chi_{w_s})$ on $\mathbb{X} := (P/B_L)^s$, and $\bar{p} := (\bar{p}_1, \dots, \bar{p}_s) \in \mathbb{X}$. Then, by Lemma (6.12) (since δ is central in L), we get

$$\begin{aligned}\mu^{\mathbb{L}}(\bar{p}, \delta) &= -\sum_{\alpha_i \in \Delta \setminus \Delta(P)} \left(\left(\left(\sum_{j=1}^s \chi_{w_j}\right) - \chi_1\right)(x_i)\right)\\ &= 0, \text{ by assumption.}\end{aligned}$$

Therefore, using Proposition (6.9)(c) for $S = P$, θ does not vanish at $\lim_{t\to 0} \delta(t)\bar{p}$. But, from the above, this limit exists as an element of $(L/B_L)^s$. Hence, $(w_1, \dots, w_s)$ is L-movable by Lemma (6.15). □

Corollary (6.17). *For any $u, v, w \in W^P$ such that $c^w_{u,v} \neq 0$ (cf. equation (11)), we have*

$$(\chi_w - \chi_u - \chi_v)(x_i) \geq 0, \quad \textit{for each } \alpha_i \in \Delta \setminus \Delta(P). \tag{20}$$

Proof. By the assumption of the corollary, $[X^P_u] \cdot [X^P_v] \cdot [X^P_{w_o w w^P_o}] = d[X^P_e]$, for some nonzero d (in fact $d = c^w_{u,v}$), where $w^P_\circ$ is the longest element of W_P. Thus, by taking $(w_1, w_2, w_3) = (u, v, w_o w w^P_o)$ in Lemma (6.15), the section θ is nonzero. Now, apply Proposition (6.9)(b) for the OPS $\delta(t) = t^{x_i}$ and Lemma (6.12) (together with the identity (16)) to get the corollary. □

Proof of Theorem (6.3): Let $\mathbb{L}$ denote the G-linearized line bundle $\mathcal{L}(\lambda_1) \boxtimes \cdots \boxtimes \mathcal{L}(\lambda_s)$ on $(G/B)^s$ and let $P_1, \dots, P_s$ be the standard parabolic subgroups such that $\mathbb{L}$ descends as an ample line bundle (still denoted by) $\mathbb{L}$ on $\mathbb{X} := G/P_1 \times \cdots \times G/P_s$. We call a point $x \in (G/B)^s$ *semistable* (with respect to, not necessarily ample, $\mathbb{L}$) if its image in $\mathbb{X}$ under the canonical map $\pi : (G/B)^s \to \mathbb{X}$ is semistable. Since the map π induces an isomorphism of G-modules:

$$H^0(\mathbb{X}, \mathbb{L}^N) \simeq H^0((G/B)^s, \mathbb{L}^N), \forall N > 0, \tag{21}$$

the condition (a) of Theorem (6.3) is equivalent to the following condition:

(c) The set of semistable points of $(G/B)^s$ with respect to $\mathbb{L}$ is nonempty.

Proof of the implication $(c) \Rightarrow (b)$ of Theorem (6.3): Let $x = (\bar{g}_1, \ldots, \bar{g}_s) \in (G/B)^s$ be a semistable point, where $\bar{g}_j = g_j B$. Since the set of semistable points is clearly open, we can choose a generic enough x such that the intersection $\cap\, g_j B w_j P$ itself is nonempty (cf. Theorem (6.5)). (By assumption, $\cap\, \overline{g_j B w_j P}$ is nonempty for any g_j.) Pick $f \in \cap\, g_j B w_j P$. Translating x by f^{-1}, we assume that $f = 1$. Consider the central OPS $\delta = t^{x_{i_P}}$ in L. Thus, applying Lemma (6.12) for $P = G$, the required inequality $I^P_{(w_1,\ldots,w_s)}$ is the same as $\mu^{\mathbb{L}}(x, \delta) \geq 0$; but this follows from Proposition (6.9), since x is semistable by assumption.

To prove the implication $(b) \Rightarrow (a)$ in Theorem (6.3), we need to recall the following result due to Kapovich-Leeb-Millson [KLM]. (For a self-contained algebro-geometric proof of this result, see [BK$_1$, §7.4].) Suppose that $x = (\bar{g}_1, \ldots, \bar{g}_s) \in (G/B)^s$ is an unstable point and $P(x)$ the Kempf's parabolic associated to $\pi(x)$. Let $\hat{\delta} = [\delta, a]$ be a Kempf's OPS associated to $\pi(x)$. Express $\delta(t) = f\gamma(t)f^{-1}$, where $\dot{\gamma} \in \mathfrak{t}_+$. Then, $P(\gamma)$ is a standard parabolic. Let P be a maximal parabolic containing $P(\gamma)$. Define $w_j \in W/W_{P(\gamma)}$ by $fP(\gamma) \in g_j B w_j P(\gamma)$ for $j = 1, \ldots, s$.

Theorem (6.18). *(i) The intersection $\bigcap_{j=1}^s g_j B w_j P \subset G/P$ is the singleton $\{fP\}$.*

(ii) For the simple root $\alpha_{i_P} \in \Delta \setminus \Delta(P)$, $\sum_{j=1}^s \lambda_j(w_j x_{i_P}) > 0$.

Now, we come to the proof of the implication $(b) \Rightarrow (a)$ in Theorem (6.3). Assume, if possible, that (a) (equivalently (c) as above) is false, i.e., the set of semistable points of $(G/B)^s$ is empty. Thus, any point $x = (\bar{g}_1, \ldots, \bar{g}_s) \in (G/B)^s$ is unstable. Choose a generic x so that for each standard parabolic $\widetilde{P}$ in G and any $(z_1, \ldots, z_s) \in W^s$, the intersection $g_1 B z_1 \widetilde{P} \cap \cdots \cap g_s B z_s \widetilde{P}$ is transverse (possibly empty) and dense in $g_1 \overline{B z_1 \widetilde{P}} \cap \cdots \cap g_s \overline{B z_s \widetilde{P}}$. Let $\hat{\delta} = [\delta, a], P, \gamma, f, w_j$ be as above associated to x. It follows from Theorem (6.18) that $\bigcap_{j=1}^s g_j B w_j P \subset G/P$ is the single point fP and, since x is generic, we get

$$[X^P_{w_1}] \cdot \ldots \cdot [X^P_{w_s}] = [X^P_e] \in H^*(G/P, \mathbb{Z}). \tag{22}$$

We now claim that the s-tuple $(w_1, \ldots, w_s) \in (W/W_P)^s$ is L-movable.

Write $g_j = f p_j w_j^{-1} b_j$, for some $p_j \in P(\gamma)$ and $b_j \in B$. Hence,

$$\delta(t)\bar{g}_j = f\gamma(t) p_j w_j^{-1} B = f\gamma(t) p_j \gamma^{-1}(t) w_j^{-1} B \in G/B.$$

Define, $l_j = \lim_{t\to 0} \gamma(t)p_j\gamma^{-1}(t)$. Then, $l_j \in L(\gamma)$, where $L(\gamma)$ is the Levi subgroup of $P(\gamma)$ containing T. Therefore,

$$\lim_{t\to 0} \delta(t)x = (fl_1w_1^{-1}B, \dots, fl_sw_s^{-1}B).$$

By Theorem (6.11), $\hat{\delta} \in \Lambda(\lim_{t\to 0} \delta(t)x)$. We further note that $fP(\gamma) \in \cap_j(fl_jw_j^{-1})Bw_jP(\gamma)$.

Applying Theorem (6.18) to the unstable point $x_o = \lim_{t\to 0} \delta(t)x$ yields: fP is the only point in the intersection $\bigcap_{j=1}^{s} fl_jw_j^{-1}Bw_jP$, i.e., translating by f, we get: $\dot{e} = eP$ is the only point in the intersection $\Omega := \bigcap_{j=1}^{s} l_jw_j^{-1}Bw_jP$. Thus, $\dim \Omega = 0$. By (22), the expected dimension of Ω is 0 as well. If this intersection Ω were not transverse at $\dot{e}$, then by [F_1, Remark 8.2], the local multiplicity at $\dot{e}$ would be > 1, each $w_j^{-1}Bw_jP$ being smooth. Further, G/P being a homogeneous space, any other component of the intersection $\bigcap l_j\overline{w_j^{-1}Bw_jP}$ contributes nonnegatively to the intersection product $[X_{w_1}^P] \cdot \ \dots \ \cdot [X_{w_s}^P]$ (cf. [F_1, §12.2]). Thus, from (22), we get that the intersection $\bigcap l_jw_j^{-1}Bw_jP$ is transverse at $\dot{e} \in G/P$, proving that $(w_1, \dots, w_s)$ is L-movable. Thus, by Proposition (6.16) and the identities (16), (22), we get $[X_{w_1}^P] \odot_0 \dots \odot_0 [X_{w_s}^P] = [X_e^P]$. Now, part (ii) of Theorem (6.18) contradicts the inequality $I^P_{(w_1,\dots,w_s)}$. Thus, the set of semistable points of $(G/B)^s$ is nonempty, proving condition (a) of Theorem (6.3). □

Remark (6.19). (1) The cone $\Gamma_s(G)_{\mathbb{R}}$ coincides with the *eigencone* under the identification of $\mathfrak{t}_+$ with $\Lambda^+_{\mathbb{R}}$ induced from the Killing form (cf. [Sj, Theorem 7.6]). The eigencone for $G = \mathrm{SL}(n)$ has extensively been studied since the initial work of H. Weyl in 1912. For a detailed survey on the subject, we refer to Fulton's article [F_2].

(2) The cone $\Gamma_3(G)_{\mathbb{R}}$ is quite explicitly determined for any semisimple G of rank 2 in [KLM, §7], any simple G of rank 3 in [KuLM] and for $G = \mathrm{Spin}(8)$ in [KKM]. It has $50, 102, 102, 306$ facets for G of type A_3, B_3, C_3, D_4 respectively.

(3) The 'explicit' determination of $\Gamma_s(G)$ via Theorem (6.3) hinges upon understanding the product $\odot_0$ in $H^*(G/P)$ in the Schubert basis, for all the maximal parabolic subgroups P. Clearly, the product $\odot_0$ is easier to understand than the usual cup product (which is the subject matter of *Schubert Calculus*) since, in general, 'many more' terms in the product $\odot_0$ in the Schubert basis drop out. For the lack of space, we do not recall various results about the product $\odot_0$, instead we refer to the papers [BK_1, §§9,10], [BK_2, §§8,9], [KKM, §4], [PS], [Ri_1], [Ri_2], [ReR], [R_3].

7. Special Isogenies and Tensor Product Multiplicities

This section is based on the work [KS] due to Kumar-Stembridge. It exploits certain 'exceptional' isogenies between semisimple algebraic groups over algebraically closed fields of char. $p > 0$ to derive relations between tensor product multiplicities of $\mathrm{Spin}_{2\ell+1}$ and $\mathrm{Sp}_{2\ell}$ and also between two different sets of multiplicities of F_4 (and also that of G_2).

Let $G = G(k)$ and $G' = G'(k)$ be connected, semisimple algebraic groups over an algebraically closed field k of char. $p > 0$, and let $f : G \to G'$ be an isogeny (i.e., a surjective algebraic group homomorphism with a finite kernel). Fix a Borel subgroup B of G and $T \subset B$ a maximal torus, and let $B' = f(B)$ and $T' = f(T)$ be the corresponding groups in G'. Then, T' (resp. B') is a maximal torus (resp. a Borel subgroup) of G'.

The map f induces a homomorphism $f^* : \Lambda(T') \to \Lambda(T)$, which extends to an isomorphism $f^*_{\mathbb{R}} : \Lambda(T')_{\mathbb{R}} \xrightarrow{\sim} \Lambda(T)_{\mathbb{R}}$, where $\Lambda(T)_{\mathbb{R}} := \Lambda(T) \otimes_{\mathbb{Z}} \mathbb{R}$. Moreover, f^* takes $\Lambda(T')^+$ to $\Lambda(T)^+$.

Letting $R = R(G,T)$ denote the root system of G with respect to T and similarly $R' = R(G',T')$, we recall the following from [C, Exposé n° 18, Definition 1].

Definition (7.1). An isomorphism $\phi : \Lambda(T')_{\mathbb{R}} \to \Lambda(T)_{\mathbb{R}}$ is called *special* if $\phi(\Lambda(T')) \subset \Lambda(T)$, and there exist integers $d(\alpha) \geq 0$ such that $R' = \{p^{d(\alpha)}\phi^{-1}(\alpha) : \alpha \in R\}$.

For any isogeny f as above, the induced map $f^*_{\mathbb{R}}$ is a special isomorphism. Conversely, for any special isomorphism $\phi : \Lambda(T')_{\mathbb{R}} \to \Lambda(T)_{\mathbb{R}}$, there exists an isogeny $f : G \to G'$ with $f^*_{\mathbb{R}} = \phi$ (cf. [C, Exposé n° 23, §3, Théorème 1]).

In the following, an important result due to Donkin, asserting the existence of good filtrations for tensor products of the space of global sections of homogeneous line bundles, has been used. (It should be noted that Donkin proved this result for almost all the cases barring a few exceptions involving small primes [D]; the result was subsequently proved uniformly by Mathieu for all primes [M_2].) This allows replacing the following inequality (23) with a cohomological statement that is independent of the char. of the field (including the char. 0 case), thereby enabling us to deduce the inequality directly from the existence of an isogeny in char. p.

Theorem (7.2). *If $f : G \to G'$ is an isogeny of connected semisimple algebraic groups over an algebraically closed field k of char. $p > 0$, then for all $\lambda'_1, \ldots, \lambda'_n \in \Lambda(T')^+$,*

$$[\lambda'_1, \ldots, \lambda'_n]^{G'(\mathbb{C})} \leq [f^*(\lambda'_1), \ldots, f^*(\lambda'_n)]^{G(\mathbb{C})}, \tag{23}$$

where $G(\mathbb{C})$ is the connected semisimple complex algebraic group with the same root datum as that of $G(k)$ and similarly for $G'(\mathbb{C})$.

Proof. The map f clearly induces a surjective morphism (of varieties) $\bar{f} : X_n \to X'_n$, where $X_n := (G/B)^{\times n}$. Consider the dominant line bundle $\mathcal{L}(\lambda'_1) \boxtimes \cdots \boxtimes \mathcal{L}(\lambda'_n)$ on X'_n. Then, the pull-back line bundle on X_n is the homogeneous line bundle $\mathcal{L}(\lambda_1) \boxtimes \cdots \boxtimes \mathcal{L}(\lambda_n)$, where $\lambda_i := f^*(\lambda'_i)$. Thus, we get an injective map

$$\bar{f}^* : H^0\big(X'_n, \mathcal{L}(\lambda'_1) \boxtimes \cdots \boxtimes \mathcal{L}(\lambda'_n)\big) \hookrightarrow H^0\big(X_n, \mathcal{L}(\lambda_1) \boxtimes \cdots \boxtimes \mathcal{L}(\lambda_n)\big).$$

Since the map $\bar{f}$ is f-equivariant under the diagonal action of G on X_n and G' on X'_n, the injection $\bar{f}^*$ induces an injection (still denoted by)

$$\bar{f}^* : H^0\big(X'_n, \mathcal{L}(\lambda'_1) \boxtimes \cdots \boxtimes \mathcal{L}(\lambda'_n)\big)^{G'} \hookrightarrow H^0\big(X_n, \mathcal{L}(\lambda_1) \boxtimes \cdots \boxtimes \mathcal{L}(\lambda_n)\big)^{G}. \quad (24)$$

We have of course

$$H^0\big(X_n, \mathcal{L}(\lambda_1) \boxtimes \cdots \boxtimes \mathcal{L}(\lambda_n)\big) \cong H^0\big(G/B, \mathcal{L}(\lambda_1)\big) \otimes \cdots \otimes H^0\big(G/B, \mathcal{L}(\lambda_n)\big).$$

By [BrK, Corollary 4.2.14], the above module $M := H^0\big(G/B, \mathcal{L}(\lambda_1)\big) \otimes \cdots \otimes H^0\big(G/B, \mathcal{L}(\lambda_n)\big)$ admits a good filtration. Hence, by [BrK, Theorem 4.2.7, identity (4.2.1.3) and Proposition 4.2.3(c)], its T-character is

$$\operatorname{ch} M = \sum_{\lambda \in \Lambda(T)^+} \dim\big[H^0\big(G/B, \mathcal{L}(\lambda)\big) \otimes M\big]^G \cdot \operatorname{ch}(V_k(\lambda)),$$

where $V_k(\lambda) := H^0(G/B, \mathcal{L}(\lambda))^*$ is the Weyl module with highest weight λ. Recall that, by the Borel-Weil Theorem, $H^0\big(G(\mathbb{C})/B(\mathbb{C}), \mathcal{L}_{\mathbb{C}}(\lambda)\big) \simeq V(\lambda)^*$, where (as earlier) $V(\lambda)$ is the (complex) irreducible $G(\mathbb{C})$-module with highest weight λ and $\mathcal{L}_{\mathbb{C}}(\lambda)$ is the homogeneous line bundle on $G(\mathbb{C})/B(\mathbb{C})$ corresponding to the character λ^{-1} of $B(\mathbb{C})$. Moreover, as is well-known, $\operatorname{ch} V_k(\lambda) = \operatorname{ch} V(\lambda)$. (This follows from the vanishing of the cohomology $H^i(G/B, \mathcal{L}(\lambda))$ for all $i > 0$.)

But, clearly, $\operatorname{ch} M = \operatorname{ch}(V_k(\lambda_1)^*) \cdots \operatorname{ch}(V_k(\lambda_n)^*)$; in particular, it is independent of the char. of the field (including char. 0). Moreover, since $\{\operatorname{ch} V(\lambda)\}_{\lambda \in \Lambda(T)^+}$ are $\mathbb{Z}$-linearly independent as elements of the group ring of $\Lambda(T)$, we deduce that $\dim\big[H^0\big(G/B, \mathcal{L}(\lambda)\big) \otimes M\big]^G$ is independent of the char. of the base field for all $\lambda \in \Lambda(T)^+$. Taking $\lambda = 0$, we obtain that $\dim M^G$ is independent of the char. Observe next that (24) implies

$$\dim[M']^{G'} \leq \dim[M]^G, \quad (25)$$

where $M' := H^0\big(G'/B', \mathcal{L}(\lambda'_1)\big) \otimes \cdots \otimes H^0\big(G'/B', \mathcal{L}(\lambda'_n)\big)$.

Thus, (25) implies

$$\begin{aligned}\dim\big[V(\lambda'_1) \otimes \cdots \otimes V(\lambda'_n)\big]^{G'(\mathbb{C})} &= \dim\big[V(\lambda'_1)^* \otimes \cdots \otimes V(\lambda'_n)^*\big]^{G'(\mathbb{C})} \\ &\leq \dim\big[V(\lambda_1)^* \otimes \cdots \otimes V(\lambda_n)^*\big]^{G(\mathbb{C})} = \dim[V(\lambda_1) \otimes \cdots \otimes V(\lambda_n)]^{G(\mathbb{C})}.\end{aligned}$$

□

Definition (7.3). An isogeny $f : G \to G'$ for a simple G is called *special* if $d(\alpha) = 0$ for some $\alpha \in R(G,T)$, where $d(\alpha)$ is as in Definition (7.1); it is *central* if $d(\alpha) = 0$ for all $\alpha \in R(G,T)$. A complete list of special non-central isogenies may be found in [BT, §3.3]. In the following, we list the resulting tensor product inequalities implied by Theorem (7.2).

Let G be the simply-connected group of type B_ℓ (i.e., $G = \mathrm{Spin}_{2\ell+1}$), and G' the simply-connected group of type C_ℓ (i.e., $G' = \mathrm{Sp}_{2\ell}$). Following the notation from the appendices of [Bo], we identify $\Lambda(T) = \{\sum_{i=1}^{\ell} a_i\varepsilon_i : a_i \pm a_j \in \mathbb{Z}\,\forall i, j\}$ and $\Lambda(T') = \bigoplus_{i=1}^{\ell} \mathbb{Z}\varepsilon_i$. This provides a canonical inclusion $\Lambda(T') \hookrightarrow \Lambda(T)$, $\varepsilon_i \mapsto \varepsilon_i$, which takes $\Lambda(T')^+ \hookrightarrow \Lambda(T)^+$. Moreover, under this identification, the image of $\Lambda(T')$ (resp. $\Lambda(T')^+$) is precisely equal to $\Lambda(\bar{T})$ (resp. $\Lambda(\bar{T})^+$), where $\bar{T}$ is the maximal torus in $\mathrm{SO}_{2\ell+1}$.

Theorem (7.2) specializes as follows.

Corollary (7.4). *(a) If $\lambda_1, \ldots, \lambda_n$ are dominant weights for $\mathrm{Sp}_{2\ell}$ ($\ell \geq 2$), then*

$$[\lambda_1, \ldots, \lambda_n]^{\mathrm{Sp}_{2\ell}(\mathbb{C})} \leq [\lambda_1, \ldots, \lambda_n]^{\mathrm{SO}_{2\ell+1}(\mathbb{C})}.$$

(b) If $\lambda_1, \ldots, \lambda_n$ are dominant weights for $\mathrm{Spin}_{2\ell+1}$ ($\ell \geq 2$), then

$$[\lambda_1, \ldots, \lambda_n]^{\mathrm{Spin}_{2\ell+1}(\mathbb{C})} \leq [2\lambda_1, \ldots, 2\lambda_n]^{\mathrm{Sp}_{2\ell}(\mathbb{C})}.$$

(c) If $\lambda_1, \ldots, \lambda_n$ are dominant weights for F_4, then

$$[\lambda_1, \ldots, \lambda_n]^{F_4(\mathbb{C})} \leq [\phi(\lambda_1), \ldots, \phi(\lambda_n)]^{F_4(\mathbb{C})},$$

where $\phi(a\omega_1 + b\omega_2 + c\omega_3 + d\omega_4) := d\omega_1 + c\omega_2 + 2b\omega_3 + 2a\omega_4$ (ω_i being fundamental weights).

(d) If $\lambda_1, \ldots, \lambda_n$ are dominant weights for G_2, then

$$[\lambda_1, \ldots, \lambda_n]^{G_2(\mathbb{C})} \leq [\phi(\lambda_1), \ldots, \phi(\lambda_n)]^{G_2(\mathbb{C})},$$

where $\phi(a\omega_1 + b\omega_2) := 3b\omega_1 + a\omega_2$.

Proof. (a) The identity map is a special isomorphism $\Lambda(T')_\mathbb{R} \to \Lambda(T)_\mathbb{R}$ giving rise to an isogeny $f : \mathrm{SO}_{2\ell+1}(k) \to \mathrm{Sp}_{2\ell}(k)$, where char. $k = 2$.

(b) In this case, the map $\mu \mapsto 2\mu$ defines a special isomorphism $\Lambda(T')_\mathbb{R} \to \Lambda(T)_\mathbb{R}$ inducing an isogeny $f : \mathrm{Sp}_{2\ell}(k) \to \mathrm{Spin}_{2\ell+1}(k)$, where char. $k = 2$.

(c) In this case, the simple roots generate $\Lambda(T)$. Numbering them $\alpha_1, \alpha_2, \alpha_3, \alpha_4$ as in [Bo], we have that α_1 and α_2 are long and α_3 and α_4 are short. Then, there is a special isomorphism $\phi : \Lambda(T)_\mathbb{R} \to \Lambda(T)_\mathbb{R}$ such that

$$\phi(\alpha_1) = 2\alpha_4,\ \phi(\alpha_2) = 2\alpha_3,\ \phi(\alpha_3) = \alpha_2,\ \phi(\alpha_4) = \alpha_1.$$

Let $G = G'$ be of type F_4 and char. $k = 2$ and apply Theorem (7.2).

(d) Letting α_1 and α_2 denote the simple roots, with α_1 short and α_2 long, there is a special isomorphism $\phi : \Lambda(T)_\mathbb{R} \to \Lambda(T)_\mathbb{R}$ such that $\phi(\alpha_1) = \alpha_2$, $\phi(\alpha_2) = 3\alpha_1$. Let $G = G'$ be of type G_2 and char. $k = 3$ and apply Theorem (7.2). $\square$

As an immediate corollary of the (a) and the (b) parts above, we have the following:

Corollary (7.5). *For any $s \geq 1$ and any $\ell \geq 2$, the saturated tensor semigroup $\Gamma_s(\mathrm{Sp}_{2\ell}(\mathbb{C})) = \Gamma_s(\mathrm{SO}_{2\ell+1}(\mathbb{C}))$ under the identification of their $\Lambda(T)^+$ as above.*

Remark (7.6). (a) Any nonspecial isogenies or central isogenies do not yield any new inequalities.

(b) There is another combinatorial proof of Theorem (7.2) based on Littelmann's Path Model for tensor product multiplicity. More specifically, Kumar-Stembridge [KS] use a variant of the Path Model (see [St_2]) in which the objects are chains in the Bruhat ordering of various Weyl group orbits, and the inequality is obtained by comparing chains related by integer renormalizations.

8. Saturation Problem

We continue to follow the notation and assumptions from Secton 2; in particular, G is a semisimple connected complex algebraic group. In Section 6, we defined the tensor product semigroup $\bar{\Gamma}_s(G)$ as well as the saturated tensor product semigroup $\Gamma_s(G)$ (for any integer $s \geq 1$) and determined $\Gamma_s(G)$ by describing its facets. The *saturation problem* aims at connecting these two semigroups.

We begin with the following definition. We take $s = 3$ as this is the most relevant case to the tensor product decomposition.

Definition (8.1). An integer $d \geq 1$ is called a *saturation factor* for G, if for any $(\lambda, \mu, \nu) \in \Gamma_3(G)$ such that $\lambda + \mu + \nu \in Q$, $(d\lambda, d\mu, d\nu) \in \bar{\Gamma}_3(G)$, where Q is the root lattice of G. Of course, if d is a saturation factor then so is its any multiple. If $d = 1$ is a saturation factor for G, we say that the *saturation property holds for G*.

The *saturation theorem* of Knutson-Tao [KT], proved by using their 'honeycomb model' asserts the following. Other proofs of their result are given by Derksen-Weyman [DK], Belkale [B_2] and Kapovich-Millson [KM_2] (cf. Theorem (8.3) below).

Theorem (8.2). *The saturation property holds for $G = \mathrm{SL}(n)$.*

The following general result (though not optimal) on saturation factor is obtained by Kapovich-Millson [KM_2] by using the geometry of geodesics in Euclidean buildings and Littelmann's path model. A weaker form of the following theorem was conjectured by Kumar in a private communication to J. Millson (also see [KT, Conjecture]).

Theorem (8.3). *For any connected simple G, $d = k_{\mathfrak{g}}^2$ is a saturated factor, where $k_{\mathfrak{g}}$ is the least common multiple of the coefficients of the highest root θ of the Lie algebra $\mathfrak{g}$ of G written in terms of the simple roots $\{\alpha_1, \ldots, \alpha_\ell\}$.*

Observe that the value of $k_{\mathfrak{g}}$ is 1 for $\mathfrak{g}$ of type $A_\ell(\ell \geq 1)$; it is 2 for $\mathfrak{g}$ of type $B_\ell(\ell \geq 2), C_\ell(\ell \geq 3), D_\ell(\ell \geq 4)$; and it is 6, 12, 60, 12, 6 for $\mathfrak{g}$ of type E_6, E_7, E_8, F_4, G_2 respectively.

Kapovich-Millson determined $\bar{\Gamma}_3(G)$ explicitly for $G = \mathrm{Sp}(4)$ and G_2 (cf. [KM_1, Theorems 5.3, 6.1]). In particular, from their description, the following theorem follows easily.

Theorem (8.4). *The saturation property does not hold for either $G = \mathrm{Sp}(4)$ or G_2. Moreover, 2 is a saturation factor (and no odd integer d is a saturation factor) for $\mathrm{Sp}(4)$, whereas both of $2, 3$ are saturation factors for G_2 (and hence any integer $d > 1$ is a saturation factor for G_2).*

It was known earlier that the saturation property fails for G of type B_ℓ (cf. [E]).

Kapovich-Millson [KM_1] made the following very interesting conjecture:

Conjecture (8.5). *If G is simply-laced, then the saturation property holds for G.*

Apart from $G = \mathrm{SL}(n)$, the only other simply-connected, simple, simply-laced group G for which the above conjecture is known so far is $G = \mathrm{Spin}(8)$, proved by Kapovich-Kumar-Millson [KKM, Theorem 5.3] by explicit calculation using Theorem (6.3).

Theorem (8.6). *The above conjecture is true for $G = \mathrm{Spin}(8)$.*

Finally, we have the following improvement of Theorem (8.3) for the groups $\mathrm{SO}(2\ell+1)$ and $\mathrm{Sp}(2\ell)$ due to Belkale-Kumar [BK_2, Theorems 25 and 26].

Theorem (8.7). *For the groups $\mathrm{SO}(2\ell+1)$ and $\mathrm{Sp}(2\ell)$, 2 is a saturation factor.*

The proof of the above theorem relies on the following theorem [BK_2, Theorem 23].

Theorem (8.8). *Let $(\lambda^1, \dots, \lambda^s) \in \bar{\Gamma}_s(\mathrm{SL}(2\ell))$. Then, $(\lambda^1_C, \dots, \lambda^s_C) \in \bar{\Gamma}_s(\mathrm{Sp}(2\ell))$, where λ^j_C is the restriction of λ^j to the maximal torus of $\mathrm{Sp}(2\ell)$.*

A similar result is true for $\mathrm{Sp}(2\ell)$ replaced by $\mathrm{SO}(2\ell+1)$.

Belkale-Kumar [BK_2, Conjecture 29] conjectured the following generalization of Theorem (8.8). Let G be a simply-connected, semisimple complex algebraic group and let σ be a diagram automorphism of G with fixed subgroup $G^\sigma = K$.

Conjecture (8.9). *Let $(\lambda^1, \dots, \lambda^s) \in \bar{\Gamma}_s(G)$. Then, $(\lambda^1_K, \dots, \lambda^s_K) \in \bar{\Gamma}_s(K)$, where λ^j_K is the restriction of λ^j to the maximal torus of K.*

(Observe that, for any dominant character λ for G, λ_K is dominant for K with respect to the Borel subgroup $B^K := B^\sigma$ of K.)

We also mention the following 'rigidity' result (conjectured by Fulton) due to Knutson-Tao-Woodward [KTW] proved by combinatorial methods. There are now geometric proofs of the theorem by Belkale [B_3] and Ressayre [R_2].

Theorem (8.10). *Let $G = \mathrm{SL}(n)$ and let $\lambda, \mu, \nu \in \Lambda^+$. If $[V(\lambda)\otimes V(\mu)\otimes V(\nu)]^G$ is one-dimensional then so is $[V(N\lambda)\otimes V(N\mu)\otimes V(N\nu)]^G$, for any $N \geq 1$.*

The direct generalization of this theorem for other groups is, in general, false. But, a certain cohomological reinterpretation of the theorem remains true for any G (cf. a forthcoming paper by Belkale-Kumar-Ressayre).

9. Generalization of Littlewood-Richardson Formula

We recall the classical Littlewood-Richardson formula for $\mathrm{GL}(n)$ (cf., e.g., [Ma, Chap. 1, §9]). Let T be the standard maximal torus of $\mathrm{GL}(n)$ consisting of invertible diagonal matrices. Then, the irreducible polynomial representations of $\mathrm{GL}(n)$ (i.e., those irreducible representations whose matrix coefficients extend as a regular function on the whole of $M(n)$) are parametrized by the partitions $\lambda : (\lambda_1 \geq \cdots \geq \lambda_n \geq 0)\,(\lambda_i \in \mathbb{Z})$, where λ is viewed as an element of Λ^+ via the character: $\mathrm{diag}(t_1, \ldots, t_n) \mapsto t_1^{\lambda_1} \ldots t_n^{\lambda_n}$. Consider the decomposition (1) in Section 1 for the tensor product of irreducible polynomial representations of $\mathrm{GL}(n)$.

Theorem (9.1). *$m^\nu_{\lambda,\mu} \neq 0$ only if both of $\lambda, \mu \subset \nu$. In this case, $m^\nu_{\lambda,\mu}$ equals the number of tableaux T of shape $\nu - \lambda$ and weight μ such that the word $w(T) = (a_1, \ldots, a_N)$ associated to T (reading the symbols in T from right to left in successive rows starting with the top row) is a lattice permutation, i.e., for all $1 \leq i \leq m-1$, and $1 \leq r \leq N$, $\#\{j \leq r : a_j = i\} \geq \#\{j \leq r : a_j = i+1\}$, where the symbols in T lie in $\{1, \ldots, m\}$.*

Littelmann generalized the above thorem for all semisimple Lie algebras $\mathfrak{g}$ by using his LS path models as below. Let G be the simply-connected complex algebraic group with Lie algebra $\mathfrak{g}$.

Definition (9.2). Let Π be the set of all piecewise-linear, continuous paths $\gamma : [0, 1] \to \Lambda_{\mathbb{R}} := \Lambda \otimes_{\mathbb{Z}} \mathbb{R}$ with $\gamma(0) = 0$ and $\gamma(1) \in \Lambda$, modulo the equivalence relation $\gamma \equiv \gamma'$ if γ' is obtained from γ by a piecewise-linear, nondecreasing, continuous reparametrization. For any simple root α_i, there are two operators $e_{\alpha_i}, f_{\alpha_i} : \Pi \sqcup \{0\} \to \Pi \sqcup \{0\}$ defined in [L_2], [L_3]. Let Π^+ be the set of those paths $\gamma \in \Pi$ such that $\mathrm{Im}\ \gamma \subset \Lambda^+_{\mathbb{R}}$. For any $\gamma \in \Pi^+$, let $\mathcal{P}_\gamma$ be the smallest subset of Π containing γ such that $\mathcal{P}_\gamma \sqcup \{0\}$ is stable under the operators $\{e_{\alpha_i}, f_{\alpha_i}; 1 \leq i \leq \ell\}$.

The following theorem due to Littelmann [L_2], [L_3] generalizes Theorem (9.1).

Theorem (9.3). *For any $\lambda, \mu \in \Lambda^+$, take any path $\gamma_\lambda, \gamma_\mu \in \Pi^+$ such that $\gamma_\lambda(1) = \lambda$ and $\gamma_\mu(1) = \mu$. Then,*

$$V(\lambda) \otimes V(\mu) = \bigoplus_{\gamma} V(\lambda + \gamma(1)),$$

where γ runs over all the paths in $\mathcal{P}_{\gamma_\mu}$ such that the cancatenation $\gamma_\lambda * \gamma \in \Pi^+$.

By [L_2, §8] (also see [L_1]), the above theorem indeed generalizes Theorem (9.1).

We now come to the tensor product multiplicity formula due to Berenstein-Zelevinsky [BZ, Theorem 2.3].

Definition (9.4). Let V be a finite-dimensional representation of G and let $\lambda, \mu \in P(V)$ (the set of weights of V), and let $\mathbf{i} = (i_1, \ldots, i_r)$ be a sequence with $1 \leq i_j \leq \ell$. An $\mathbf{i}$-*trail* from λ to μ in V is a sequence of weights $\mathcal{T} = (\lambda_0 = \lambda, \lambda_1, \ldots, \lambda_r = \mu)$ in $P(V)$ such that

(1) for all $1 \leq j \leq r$, we have $\lambda_{j-1} - \lambda_j = c_j(\mathcal{T})\alpha_{i_j}$, for some $c_j = c_j(\mathcal{T}) \in \mathbb{Z}_+$, and

(2) $e_{i_1}^{c_1} \ldots e_{i_r}^{c_r} : V_\mu \to V_\lambda$ is a nonzero map, where e_{i_j} is a nonzero simple root vector as in Section 2 and V_μ is the weight space of V corresponding to the weight μ.

Fix a reduced word for the longest element $w_o = s_{i_1} \ldots s_{i_N}$ and let $\mathbf{i_o} = (i_1, \ldots, i_N)$.

Theorem (9.5). *For $\lambda, \mu, \nu \in \Lambda^+$, the tensor product multiplicity $m_{\lambda,\mu}^\nu$ equals the number of N-tuples $(d_1, \ldots, d_N)$ of nonnegative integers satisfying the following conditions:*

(a) $\sum_{j=1}^N d_j s_{i_1} \ldots s_{i_{j-1}} \alpha_{i_j} = \lambda + \mu - \nu,$

(b) $\sum_j c_j(\mathcal{T}) d_j \geq (s_i\lambda + \mu - \nu)(\omega_i^\vee)$, *for any $1 \leq i \leq \ell$ and any $\mathbf{i_o}$-trail $\mathcal{T}$ from $s_i\omega_i^\vee$ to $w_o\omega_i^\vee$ in $V(\omega_i^\vee)$, and*

(c) $\sum_j c_j(\mathcal{T}) d_j \geq (\lambda + s_i\mu - \nu)(\omega_i^\vee)$, *for any $1 \leq i \leq \ell$ and any $\mathbf{i_o}$-trail $\mathcal{T}$ from $\omega_i^\vee$ to $w_o s_i \omega_i^\vee$ in $V(\omega_i^\vee)$,*

where $V(\omega_i^\vee)$ is the i-th fundamental representation for the Langlands dual Lie algebra $\mathfrak{g}^\vee$.

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